

Kazhdan-Lusztig-Stanley polynomials and quadratic algebras I

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INTRODUCTION

In [KL], Kazhdan and Lusztig associate a polynomial to each Bruhat interval in a Coxeter group; some of these “Kazhdan-Lusztig” polynomials now play an important role in several areas of representation theory. A similar family of polynomials has been associated to face lattices of convex polytopes (see Stanley [St]). Both families of polynomials remain poorly understood in general, though they admit remarkable representation-theoretic or intersection cohomological interpretations in the special cases of crystallographic Coxeter groups and rational polytopes.

In [Dy3], it was shown that these two families of polynomials may be given a common definition involving natural labellings, by elements of a vector space, of the associated Bruhat interval or face lattice. This paper is one of a series presenting some rudimentary results arising from an attempt to provide Kazhdan-Lusztig-Stanley polynomials with interpretations in the representation theory of finite-dimensional quadratic algebras which can conjecturally be constructed from these labelled posets. The categories of finite-dimensional modules for such algebras are expected to be analogous to categories of infinite-dimensional representations of a semisimple complex Lie algebra (regular blocks of category $\mathcal{O}$). It is known (c.f. [BGS]) that each block of $\mathcal{O}$ is equivalent to the category of finite-dimensional modules for some basic quasi-hereditary Koszul algebra $\mathcal{A}$ (with an anti-involution) such that the quadratic dual algebra $\mathcal{A}^!$ is also quasi-hereditary; we call any algebra $\mathcal{A}$ (over a field) with these properties an $\mathcal{O}$ algebra. The algebras conjecturally associated to labelled Bruhat intervals or face lattices should be $\mathcal{O}$ algebras. Each should also satisfy an analogue of Beilinson-Ginsburg duality in the sense that its quadratic dual algebra should be the algebra associated to a “dual” labelled poset. In the case of a Coxeter group, it should also be possible to construct “singular” $\mathcal{O}$ algebras analogous to the algebras of singular blocks of $\mathcal{O}$, with simple modules indexed by intervals of distinguished coset representatives for a parabolic subgroup. One would also have “parabolic” algebras (analogous to the algebras of parabolic category $\mathcal{O}$, see [BGS]) arising as the quadratic duals of the singular $\mathcal{O}$ algebras.

The idea of this work is to attempt to define such algebras so that coefficients of Kazhdan-Lusztig-Stanley polynomials associated to an interval would have interpretations as multiplicities of irreducible modules in (graded) Verma modules for these algebras (for finite Weyl groups, for instance, the strong form of the Kazhdan-Lusztig conjecture implies that the Kazhdan-Lusztig polynomials have a similar interpretation involving the Verma modules for semisimple complex Lie algebras; the proof of this uses D -modules and intersection cohomology, see [L, 4]). Assuming that algebras with the above mentioned properties may be constructed in general, the problem of showing that the multiplicities are as expected would remain subtle since in general there are no obviously associated geometric objects to which intersection cohomological methods could be applied. In the Coxeter group case, methods based on Hecke algebra actions may be applicable (see 3.25(c)), but such methods will not be available for face lattices. It is expected that there should in fact be a uniform explanation of why multiplicities are given by the corresponding Kazhdan-Lusztig-Stanley polynomial, at the same time giving a possible approach to conjectures [Dy1, 4.19] and [St, 5.5(a)] on the positivity of members of certain families of polynomials generalizing the Kazhdan-Lusztig-Stanley polynomials. Namely, there should exist families of modules “interpolating” between a Verma module and its dual, these modules being related by various exact sequences and dualities parametrized by certain combinatorial data. The exact sequences are graded analogues of the Duflo-Zelevenko four-term exact sequence for Harish-Chandra modules [J, 4.7], [I3], and the data is expected to be a refinement of the combinatorial notion of a recursive atom order [BW, 3.1].

The conjectures described above are based on a study of examples obtained by a complex procedure which will be described in a subsequent paper (this recursive “construction” determines generators and relations for an algebra associated to a labelled poset from those associated to its (proper) labelled subposets, under some stringent global hypotheses, which have so far only been proved in very special cases). This paper describes the base step of this recursive procedure and studies the algebras which can be obtained using this part of the construction only. These have as weight posets the intervals for which the Kazhdan-Lusztig-Stanley polynomial associated to every (non-empty) closed subinterval is equal to one, and Verma modules for the associated $\mathcal{O}$ algebra are therefore multiplicity free. In this paper, we verify all the above conjectures in this very special case.

We mention now some questions that can be asked concerning these algebras in any case where their existence has been established e.g. for the algebras constructed in this paper. Firstly, the results in this paper associate a “dihedral” $\mathcal{O}$ algebra to any finite dihedral group, and it can be shown that the family of (regular and singular) algebras for dihedral groups of orders 4, 6, 8 and 12 is just the set of algebras of blocks of category $\mathcal{O}$ for the rank 2 semisimple complex Lie algebras. One can ask more generally if, in the case of crystallographic Coxeter groups, the algebras associated here to certain twisted Bruhat intervals are similarly related to certain special quasi-hereditary algebras associated to $\mathcal{O}$ for Kac-Moody Lie algebras (see 3.34(a)). Also, multiplicities of irreducibles as composition factors of Verma modules for the parabolic algebras defined here are given by inverting a matrix of Kazhdan-Lusztig polynomials (with entries indexed by shortest coset representatives of a parabolic subgroup); this result is reminiscent of conjectures of Lusztig [L, 9] on multiplicities of irreducibles in Weyl modules for algebraic groups and quantum groups. The parabolic algebras (at least in the multiplicity-free case)

exist both in characteristic 0 and in sufficiently large prime characteristic, and one may ask if, in the case of affine Weyl groups, they are related to highest-weight categories associated to algebraic groups or quantum groups. Another natural question, in view of [St, 3] is whether algebras associated in this paper to simplicial cones are related, for rational simplicial cones, to categories of perverse sheaves on associated toric varieties.

The connection between the “dihedral” algebras and algebras of blocks of category $\mathcal{O}$ in rank 2 could in principle be established by direct computation, but we will give an argument in a future paper which should apply in outline to other situations as well. One first defines certain functors between graded module categories for the various regular and singular algebras associated to an interval; these are analogous to translation functors for category $\mathcal{O}$ for complex semisimple Lie algebras. Using these, one can develop a theory of “graded projective functors”; the results, though not the proofs, are closely analogous to those in the classical theory of projective functors [BeGe]. Using this, one can give an alternative characterization of the dihedral algebras (in terms of the coinvariant algebra of the dihedral group in its natural reflection representation) which coincides with Soergel’s description [So2] of the algebras of blocks of $\mathcal{O}$ for semisimple complex Lie algebras.

The connection with the coinvariant algebra is only expected for algebras associated to (untwisted) Bruhat order (and for infinite Coxeter groups, one should use instead a certain algebra defined by Kostant and Kumar [KoKu], which for crystallographic Coxeter groups is the cohomology ring of the flag variety of an associated Kac-Moody group). However, the projective functor machinery is expected to exist quite generally for algebras associated to twisted Bruhat orders. Crudely, in cases so far studied, composition of projective functors corresponds to multiplication in the Hecke algebra, and action of projective functors on projective modules corresponds to Hecke algebra action on a module associated in [Dy1] to the twisted Bruhat order. It is hoped that these ideas will eventually provide an approach to a conjecture [Dy1, 4.19] generalizing the conjectures that when one expresses a product $C'_x C'_y$ (resp., $C'_x C_y$), in H as a linear combination of the Kazhdan-Lusztig basis elements C'_z (resp., C_z), the coefficients are integral Laurent polynomials with non-negative coefficients.

During the course of this work, I have benefitted from conversations with many people, and I particularly wish to thank Jens Jantzen, Ron Irving and George Lusztig for helpful ideas and references. Some of the results here were established while I was visiting the University of Sydney, and I wish to thank the Department there for its hospitality.

The broad arrangement of this paper is as follows. Section 1 outlines the main conjecture motivating this work, on the existence of $\mathcal{O}$ algebras associated to certain labelled posets. Section 2 describes in some detail the main properties of $\mathcal{O}$ algebras. Section 3 checks the conjecture in the simplest case of intervals isomorphic to a product of Bruhat intervals in dihedral groups, then constructs the singular and parabolic algebras for these cases and concludes with a question on the relationship of these algebras (for crystallographic Coxeter groups) to Kac-Moody Lie algebras.

We fix some notation and terminology used throughout. For any subset A of a F -vector space V , we will write $F\langle A \rangle$ to denote the F -linear span of A , and, if $F = \mathbb{R}$, $\langle A \rangle_+$ for $\{ \sum_{\alpha \in A} \lambda_\alpha \alpha \mid \text{all } \lambda_\alpha \geq 0 \}$ (we set $F\langle \emptyset \rangle = \langle \emptyset \rangle_+ = 0$). The dual space of V will be denoted $V^\#$. By a module for a ring, we will mean a finitely

generated left module unless otherwise stated; we will denote the isomorphism class of a module F by $\{F\}$. The cardinality of a finite set X will be denoted $\sharp(X)$ or $\sharp X$. We denote the transpose of a matrix B by B^T .

For a poset X , we write $x \triangleleft y$ to indicate that $x < y$ and no $z \in X$ satisfies $x < z < y$ (we say that y covers x). For x, y in X , we have a closed interval $[x, y] = \{z \in X \mid x \leq z \leq y\}$. The opposite poset of X will be denoted X^{op} . An ideal of X is a subset Y of X such that $y \in Y$, $x \in X$ and $x \leq y$ imply $x \in Y$; coideals are defined similarly with $\geq$ in place of $\leq$. We will say that a subset Y of X is closed if it is the intersection of an ideal with a coideal. For a subset Y of X , we denote the least upper bound (resp, greatest lower bound) of Y by $\text{lub}(Y)$ (resp., $\text{glb}(Y)$) when it exists; we will be dealing with posets which aren't necessarily lattices, and use of this notation carries the implication that the lub or glb actually exists. For a finite poset X , the order complex of X , which is the simplicial complex which has as simplexes the totally ordered subsets of X , will be denoted $\Delta(X)$. If X is a finite graded poset and $x \leq y$ in X , we denote the common length n of all maximal chains $x = x_0 \triangleleft x_1 \triangleleft \dots \triangleleft x_n = y$ in a subinterval $[x, y]$ by $l(x, y) = n$. Finally, $H^j(\mathcal{K})$ denote the cohomology groups of a cochain complex $\mathcal{K}$.

