

Geometry of Newton homotopies: bivariate case

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Abstract

A standard question in computational real algebraic geometry is to compute all real solutions to a system of polynomial equations with real coefficients. One classical and promising approach is to track along a connected component of a real curve defined by a Newton homotopy, which is dependent upon the selected start point. As the start point varies, different subsets of real solutions may be obtained. This yields a partition of the space of start points into cells, and it is important to understand the structure of this partition in order to develop efficient algorithms based on Newton homotopies. The structure of the boundary of such cells and the number of cells in the corresponding partition are investigated for bivariate systems. Several examples are included to demonstrate the results.

Keywords: Newton homotopies, real solutions, bivariate systems, homotopy continuation, real numerical algebraic geometry

1 Introduction

Given polynomials $f_1, \dots, f_n \in \mathbb{R}[x_1, \dots, x_n]$, a standard question in computational real algebraic geometry is to find the solution set in $\mathbb{R}^n$ to $f_1 = \dots = f_n = 0$, namely

$$V_{\mathbb{R}}(f_1, \dots, f_n) = \{x \in \mathbb{R}^n \mid f_1(x) = \dots = f_n(x) = 0\}.$$

As usual, we will assume that the corresponding solution set in $\mathbb{C}^n$, namely

$$V_{\mathbb{C}}(f_1, \dots, f_n) = \{x \in \mathbb{C}^n \mid f_1(x) = \dots = f_n(x) = 0\},$$

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is finite, i.e., $V_{\mathbb{C}}(f_1, \dots, f_n)$ is zero dimensional.

One obvious approach to find all the real solutions could be to find all of the complex solutions and then sort through to select the real solutions. The downside is that the number of real solutions may be small compared with the number of complex solutions. For example, counting multiplicity in the univariate case, the fundamental theorem of algebra posits that the number of complex solutions is the degree while Descartes' rule of signs can be used to bound the number of real solutions based on the number of terms.

A classical (e.g., already commonly used in 1968, see [18]) and promising approach for computing only the real solutions is via Newton homotopies. Given a start point $r \in \mathbb{R}^n$ with $f(r) \neq 0$, the corresponding Newton homotopy is

$$H(x, r; t) = f(x) - t \cdot f(r) = 0.$$

Some applications of Newton homotopies include global optimization [7], potential energy landscapes [16, 17, 19, 20], and decomposing positive-dimensional solution sets using monodromy [22]. The simple structure of Newton homotopies is conducive to efficient certified path trackers [11, 13]. Moreover, the geometry of Newton homotopies for bivariate quadratics was studied in [12].

When $n = 1$, the Newton homotopy is very special. For a given $r \in \mathbb{R}$ with $f(r) \neq 0$, the real solution set of the Newton homotopy H corresponds with the graph of f over $\mathbb{R}$, which is connected. Since the solutions of $f = 0$ are precisely where $t = 0$, one can start from r at $t = 1$ and track along the graph of f as a curve in $(x, t) \in \mathbb{R}^2$ to obtain every real solution of $f = 0$.

When $n \geq 2$, the real solution set of the Newton homotopy need not be connected. Let Z_r denote the set of real roots of f on the same connected component as r . This naturally induces a partition on the space of start points into cells, a path connected region where $Z_r = Z_{r'}$ when r and r' are in the same cell. Therefore, in the study of Newton homotopies, it is imperative to understand the structure of such a partition for $n \geq 2$.

One naturally should begin with $n = 2$ in the hope of gaining insights to tackle larger n and eventually arbitrary n . In this paper, we focus on $n = 2$ since this is already nontrivial and interesting. Our main contributions are an investigation of the boundaries of the cells (Theorems 21, 35, and 39) and a bound on the number of cells in the partition (Theorems 46 and 51). Although we tackle fundamental structural questions related to Newton homotopies here, we hope this foundational study will guide and inspire development of efficient algorithms for computing the real solutions.

The rest of the paper is structured as follows. Section 2 precisely states the problems under consideration in this paper while Section 3 summarizes the genericity assumptions used throughout. Section 4 looks at the geometry of the boundary. Section 5 algebraically describes the extended boundary of the cells which arise from singularities. The tangent cone at these singularities are investigated in Sections 6. The number of singularities and the number of cells are considered in Section 7, along with several examples. A short conclusion is provided in Section 8.

2 Problems

In this section, we will identify and precisely state some fundamental problems that we will tackle in this paper. We first recall or introduce various notions and illustrate them with the following running example that will use throughout.

Example 1 (Running Example).

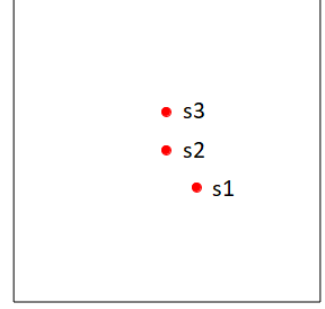
The following system f will be used as a running illustrative example:

$$f = \begin{bmatrix} x_1^3 - x_2^2 - x_1x_2 + 1 \\ x_2^3 - x_1^2 + x_1x_2 + 1 \end{bmatrix}.$$

It is known that $V_{\mathbb{C}}(f)$ consists of 9 points while $V_{\mathbb{R}}(f) = \{s_1, s_2, s_3\}$ where

$$s_1 = (0, -1), \quad s_2 = (-1, 0), \quad s_3 = (-1, 1)$$

as shown in the figure on the right.



Definition 2 (Newton homotopy). The *Newton homotopy* H of $f \in \mathbb{R}[x_1, \dots, x_n]^n$ with $r \in \mathbb{R}^n$ is defined by

$$H(x, r; t) = f(x) - t f(r).$$

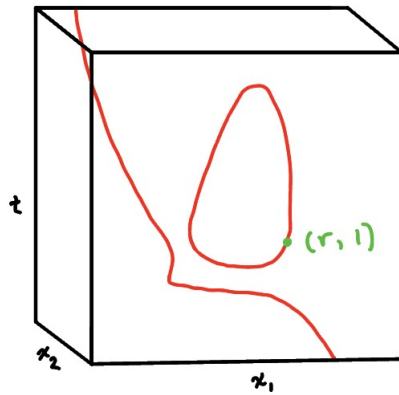
Example 3 (Running Example continued). For the running example (Example 1), the Newton homotopy H of f is

$$H(x, r; t) = \begin{bmatrix} x_1^3 - x_2^2 - x_1x_2 + 1 - t(r_1^3 - r_2^2 - r_1r_2 + 1) \\ x_2^3 - x_1^2 + x_1x_2 + 1 - t(r_2^3 - r_1^2 + r_1r_2 + 1) \end{bmatrix}.$$

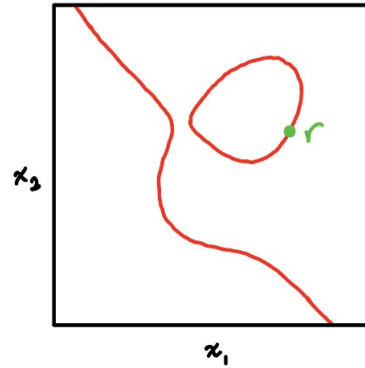
Definition 4 (Newton homotopy curve). For $f \in \mathbb{R}[x_1, \dots, x_n]^n$ and $r \in \mathbb{R}^n \setminus V_{\mathbb{R}}(f)$, the *Newton homotopy curve* C_r of f and r is defined by

$$C_r = \left\{ x \in \mathbb{R}^n : \exists_{t \in \mathbb{R}} (x, t) \in V_r \right\} \quad \text{where} \quad V_r = \left\{ (x, t) \in \mathbb{R}^{n+1} : H(x, r; t) = 0 \right\}.$$

Example 5 (Running Example continued). For the running example (Example 1), consider the Newton homotopy with $r = (1.47, 2)$. The figure on the left depicts $V_r \subset \mathbb{R}^3$. By projecting V_r onto x -space, we obtain $C_r \subset \mathbb{R}^2$ as shown in the figure below.



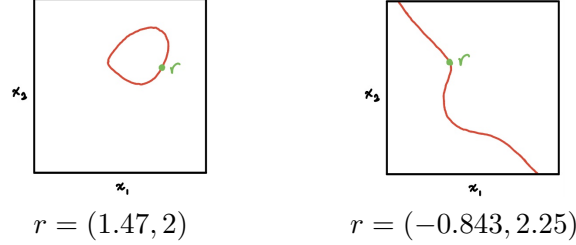
V_r where $r = (1.47, 2)$



C_r where $r = (1.47, 2)$

Definition 6 (Newton homotopy start component). The *Newton homotopy start component* ζ_r of f with r is the maximally connected component of C_r that contains r .

