1. Partitions of rectangles

A *rectangle* $I \subset \mathbf{R}^n$ is a product $I = [a_1, b_1] \times \cdots \times [a_n, b_n]$ of compact intervals $[a_j, b_j] \subset \mathbf{R}$. We refer to $I_j = [a_j, b_j]$ as the *jth component* of I. The *volume* of I is the non-negative number $\operatorname{Vol} I := \Pi_{1 \le j \le n}(b_j - a_j)$. We call I *non-degenerate* if $\operatorname{Vol} I \neq 0$.

We say that two rectangles I, I' *overlap* if their interiors intersect. We say that a finite collection $\mathcal{C}$ of rectangles *fills* I if $I = \bigcup_{K \in \mathcal{C}} K$. A *partition* of I is a finite collection $\mathcal{P}$ of mutually non-overlapping rectangles that fills I. We call $\mathcal{P}$ *non-degenerate* if all its elements are non-degenerate. Of course, if a partition $\mathcal{P}$ of I is non-degenerate, then so is I. A non-degenerate partition $\mathcal{P}$ of an interval $[a, b] \subset \mathbf{R}$ is equivalent to the finite increasing sequence

$$a = x_0 < x_1 < \cdots < x_m = b$$

of endpoints of successive intervals $[x_{j-1}, x_j] \in \mathcal{P}$.

We will need the following reality check on our definition of volume of a rectangle.

Theorem 1.1 (Consistency)**.** *If $I \subset \mathbf{R}^n$ is a rectangle and $\mathcal{P}$ is a partition of I, then $\operatorname{Vol} I = \sum_{J \in \mathcal{P}} \operatorname{Vol} J$.*

Though this might seem intuitively obvious, it is nevertheless a bit awkward to justify logically. In order to prove it, we introduce an especially simple sort of partition and show that any partition can be replaced by one of this type. Specifically, suppose we have separate partitions $\mathcal{P}_j$ of each component $[a_j, b_j]$ of I. Then the associated *product partition* $\mathcal{P}$ of I is the one consisting of all rectangles $J = J_1 \times \cdots \times J_n$ where $J_j \in \mathcal{P}_j$.

Definition 1.2. *If $\mathcal{C}$ is another collection of rectangles that fills I, then we say that the partition $\mathcal{P}$ of I refines $\mathcal{C}$ if for each $J \in \mathcal{P}$ and $K \in \mathcal{C}$, either $J \subset K$ or J and K do not overlap.*

If $\mathcal{P}$ is a partition refining $\mathcal{C}$, then for each rectangle $K \in \mathcal{C}$, the subcollection $\mathcal{P}_K := \{J \in \mathcal{P} : J \subset K\}$ is a partition of K. If, moreover, $\mathcal{P}$ is a product partition, then so is $\mathcal{P}_K$.

Lemma 1.3 (Refinement)**.** *Let $I \subset \mathbf{R}^n$ be a non-degenerate rectangle and $\mathcal{C}$ be a collection of rectangles with $\bigcup_{J \in \mathcal{C}} = I$. Then there is a non-degenerate product partition $\mathcal{P}$ of I that refines $\mathcal{C}$.*

As will be fairly clear from the proof, the partition $\mathcal{P}$ we construct is the 'coarsest' possible in the sense that any other product partition $\tilde{\mathcal{P}}$ refining $\mathcal{C}$ must also refine $\mathcal{P}$.

Proof. Suppose first that we are in dimension $n = 1$. Then we can take $\mathcal{P}$ to be the partition associated to the ordered sequence $x_0 < \cdots < x_m$ of distinct endpoints x_j of intervals in $\mathcal{C}$. To see that $\mathcal{P}$ refines $\mathcal{C}$, note that for each interval $J = [x_{j-1}, x_j] \in \mathcal{P}$, there is an interval $K = [x_k, x_\ell] \in \mathcal{C}$ containing the midpoint of J. Necessarily then, we have $k \le j - 1$ and $\ell \ge j$, i.e. $J \subset K$.

Now suppose $n > 1$. For each $1 \le j \le n$ we let $I_j \subset \mathbf{R}$ denote the jth component of I and set

$$\mathcal{C}_j := \{K_j \subset I_j : \text{ there exists } K \in \mathcal{C} \text{ with } j\text{th component } K_j\}.$$

Then $I_j = \bigcup_{K_j \in \mathcal{C}_j} K_j$, so the previous paragraph tells us that there is a partition $\mathcal{P}_j$ of I_j that refines $\mathcal{C}_j$. The proof is concluded by observing that the product partition $\mathcal{P}$ of I associated to $\mathcal{P}_1, \ldots, \mathcal{P}_n$ refines $\mathcal{C}$. $\square$

Now we can justify the Consistency Theorem above in a relatively painless fashion.

Proof. Let us first prove the theorem assuming that the given partition $\mathcal{P}$ is the non-degenerate product partition of I associated to partitions $\mathcal{P}_j$ of the components $[a_j, b_j]$ of I. For each j, let $a_j = x_{j,0} < \cdots < x_{j,m_j} = b_j$ be the sequence of endpoints of intervals in $\mathcal{P}_j$. Then

$$\begin{aligned}\sum_{J\in\mathcal{P}} \mathrm{Vol}\, J &= \sum_{i_n=1}^{m_n} \cdots \sum_{i_1=0}^{m_1} (x_{1,i_1} - x_{1,i_1-1}) \dots (x_{n,i_n} - x_{n,i_n-1}) \\ &= (b_1 - a_1) \sum_{i_n=1}^{m_n} \cdots \sum_{i_2=0}^{m_2} (x_{2,i_2} - x_{2,i_2-1}) \dots (x_{n,i_n} - x_{n,i_n-1}) \\ &= \cdots = (b_1 - a_1) \dots (b_n - a_n) = \mathrm{Vol}\, I,\end{aligned}$$

where all but the first equality follow from the fact that the inner sum telescopes.