A CONJECTURE

1.1 Let $(X, \leq)$ denote either a Bruhat interval or reverse Bruhat interval in a finitely generated Coxeter system (W, S) , or the face lattice of a polyhedral cone. We recall the definition from [Dy3] of a labelling function ℓ from the edge set $E_0 := \{(x, y) \in X \times X \mid x \triangleleft y\}$ of the Hasse diagram of X to a real vector space V .

1.2 First, consider the case when X is the face lattice of a polyhedral cone C in a real Euclidean space V . For two faces $A < B$ of C with $\dim B = \dim A + 1$, there is a unique unit vector α in the subspace spanned by B such that α is orthogonal to A and has non-negative inner product with every element of B . Set $\ell(A, B) = \alpha$.

1.3 Now suppose that X is a Bruhat interval in W , where W is a Coxeter group in its natural reflection representation on a real vector space V . For any pair of elements x, y of X such that $x < y$ and yx^{-1} is a reflection, define $\ell(x, y)$ to be the unique positive root α such that yx^{-1} is the reflection in α . Note that in this case, the labelling function ℓ is defined on a set E containing E_0 .

1.4 Let $\mathcal{R} = \mathbb{Z}[v, v^{-1}]$ denote the ring of integral Laurent polynomials in an indeterminate v and $\mathcal{R}X$ denote the free $\mathcal{R}$ -module on basis X . The ring involution of $\mathcal{R}$ with $v \mapsto v^{-1}$ induces a $\mathcal{R}$ -antilinear map $f \mapsto \bar{f}$ on $\mathcal{R}X$ with $\bar{\bar{x}} = x$ for all $x \in X$. One has similarly a ring involution $p \mapsto \bar{p}$ given by $\bar{p}_{ij} = \overline{p_{ij}}$ on the ring $\text{Mat}_n(\mathcal{R})$ of $n \times n$ -matrices over $\mathcal{R}$.

For each $z \in X$, let $\ell_z = \{\ell(x, z) \mid (x, z) \in E\}$ and let I_z denote the set of intersections $\ell_z \cap P$ where P ranges over the closed half-spaces of V determined by the hyperplanes of V containing the origin. It is known that if $F \in I_z$, then $\ell_z \setminus F \in I_z$.

The following result is a trivial reformulation of [Dy3, 1.7]. The coefficient in $\mathcal{R}$ of $x \in X$ in $P(z, \ell_z)$ below is, up to normalization, a Kazhdan-Lusztig-Stanley polynomial associated to the interval $[x, z]$ (see [loc.cit., 1.8]).

Theorem 1.5. *There are unique elements $P(z, F) \in \mathcal{R}X$ (where $z \in X$ and $F \in I_z$) satisfying the conditions (a)–(c) below:*

- (a) For $z \in X$, one has $P(z, \emptyset) \in z + \sum_{y < z} v\mathbb{Z}[v]y$.
- (b) For $z \in X$ and $F \in I_z$ one has $\overline{P(z, F)} = P(z, \ell_z \setminus F)$.
- (c) Let $z, w \in X$ and $F \in I_z$ be such that $(w, z) \in E$ and $F \neq G := F \cup \{\alpha\} \in I_z$ where $\alpha = \ell(w, z)$. Then $H := \ell_w \cap \langle G \cup \{-\alpha\} \rangle_+ \in I_w$ and

$$P(z, G) = P(z, F) + (v^{-1} - v)P(w, H).$$

1.6 A Hecke algebra calculation suggests the possibility that in 1.5(c), the Laurent polynomials arising as coefficients of x in $P(z, F) - vP(w, H) = P(z, G) - v^{-1}P(w, H)$ might have non-negative coefficients (see [Dy4, 3.10]). The conjecture below arose from an attempt to find a natural interpretation of these polynomials making their positivity obvious; in it, $K = \mathbb{C}$ and σ, δ denote certain shift and duality functors on a category of graded modules (see Section 2, especially 2.1, 2.5, 2.11 for details).

Main Conjecture. *There is a basic quasi-hereditary Koszul algebra $\mathcal{A} = \mathcal{A}_X$ over K with weight poset X and a family $M(z, F)$ of graded $\mathcal{A}_X$ -modules, defined for $z \in X$ and $F \in I_z$, satisfying conditions (a)–(c) below:*

- (a) *For any $z \in X$, $M(z, \emptyset)$ is the graded Verma module of highest weight z .*
- (b) *For $z \in X$ and $F \in I_z$ one has $\delta M(z, F) \cong M(z, \ell_z \setminus F)$.*
- (c) *In the situation of 1.5 (c), there exists an exact sequence*

$$0 \longrightarrow \sigma M(w, H) \longrightarrow M(z, F) \longrightarrow M(z, G) \longrightarrow \sigma^{-1} M(w, H) \longrightarrow 0$$

of graded $\mathcal{A}_X$ -modules.

Identifying the Grothendieck group $\text{gr } \mathcal{A}_X$ of graded $\mathcal{A}_X$ -modules with $\mathcal{R}X$, 1.5 would imply that $[M(z, F)] = P(z, F)$.

Remarks 1.7. (a) An additional part of the conjecture, to be described in a future paper, would imply that there is a presentation of such an algebra $\mathcal{A} = \mathcal{A}_X$ involving (spaces of) generators and quadratic relations determined (by an explicit recursive construction) from the poset X and its labelling ℓ alone.

The quadratic dual algebra $\mathcal{A}^!$ would then be obtained by applying the same construction to the reverse poset X^{op} with a “dual labelling” ℓ' , and should provide a solution to the conjecture for the reverse poset; this is an analogue of Beilinson-Ginsburg duality [So2, 3.5]. There should also be another pair of quadratic dual algebras satisfying the conjecture for X and X^{op} , one constructed from X labelled by ℓ' , the other from X^{op} labelled by ℓ .

(b) Suppose that for some interval X one has an associated algebra $\mathcal{A}$ satisfying all aspects of the conjecture including Beilinson-Ginsburg duality as in (a). Then for any subinterval Y of X , there would be a canonically associated algebra $\mathcal{A}_Y$ which satisfies the conjecture for Y (see 2.14 and 2.22). If the main conjecture and the extension alluded to in (a) are both correct, it would follow that $\mathcal{A}$ is determined up to isomorphism by the conditions of the conjecture (in fact, by much weaker conditions) and the following relation to the edge-labelling; there should exist a “compatible” family of identifications of the algebras $\mathcal{A}_Y$ for closed subintervals Y of X of length at most two with the algebras $\mathcal{A}_Y$ defined in 3.8 from the edge-labelling of X restricted to Y .

(c) The modules $M(x, F)$ and the exact sequences occurring in the conjecture should be unique up to isomorphism. They should be part of a larger family of

modules and exact sequences parametrized by data depending only on the isomorphism type of the (unlabelled) poset X (see 2.24). This would imply in particular that a Kazhdan-Lusztig polynomial is determined by the isomorphism type of the associated interval (as unlabelled poset).

(d) According to the conjecture and (a), $\mathcal{A}$ would be analogous in some respects to the algebra of a regular block of category $\mathcal{O}$ for a semisimple complex Lie algebra (see [BG], [So1–3], [I3]). In the Coxeter group case, one expects that it should be possible to construct from $\mathcal{A}$ (by the procedure of 3.27) other algebras analogous to the algebras of singular blocks of $\mathcal{O}$, with weight poset given by an interval of shortest coset representatives of a standard parabolic subgroup. The quadratic duals of these singular algebras would then be analogues of algebras of parabolic category $\mathcal{O}$ (see [BG]). Moreover, all the above should apply as well to (spherical) subintervals of twisted Bruhat orders on W [Dy3].

(e) The regular and singular algebras (conjecturally) associated to finite Weyl groups should coincide with the algebras of blocks of $\mathcal{O}$ for complex semisimple Lie algebras (more generally, see 3.34(a) for a possible connection between algebras associated to twisted Bruhat orders on crystallographic Coxeter groups and $\mathcal{O}$ for Kac-Moody Lie algebras).

Example 1.8. If $X = \{0 < 1\}$ is a length one chain (the face lattice of a single closed ray), one may take $\mathcal{A}_X$ to be the algebra of 3×3 upper triangular matrices (a_{ij}) over $\mathbb{C}$ such that $a_{11} = a_{33}$ (this is the algebra of a regular integral block of category $\mathcal{O}$ for $\mathfrak{sl}(2, \mathbb{C})$).

2. $\mathcal{O}$ ALGEBRAS

Throughout this entire section, K denotes a fixed field.

Graded Quasi-hereditary Algebras. This subsection recalls the notion [CPS1] of a quasi-hereditary algebra (c.f. also [I1]), and fixes some notation and terminology concerning graded modules.

Definition 2.1. Let Y be a finite poset. A finite-dimensional K -algebra $\mathcal{A}$ is said to be quasi-hereditary with weight poset Y if the conditions (a)–(c) below hold:

(a) There is a bijective indexing $y \mapsto \{L(y)\}$ of the (isomorphism classes of) simple $\mathcal{A}$ -modules by elements of Y .

(b) There is a collection $\{M(y)\}_{y \in Y}$ of $\mathcal{A}$ -modules and for $y \in Y$ a surjection $\theta: M(y) \rightarrow L(y)$ such that all composition factors $L(x)$ of $\ker \theta$ satisfy $x < y$.