Example 7 (Running Example continued). For the running example (Example 1), the figure on the left depicts ζ_r for $r = (1.47, 2)$ while the figure on the right depicts ζ_r for $r = (-0.843, 2.25)$.



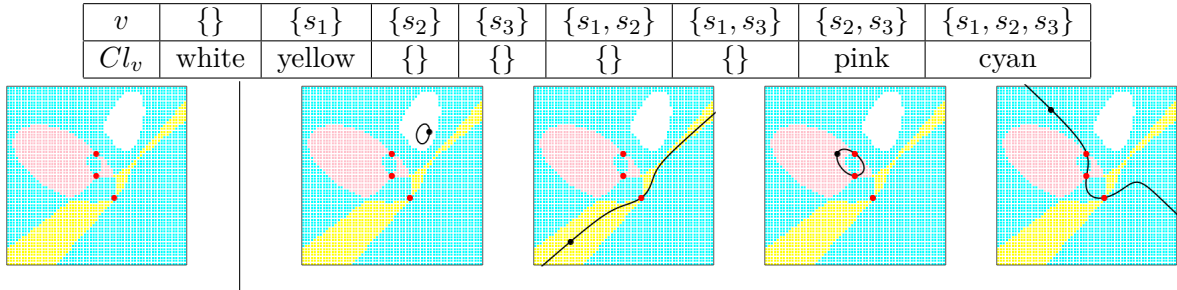
Clearly, $V_{\mathbb{R}}(f) \subset C_r$. As summarized in the Introduction, in the univariate case, $C_r = \zeta_r$ for every $r \in \mathbb{R} \setminus V_{\mathbb{R}}(f)$ so that one obtains every point in $V_{\mathbb{R}}(f)$ by simply tracking along $C_r = \zeta_r$. Since this does not necessarily hold when $n \geq 2$, we note that the set $V_{\mathbb{R}}(f) \cap \zeta_r$ is the set of points in $V_{\mathbb{R}}(f)$ that are obtained by tracking along ζ_r starting from r .

For a subset of $V_{\mathbb{R}}(f)$, the following describes all of the points r for which the start component ζ_r contains exactly that subset of real solutions.

Definition 8 (Class). For $v \subset V_{\mathbb{R}}(f)$, the *class* of v , denoted as Cl_v , is defined as

$$Cl_v = \{r \in \mathbb{R}^n \setminus V_{\mathbb{R}}(f) : V_{\mathbb{R}}(f) \cap \zeta_r = v\}.$$

Example 9 (Running Example continued). The following illustrates the various classes for the running example (Example 1) along with a start component from each of the four nonempty classes which are colored as follows:

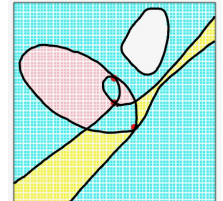


One key object of study is the boundary of the classes, which is defined next.

Definition 10 (Boundary). The *boundary* of the classes, denoted as B , is defined by

$$B = \bigcup_{v \subset V_{\mathbb{R}}(f)} \partial Cl_v \setminus V_{\mathbb{R}}(f)$$

Example 11 (Running Example continued). The boundary B is shown in black for the the running example (Example 1) in the figure on the right.



In Definition 10, the boundary B is characterized using point-set topology. Since the underlying Newton homotopy curves are defined in terms of polynomial equations, we aim to characterize the boundary B using algebraic geometry. For this, as usual, we will characterize $\overline{B}$, the Zariski closure of B .

Definition 12 (Extended Boundary). The *extended boundary*, denoted as $\overline{B}$, is defined as

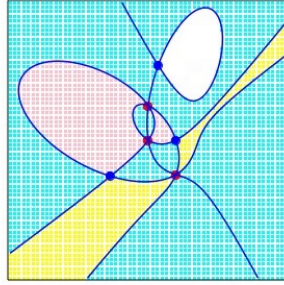
$$\overline{B} = (\text{Zariski Closure of } B) \cap \mathbb{R}^n$$

Equivalently, it can be defined as

$$\overline{B} = \text{Real Zariski Closure of } B$$

where Real Zariski Closure of B stands for the intersection of all real algebraic supersets of B .

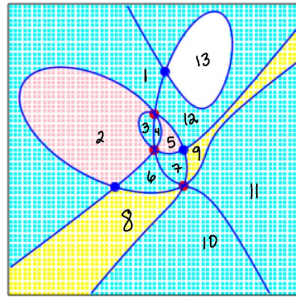
Example 13 (Running Example continued). For the running example (Example 1), the extended boundary is given by the blue curve in the figure below.



The extended boundary $\overline{B}$ is an algebraic object whose complement consists of connected components, which are objects of interest.

Definition 14 (Cells). The *cells* are the maximally connected components of $\mathbb{R}^n \setminus \overline{B}$.

Example 15 (Running Example continued). For the running example (Example 1), the figure below labels the 13 cells obtained by removing the blue curve, the extended boundary, from the real plane.



Since $B \subset \overline{B}$, each connected component of $\mathbb{R}^n \setminus \overline{B}$ is contained in a connected component of $\mathbb{R}^n \setminus B$. Moreover, each maximally connected component of $\mathbb{R}^n \setminus B$ arises as the union of cells along with parts of $\overline{B} \setminus B$. For example, cells 10 and 11 together with their common boundary would form a maximally connected component of $\mathbb{R}^2 \setminus B$. Similar statements hold for the pairs 1 and 12, 3 and 4, and 6 and 7. In particular, $\mathbb{R}^2 \setminus B$ has $13 - 4 = 9$ maximally connected components.

Using the above characterizations, we now state a few problems that we will consider throughout the rest of the paper.

Problem 16. *Find a geometric characterization of the boundary.*

Problem 17. *Find an algebraic characterization of the extended boundary – that is, find polynomials that vanish on the extended boundary.*

Problem 18. *Characterize the local structure of the extended boundary at singularities.*

Problem 19. *Find a “reasonable” upper bound on the number of cells.*

Focus on bivariate systems: The above problems are either trivial or meaningless when $n = 1$. Thus, one naturally should begin with $n = 2$ in the hope of gaining insights to tackle larger n and eventually arbitrary n . This paper will focus on $n = 2$ since it is already non-trivial and interesting. Bivariate systems are well-studied and arise naturally in various contexts such as computer graphics [15] and Gale dual transformations of certain fewnomial systems [2, 4]. Some other approaches for computing real solutions to bivariate systems are Sturm-Habicht sequences [8] and generalized eigenvalue decomposition [23]. Nonetheless, the focus here is foundational in terms of analyzing Newton homotopies for solving bivariate systems summarized in the problems listed above.

3 Generic assumptions

Let $f_1, f_2 \in \mathbb{R}[x_1, x_2]$ such that $d_j = \deg f_j > 0$ where $d_1 \geq d_2$ and $d_1 \geq 2$. Although we consider f_1 and f_2 with real coefficients, we assume that f_1 and f_2 satisfy the following genericity conditions over the complex numbers throughout the rest of the paper.

Assumption 20 (Genericity conditions).

1. $V_{\mathbb{C}}(f_j)$ is a smooth curve of degree d_j .
2. $f_j^* = 0$ has d_j nonsingular solutions in the complex projective space, where f_j^* denotes the homogeneous part of f_j of highest degree (which has degree d_j).
3. $V_{\mathbb{C}}(f_1, f_2)$ is a complete intersection with $d_1 d_2$ nonsingular solutions in $\mathbb{C}^2$.
4. For $r \in \mathbb{C}^2 \setminus V_{\mathbb{C}}(f)$ and $C_r^{\mathbb{C}} = \left\{ x \in \mathbb{C}^n : \exists_{t \in \mathbb{C}} (x, t) \text{ s.t. } H(x, r; t) = 0 \right\}$, the complex Newton homotopy curve:
 - (a) $C_r^{\mathbb{C}}$ is a curve of degree d_1 whenever $f_2(r) \neq 0$;
 - (b) $C_r^{\mathbb{C}}$ is smooth at infinity except possibly when $f_2(r) = 0$ and $d_1 > d_2$;
 - (c) $C_r^{\mathbb{C}}$ has at most one singularity and that singularity (if it exists) is an ordinary double point, in other words, a quadratic singularity.

The above assumptions are *generic* in that almost all $f \in \mathbb{R}[x_1, x_2]^2$ satisfy all them. Items 1, 2, 3, and 4a are well known generic properties. Item 4b follows from Items 2 and 4a. The genericity of Item 4c follows from [21, Thm. 22 & Cor. 27] arising from the well known Lefschetz pencil.

4 Geometric characterization of boundary

The following tackles Problem 16 by determining a geometric characterization of the boundary.