For a more general partition $\mathcal{P}$, we can choose a non-degenerate product partition $\tilde{\mathcal{P}}$ that refines $\mathcal{P}$. For each $J \in \mathcal{P}$ we let (as above) $\tilde{\mathcal{P}}_J$ be the product partition of J consisting of rectangles $\tilde{J} \in \tilde{\mathcal{P}}$ contained in J. Then

$$\sum_{J\in\mathcal{P}} \mathrm{Vol}\, J = \sum_{J\in\mathcal{P}} \sum_{\tilde{J}\in\tilde{\mathcal{P}}_J} \mathrm{Vol}\, \tilde{J} = \sum_{\tilde{J}\in\tilde{\mathcal{P}}} \mathrm{Vol}\, \tilde{J} = \mathrm{Vol}\, I,$$

where we have applied the first part of the proof in both the first and last equalities. □

Another important implication of the Refinement Lemma is the following.

Corollary 1.4 (Common Refinement)**.** *If $\mathcal{P}, \mathcal{P}'$ are two partitions of the same rectangle $I \subset \mathbf{R}^n$, then there is a non-degenerate product partition refining both $\mathcal{P}$ and $\mathcal{P}'$.*

Proof. Apply the Refinement Lemma to $\mathcal{C} = \mathcal{P} \cup \mathcal{P}'$. □

2. Integrals of step functions

To any subset $S \subset \mathbf{R}^n$, one associates the *indicator function* $\mathbf{1}_S : \mathbf{R}^n \to \mathbf{R}$ given by $\mathbf{1}(x) = 1$ if $x \in S$ and $\mathbf{1}(x) = 0$ otherwise.

Definition 2.1. *A* step function $f : I \to \mathbf{R}$ *on a rectangle $I \subset \mathbf{R}^n$ is a linear combination $\sum_{J\in\mathcal{P}} c_J \mathbf{1}_J$, where $\mathbf{P}$ is a partition of I. The* integral *of f is*

$$\int f = \sum_{J\in\mathcal{P}} c_J \mathrm{Vol}\, J.$$

We will more or less disregard the values of the step function on the boundaries of the rectangles $J \in \mathcal{P}$. For instance, if $g : I \to \mathbf{R}$ is another function, we will say that $f \le g$ if $f(x) \le g(x)$ for all $x \notin \bigcup_{J\in\mathcal{P}} \partial J$. As with volumes, one needs to perform a consistency check for the definition of integral for a step function, showing that it is independent of the decomposition of f into indicator functions.

Theorem 2.2 (Consistency for Integrals)**.** *Suppose that $f = \sum_{J\in\mathcal{P}} c_J \mathbf{1}_J = \sum_{\tilde{J}\in\tilde{\mathcal{P}}} c_{\tilde{J}} \mathbf{1}_{\tilde{J}}$ are two different ways of defining the same step function. Then*

$$\sum_{J\in\mathcal{P}} c_J \mathrm{Vol}\, J = \sum_{\tilde{J}\in\tilde{\mathcal{P}}} c_{\tilde{J}} \mathrm{Vol}\, \tilde{J}.$$

Proof. Applying the Common Refinement Corollary above to the partitions $\mathcal{P}$ and $\tilde{\mathcal{P}}$ and replacing $\tilde{\mathcal{P}}$ with the refinement given by the corollary, we reduce to the case where $\tilde{\mathcal{P}}$ refines $\mathcal{P}$. Then for each $\tilde{J} \in \tilde{\mathcal{P}}$ we have $c_{\tilde{J}} = c_J$ where J is the unique rectangle in $\mathcal{P}$ that contains $\tilde{J}$. The integral of f is therefore given by

$$\sum_{J \in \mathcal{P}} c_J \mathrm{Vol}\, J = \sum_{J \in \mathcal{C}} \sum_{\tilde{J} \in \mathcal{P}_J} c_J \mathrm{Vol}\, \tilde{J} = \sum_{\tilde{J} \in \tilde{\mathcal{P}}} c_{\tilde{J}} \mathrm{Vol}\, \tilde{J},$$

where the first equality comes from the Refinement Theorem and the second from the facts that $c_{\tilde{J}} = c_J$ and that each $\tilde{J} \in \tilde{\mathcal{P}}$ is contained in exactly one $J \in \mathcal{P}$. $\square$

The Common Refinement Corollary is also useful for establishing many other basic properties of step functions.

Proposition 2.3. *Suppose $f, g : I \to \mathbf{R}$ are step functions and $c \in \mathbf{R}$ is a constant. Then cf, $f + g$, fg, $|f|$, $\max\{f, g\}$, $\min\{f, g\}$ are all step functions. If $f(x) \neq 0$ for all $x \in I$, then the reciprocal $1/f$ is a step function. Moreover,*

- *$f \leq g$ implies $\int f \leq \int g$;*
- *$\int cf = c \int f$;*
- *$\int f + g = \int f + \int g$;*
- *$\int |f| \geq \left|\int f\right|$.*

Proof. Good Exercise. $\square$