(c) The projective cover $P(y)$ of $L(y)$ has a filtration

$$P(y) = F^0 \supseteq F^1 \supseteq \dots \supseteq F^n = 0$$

by $\mathcal{A}$ -modules F^i such that $F^0/F^1 \cong M(y)$ and for $i > 0$ one has $F^i/F^{i+1} \cong M(x)$ for some $x > y$ in Y .

The following result is a partial converse of Brauer-Humphreys (or BGG) reciprocity ([CPS1, 3.11], [I1]). It will be convenient to call a quasi-hereditary algebra $\mathcal{A}$ satisfying 2.2(a)–(d) a qh algebra with weight poset Y .

Proposition 2.2. *Let Y be a finite poset. Let $\mathcal{A} = \bigoplus_{n \in \mathbb{N}} \mathcal{A}_n$ be a graded K -algebra, with $\mathcal{A}_n$ finite-dimensional for $n = 0, 1$, which satisfies the following conditions (a)–(d):*

(a) $\mathcal{A}_0 \cong K^n$, a product of fields, and Y is identified with the complete set of set of primitive idempotents of $\mathcal{A}_0$.

(b) Let $\mathcal{A}^\pm$ denote the subalgebra of $\mathcal{A}$ generated by $\mathcal{A}_0 \cup \mathcal{A}_1^\pm$ where $\mathcal{A}_1^+ = \sum_{y>x} y\mathcal{A}_1x$ and $\mathcal{A}_1^- = \sum_{y>x} x\mathcal{A}_1y$. Then $\mathcal{A} = \mathcal{A}^- \mathcal{A}^+$.

(c) One has $\dim \mathcal{A} \geq \sum_{y \in Y} (\dim M(y))^2$ where $M(y)$ denotes the left $\mathcal{A}$ -module

$$M(y) := \mathcal{A}y / \mathcal{A}\mathcal{A}_1^+y.$$

(d) There is an involutory graded algebra anti-automorphism ω of $\mathcal{A}$ with $\omega(y) = y$ for all $y \in Y$.

Then $\mathcal{A}$ is quasi-hereditary.

Proof. Note that $\mathcal{A}^\pm$ is a finite-dimensional subalgebra of $\mathcal{A}$, so $\mathcal{A}$ is finite-dimensional by (b). Moreover, $J(\mathcal{A}) := \bigoplus_{n>0} \mathcal{A}_n$ is the radical (largest nilpotent two-sided ideal) of $\mathcal{A}$, and $\mathcal{A}/J(\mathcal{A}) \cong \mathcal{A}_0$. Hence the irreducible $\mathcal{A}$ -modules are just the irreducible $\mathcal{A}_0$ -modules, regarded as $\mathcal{A}$ -modules. That is, for $x \in Y$ there is a unique up to isomorphism one dimensional $\mathcal{A}$ -module $L(x)$ with $xL(x) = L(x)$, and any simple $\mathcal{A}$ -module is isomorphic to $L(x)$ for a unique $x \in Y$.

Note that for any $\mathcal{A}$ -module M , one has a “weight space decomposition”

$$(e) \quad M = \bigoplus_{x \in Y} xM \quad \text{where } xM = \{v \in M \mid xv = v\}.$$

Now by (b), $M(x)$ is generated by $v_x := x + \mathcal{A}\mathcal{A}_1^+x \in \mathcal{A}x / \mathcal{A}\mathcal{A}_1^+x$ as $\mathcal{A}^-$ -module, so $M(x) = \bigoplus_{y \leq x} yM(x)$. One sees immediately that $M'(x) := \sum_{y < x} yM(x)$ is the unique maximal submodule of $M(x)$; all composition factors of $M'(x)$ are of the form $L(y)$ with $y < x$ and $M(x)/M'(x) \cong L(x)$. It is clear that 2.1(a)–(b) hold. For any $x \in Y$, define the left $\mathcal{A}$ -module $P(x) = \mathcal{A}x$. One has

$$(f) \quad \mathcal{A} = \bigoplus_{x \in Y} P(x)$$

and $P(x)$ is a projective cover of $L(x)$. We will establish 2.1(c) during the proof of the following result.

Proposition 2.3. *Let $\mathcal{A}$ be as in 2.2. For $y \leq x$ in Y , choose $B_{yx} \subseteq y\mathcal{A}^-x$ so that the elements bv_x with $b \in B_{yx}$ form a basis of $yM(x)$. Then the elements $b'\omega(b)$ with $b \in B_{zx}$, $b' \in B_{yx}$ (where $z \leq x$, $y \leq x$ in Y) form a K -basis of $\mathcal{A}$.*

Proof. Fix $z \in X$ and let $x_1, \dots, x_r$ denote the elements of $\{x \in Y \mid z \leq x\}$ arranged so that $x_i > x_j$ implies $i < j$. For $i = 0, \dots, r$, let N_i denote the $\mathcal{A}^-$ submodule of $P(z)$ generated by $\bigcup_{j=1}^i \omega(B_{zx_j})$ and N'_i denote the left ideal $\sum_{j=1}^i \mathcal{A}^-x_j\mathcal{A}^+z$ of $\mathcal{A}$. It is clear that $N_i \subseteq N'_i$, and we prove $N'_i = N_i$ by induction on i . Suppose $N_{i-1} = N'_{i-1}$ where $r \geq i \geq 1$. Let $b \in x_i\mathcal{A}^+z$. Considering $\omega(b)v_{x_i} \in M(x_i) = \mathcal{A}x / \mathcal{A}\mathcal{A}_1^+x_i$, one finds

$$\omega(b) \in \langle B_{zx_i} \rangle + \sum_{j=1}^{i-1} z\mathcal{A}^-x_j\mathcal{A}^+x_i.$$

By induction, one gets $b \in \langle \omega B_{zx_i} \rangle \pmod{N_{i-1}}$ which implies $N'_i \subseteq N_i$.

Now let $b_1, \dots, b_{k_z}$ denote the elements of $\bigcup_{j=1}^r \omega(B_{zx_j})$ ordered in such a way that $b_{n_{j-1}+1}, \dots, b_{n_j} \in \omega(B_{zx_j})$, where $0 = n_0 < \dots < n_r = k_z$. For $j = 0, \dots, k_z$,

let F^j denote the left ideal $F^j = \mathcal{A}b_1 + \dots + \mathcal{A}b_j$ of $\mathcal{A}$. One has $0 = F^0 \subseteq \dots \subseteq F^{k_z} = P(z)$, and if $n_{j-1} + 1 \leq m \leq n_j$, then $\mathcal{A}_1^+ b_m \subseteq N_{j-1} \subseteq F^{m-1}$. It follows that F^m/F^{m-1} is a quotient of $M(x_j)$, and so $\dim F^m/F^{m-1} \leq \dim M(x_j)$ with equality iff $F^m/F^{m-1} \cong M(x_j)$. Now

$$(*) \quad \dim P(z) = \sum_{m=1}^{k_z} \dim F^m/F^{m-1} \\ \leq \sum_{j=1}^r \sum_{i=n_{j-1}+1}^{n_j} \dim M(x_j) = \sum_{y \geq z} \dim B_{zy} \dim M(y).$$

Summing over $z \in Y$, one has by 2.2(c) and 2.2(f) that equality holds in (*) and hence $F^m/F^{m-1} \cong M(x_j)$ if $n_{j-1} + 1 \leq m \leq n_j$ (this completes the proof of 2.2). It follows that one has vector space direct sums

$$F^m = F^{m-1} \oplus \langle b'b_m \mid b' \in B_{yx_j} \rangle$$

for $n_{j-1} + 1 \leq m \leq n_j$, and the proposition follows.

Remark 2.4. Except for 2.7 below, most of the results of this subsection and the next can be formulated so as to hold without the assumption that $\mathcal{A}$ has an anti-involution. For instance, 2.2 holds if one drops (c) and replaces (d) by $\dim \mathcal{A} \geq \sum_{y \in Y} (\dim M(y))(\dim N(y))$ where $N(y)$ denotes the right $\mathcal{A}$ -module $N(y) := y\mathcal{A}/y\mathcal{A}_1^-\mathcal{A}$. Since all the examples considered in this paper will have an anti-involution, and since the anti-involution is needed to define interpolating modules, we leave the interested reader to make the appropriate modifications.

2.5 We fix some notation concerning graded modules. To begin, let $\mathcal{A} = \bigoplus_{n \in \mathbb{N}} \mathcal{A}_n$ denote a finite-dimensional graded K -algebra with $\mathcal{A}_0 \cong K^n$.

A graded $\mathcal{A}$ -module M_* is defined to be a (finitely generated) $\mathcal{A}$ -module M equipped with a vector space decomposition $M = \bigoplus_{i \in \mathbb{Z}} M_i$ such that $\mathcal{A}_i M_j \subseteq M_{i+j}$ for all $i \in \mathbb{N}$, $j \in \mathbb{Z}$. We let $\text{gr } \mathcal{A}$ denote the category of graded $\mathcal{A}$ -modules; this is an abelian category, the morphisms $M_* \rightarrow N_*$ being $\mathcal{A}$ -module homomorphisms $f: M \rightarrow N$ with $f(M_i) \subseteq N_i$.

For any $i \in \mathbb{Z}$, there is a shift functor σ^i on $\text{gr } \mathcal{A}$ defined on objects by $(\sigma^i M)_j = M_{j-i}$. For any left graded $\mathcal{A}$ -module M_* , the dual space $M^\#$ may be regarded as a graded right $\mathcal{A}$ -module $(M^\#)_*$ with $(M^\#)_n = (M_{-n})^\#$ and $(fa)(m) = f(am)$ for $f \in M^\#$, $a \in \mathcal{A}$ and $m \in M$. If $\mathcal{A}$ has an anti-involution ω as graded K -algebra, the graded space M_* may be regarded as a right $\mathcal{A}$ -module $(M^t)_*$ with action $ma = \omega(a)m$ for $m \in M$ and $a \in \mathcal{A}$. Similarly, one defines $\sigma^k M_*$, $(M^\#)_*$ etc if M_* is a graded right $\mathcal{A}$ -module or graded $(\mathcal{A}, \mathcal{A})$ -bimodule.