Theorem 21 (Geometric characterization of boundary). *We have*

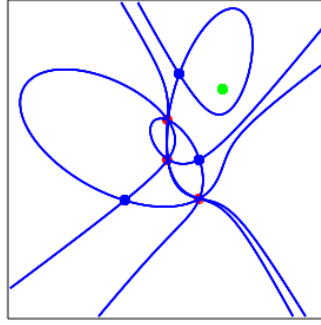
C1: If $d_1 = d_2$, then $B \subset \bigcup_{r: C_r \text{ is singular}} C_r$.

C2: If $d_1 > d_2$, then $B \subset \bigcup_{r: C_r \text{ is singular}} C_r \cup V_{\mathbb{R}}(f_2)$.

Before we show the proof of the above theorem, we will first illustrate what the theorem is saying using two examples.

Example 22.

1. For the running example (Example 1), $d_1 = d_2 = 3$. There are four values of $r \in \mathbb{R}^2$ such that C_r is singular: the three blue points and the green point in the figure below.

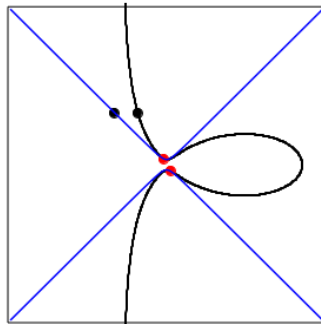


The Newton homotopy curves arising from the three blue points, which are singular points that are non-isolated over the real numbers, contain the boundary B , and actually form $\bar{B}$ as in Example 13. The green point is a singular point that is isolated over the real numbers and does not contribute to B ; rather, it arises as the limit of a loop that contracts to a point.

2. Consider the system

$$f = \begin{bmatrix} x_1^3 + 3x_1x_2^2 + x_2^3 + 3x_2 - 1 \\ x_1^2 - x_2^2 + 1 \end{bmatrix}$$

where $d_1 = 3$ and $d_2 = 2$ with $V_{\mathbb{R}}(f)$ consisting of two red points in the figure below.



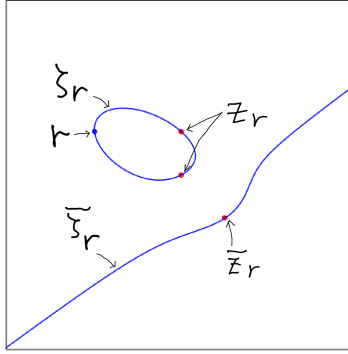
The Newton homotopy curves associated with the two black points are shown in the above image. For the point $r \in V_{\mathbb{R}}(f_2)$, the curve C_r is a blue hyperbola (degree 2) with 2 connected components, each containing one point in $V_{\mathbb{R}}(f)$. However, for a slight perturbation, the Newton homotopy curve is the black connected cubic curve that passes through both points in $V_{\mathbb{R}}(f)$ showing that the boundary B can be contained in $V_{\mathbb{R}}(f_2)$ when $d_1 > d_2$.

Now we will begin to prove Theorem 21. For this, we need a few notations, definitions, and lemmas.

Notation 23.

1. $Z_r = V_{\mathbb{R}}(f) \cap \zeta_r$ “the real solutions of f on the start component of C_r ”
2. $\bar{Z}_r = V_{\mathbb{R}}(f) \setminus \zeta_r$ “the real solutions of f on the other components of C_r ”
3. $\bar{\zeta}_r = C_r \setminus \zeta_r$ “the union of the other components”

Example 24 (Running Example continued). The previous notation is illustrated in the figure below using the running example (Example 1). For the choice of r , C_r has two connected components with ζ_r being the one containing r and the other being $\bar{\zeta}_r = C_r \setminus \zeta_r$. Although $V_{\mathbb{R}}(f) \subset C_r$, $Z_r = V_{\mathbb{R}}(f) \cap \zeta_r$ consists of two points while $\bar{Z}_r$ consists of the other one.



Definition 25 (Distances).

- For $p \in \mathbb{R}^2$, let $\hat{p} = \frac{p}{\|(p,1)\|_2} \in \mathbb{R}^2$. This is essentially the Poincaré compactification.
- For $a, b \in \mathbb{R}^2$, let $d(a, b) = \|\hat{a} - \hat{b}\|_2$. Lemma 26 shows that this is a distance function.
- For $A, B \subset \mathbb{R}^2$, let

1. $d_{\inf}(A, B) = \inf_{(a,b) \in A \times B} d(a, b),$

2. $d_{\sup}(A, B) = \sup \left\{ \sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b) \right\}$ which is the Hausdorff distance between A and B .

Since we are working with nonnegative numbers, we define $\sup \emptyset = 0$ and $\inf \emptyset = \infty$.

Lemma 26. *The function $d(a, b) = \|\hat{a} - \hat{b}\|_2$ is a distance function satisfying the metric axioms.*

Proof. Let $a, b \in \mathbb{R}^2$. First, we trivially have $d(a, a) = \|\hat{a} - \hat{a}\|_2 = \|0\|_2 = 0$. Moreover, if $a \neq b$, it is easy to see $\hat{a} \neq \hat{b}$ so that $d(a, b) > 0$. Symmetry and the triangle inequality follow from the same properties of $\|\cdot\|_2$. $\square$

Since Hausdorff distance $d_{\sup}$ is a metric, $d_{\sup}$ satisfies the triangle inequality. Although $d_{\inf}$ need not satisfy the triangle inequality, the following shows that there is mixed triangle inequality involving $d_{\inf}$ and $d_{\sup}$.

Lemma 27 (Mixed Triangle Inequality). *If $U, V, W \subset \mathbb{R}^2$ are nonempty, then*

$$d_{\inf}(U, W) \leq d_{\sup}(U, V) + d_{\inf}(V, W).$$

Proof. The diagram on the right provides a visual reference. In order to yield the result, we show the following equivalent statement holds, which arises using a standard technique involving perturbations to prove inequalities involving infima and suprema:

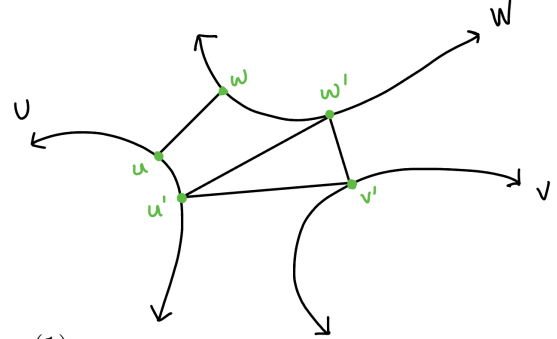
$$\forall_{\delta > 0} (d_{\inf}(U, W) - \delta \leq d_{\sup}(U, V) + d_{\inf}(V, W)).$$

To that end, let $\delta > 0$. Since U, V, W are nonempty, there exists $u, u' \in U$, $v' \in V$, and $w, w' \in W$ such that

$$d_{\inf}(U, W) \leq d(u, w) \leq d_{\inf}(U, W) + \delta/4, \quad (1)$$

$$d_{\inf}(V, W) \leq d(v', w') \leq d_{\inf}(V, W) + \delta/4, \quad (2)$$

$$d_{\inf}(U, v') \leq d(u', v') \leq d_{\inf}(U, v') + \delta/4. \quad (3)$$



Hence, we have

$$\begin{aligned} d_{\inf}(U, W) - \delta &\leq d(u, w) - 3\delta/4 && \text{from (1)} \\ &\leq d(u', w') - \delta/2 && \text{since } d(u, w) - \delta/4 \leq d_{\inf}(U, W) \leq d(u', w') \\ &\leq d(u', v') + d(v', w') - \delta/2 && \text{from the triangle inequality of } d \\ &\leq d_{\inf}(U, v') + d_{\inf}(V, W) && \text{from (2) and (3)} \\ &\leq d_{\sup}(U, V) + d_{\inf}(V, W) && \text{since } d_{\inf}(U, v') \leq d_{\sup}(U, V). \end{aligned}$$

$\square$

Now we state and prove the *key lemma*, from which Theorem 21 will follow immediately.

Lemma 28 (Key Lemma). *Let $p \in \mathbb{R}^2 \setminus V_{\mathbb{R}}(f)$ be such that C_p is smooth at infinity. If $p \in B$, then C_p is singular.*

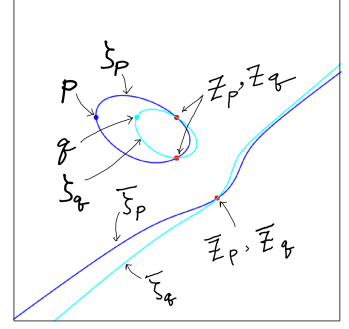
Proof. Let $p \in \mathbb{R}^2 \setminus V_{\mathbb{R}}(f)$ be such that C_p is smooth at infinity. It suffices to show that if $p \in B$, then C_p is singular. We will prove the contrapositive: Assume C_p is nonsingular and show $p \notin B$. If all points q sufficiently close to p lie in the same class, that is, $Z_p = Z_q$, then $p \notin B$. Hence, it suffices to show

$$\exists_{\varepsilon > 0} \forall_{q \neq p} \|p - q\|_2 < \varepsilon \implies Z_p = Z_q. \quad (4)$$

The annotated plot on the right can help aid in the geometric meaning of the following argument.