Define a duality on an abelian category $\mathcal{A}$ to be a contravariant, exact functor $\delta': \mathcal{A} \rightarrow \mathcal{A}$ with δ'^2 naturally equivalent to $1_{\mathcal{A}}$. Note that any graded K -algebra anti-involution ω of $\mathcal{A}$ determines a duality δ on $\text{gr } \mathcal{A}$ defined on objects by $\delta M_* = ((M^t)^\#)_*$. Note that one has $\delta \circ \sigma^k = \sigma^{-k} \circ \delta$ for any $k \in \mathbb{Z}$. The duality functors defined similarly on the categories of graded right $\mathcal{A}$ -modules and graded $(\mathcal{A}, \mathcal{A})$ -bimodules will also be denoted δ .

Now suppose that $\mathcal{A}$ is a qh algebra with weight poset Y . Note that a simple $\mathcal{A}$ -module $L(y)$ has a graded version $L(y)_*$ with $L(y)_i = 0$ for $i \neq 0$, and that $P(y) = \mathcal{A}y \cong \mathcal{A}/\mathcal{A}(y-1)$ and $M(y) \cong \mathcal{A}y/\mathcal{A}\mathcal{A}_1^+y$ have natural graded versions $P(y)_*$,

$M(y)_*$ with $L(y)_*$ as unique simple homomorphic image in $\text{gr } \mathcal{A}$; one has $P(y)_n = \mathcal{A}_n y$ and $M(y)_n = \mathcal{A}_n v_y$ where $yM(y) = Kv_y$. Moreover, one has $\delta L(x)_* \cong L(x)_*$ for any $x \in Y$.

The split Grothendieck group $\text{Gr}_*(\mathcal{A})$ of $\text{gr } \mathcal{A}$ is defined as the abelian group generated by isomorphism classes of graded $\mathcal{A}$ -modules subject to a relation $[M_*] = [M'_*] + [M''_*]$ for each exact sequence

$$0 \longrightarrow M'_* \longrightarrow M_* \longrightarrow M''_* \longrightarrow 0$$

in $\text{gr } \mathcal{A}$. It is a free abelian group on basis $[\sigma^i L(y)_*]$, for $i \in \mathbb{Z}$ and $y \in Y$, and hence may be identified with the free $\mathcal{R}$ -module $\mathcal{R}Y$ on basis Y by means of the isomorphism $[\sigma^i L(y)_*] \mapsto v^i y$. Under this identification, one has $[\delta M_*] = [\overline{M_*}]$ (where $f \mapsto \bar{f}$ is the $\mathcal{R}$ antilinear map on $\mathcal{R}Y$ fixing all $y \in Y$), $[\sigma^k M_*] = v^k [M_*]$ and $[M(z)_*] \in z + \sum_{y < z} v\mathbb{Z}[v]y$. One has an obvious specialization map $[M] \mapsto [M]_{v=1}$, $\mathcal{R}Y \rightarrow \oplus_{y \in Y} \mathbb{Z}y \subseteq \mathcal{R}Y$.

Since we will be concerned primarily with graded modules in this paper, we will usually write M_* simply as M . The following proposition is a consequence of graded Brauer-Humphreys (or BGG) reciprocity [I, 3.5] but we include a direct proof.

Proposition 2.6. (a) *For any $y \in Y$, the graded module $P(y)$ has a filtration $P(y) = F^0 \supseteq F^1 \supseteq \dots \supseteq F^n = 0$ in $\text{gr } \mathcal{A}$ such that $F^0/F^1 \cong \mathcal{M}(y)$ and for $i > 0$ one has $F^i/F^{i+1} \cong \sigma^k M(x)$ for some $x > y$ in Y and $k > 0$. The number of times that $\sigma^k M(x)$ appears as a quotient F^i/F^{i+1} in any such filtration is $\dim yM(x)_k$.*

(b) *Define the matrix Poincaré series $p'(\mathcal{A}, v)$ of $\mathcal{A}$ by*

$$p'_{x,y}(\mathcal{A}, v) = \sum_{i \in \mathbb{N}} \dim(x\mathcal{A}_i y) t^i \in \mathbb{Z}[[v]] \quad \text{for } x, y \in Y.$$

Then $p'(\mathcal{A}, v) = p(\mathcal{A}, v)p(\mathcal{A}, v)^T$ where $p_{x,y}(\mathcal{A}, v) := \sum_{i \in \mathbb{N}} \dim xM(y)_i v^i$.

Proof. In 2.3, one may choose B_{yx} as a union of sets $B_{yxi} \subseteq (y\mathcal{A}^-x)_i$ with $i \in \mathbb{N}$ so that the elements bv_x with $b \in B_{yxi}$ form a basis of $yM(x)_i$. Then the basis given by 2.3 is compatible with the grading of $\mathcal{A}$ (proving (b)), and the submodules F^m occurring in the proof of 2.3 are graded (proving the existence of a filtration as in (a)). The other assertion in (a) follows since the elements $[\sigma^k M(y)]$ with $k \in \mathbb{Z}$ and $y \in Y$ form a $\mathbb{Z}$ -basis for $\text{Gr}_*(\mathcal{A})$.

Remark 2.7. Suppose that $y \in Y$ satisfies $p_{y,z}(\mathcal{A}, 1) \leq 1$ for all $z \in Y$ i.e. $L(y)$ is multiplicity-free in any $M(z)$ in which it appears as composition factor. Then 2.3 implies that $y\mathcal{A}y$ has a basis fixed under the anti-involution ω and hence $\text{End}_{\mathcal{A}}(P(y))$ is a commutative K -algebra.

$\mathcal{O}$ Algebras. This subsection is concerned with general properties of a class of algebras to which algebras satisfying the main conjecture 1.6 (and 1.7(a)) would belong. The name used here for this class of algebras, $\mathcal{O}$ algebras, and many of the results we give on these algebras, are suggested by work of Beilinson-Ginsburg-Soergel [BGS] on the algebras of blocks of category $\mathcal{O}$ for semisimple complex Lie algebras.

We will make extensive use of results from [BGS] on Koszul algebras. First, we recall the definition of (basic) quadratic algebras over K and their quadratic duals (see [BGS, Definition 2 and 4]; dealing only with basic algebras permits some simplifications in the definitions). Tensor products are over $\mathcal{A}_0$ unless otherwise stated.

Definition 2.8. Consider a K -algebra $\mathcal{A}_0 \cong K^n$. Let V be a finite-dimensional $\mathcal{A}_0$ bimodule, and form the tensor algebra $T(V) := T_{\mathcal{A}_0}(V) := \bigoplus_{n \in \mathbb{N}} T^n(V)$ over $\mathcal{A}_0$, where $T^n(V) := V \otimes \dots \otimes V$ denotes the n -fold tensor product (and $T^0(V) = \mathcal{A}_0$).

(a) Any algebra of the form $\mathcal{A} = T(V)/\langle W \rangle$, where $\langle W \rangle$ is the two-sided ideal of $T(V)$ generated by some $\mathcal{A}_0$ -sub-bimodule W of $T^2(V)$, is called a (basic) quadratic algebra. Such an algebra has a natural grading $\mathcal{A} = \bigoplus_{n \in \mathbb{N}} \mathcal{A}_n$, $A_i A_j \subseteq A_{i+j}$.

(b) Regard the dual space $V^\#$ as an $\mathcal{A}_0$ -module with the action $(afb)(v) = f(bva)$, where $a, b \in \mathcal{A}_0$ and $f \in V^\#$. The dual quadratic algebra $\mathcal{A}^\dagger$ is defined by $\mathcal{A}^\dagger = T(V^\#)/\langle W^\perp \rangle$ where $W^\perp$ is the annihilator $W^\perp := \{f \in V^\# \otimes V^\# \mid f(W) = 0\}$ of W (we identify $V^\# \otimes V^\#$ with $(V \otimes V)^\#$ and $(\mathcal{A}_0)^\#$ with $\mathcal{A}_0$).

We now define certain complexes associated to a quadratic algebra $\mathcal{A}$ as above. In 2.9–2.11 only, graded module will mean “not necessarily finitely generated graded module M ” and the dual of M will mean “continuous dual” $(M^\#)_n = (M_{-n})^\#$. In applications in this paper, we will consider only finite-dimensional modules for quadratic algebras $\mathcal{A}$ with both $\mathcal{A}$ and $\mathcal{A}^\dagger$ finite-dimensional, so the complexes below will be finite-dimensional.

2.9 Let $\mathcal{A}$ be a quadratic algebra as in 2.8. For a graded left $\mathcal{A}^\dagger$ -module M and graded right $\mathcal{A}$ -module N , consider the K -vector space $\mathcal{K}_{\mathcal{A}}(N, M) = \mathcal{K}(N, M) := N \otimes M$ endowed with the map $\partial: \mathcal{K}(N, M) \rightarrow \mathcal{K}(N, M)$ defined by

$$\partial(n \otimes m) = \sum_{\nu} n e_{\nu} \otimes e_{\nu}^{\#} m$$

for $n \in N$ and $m \in M$, where $\{e_{\nu}\}$ is a basis of $\mathcal{A}_1 = V$ and $\{e_{\nu}^{\#}\}$ is the dual basis of $\mathcal{A}_1^{\dagger} = V^{\#}$. It is easily checked that ∂ is independent of the choice of basis and is a differential (i.e. $\partial^2 = 0$; cf [BGS, 2.7]. Note that $\mathcal{K}$ is a functor of M and N , exact in each variable, and that the space $(\mathcal{K}_{\mathcal{A}}(N, M))^{\#}$ with differential $\partial^{\#}$ may be identified with $\mathcal{K}_{\mathcal{A}^{\dagger}}(N^{\#}, M^{\#})$ if N, M are finite-dimensional.

One may regard $\mathcal{K}(N, M)$ as a complex of graded vector spaces

$$\dots \mathcal{K}(N, M)^p \rightarrow \mathcal{K}(N, M)^{p+1} \rightarrow \dots$$

where for $p, q \in \mathbb{Z}$, $\mathcal{K}(N, M)_q^p = N_{p+q} \otimes M_p$. If N is a graded $(\mathcal{A}, \mathcal{A})$ -bimodule, $\mathcal{K}(N, M)$ may be regarded as a complex of graded left $\mathcal{A}$ -modules, and it has a decomposition $\mathcal{K}(N, M) = \bigoplus_{y \in Y} \mathcal{K}(N, My)$ as a direct sum of complexes of graded vector spaces. If in addition M is a graded $(\mathcal{A}^{\dagger}, \mathcal{A}^{\dagger})$ -bimodule, one has $\mathcal{K}(N, M) = \bigoplus_{x, y \in Y} \mathcal{K}(xN, My)$ where $H^i(\mathcal{K}(xN, My)) \cong xH^i(N, M)y$.

The Koszul complex of $\mathcal{A}$ is defined to be $\mathcal{K}(\mathcal{A}, (\mathcal{A}^{\dagger})^{\#})$ (see [BGS, Section 2]; it may be regarded as a complex of graded projective left $\mathcal{A}$ -modules, with a natural augmentation $\mathcal{K}(\mathcal{A}, (\mathcal{A}^{\dagger})^{\#}) \cong \mathcal{A} \rightarrow \mathcal{A}_0$).

We now collect together some facts about Koszul algebras; we need only special cases of results from [BGS, Section 2].

Theorem 2.10. Let $\mathcal{A} = \bigoplus_{n \in \mathbb{N}} \mathcal{A}_n$ be a graded K -algebra with $\mathcal{A}_0 \cong K^n$ and $\mathcal{A}_1$ finite-dimensional. Then the conditions (a)–(b) below are equivalent, and imply (c):

- (a) $\mathcal{A}$ is quadratic and $\mathcal{K}(\mathcal{A}, (\mathcal{A}^{\dagger})^{\#})$ is a resolution of $\mathcal{A}_0$.
 - (b) one has $\text{Ext}_{\text{gr } \mathcal{A}}^n(\sigma^{-m} \mathcal{A}_0, \mathcal{A}'_0) = 0$ unless $n = m$.
 - (c) there is an isomorphism $(\mathcal{A}^{\dagger})^{\text{op}} \cong \text{Ext}_{\mathcal{A}}^{\bullet}(\mathcal{A}_0, \mathcal{A}_0)$ of graded K -algebras, where $\text{Ext}_{\mathcal{A}}^{\bullet}$ denotes the Yoneda Ext-algebra (Ext in the category of ungraded $\mathcal{A}$ -modules).
- Moreover, if the conditions (a)–(b) are satisfied by $\mathcal{A}$, they are also satisfied by $\mathcal{A}^{\dagger}$ and $\mathcal{A}^{\text{op}}$.

Definition 2.11. (a) A (basic) quadratic algebra $\mathcal{A}$ as in 2.8 satisfying the equivalent conditions 2.10(a)–(b) is said to be formal quadratic, or a Koszul algebra.

(b) A Koszul algebra $\mathcal{A}$ over a field K will be called an $\mathcal{O}$ K -algebra with weight poset Y if $\mathcal{A}, \mathcal{A}^!$ are qh algebras with weight posets Y and Y^{op} respectively.

Note that this implies that $\mathcal{A}^!$ is also an $\mathcal{O}$ -algebra. We will always assume that the restriction of the anti-involution $\omega^!$ of an $\mathcal{O}$ -algebra $\mathcal{A}^!$ to $\mathcal{A}_1^! = V^\# = (\mathcal{A}_1)^\#$ coincides with that of $\omega^\#$. If the only $\mathcal{O}$ -algebras under consideration are $\mathcal{A}$ and $\mathcal{A}^!$, we will write $M_!(y), P_!(y), L_!(y)$ for the Verma modules, projective indecomposables and simple modules of $\mathcal{A}^!$ respectively. If it becomes necessary to distinguish modules for additional such algebras, we will write $M_{\mathcal{A}}(y)$ etc.

Remarks 2.12. (a) Suppose that $\mathcal{A}$ is a quadratic algebra as in 2.8 and that $\mathcal{A}^!$ is a qh algebra with weight poset Y^{op} . Using 2.3, one easily sees that

$$(*) \quad \mathcal{A}_1^+ \mathcal{A}_1^- \subseteq \mathcal{A}_1^+ \mathcal{A}_1^+ + \mathcal{A}_1^- \mathcal{A}_1^+ + \mathcal{A}_1^- \mathcal{A}_1^-$$

and hence $\mathcal{A}$ satisfies 2.2(b).

(b) More generally, (a) remains true assuming only that $\mathcal{A}$ is (basic) quadratic as above and $\mathcal{A}^!$ is quasi-hereditary; this can be shown (for instance) using [CPS1, 3.2(b)] and graded Brauer-Humphreys reciprocity. It follows that if $\mathcal{A}$ is a basic quadratic algebra and $\mathcal{A}, \mathcal{A}^!$ are quasi-hereditary with weight posets Y and Y^{op} respectively (and anti-involution as in 2.2(d)), then $\mathcal{A}$ and $\mathcal{A}^!$ are qh algebras, and that all the statements in the remainder of this subsection remain true with qh replaced by “basic quasi-hereditary with anti-involution as in 2.2(d)”.

(c) The algebras of (regular and singular) blocks of category $\mathcal{O}$ for complex semisimple Lie algebras are $\mathcal{O}$ algebras in the above sense, see [BGS, Section 1] and [So2, Theorem 11].

The following result is an analogue for $\mathcal{O}$ algebras of a result [CPS1, 3.5] on quasi-hereditary algebras.

Proposition 2.13. *Let $\mathcal{A}$ be an $\mathcal{O}$ algebra with weight poset Y . Let X be a coideal of Y and set $e = \sum_{y \in Y} y \in \mathcal{A}_0, f = 1 - e$. Then*

(a) $\mathcal{B} := e\mathcal{A}e$ is an $\mathcal{O}$ algebra with weight poset X . For $x \in X$, one may identify $M_{\mathcal{B}}(x)$ with $eM_{\mathcal{A}}(x)$ in $\text{gr } \mathcal{B} = \text{gr } e\mathcal{A}e$.

(b) $\mathcal{C} := \mathcal{A}^!/\mathcal{A}^!f\mathcal{A}^!$ is an $\mathcal{O}$ algebra with weight poset X^{op} . For $x \in X$, the Verma module $M_{\mathcal{C}}(x)$ is just $M_{\mathcal{A}^!}(x)$ regarded as a graded $\mathcal{C}$ -module.

(c) $\mathcal{B}$ and $\mathcal{C}$ are dual quadratic algebras; there is an isomorphism $\mathcal{B}^! \cong \mathcal{C}$ which restricts to the identity on $\mathcal{B}_0^! = \mathcal{C}_0$.

(d) one may identify $\mathcal{B}$'s Koszul complex $\mathcal{K}_{\mathcal{B}}(\mathcal{B}, \delta\mathcal{C})$ with the subcomplex $\mathcal{K}_{\mathcal{A}}(e\mathcal{A}, \delta\mathcal{C})$ of the Koszul complex of $\mathcal{A}$.

Proof. We sketch a direct proof for the portion of the proposition that could be deduced from [loc cit]. Choose a basis $\{b'\omega(b)\}$ for $\mathcal{A}$ as in 2.3. Then the elements $b'\omega(b)$ with $b \in B_{zx}, b' \in B_{yx}$ (where $z \leq x, y \leq x$ in X) form a K -basis of $\mathcal{B}$. One sees that for any $x \in X$, one has

$$M_{\mathcal{B}}(x) := \mathcal{B}x/\mathcal{B}\mathcal{B}_1^+x \cong e\mathcal{A}x/e\mathcal{A}\mathcal{A}_1^+x \cong eM_{\mathcal{A}}(x)$$

and it follows immediately using 2.3 that $\mathcal{B}$ is a qh algebra with Verma modules as claimed. Similarly, one checks that $\mathcal{C}$ is qh and $M_{\mathcal{C}}(x) = M_{\mathcal{A}^!}(x)$ for $x \in X$.

Note that $\mathcal{C}$ is obviously a quadratic algebra; we now check that $\mathcal{B} \cong \mathcal{C}^!$. Write $\mathcal{A} \cong T_{\mathcal{A}_0}(V)/\langle W \rangle$ where $V = \mathcal{A}_1$ and W is an $\mathcal{A}_0$ -sub-bimodule of $T_{\mathcal{A}_0}^2(V)$. Set $\mathcal{A}'_0 = e\mathcal{A}_0e$, $V' = eVe$ (an $\mathcal{A}'_0$ -bimodule) and regard $V' \otimes_{\mathcal{A}'_0} V'$ as a subspace of $V \otimes_{\mathcal{A}_0} V$ in the natural way. Using 2.12(*) for $\mathcal{A}$ and $\mathcal{A}^!$, one easily checks that $\mathcal{C}^! \cong T_{\mathcal{A}'_0}(V')/\langle W' \rangle$ where $W' = W \cap (V' \otimes_{\mathcal{A}'_0} V')$. But $\mathcal{B}$ is generated by $\mathcal{B}_0 \cup \mathcal{B}_1$, and W' is the space of quadratic relations satisfied by $\mathcal{B}$, so $\mathcal{B} \cong \mathcal{C}^!$ will follow if we show that $\mathcal{B}$ is Koszul. To do this, we may assume without loss of generality that $f = x$, where x is a minimal element of X .