Recall that C_p is nonsingular and smooth at infinity. Thus, there is a one-to-one correspondence between the connected components of C_p and C_q for all q sufficiently close to p .

In fact, by using the scaled distance metric d in Definition 25, which essentially provides a Poincaré compactification, continuity ensures that the corresponding connected components of C_p and C_q do not vary too much when q is close to p . That is,



$$\forall_{\delta > 0} \exists_{\varepsilon > 0} \forall_{q \neq p} \left(\|p - q\|_2 < \varepsilon \implies d_{\text{sup}}(\zeta_q, \zeta_p) < \delta \wedge d_{\text{sup}}(\bar{\zeta}_q, \bar{\zeta}_p) < \delta \right). \quad (5)$$

Now, the finiteness of Z_p and $\bar{Z}_p$ along with disjointness ensures that

$$d_{\text{inf}}(Z_p, \bar{\zeta}_p) > 0$$

and

$$d_{\text{inf}}(\bar{Z}_p, \zeta_p) > 0.$$

To show that $Z_p = Z_q$ for all q sufficiently close to p , we will consider containment in both directions separately.

- Claim 1: $\exists_{\varepsilon > 0} \forall_{q \neq p} (\|p - q\|_2 < \varepsilon \implies Z_p \supset Z_q)$.

If $\bar{Z}_p = \emptyset$, then $Z_p = V_{\mathbb{R}}(f)$ and the claim is trivial. Thus, assume that $\bar{Z}_p \neq \emptyset$. The condition $Z_q \subset Z_p$ is equivalent to $d_{\text{inf}}(\bar{Z}_p, \zeta_q) > 0$, i.e., the finite points in $\bar{Z}_p = V_{\mathbb{R}}(f) \setminus Z_p$ are disjoint from ζ_q shows that ζ_q could possibly only contain the points in Z_p . Since $p \in \zeta_p$, we have both $\bar{Z}_p$ and ζ_p are nonempty. Let $\delta = d_{\text{inf}}(\bar{Z}_p, \zeta_p) > 0$ and take $\varepsilon > 0$ that satisfies (5), i.e.,

$$\forall_{q \neq p} \left(\|p - q\|_2 < \varepsilon \implies d_{\text{sup}}(\zeta_q, \zeta_p) < d_{\text{inf}}(\bar{Z}_p, \zeta_p) \right). \quad (6)$$

Fix $q \neq p$ with $\|p - q\|_2 < \varepsilon$. Since $p \in \zeta_p$ and $q \in \zeta_q$, Lemma 27 and (6) yield

$$d_{\text{inf}}(\zeta_p, \bar{Z}_p) \leq d_{\text{sup}}(\zeta_p, \zeta_q) + d_{\text{inf}}(\zeta_q, \bar{Z}_p) < d_{\text{inf}}(\bar{Z}_p, \zeta_p) + d_{\text{inf}}(\zeta_q, \bar{Z}_p)$$

so that $d_{\text{inf}}(\zeta_q, \bar{Z}_p) > 0$.

- Claim 2: $\exists_{\varepsilon > 0} \forall_{q \neq p} (\|p - q\|_2 < \varepsilon \implies Z_p \subset Z_q)$.

If $C_p = \zeta_p$, i.e., C_p is connected, then C_q is also connected for q sufficiently close to p by continuity so that $Z_p = Z_q = V_{\mathbb{R}}(f)$. If $\zeta_p \subsetneq C_p$, then the same holds for all q sufficiently close to p . In this case, one can follow essentially the same argument as above to show $\bar{Z}_p \supset \bar{Z}_q$, which is equivalent to $Z_p \subset Z_q$.

The statement (4) follows from Claim 1 and Claim 2 by choosing the ε in (4) to be the minimum of the ε 's in Claims 1 and 2. Thus, we have proved the key lemma (Lemma 28). $\square$

Finally, using the key lemma (Lemma 28), we will prove Theorem 21.

Proof of Theorem 21.

C1: If $d_1 = d_2$, then $B \subset \bigcup_{r: C_r \text{ is singular}} C_r$.

Assume $d_1 = d_2$. Let $p \in B$. It suffices to show that C_p is singular. From Assumption 20(4b) and $d_1 = d_2$, we see that C_p is smooth at infinity. Thus, from the key lemma (Lemma 28), it is immediate that C_p is singular.

C2: If $d_1 > d_2$, then $B \subset \bigcup_{r: C_r \text{ is singular}} C_r \cup V_{\mathbb{R}}(f_2)$.

Assume $d_1 > d_2$. Let $p \in B$ and $p \notin V_{\mathbb{R}}(f_2)$. It suffices to show that C_p is singular. From Assumption 20(4b) and $p \notin V_{\mathbb{R}}(f_2)$, we see that C_p is smooth at infinity. Thus, from the key lemma (Lemma 28), it is immediate that C_p is singular.

□

5 Algebraic characterization of extended boundary

In this section, we will tackle Problem 17 by finding an algebraic characterization of the extended boundary – that is, finding polynomials that vanish on the extended boundary. Building on the geometric characterization described in Theorem 21, we start by considering C_r , a Newton homotopy curve (Definition 4). Then, we determine a condition on r to determine if C_r is singular.

Although C_r is defined via a projection, it is actually algebraic as shown in the following.

Proposition 29. *If $r \in \mathbb{R}^2 \setminus V_{\mathbb{R}}(f)$, then $C_r = V_{\mathbb{R}}(h(x, r))$ where $h(x, r) = \det \begin{bmatrix} f(x) & f(r) \end{bmatrix}$.*

Proof. Fix $r \in \mathbb{R}^2 \setminus V_{\mathbb{R}}(f)$ and consider $h(x, r) = \det \begin{bmatrix} f(x) & f(r) \end{bmatrix}$. In particular, since $f(r) \neq 0$, $h(x, r)$ is not the zero polynomial, so $V_{\mathbb{C}}(h(x, r))$ is indeed a curve in $\mathbb{C}^2$. Thus, we aim to show that C_r is precisely the real points of this complex curve. Hence, it suffices to show

$$C_r = V_{\mathbb{R}}(h(x, r)).$$

Claim: $LHS \supset RHS$. Let $x \in RHS$. Hence, we have $h(x, r) = 0$. Thus, there exists $v \in \mathbb{R}^2$ with $\|v\|_2 = 1$ such that $f(x)v_1 + f(r)v_2 = 0$. If $v_1 = 0$, then $|v_2| = 1$ which implies $f(r) = 0$, which is a contradiction. Hence, $v_1 \neq 0$ so that $t = -v_2/v_1 \in \mathbb{R}$ with $f(x) - tf(r) = 0$. Thus, by definition, $x \in LHS$.

Claim: $LHS \subset RHS$. Let $x \in LHS$. Hence, by definition, there exists $t \in \mathbb{R}$ such that $f(x) - tf(r) = 0$. Hence, $v = \begin{bmatrix} 1 & -t \end{bmatrix}^T$ is a nonzero null vector of $\begin{bmatrix} f(x) & f(r) \end{bmatrix}$ showing that $h(x, r) = \det \begin{bmatrix} f(x) & f(r) \end{bmatrix} = 0$. Thus, $x \in RHS$. □

Next, we consider the *singular* Newton homotopy curves. To do this, we need a consistent way to define a Newton homotopy curve, while the following shows that a Newton homotopy curve can be defined in many equivalent ways.

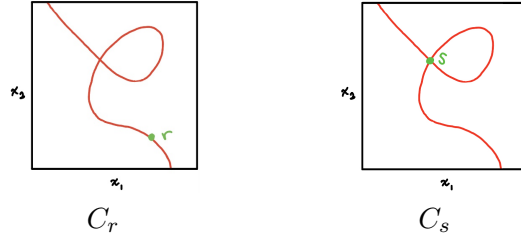
Proposition 30. *If $r \in \mathbb{R}^2 \setminus V_{\mathbb{R}}(f)$ and $s \in C_r \setminus V_{\mathbb{R}}(f)$, then $C_r = C_s$.*

Proof. Let $r \in \mathbb{R}^2 \setminus V_{\mathbb{R}}(f)$ and $s \in C_r \setminus V_{\mathbb{R}}(f)$. Then for some $t_s \in \mathbb{R}$ we have $f(s) = t_s f(r)$. Since $f(s) \neq 0$, we have $t_s \neq 0$. Thus $f(r) = \frac{1}{t_s} f(s)$. Let $x \in \mathbb{R}^2$ be arbitrary but fixed. Note

$$x \in C_r \iff \exists_{t \in \mathbb{R}} f(x) = t f(r) \iff \exists_{t \in \mathbb{R}} f(x) = \frac{t}{t_s} f(s) \iff \exists_{t' \in \mathbb{R}} f(x) = t' f(s) \iff x \in C_s$$

where $t' = \frac{t}{t_s}$. □

This result shows that we are able to describe a Newton homotopy curve by selecting any point on the curve that is not contained in $V_{\mathbb{R}}(f)$. By the genericity assumptions (Assumption 20(4c)), a singular Newton homotopy curve has exactly one singular point and this point is not contained in $V_{\mathbb{R}}(f)$. Hence, we choose to describe each singular Newton homotopy curve by taking the start point to be the singular point as illustrated in the following.



Definition 31 (Set of singular points). Define

$$S = \{r \in \mathbb{R}^n : f(r) \neq 0 \text{ and } r \text{ is a singular point of } C_r\}. \quad (7)$$

Example 32 (Running Example continued). For the running example (Example 1), a look at the first figure from Example 22 gives us an example of this set of singular points S for a bivariate system with $d_1 = d_2 = 3$. Here, the three blue points and one green point make up 4 singular points total with (r_1, r_2) coordinates approximately $(0, 0)$, $(-0.6393, 2.1722)$, $(-2.3650, -1.0466)$, and $(0.7463, 1.7978)$, where this last point is the green center in the figure.

The following lemma provides a *constructible* description of S .

Lemma 33. *The set of singular points is given by $S = V_{\mathbb{R}}(\gamma) \setminus V_{\mathbb{R}}(f)$ where*

$$\gamma = \begin{bmatrix} \det \begin{bmatrix} \partial_{x_1} f & f \end{bmatrix} \\ \det \begin{bmatrix} \partial_{x_2} f & f \end{bmatrix} \end{bmatrix}. \quad (8)$$

Proof. Note

$$\begin{aligned} S &= \{r \in \mathbb{R}^n : f(r) \neq 0 \text{ and } r \text{ is a singular point of } C_r\} && \text{by (7)} \\ &= \{r \in \mathbb{R}^n : r \text{ is a singular point of } C_r\} \setminus V_{\mathbb{R}}(f) && \text{since } f(r) \neq 0 \\ &= \{r \in \mathbb{R}^n : h(r, r) = 0 \text{ and } (\partial_x h)(r, r) = 0\} \setminus V_{\mathbb{R}}(f) && \text{since } C_r = V_{\mathbb{R}}(h(x, r)) \\ &= \{r \in \mathbb{R}^n : (\partial_x h)(r, r) = 0\} \setminus V_{\mathbb{R}}(f) && \text{since } h(r, r) = 0 \text{ for any } r \end{aligned}$$

Note that $\gamma(p)$ is the transpose of the gradient of $h(x, r) = \det \begin{bmatrix} f(x) & f(r) \end{bmatrix}$ with respect to x evaluated at $(x, r) = (p, p)$ and $h(p, p) = 0$, using the convention that the gradient is a row vector. In particular,

$$\begin{aligned} S &= \{r \in \mathbb{R}^n : (\partial_x h^T)(r, r) = 0\} \setminus V_{\mathbb{R}}(f) && \text{since } V_{\mathbb{R}}((\partial_x h^T)(r, r)) = V_{\mathbb{R}}((\partial_x h)(r, r)) \\ &= \{r \in \mathbb{R}^n : \gamma(r) = 0\} \setminus V_{\mathbb{R}}(f) && \text{by the note above} \\ &= V_{\mathbb{R}}(\gamma) \setminus V_{\mathbb{R}}(f) \end{aligned}$$

□

Next, we will give an *algebraic* description of S .

Lemma 34. *Let γ as in (8) and $\partial_x f$ be the Jacobian matrix of f . Then, $S = V_{\mathbb{R}}(\gamma, \det(\partial_x f))$.*

Proof. Recall that, from Lemma 33, we have that $S = V_{\mathbb{R}}(\gamma) \setminus V_{\mathbb{R}}(f)$. Thus, it suffices to show

$$V_{\mathbb{R}}(\gamma) \setminus V_{\mathbb{R}}(f) = V_{\mathbb{R}}(\gamma, \det(\partial_x f)).$$

Claim: $LHS \supset RHS$. Let $p \in RHS$. Then, $\gamma(p) = 0$ and $\det(\partial_x f)(p) = 0$. By the genericity assumptions (Assumption 20(3)), $f(p) \neq 0$. Hence, $p \in LHS$.

Claim: $LHS \subset RHS$. Let $p \in LHS$. Then $\gamma(p) = 0$ and $f(p) \neq 0$. By writing

$$h(x, r) = \det \begin{bmatrix} f(x) & f(r) \end{bmatrix} = f_2(r)f_1(x) - f_1(r)f_2(x),$$

the vanishing of the transpose of the gradient of $h(x, r)$ with respect to x evaluated at (p, p) is equivalent to $\begin{bmatrix} f_2(p) & -f_1(p) \end{bmatrix}$ being a left null vector of $\partial_x f(p)$. Since $LHS \subset \mathbb{R}^2 \setminus V_{\mathbb{R}}(f)$, such a null vector is nonzero. Thus, $\det(\partial_x f)(p) = 0$. Hence, $p \in RHS$. □

Finally we are ready to state the main result of this section.

Theorem 35 (Algebraic characterization of the extended boundary). *For S as in (7), define $\mathcal{H}(x) = \prod_{r \in S} h(x, r)$ and $\mathcal{H}_2 = \mathcal{H} \cdot f_2$. Then, we have the following.*

C1: *If $d_1 = d_2$, then $\overline{B} \subset V_{\mathbb{R}}(\mathcal{H})$.*

C2: *If $d_1 > d_2$, then $\overline{B} \subset V_{\mathbb{R}}(\mathcal{H}_2)$.*

Proof. Immediate from Proposition 29, Lemmas 33 and 34, and Theorem 21. □

Example 36 (Running Example continued). For the running example (Example 1) in which $d_1 = d_2 = 3$, we can define the system γ as follows:

$$\begin{aligned} \gamma &= \begin{bmatrix} \det[\partial_{x_1} f & f] \\ \det[\partial_{x_2} f & f] \end{bmatrix} = \begin{bmatrix} \det \begin{bmatrix} 3x_1^2 - x_2 & x_1^3 - x_1x_2 - x_2^2 + 1 \\ x_2 - 2x_1 & -x_1^2 + x_1x_2 + x_2^3 + 1 \end{bmatrix} \\ \det \begin{bmatrix} -x_1 - 2x_2 & x_1^3 - x_1x_2 - x_2^2 + 1 \\ 3x_2^2 + x_1 & -x_1^2 + x_1x_2 + x_2^3 + 1 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} -x_1^4 + 2x_1^3x_2 + 3x_1^2x_2^2 - x_1^2x_2 + 3x_1^2 - 2x_1x_2^2 + 2x_1 - x_2^4 + x_2^3 - 2x_2 \\ -x_1^4 - 3x_1^3x_2^2 + x_1^3 + 2x_1^2x_2 + 2x_1x_2^3 - x_1x_2^2 - 2x_1 + x_2^4 - 3x_2^2 - 2x_2 \end{bmatrix} \end{aligned}$$

This system has 21 solutions of which 7 are real and 3 of these satisfy f . Thus, we are left with the four singular points as previously shown in Example 32.

6 Local structure of extended boundary at singularity

We now consider Problem 18 for characterizing the local structure of the extended boundary at a singularity.

Lemma 37. *Let $r \in S$. Let T_r be the tangent cone of C_r at r . Then, we have*

1. $T_r = \{v \in \mathbb{C}^2 : v^T(\partial_x \gamma)(r)v = 0\}$
2. *If r is not an isolated point in C_r , then T_r consists of two real lines.*

Proof.

1. As in the proof of Lemma 33, $\gamma(p)$ is the transpose of the gradient of $h(x, r)$ with respect to x evaluated at $(x, r) = (p, p)$, using the convention that the gradient is a row vector. Hence, $\partial_x \gamma$ is the corresponding Hessian matrix of $h(x, r)$ since the Hessian of a polynomial is a symmetric matrix. Thus, this claim follows from the genericity assumptions (Assumption 20(4c)) in that r is an ordinary double point.
2. As r is an ordinary double point, $(\partial_x \gamma)(r)$ is a full rank matrix. Hence, T_r consists of two lines, which either are both real or complex conjugates of each other. When r is not an isolated point in C_r , at least one of the lines in T_r must be real since nonisolated implies the existence of real secant lines arbitrarily close to a line in T_r . Hence, it follows that both of the lines in T_r must be real when r is not an isolated point in C_r .