The exact sequence $0 \rightarrow \mathcal{A}^!x\mathcal{A}^! \rightarrow \mathcal{A}^! \rightarrow \mathcal{A}^!/ \mathcal{A}^!x\mathcal{A}^! \rightarrow 0$ of left $\mathcal{A}^!$ -modules gives an exact sequence of complexes

$$0 \rightarrow \mathcal{K}(e\mathcal{A}, \delta\mathcal{C}) \rightarrow \mathcal{K}(e\mathcal{A}, \delta\mathcal{A}^!) \rightarrow \mathcal{K}(e\mathcal{A}, \delta\mathcal{A}^!x\mathcal{A}^!) \rightarrow 0.$$

Since $\mathcal{A}$ is Koszul, $H^i(\mathcal{K}(\mathcal{A}, \delta P_!(x)))$ is zero for $i \neq 0$ and equal to $L(x)$ for $i = 0$. Since $eL(x) = 0$, one gets that $\mathcal{K}(e\mathcal{A}, \delta P_!(x))$ is acyclic. But since x is maximal in X^{op} , one has easily from the proof of 2.3 that $\mathcal{A}^!x\mathcal{A}^!$ is isomorphic to a direct sum of degree shifts of $P_!(x)$ as left $\mathcal{A}^!$ -module, and so $\mathcal{K}(e\mathcal{A}, \delta\mathcal{A}^!x\mathcal{A}^!)$ is acyclic. It follows that $H^i(e\mathcal{A}, \delta\mathcal{C}) \cong H^i(e\mathcal{A}, \delta\mathcal{A}^!)$ is zero for $i \neq 0$ and isomorphic as graded $e\mathcal{A}e = \mathcal{B}$ -module to $e\mathcal{A}_0 = \mathcal{A}'_0$ in degree 0. By means of the natural augmentation $\mathcal{K}(e\mathcal{A}, \delta\mathcal{C})^0 \cong \mathcal{B} \rightarrow \mathcal{A}'_0 = \mathcal{B}_0$, one may regard $\mathcal{K}(e\mathcal{A}, \delta\mathcal{C})$ as a resolution of $\mathcal{B}_0$ in $\text{gr } \mathcal{B}$. Now for any $n \in \mathbb{Z}$, $\mathcal{K}(e\mathcal{A}, \delta\mathcal{C})^n \cong \mathcal{B} \otimes_{\mathcal{A}'_0} (\delta\mathcal{C})_n$ is a direct sum of modules of the form $\sigma^{-n}P_{\mathcal{B}}(z)$ with $z \in X$; it follows for $y, z \in X$, one has that $\text{Ext}_{\text{gr } \mathcal{B}}^n(\sigma^{-m}L_{\mathcal{B}}(y), L_{\mathcal{B}}(z)) = 0$ unless $n = m$. Now 2.10 implies that $\mathcal{B}$ is Koszul. This completes the proof of (a)–(c), and shows that (d) holds in this special case $\sharp(Y \setminus X) = 1$. Finally, an obvious induction on $\sharp(Y \setminus X)$ completes the proof of (d) for arbitrary coideals X .

Corollary 2.14. *Consider an $\mathcal{O}$ algebra $\mathcal{A}$ with weight poset Y . For a closed subset X of Y , let $\mathcal{A}_X$ denote the graded algebra $\mathcal{A}_X := e(\mathcal{A}/\mathcal{A}f\mathcal{A})e$ where $f = \sum_{z \in \mathcal{A}_0} z \in \mathcal{A}_0$ (sum over $z \in X$ such that $z \not\leq x$ for all $x \in X$) and $e = \sum_{y \in Y} y \in (\mathcal{A}/\mathcal{A}f\mathcal{A})_0$ (with the anti-involution induced by that of $\mathcal{A}$). Then $\mathcal{A}_X$ is a $\mathcal{O}$ algebra, $(\mathcal{A}_X)^! \cong (\mathcal{A}^!)_{X^{\text{op}}}$ and for any closed subset Z of X , one has $(\mathcal{A}_X)_Z \cong \mathcal{A}_Z$.*

Proof. This is a consequence of 2.13. The isomorphisms may be proved, for instance, by comparing the (quadratic) relations for the algebras involved.

We will say that the $\mathcal{O}$ algebras $\mathcal{A}_X$ above are obtained by truncation of the weight poset of $\mathcal{A}$.

We now give some equivalent characterizations of $\mathcal{O}$ algebras. Condition (d) below has been studied by Irving [I2], who calls it purity. The implication (d) $\Rightarrow$ (a) below, under the additional hypothesis that $\mathcal{A}$ is Koszul, is a minor variation of a result on graded Kazhdan-Lusztig theories described by Cline, Parshall and Scott [CPS2, Appendix to Section 3]. The proof here amounts to little more than giving a graded version of [CPS2, 2.4] and its proof, using purity in place of the even-odd Ext-vanishing assumed for (graded) Kazhdan-Lusztig theories.

Proposition 2.15. *Let $\mathcal{A}$ be a qh algebra with weight poset Y . Then the conditions (a)–(e) below are equivalent:*

- (a) $\mathcal{A}$ is a $\mathcal{O}$ algebra.
- (b) $\mathcal{A}$ is quadratic and for all $x \neq y$ in Y , the complex $\mathcal{K}(M(x)^t, \delta M_!(y))$ of graded vector spaces is acyclic.

(c) $\mathcal{A}$ is quadratic and for every $x \in Y$, the complex $\mathcal{K}_x(\mathcal{A}) := \mathcal{K}(\mathcal{A}, \delta M_!(x))$ with augmentation $\mathcal{K}_x(\mathcal{A})^0 = P(x) \rightarrow M(x)$ is a projective resolution of $M(x)$ in $\text{gr } \mathcal{A}$.

(d) for all $x, y \in Y$, $\text{Ext}_{\text{gr } \mathcal{A}}^n(\sigma^k M(x), L(y)) = 0$ unless $n = -m$.

(e) $\mathcal{A}$ is Koszul and $p(\mathcal{A}, v)p(\mathcal{A}^!, -v)^T = \text{Id}$.

Proof. The proof that (d) $\Rightarrow$ (a) will be given in the proof of the next proposition. Hence it will be sufficient to show here that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) and that (e) $\Leftrightarrow$ (a).

First, suppose that $\mathcal{A}$ is an $\mathcal{O}$ algebra. Let X be any closed subset of Y , and set $\mathcal{B} = \mathcal{A}_X$ (notation as in 2.14). Note that, by 2.13, for any $x, y \in X$, one may identify $\mathcal{K}_{\mathcal{A}}(M_{\mathcal{A}}(x)^t, \delta M_{\mathcal{A}^!}(y))$ with $\mathcal{K}_{\mathcal{B}}(M_{\mathcal{B}}(x)^t, \delta M_{\mathcal{B}^!}(y))$. We use this observation to prove that (b) holds by induction on $\sharp(Y)$. Fix distinct x, y in Y . Then $\mathcal{K}(M(x)^t, \delta M_!(y))$ is acyclic by induction and the above remark, unless perhaps x is a maximal element of Y and y is a minimal element of Y . But in this case,

$$H^i(\mathcal{K}(M(x)^t, \delta M_!(y))) = H^i(\mathcal{K}(P(x)^t, \delta P_!(y))) \cong xH^i(\mathcal{K}(\mathcal{A}, \delta \mathcal{A}^!))y$$

is zero if $i \neq 0$ and isomorphic to $x\mathcal{A}_0 y = 0$ if $i = 0$. Hence (a) implies (b)

Assume that (b) holds. Choose a filtration $\mathcal{A} = F^0 \supseteq F^1 \supseteq \dots \supseteq F^n = 0$ of $\mathcal{A}$ by right $\mathcal{A}$ -modules with subquotients $F^{i-1}/F^i \cong \sigma^{k_i} M(x_i)^t$. Then one has a filtration $\mathcal{K}_x(\mathcal{A}) = G^0 \supseteq G^1 \supseteq \dots \supseteq G^n = 0$ of $\mathcal{K}_x(\mathcal{A})$ by subcomplexes $G^i := \mathcal{K}(F^i, \delta M_!(x))$ (of graded vector spaces), and the subquotient complexes $G^{i-1}/G^i \cong \mathcal{K}(\sigma^{k_i} M(x_i)^t, \delta M_!(x))$ are acyclic except perhaps in degree 0. Hence $H^j(\mathcal{K}_x(\mathcal{A})) = 0$ for $j \neq 0$. A direct check shows that $H^0(\mathcal{K}_x(\mathcal{A})) \cong \mathcal{A}x/\mathcal{A}\mathcal{A}_1^+ x = M(x)$, so (c) holds.

Now assume that (c) holds. Then for any $x \in Y$ one has a resolution

$$0 \rightarrow P^n \rightarrow \dots \rightarrow P^1 \rightarrow P^0 \rightarrow M(x) \rightarrow 0$$

in which P^i is a direct sum of projective modules of the form $\sigma^i P(x)$ for various $x \in Y$. Then (d) follows immediately using this resolution to compute Ext noting that

$$\dim \text{Hom}_{\text{gr } \mathcal{A}}(\sigma^i M(y), M) = \dim yM_i$$

for any integer i , $y \in Y$ and M in $\text{gr } \mathcal{A}$.

Now we show (e) $\Leftrightarrow$ (a). Assume that (e) holds. Note that $\mathcal{A}^!$ satisfies 2.2(b) (see 2.12(a)) and 2.2(a),(d). Moreover, by [BGS, Lemma 7], the Euler-Poincaré principle applied to the Koszul complex of $\mathcal{A}$ gives $p'(\mathcal{A}^!, v) = (p'(\mathcal{A}, -v)^T)^{-1} = \text{Id}$. Hence

$$p'(\mathcal{A}^!, v) = (p'(\mathcal{A}, -v)^T)^{-1} = [p(\mathcal{A}, -v)p(\mathcal{A}, -v)^T]^{-1} = p(\mathcal{A}^!, v)p(\mathcal{A}^!, v)^T.$$

Put $v = 1$; summing the entries of these matrices gives $\dim \mathcal{A}^! = \sum_{y \in Y} (\dim M_!(y))^2$ and so $\mathcal{A}^!$ is qh by 2.2.

Finally, assume (a) holds. Then $\mathcal{A}$ is Koszul, (b) holds and the Euler-Poincaré principle applied to the complexes in (b) give the matrix equation in (e).

One could substitute 1 for v in (e) to have another equivalent condition. I don't know whether the matrix equation in (e) above could be omitted altogether, or whether the assumption that $\mathcal{A}$ is Koszul in (e) could be replaced by the assumption that $\mathcal{A}$ is quadratic and $\mathcal{A}^!$ is qh (leaving the matrix equation).

Proposition 2.16. *Let $\mathcal{A}$ be an $\mathcal{O}$ algebra. Using 2.10, identify $(\mathcal{A}^!)^{\text{op}}$ with $E = \text{Ext}_{\mathcal{A}}^{\bullet}(\mathcal{A}_0, \mathcal{A}_0)$. Then, identifying left $\mathcal{A}^!$ -modules and right E -modules, one has isomorphisms of graded left $\mathcal{A}^!$ -modules $L_!(x) \cong \text{Ext}_{\mathcal{A}}^{\bullet}(\mathcal{A}_0, \delta P(x))$, $M_!(x) \cong \text{Ext}_{\mathcal{A}}^{\bullet}(\mathcal{A}_0, \delta M(x))$ and $P_!(x) \cong \text{Ext}_{\mathcal{A}}^{\bullet}(\mathcal{A}_0, \delta L(x))$.*

Proof. If $\mathcal{A}$ is an $\mathcal{O}$ algebra, then it is a pure qh algebra by the (proven) implication (a) $\Rightarrow$ (d) of 2.15. To complete the proof of 2.15 and of this Proposition, it will therefore suffice to show that if $\mathcal{A}$ is a pure qh algebra, then $\mathcal{A}$ is an $\mathcal{O}$ algebra and one has the identifications of modules described in 2.16.

Fix a pure, qh K -algebra $\mathcal{A}$. Then $\mathcal{A}$ has finite global dimension [CPS1, Section 3], which ensures that all the algebras and modules occurring in the proof below are finite-dimensional. The bounded derived category of complexes of graded $\mathcal{A}$ -modules will be denoted $\mathcal{D} = D^b(\text{gr } \mathcal{A})$; it is equipped with a translation functor $X \mapsto X[n]$, and a twist $X \mapsto X(n)$ induced by σ^n , for each $n \in \mathbb{Z}$. Regard $\text{gr } \mathcal{A}$ as embedded in $D^b(\text{gr } \mathcal{A})$ in the usual way (complexes with cohomology concentrated in degree 0). Recall that the bifunctor $\text{Hom}_{\mathcal{D}}(-, -)$ is cohomological in each variable. Moreover, for M, N in $\text{gr } \mathcal{A}$ one has $\text{Ext}_{\text{gr } \mathcal{A}}^n(M, N) \cong \text{Hom}_{\mathcal{D}}(M, N[n])$. Composition of $\text{Hom}_{\mathcal{D}}$ induces a bilinear multiplication $\text{Ext}_{\text{gr } \mathcal{A}}^n(M, N) \times \text{Ext}_{\text{gr } \mathcal{A}}^m(L, M) \rightarrow \text{Ext}_{\text{gr } \mathcal{A}}^{n+m}(L, N)$ which is associative for triple products. Using the well-known identifications

$$(a) \quad \text{Ext}_{\mathcal{A}}^n(M, N) \cong \bigoplus_{k \in \mathbb{Z}} \text{Ext}_{\text{gr } \mathcal{A}}^n(\sigma^k M, N),$$

one obtains a bilinear multiplication (Yoneda product) $\text{Ext}_{\mathcal{A}}^n(M, N) \times \text{Ext}_{\mathcal{A}}^m(L, M) \rightarrow \text{Ext}_{\mathcal{A}}^{n+m}(L, N)$, and triple products are again associative when defined. In particular, one has a graded associative algebra E (Yoneda Ext algebra) with $E_n := \text{Ext}_{\mathcal{A}}^n(\mathcal{A}_0, \mathcal{A}_0)$, and for X in $\mathcal{D}$, one has a graded right E -module $\tilde{X}$ with $\tilde{X}_k := \bigoplus_j \text{Hom}_{\mathcal{D}}(\mathcal{A}_0(j), X[k])$.

It will be convenient to denote the “induced module” $\delta M(x)$ in $\text{gr } \mathcal{A}$ by $A(x)$, and the injective hull of $L(x)$ by $I(x) := \delta P(x)$. Note that under the natural identification $E_0 = \text{Hom}_{\mathcal{A}}(\mathcal{A}_0, \mathcal{A}_0) \cong \mathcal{A}_0^{\text{op}}$, the idempotent $x \in Y$ identifies with the projection $\mathcal{A}_0 \rightarrow Kx \rightarrow \mathcal{A}_0$. It follows immediately that the simple right E -module not annihilated by the idempotent x is $\widetilde{I(x)}$ and that its projective cover is $\widetilde{L(x)}$. Now $\text{Ext}_{\mathcal{A}}^n(L(y), A(x))$ is zero unless $y \geq x$, and if $y = x$ it is zero unless $n = 0$ in which case it is one-dimensional (see [CPS1, 3.8(b)] or [I1, 4.5]). Also, the injection $A(x) \rightarrow I(x)$ induces a surjection $\widetilde{A(x)} \rightarrow \widetilde{I(x)}$ of right E -modules, and the above remarks imply that all the composition factors of the kernel are of the form $\sigma^k \widetilde{I(y)}$ for various $y > x$ and integers k . This verifies that E^{op} satisfies defining conditions 2.1(a)–(b) of a quasi-hereditary algebra (with Y replaced by Y^{op}). To show that 2.1(c) holds and that $\mathcal{A}$ is Koszul, we now repeat in the graded setting some arguments from [CPS2, Section 2].

Define $\mathcal{E}_L$ to be the smallest full additive subcategory of $\mathcal{D}$ which contains all objects $M(x)k$ for all $x \in Y$ and $k \in \mathbb{Z}$, and is closed under extension (i.e if X_1, X_3 are objects of $\mathcal{E}_L$ and there is a distinguished triangle $X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow$ in $\mathcal{D}$, then X_2 is an object of $\mathcal{E}_L$). Similarly, define $\mathcal{E}_R$ to be the full additive subcategory of $\mathcal{D}$ generated in the same way by the objects $A(x)k$ with $x \in Y$ and $k \in \mathbb{Z}$. Now one has that

$$(a) \quad \text{Hom}_{\mathcal{D}}(M(x), A(y)[n](k)) \cong \text{Ext}_{\text{gr } \mathcal{A}}^n(M(x), \sigma^k A(y))$$

is zero unless $x = y$ in Y and $n = k$ in $\mathbb{Z}$, when it is isomorphic to K . This follows either from the ungraded version [CPS, 2.2] of the same result and (a), or by the same proof as [I2, 6.4], using $\dim L(x) = 1$ in place of the assumption there that K is algebraically closed. It follows that for objects X of $\mathcal{E}_L$ and Z of $\mathcal{E}_R$, one has

$$(c) \quad \text{Hom}_{\mathcal{D}}(X, Z[n](k)) = 0 \text{ unless } n = k.$$

In turn, this implies that for any distinguished triangle $U \rightarrow V \rightarrow W \rightarrow$ with U, V, W in $\mathcal{E}_L$, and any Z in $\mathcal{E}_R$, the sequence

$$(d) \quad 0 \rightarrow \text{Hom}_{\mathcal{D}}(W, Z[n](k)) \rightarrow \text{Hom}_{\mathcal{D}}(V, Z[n](k)) \rightarrow \text{Hom}_{\mathcal{D}}(U, Z[n](k)) \rightarrow 0$$

is exact for any integers n, k . One can show, by exactly the same argument as [CPS2, 2.4], that

(e) an object X of $\mathcal{D}$ belongs to $\mathcal{E}_L$ (resp., to $\mathcal{E}_R$) iff for all $y \in Y$ and integers k, n , one has $\text{Hom}_{\mathcal{D}}(X, A(y)[n](k)) = 0$ unless $n = k$ (resp., $\text{Hom}_{\mathcal{D}}(M(x)[n](k), X) = 0$ unless $n = k$).

Since the remainder of the proof essentially involves a reinterpretation of the proof of this claim, we recall the argument. The verification for $\mathcal{E}_L$ is dual to that for $\mathcal{E}_R$, so by (c) one needs only check that if $\text{Hom}_{\mathcal{D}}(M(y)[n](k), X) = 0$ unless $n = k$, then X is in $\mathcal{E}_L$. Let Y' be an ideal of Y and $\mathcal{A}' = \mathcal{A}/J$ where J is the ideal of $\mathcal{A}$ generated by idempotents $x \in Y \setminus Y'$. One has by [CPS3, 4.5] and [CPS1, 3.7 and 1.3] a full embedding $D^b(\text{gr } \mathcal{A}') \rightarrow \mathcal{D}$ which has as strict image the full subcategory of $D^b(\text{gr } \mathcal{A})$ consisting of all complexes in $D^b(\text{gr } \mathcal{A})$ with all cohomology objects in the full subcategory $\text{gr } \mathcal{A}'$ of $\text{gr } \mathcal{A}$. One may therefore suppose that the poset Y is generated as ideal by the $x \in Y$ such that some $\sigma^k L(x)$ appears as a composition factor of some cohomology object of X , and we proceed by induction on the cardinality of Y .

Choose $x \in Y$ maximal; we proceed now by induction on the total of the multiplicities of all $\sigma^k L(x)$ as composition factor of all cohomology objects $H^n(X)$, for all integers k and n . Choose integers k, n such that $\sigma^k L(x)$ is a composition factor of $H^n(X)$. We use repeatedly below the fact that $M(x)$ (resp., $A(x)$) is projective (resp., injective) in $\text{gr } \mathcal{A}$. One has

$$\text{Hom}_{\mathcal{D}}(M(x)[-n](k), X) \cong \text{Hom}_{\text{gr } \mathcal{A}}(\sigma^k M(x), H^n(X)) \neq 0,$$

so $k = -n$. One may choose a morphism $X \rightarrow A(x)k$ inducing a surjection

$$\text{Hom}_{\mathcal{D}}(M(x)k, X) \rightarrow \text{Hom}_{\mathcal{D}}(M(x)k, A(x)k) \cong K.$$

Form the distinguished triangle

$$(f) \quad X' \rightarrow X \rightarrow A(x)k \rightarrow .$$

For any $y \in Y$ and integers p, q , this triangle induces an embedding

$$\text{Hom}_{\mathcal{D}}(M(y)[p](q), X') \rightarrow \text{Hom}_{\mathcal{D}}(M(y)[p](q), X),$$

and if $y = x, p = q = k$ the embedding is not an isomorphism. It follows by induction that X' belongs to $\mathcal{E}_L$, and hence so does X .