□

Example 38 (Running Example continued). Let us consider the local structure for the running example (Example 1). From Example 36, we have obtained

$$\gamma = \begin{bmatrix} -x_1^4 + 2x_1^3x_2 + 3x_1^2x_2^2 - x_1^2x_2 + 3x_1^2 - 2x_1x_2^2 + 2x_1 - x_2^4 + x_2^3 - 2x_2 \\ -x_1^4 - 3x_1^3x_2^2 + x_1^3 + 2x_1^2x_2 + 2x_1x_2^3 - x_1x_2^2 - 2x_1 + x_2^4 - 3x_2^2 - 2x_2 \end{bmatrix}$$

We will compute the tangent cone of $C_{(0,0)}$ at $(0,0)$ using Lemma 37. Note

$$\begin{aligned} T_{(0,0)} &= \{v : v^T(\partial_x \gamma)(0,0)v = 0\} && \text{from Lemma 37} \\ &= \left\{ v : v^T \begin{bmatrix} 2 & -2 \\ -2 & -2 \end{bmatrix} v = 0 \right\} && \text{by evaluating } (\partial_x \gamma)(0,0) \\ &= \text{span} \begin{bmatrix} 1 + \sqrt{2} \\ 1 \end{bmatrix} \cup \text{span} \begin{bmatrix} 1 - \sqrt{2} \\ 1 \end{bmatrix} && \text{by solving for } v. \end{aligned}$$

Note that $(0,0)$ is not an isolated point. Thus, as Lemma 37 claims, $T_{(0,0)}$ consists of two real lines.

Theorem 39 (Local structure of extended boundary at a singularity). *If $r \in S$ and γ as in (8), then the tangent cone T_r of C_r at r is equal to the union of the eigenspaces of $(\partial_x P\gamma)(r)$ where*

$$P = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \text{a rotation by } 90^\circ. \quad (9)$$

Proof. Note

$$\begin{aligned} T_r &= \{v \in \mathbb{C}^2 : v^T (\partial_x \gamma)(r) v = 0\} && \text{from Lemma 37} \\ &= \{v \in \mathbb{C}^2 : v^T (P^T P) (\partial_x \gamma)(r) v = 0\} && \text{since } P \text{ is orthonormal} \\ &= \{v \in \mathbb{C}^2 : (Pv)^T (\partial_x P\gamma)(r) v = 0\} && \text{by rearranging} \\ &= \{v \in \mathbb{C}^2 : (Pv)^T M v = 0\} && \text{by using a shorthand } M = (\partial_x P\gamma)(r) \\ &= \{v \in \mathbb{C}^2 : \det \begin{bmatrix} v & Mv \end{bmatrix} = 0\} && \text{from } \forall w (Pv)^T w = \det \begin{bmatrix} v & w \end{bmatrix} \\ &= \{v \in \mathbb{C}^2 : v \text{ and } Mv \text{ are linearly dependent}\} && \text{from a basic property of determinant} \quad (10) \\ &= \{v \in \mathbb{C}^2 : \exists \lambda Mv = \lambda v\} && \text{see below} \quad (11) \\ &= \text{union of eigenspaces of } (\partial_x P\gamma)(r). \end{aligned}$$

We remark on the claimed equality between the two sets (10) and (11). The inclusion $(10) \supset (11)$ is immediate from the definition of linear dependence. The inclusion $(10) \subset (11)$ is subtle but still follows from the special relationship between the two vectors v and Mv . $\square$

Example 40 (Running Example continued). Let us consider computing the tangent cone of $C_{(0,0)}$ at $(0,0)$ using Theorem 39 for the running example (Example 1). Note

$$\begin{aligned} (\partial_x P\gamma)(0,0) &= P(\partial_x \gamma)(0,0) && \text{since } P \text{ is a constant matrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & -2 \end{bmatrix} && \text{from 9 and by evaluating } (\partial_x \gamma)(0,0) \\ &= \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix} \end{aligned}$$

One can directly find that the above matrix has the following eigenspaces:

$$\text{span} \begin{bmatrix} 1 + \sqrt{2} \\ 1 \end{bmatrix} \quad \text{and} \quad \text{span} \begin{bmatrix} 1 - \sqrt{2} \\ 1 \end{bmatrix}.$$

Thus, from Theorem 39, we conclude that the tangent cone $T_{(0,0)}$ is the union of these two eigenspaces. Note that this is identical to the tangent cone computed in Example 38.

7 Upper bound on the number of cells

Finally, we consider Problem 19 regarding bounding the number of connected components of $\mathbb{R}^2 \setminus \overline{B}$. For a given f , one could directly compute the number of cells exactly by applying a known method such as [1, 5, 6, 14]. However, the aim here is to determine an upper bound in terms of the degrees of the polynomials in f . Since $\mathcal{H}$ in Theorem 35 is defined in terms of S , we first find an upper bound on $\#S$ and then derive bounds separately for when the degrees are equal and when the degrees are different.

7.1 Upper bound on number of singular points

Theorem 41 (Upper bound on number of singular points). *We have*

C1: *If $d_1 = d_2$, then $\#S \leq 3(d_1 - 1)^2$.¹*

C2: *If $d_1 > d_2$, then $\#S \leq (d_1 + d_2 - 1)^2 - d_1 d_2$.*

Proof. We begin with a few notations. For a polynomial system F , let $\mathcal{N}_{\mathbb{C}}(F)$ denote the number of solutions of F in $\mathbb{C}^2$ counting multiplicity and $\mathcal{N}_{\infty}(F)$ denote the same at infinity.

From Lemma 33, we have $S = V_{\mathbb{R}}(\gamma) \setminus V_{\mathbb{R}}(f)$. Since $V_{\mathbb{C}}(f) \subset V_{\mathbb{C}}(\gamma)$ and $\#V_{\mathbb{C}}(\gamma) \leq \mathcal{N}_{\mathbb{C}}(\gamma)$, we have

$$\#S = \#V_{\mathbb{R}}(\gamma) - \#V_{\mathbb{R}}(f) \leq \#V_{\mathbb{C}}(\gamma) - \#V_{\mathbb{C}}(f) \leq \mathcal{N}_{\mathbb{C}}(\gamma) - \#V_{\mathbb{C}}(f).$$

From the genericity condition (Assumption 20(3)), we have $\#V_{\mathbb{C}}(f) = d_1 d_2$. Thus,

$$\#S \leq \mathcal{N}_{\mathbb{C}}(\gamma) - d_1 d_2.$$

Since γ consists of two polynomials of degree at most $d_1 + d_2 - 1$, Bézout's bound provides

$$\#S \leq (d_1 + d_2 - 1)^2 - \mathcal{N}_{\infty}(\gamma) - d_1 d_2. \quad (12)$$

It remains to estimate $\mathcal{N}_{\infty}(\gamma)$. To that end, we compute the number of solutions in $\mathbb{P}^1$ of the highest degree homogeneous part of γ , denoted γ^* . Since $*$ is a homomorphism, (8) yields

$$\gamma^* = \left[\begin{array}{cc} \det \begin{bmatrix} \partial_{x_1} f^* & f^* \\ \partial_{x_2} f^* & f^* \end{bmatrix} \\ \det \begin{bmatrix} \partial_{x_1} f^* & f^* \\ \partial_{x_2} f^* & f^* \end{bmatrix} \end{array} \right].$$

Euler's theorem on homogeneous functions provides that

$$x_1 \gamma_1^* + x_2 \gamma_2^* = (d_1 - d_2) f_1^* f_2^*. \quad (13)$$

We consider the two cases.

- Case: $d_1 = d_2$. We claim that $\mathcal{N}_{\infty}(\gamma) = 2(d_1 - 1)$. By (13), $x_1 \gamma_1^* = -x_2 \gamma_2^*$ so that x_1 is a factor of γ_2^* . Hence, $\phi = \gamma_2^*/x_1$ is a polynomial of degree $d_1 + d_2 - 2 = 2(d_1 - 1)$ such that $\gamma_1^* = -x_2 \phi$ and $\gamma_2^* = x_1 \phi$. Therefore, by the Fundamental Theorem of Algebra, $\mathcal{N}_{\infty}(\gamma) = 2(d_1 - 1)$. From the claim and (12), we finally have

$$\#S \leq (d_1 + d_2 - 1)^2 - 2(d_1 - 1) - d_1 d_2 = (2d_1 - 1)^2 - 2(d_1 - 1) - d_1^2 = 3(d_1 - 1)^2.$$

- Case: $d_1 > d_2$. We claim that $\mathcal{N}_{\infty}(\gamma) = 0$. We will prove it by contradiction. Assume that γ has a solution at infinity. By (13), either $f_1^*(x) = 0$ or $f_2^*(x) = 0$, but not both by genericity assumptions (Assumption 20(3)) and any such solution at infinity is nonsingular. Assume that $f_1^*(x) = 0$ so that $f_2^*(x) \neq 0$. Then, $0 = \gamma^*(x) = f_2^*(x) \partial_x f_1^*(x)$ which is a contradiction since $f_2^*(x) \neq 0$ and $\partial_x f_1^*(x) \neq 0$. Repeating a similar argument assuming that $f_2^*(x) = 0$ also yields a contradiction completing the proof. From the claim and (12), we finally have

$$\#S \leq (d_1 + d_2 - 1)^2 - 0 - d_1 d_2 = (d_1 + d_2 - 1)^2 - d_1 d_2.$$

¹Note that the case when $d_1 = d_2$ corresponds with the degree of the set of singular curves in $\mathbb{C}^2$ of degree d_1 , which is a special case of a Boole's formula, e.g., see [9, Chap. 9].

□

In the following, we will give four examples with degrees $(2, 1)$, $(2, 2)$, $(3, 1)$ and $(3, 2)$ respectively where the bounds in Theorem 41 are exact. We leave it as an open problem to check whether for all values of $d_1 \geq d_2$ with $d_1 \geq 2$, there exists an example where the bounds are exact.

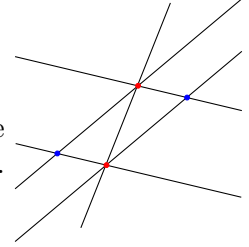
Example 42. Consider the example:

$$\begin{aligned} f_1 &= x_1^2 + 3x_1x_2 - 5x_2^2 + 9x_1 - 2x_2 + 40 \\ f_2 &= 5x_1 - 2x_2 + 1. \end{aligned}$$

Note that $d_1 = 2$ and $d_2 = 1$. From Theorem 41, we have

$$\#S \leq (d_1 + d_2 - 1)^2 - d_1d_2 = 2.$$

Look at the figure on the right. The red dots are the real solutions of f and the blue dots are elements of S . Note that there are *two* blue dots, that is, $\#S = 2$. Thus, for this example, the upper bound for $\#S$ is *exact*.



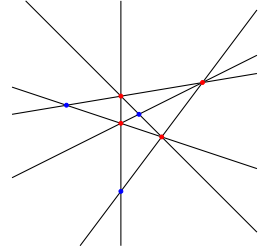
Example 43. Consider the example (which was studied in [12]).

$$\begin{aligned} f_1 &= -11x_1^2 + 8x_1x_2 - 2x_2^2 + 63x_1 - 112 \\ f_2 &= -8x_1^2 + 5x_1x_2 - 8x_2^2 + 54x_1 + 54x_2 - 196. \end{aligned}$$

Note that $d_1 = 2$ and $d_2 = 2$. From Theorem 41, we have

$$\#S \leq 3(d_1 - 1)^2 = 3.$$

In the figure on the right, the red dots are the real solutions of f and the blue dots are elements of S . Note that there are *three* blue dots, that is, $\#S = 3$. Thus, for this example, the upper bound for $\#S$ is *exact*.



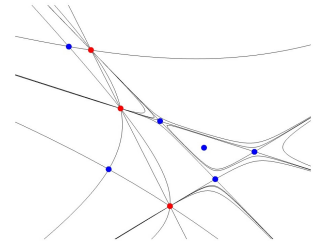
Example 44. Consider the example:

$$\begin{aligned} f_1 &= 37x_1^3 - 39x_1^2x_2 - 27x_1x_2^2 - 6x_2^3 - 4x_1^2 + 95x_1x_2 + 55x_2^2 - 68x_1 + 97x_2 - 64 \\ f_2 &= 18x_1 + 9x_2 - 94. \end{aligned}$$

Note that $d_1 = 3$ and $d_2 = 1$. From Theorem 41, we have

$$\#S \leq (d_1 + d_2 - 1)^2 - d_1d_2 = 6.$$

Looking at the figure on the right, the red dots are the real solutions of f and the blue dots are elements of S . Note that there are *six* blue dots, that is, $\#S = 6$. Thus, for this example, the upper bound for $\#S$ is *exact*.



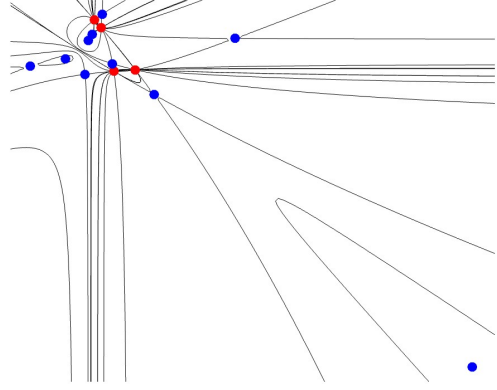
Example 45. Consider the example:

$$\begin{aligned} f_1 &= 30x_1^2x_2 - 50x_1x_2^2 + 36x_1^2 + 49x_1x_2 - 2x_2^2 + 50x_1 - 10x_2 + 37 \\ f_2 &= 23x_1^2 + 37x_1x_2 + 14x_2^2 - 28x_1 - 33x_2 - 8 \end{aligned}$$

Note that $d_1 = 3$ and $d_2 = 2$. From Theorem 41, we have

$$\#S \leq (d_1 + d_2 - 1)^2 - d_1 d_2 = 10.$$

In the figure on the right, the red dots are the real solutions of f and the blue dots are elements of S . Note that there are *ten* blue dots, that is, $\#S = 10$. Thus, for this example, the upper bound for $\#S$ is *exact*.



7.2 Upper bound on number of cells when degrees are equal

Theorem 46 (Upper bound on number of cells when $d_1 = d_2$). *When $d_1 = d_2$, we have*

$$\#\text{cells} \leq \#S(2d_1^2 - 1) - d_1^2 + 1 \leq 6d_1^4 - 12d_1^3 + 2d_1^2 + 6d_1 - 2. \quad (14)$$

Proof. Let $d_1 = d_2$. The upper bound in (14) arises from a bound on the number of routing points [6, 14], which provides an upper bound on the number of cells since each cell must contain at least one routing point [6, Prop. 3.2].

Let

$$\mathcal{H}(x) = \prod_{r \in S} h(x, r) \quad \text{as in Theorem 35} \quad (15)$$

$$\rho = \prod_{r \in S} f_2(r) \quad (16)$$

$$p(x) = \frac{\mathcal{H}(x)}{\rho} = \prod_{r \in S} \left(f_1(x) - \frac{f_1(r)}{f_2(r)} f_2(x) \right). \quad (17)$$

Claim: $p(x)$ is well-defined. It suffices to show that $\rho \neq 0$. Let $r \in V_{\mathbb{R}}(f_2)$ be arbitrary. It suffices to show that $r \notin S$.

- When $r \in V_{\mathbb{R}}(f_1)$: From (7), we have $r \notin S$.
- When $r \notin V_{\mathbb{R}}(f_1)$: From Proposition 29, we immediately have $C_r = V_{\mathbb{R}}(f_2)$. From the genericity assumption (Assumption 20(1)), we have that $V_{\mathbb{C}}(f_2)$ is smooth. Hence C_r is smooth. Thus $r \notin S$.

The claim has been proved.

It is obvious that $\deg p \leq (\#S)d_1$. Due to the genericity assumption (Assumption 20(4a)), we have

$$\deg p = (\#S)d_1. \quad (18)$$

Let

$$e = \lceil (\deg p + 1)/2 \rceil. \quad (19)$$

Note that $2e > \deg p$. For a general $c \in \mathbb{R}^2$, we consider the routing function

$$u(x) = \frac{p(x)}{q(x)^e}$$

where

$$q(x) = 1 + (x_1 - c_1)^2 + (x_2 - c_2)^2. \quad (20)$$

The set of routing points is $V_{\mathbb{R}}(\partial_x u) \setminus V_{\mathbb{R}}(p)$ where

$$\partial_x u = \frac{\partial_x p}{q^e} - e \frac{(\partial_x q)p}{q^{e+1}} = \frac{(\partial_x p)q - e(\partial_x q)p}{q^{e+1}} = 0. \quad (21)$$

Since the numerator of $\partial_x u$ has degree $(\#S)d_1 + 1$, Bézout's bound provides that the number of routing points is at most $((\#S)d_1 + 1)^2$. The following sharpens this bound by identifying several cases that can not yield routing points.

- S : For every $r \in S$, r is a singular point of $h(x, r)$ and thus the numerator of $\partial_x u$ will vanish at r . Hence, we can reduce the bound by $\#S$.
- p, q : Since q is the denominator of $\partial_x u$ and $V_{\mathbb{R}}(q) = \emptyset$, we can ignore any points arising from the vanishing of q . Since the simultaneous vanishing of p and q contributes $2(\#S)d_1$, we can reduce the bound by $2(\#S)d_1$.
- $V_{\mathbb{C}}(f)$: For every $r \in S$, we know that $h(x, r)$ vanishes on $V_{\mathbb{C}}(f)$, which consists of d_1^2 distinct points by genericity assumptions (Assumption 20(3)). Since p arises as a product of $\#S$ bivariate polynomials, each point in $V_{\mathbb{C}}(f)$ has multiplicity $(\#S - 1)^2$ with respect to the numerator of $\partial_x u$. Thus, we can reduce the bound by $d_1^2(\#S - 1)^2$.

Since each of the cases above are disjoint, the number of routing points is at most

$$((\#S)d_1 + 1)^2 - \#S - 2(\#S)d_1 - d_1^2(\#S - 1)^2 = \#S(2d_1^2 - 1) - d_1^2 + 1.$$

The last inequality in (14) follows from $\#S \leq 3(d_1 - 1)^2$ by Theorem 41. $\square$

In the following, we give an example where the bound in Theorem 46 is exact.

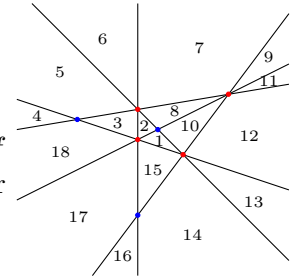
Example 47. Consider the example (which was studied in [12] and also in Example 43):

$$\begin{aligned} f_1 &= -11x_1^2 + 8x_1x_2 - 2x_2^2 + 63x_1 - 112 \\ f_2 &= -8x_1^2 + 5x_1x_2 - 8x_2^2 + 54x_1 + 54x_2 - 196. \end{aligned}$$

Note that $d_1 = 2$ and $d_2 = 2$. From Theorem 46, we have

$$\# \text{cells} \leq 6d_1^4 - 12d_1^3 + 2d_1^2 + 6d_1 - 2 = 18.$$

Consider the figure on the right. The red dots are the real solutions of f and the blue dots are elements of S . Note that there are 18 cells. Thus, for this example, the upper bound for $\#$ cells is *exact*.



7.3 Upper bound on number of cells when degrees are different

Before we state the main result, we first briefly go over the overall strategy. Let p be as in (17); we consider $p_2 = p \cdot f_2$ as one would expect from Theorem 35. The proof of Theorem 46 identified three cases that are not routing points. With appropriate modification, these extend to the case when $d_1 > d_2$. However, there is one additional type of points to remove: solutions at infinity. The reason is that for every r with $f_2(r) \neq 0$, $h(x, r)$ now defines the same d_1 points at infinity. In fact, the terms of degree $d_1, d_1 - 1, \dots, d_1 - d_2 + 1$ are the same, i.e., the top $d_1 - d_2$ degrees. Therefore, multiplying $\#S$ polynomials together with the same $d_1 - d_2$ leading degrees yields a scheme at infinity that can be ignored when bounding the number of cells.

Definition 48. We say that f is *infinity complete* if there exists $c \in \mathbb{R}^2$ such that

$$(\partial_x p_2)q - e(\partial_x q)p_2 = 0$$

has only finitely many solutions and the scheme at infinity has degree $\#S(\#S - 1)d_1(d_1 - d_2)$, where p, q and e are as defined in (17), (20) and (19), respectively.

Remark 49. When $d_1 = d_2$, the systems f under consideration are infinity complete with no corresponding solutions at infinity due to the genericity assumptions (Assumption 20).

Note that infinity completeness is an open condition on the system f . We selected random cases with $1 \leq d_2 < d_1 \leq 3$ with $\#S \leq 4$ and all were verified to be infinity complete using Macaulay2 [10] and Bertini [3]. This suggests the following.

Conjecture 50. *Almost all f 's are infinity complete.*

Now we are ready to state the main result.

Theorem 51 (Upper bound on number of cells when $d_1 > d_2$). *When $d_1 > d_2$, we have*

$$\#\text{cells} \leq (\#S)^2 d_1(d_1 - d_2) + \#S(2d_1 d_2 - 1) + d_2^2 + 1$$

and, if f is infinity complete,

$$\#\text{cells} \leq \#S(d_1^2 + d_1 d_2 - 1) + d_2^2 + 1 \leq (d_1 + d_2)(d_1^3 + d_1^2 d_2 - 2d_1^2 + d_1 d_2^2 - 2d_1 d_2 + 2). \quad (22)$$

Proof. Similar to the proof of Theorem 46, Bézout's theorem yields a bound of $((\#S)d_1 + d_2 + 1)^2$ which is reduced by the same three cases. The first is the same, removing $\#S$ points. The second removes $2((\#S)d_1 + d_2)$ points. The third becomes $d_1 d_2 (\#S)^2$ since f_2 also vanishes at every point in $V_{\mathbb{C}}(f)$. Subtracting these off yields the first inequality. The second inequality arises from subtracting an additional $\#S(\#S - 1)d_1(d_1 - d_2)$ and utilizing the bound on $\#S$ from Theorem 41. $\square$

In the following, we give an example where the bound in Theorem 51 is exact.

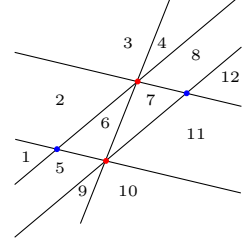
Example 52. Consider the example (which was studied in Example 42):

$$\begin{aligned} f_1 &= x_1^2 + 3x_1 x_2 - 5x_2^2 + 9x_1 - 2x_2 + 40 \\ f_2 &= 5x_1 - 2x_2 + 1. \end{aligned}$$

Note that $d_1 = 2$ and $d_2 = 1$. It turns out that f is infinity complete. From Theorem 51, we have

$$\# \text{cells} \leq (d_1 + d_2)(d_1^3 + d_1^2 d_2 - 2d_1^2 + d_1 d_2^2 - 2d_1 d_2 + 2) = 12.$$

In the figure on the right, the red dots are the real solutions of f and the blue dots are elements of S . Note that there are 12 cells. Thus, for this example, the upper bound for $\# \text{cells}$ is *exact*.



In the following, we will give two examples where the bound in Theorem 51 is not exact over the real numbers, but *is* exact over the complex numbers in terms of counting critical points.

Example 53. Consider the example:

$$\begin{aligned} f_1 &= -19x_1^3 - 37x_1^2 x_2 + 43x_1 x_2^2 + 46x_2^3 + 49x_1^2 - 28x_1 x_2 - 13x_2^2 + 22x_1 - 48x_2 + 10 \\ f_2 &= -15x_1 - 6x_2 - 36 \end{aligned}$$

Note that $d_1 = 3$ and $d_2 = 1$. It turns out that f is infinity complete. From Theorem 51, we have

$$\# \text{cells} \leq (d_1 + d_2)(d_1^3 + d_1^2 d_2 - 2d_1^2 + d_1 d_2^2 - 2d_1 d_2 + 2) = 68.$$

A calculation with **Bertini** [3] confirms that there is exactly 68 critical points over the complex numbers. In particular, of the 68 points, 46 are real (and thus routing points) and the other 22 arise in complex conjugate pairs.

Example 54. Consider the example:

$$\begin{aligned} f_1 &= 30x_1^2 x_2 - 50x_1 x_2^2 + 36x_1^2 + 49x_1 x_2 - 2x_2^2 + 50x_1 - 10x_2 + 37 \\ f_2 &= 23x_1^2 + 37x_1 x_2 + 14x_2^2 - 28x_1 - 33x_2 - 8 \end{aligned}$$

Note that $d_1 = 3$ and $d_2 = 2$. It turns out that f is infinity complete. From Theorem 51, we have

$$\# \text{cells} \leq (d_1 + d_2)(d_1^3 + d_1^2 d_2 - 2d_1^2 + d_1 d_2^2 - 2d_1 d_2 + 2) = 145.$$

A calculation with **Bertini** [3] confirms that there is exactly 145 critical points over the complex numbers. In particular, of the 145 points, 99 are real (and thus routing points) and the other 46 arise in complex conjugate pairs.

8 Conclusion

Newton homotopies are an approach to compute real solutions to real polynomial systems. The set of real solutions obtained by tracking from a start point depends upon which cell the start point is selected from, with the boundaries of the cells being of particular interest for analyzing the behavior of Newton homotopies. By focusing on the bivariate case, we found a geometric description of the boundary, an algebraic characterization of the Zariski closure of the boundary, and characterized the local structure of the boundary at singularities by looking at the tangent cone. From a bound on the number of singular Newton homotopy curves, we derived an upper bound on the number of cells in terms of the degrees of the polynomials and demonstrated several examples where the bounds are exact. Future work involves extending to an arbitrary number of variables.

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