We now show that $\mathcal{A}$ is Koszul. The purity assumption and the existence of a duality imply that for x, y in Y and integers n, k one has that

$$\mathrm{Ext}_{\mathrm{gr} \mathcal{A}}^n(L(x), \sigma^k A(y)) \cong \mathrm{Ext}_{\mathrm{gr} \mathcal{A}}^n(\sigma^{-k} M(y), L(x))$$

is zero unless $n = k$. It follows by (e) that for all $x \in Y$, $L(x)$ is an object of both $\mathcal{E}_L$ and $\mathcal{E}_R$, and (c) immediately gives that 2.10(b) is satisfied. Hence $\mathcal{A}$ is Koszul by Definition 2.11. Henceforward, we identify $(\mathcal{A}^!)^{\mathrm{op}}$ with the Yoneda Ext algebra E ; the identifications $L_!(x) \cong \widetilde{P(x)}$ and $P_!(x) \cong \widetilde{L(x)}$ follow immediately by previous remarks.

Now applying the functors $\mathrm{Hom}_{\mathcal{D}}(\mathcal{A}_0(j), -)$ for integers j to the distinguished triangle (f) and taking the direct sum over j gives by (d) an exact sequence

$$0 \rightarrow \tilde{X}' \rightarrow \tilde{X} \rightarrow \sigma^{-k} \widetilde{A(x)} \rightarrow 0$$

of graded left $\mathcal{A}^!$ -modules. It follows by induction that for any object X of $\mathcal{E}_L$, there is a filtration $\tilde{X} = F^0 \supseteq F^1 \supseteq \dots \supseteq F^m = 0$ in $\mathrm{gr} \mathcal{A}^!$ such that each F^i / F^{i+1} is isomorphic to $\sigma^{k_i} \widetilde{A(x_i)}$ for some integer k_i , and $x_i \in X$. Moreover, if $X = L(x)$ for some $x \in X$, it is clear from the above proof of (e) that one may take $x_0 = x$ and $x_i < x$ for all $i = 1, \dots, n$. We have now verified that $A^! \cong E^{\mathrm{op}}$ satisfies the conditions 2.1(a)–(c) with Y replaced by Y^{op} , so $\mathcal{A}^!$ is quasi-hereditary with weight poset Y^{op} and Verma modules $\widetilde{A(x)}$. Finally, $\mathcal{A}^!$ is qh by 2.12(b), and one must have $\widetilde{A(x)} \cong M_!(x)$.

The conclusion of the following result has been obtained under slightly different assumptions in [I, 4.4 and 4.6] and [CPS2, 3.7 and 3, Appendix].

Corollary 2.17. *Let $\mathcal{A}$ be a $\mathcal{O}$ algebra, and let C, D denote either L, L or M, L so that for $y, w \in Y$, one has a pair of modules $C(y), D(w)$. Define the matrix Poincaré series $E^{C,D} = E^{C,D}(\mathcal{A}, v)$ with (y, w) -entry*

$$E_{y,w}^{C,D} = \sum_{n \in \mathbb{Z}} \dim \mathrm{Ext}_{\mathcal{A}}^n(C(y), D(w)) v^n.$$

Then $E^{L,L} = p(\mathcal{A}^!, v) p(\mathcal{A}^!, v)^T$ and $E^{M,L} = p(\mathcal{A}^!, v)^T$.

Proof. The result follows by comparing the two expressions for the dimensions of $y M_!(x)_n$ and $(x \mathcal{A}^! y)_n$ from 2.16, and using graded Brauer-Humphreys reciprocity 2.6(b).

The preceding result is also a consequence of the following refinement for $\mathcal{O}$ algebras of Beilinson-Ginsburg's Koszul duality theorem [BGS, Theorems 14, 16] for Koszul algebras. The refinement is proved for the algebras of regular and singular blocks of $\mathcal{O}$ in [loc cit, Theorem 24] and could be proved in general by the same argument.

Theorem 2.18. *Let $\mathcal{A}$ be an $\mathcal{O}$ algebra with weight poset Y . Then there is an equivalence of triangulated categories $\kappa: D^b(\mathrm{gr} \mathcal{A}^!) \rightarrow D^b(\mathrm{gr} \mathcal{A})$ satisfying (a) and (b) below:*

- (a) $\kappa(X[n]) \cong (\kappa X)[n]$, $\kappa(X(n)) \cong (\kappa X)-n$ for X in $D^b(\mathrm{gr} \mathcal{A}^!)$
- (b) $\kappa(\delta L_!(x)) \cong P(x)$, $\kappa(\delta M_!(x)) \cong M(x)$, $\kappa(\delta P_!(x)) \cong L(x)$ for all $x \in Y$.

Proof. The only part which is not immediate from [loc cit, Theorem 15–16] is the second assertion in (b); this is an immediate consequence of 2.15(c) and the definition of the equivalence of categories in the proof of [loc cit], or could be proved by noting that [loc cit, Lemmas 21–22] extend to this situation.

A result due to Irving [I2, 6.6] gives a sufficient condition for a qh algebra to be a pure Koszul algebra, in terms of the existence of "BGG-resolutions" for all of its simple modules. The following result gives a necessary and sufficient condition for the existence of a BGG-resolution of a simple module in a $\mathcal{O}$ algebra (the condition (b) seems unlikely to hold for general $\mathcal{O}$ algebras, or even for those satisfying Conjecture 1.6, but results in [I2] imply that BGG resolutions always exist in the case of $\mathcal{O}$ algebras with weight posets of length at most three).

Proposition 2.19. *Let $\mathcal{A}$ be a $\mathcal{O}$ algebra. Then for any $y \in Y$, the following conditions (a)–(b) are equivalent:*

(a) *there is a BGG-resolution $0 \rightarrow M^{-n} \rightarrow \dots \rightarrow M^{-1} \rightarrow M^0 \rightarrow L(y) \rightarrow 0$ in $\text{gr } \mathcal{A}$ i.e. a resolution in which M^{-i} has a filtration with successive subquotients of the form $\sigma^i M(x)$ for various $x \in Y$.*

(b) *there exists a filtration $P_!(y) = F^0 \supseteq F^1 \supseteq \dots \supseteq F^k = 0$ in $\text{gr } \mathcal{A}^!$ such that each F^i/F^{i+1} itself has a filtration with all successive subquotients of the form $\sigma^i M_!(x)$ for various $x \in Y$.*

Proof. Let $\mathcal{C} = \mathcal{K}(\mathcal{A}, \delta P_!(y))$ and assume that (b) holds. Then $\delta P_!(y)$ has a filtration

$$\delta P_!(y) = G^{-m} \supseteq \dots \supseteq G^0 \supseteq G^1 = 0$$

in which G^i/G^{i+1} has a filtration with successive subquotients all of the form $\delta(\sigma^{-i} M_!(x))$ for various $x \in Y$. This induces a filtration $\mathcal{C} = \mathcal{C}^{-m} \supseteq \dots \supseteq \mathcal{C}^0 \supseteq \mathcal{C}^1 = 0$ of $\mathcal{C}$ by subcomplexes $\mathcal{C}^i := \mathcal{K}(\mathcal{A}, G^i)$. There is a spectral sequence with $E_1^{in} = H^{n+i}(\mathcal{C}^i/\mathcal{C}^{i+1})$. Now the complex $\mathcal{C}^i/\mathcal{C}^{i+1} \cong \mathcal{K}(\mathcal{A}, G^i/G^{i+1})$ has a filtration with successive subquotient complexes $\mathcal{K}(\mathcal{A}, \delta(\sigma^{-i} M_!(x)))$ for various $x \in Y$, and one has by 2.15 that $H^{n+i}(\mathcal{K}(\mathcal{A}, \delta(\sigma^{-i} M_!(x))))$ is zero unless $n = 0$, when it is isomorphic to $\sigma^{-i} M(x)$. Hence the spectral sequence collapses; $E_1^{in} = 0$ unless $n = 0$, $E_2 = E_\infty$ and moreover E_1^{i0} has a filtration with successive subquotients all of the form $\sigma^{-i} M(x)$ for various $x \in Y$. But since $\mathcal{A}$ is Koszul, $H^n(\mathcal{C})$ is zero for $n \neq 0$ and isomorphic to $L(y)$ if $n = 0$. A suitable resolution as in (a) is therefore given by setting $M^i = E_1^{i0}$ with the differential of the E_1 term of the spectral sequence.

Conversely, assume that there is a resolution as in (a), and set $\mathcal{C}^i = \mathcal{K}'(\mathcal{A}^!, \delta M^{-i})$, where we write $\mathcal{K}'$ for $\mathcal{K}_{\mathcal{A}^!}$. Now δM^{-i} has a filtration with all subquotients of the form $\sigma^{-i} \delta M(x)$ for various $x \in Y$, and, by 2.15 applied to $\mathcal{A}^!$, $H^n(\mathcal{K}'(\mathcal{A}^!, \sigma^{-i} \delta M(x)))$ is zero unless $n = -i$ when it is isomorphic to $\sigma^i M_!(x)$. Hence $H^n(\mathcal{C}^i)$ is zero unless $n = -i$, when it has a filtration with successive subquotients all of the form $\sigma^i M_!(x)$ for various $x \in Y$. Now one has an exact sequence of complexes

$$0 \rightarrow \mathcal{K}'(\mathcal{A}^!, \delta L(y)) \rightarrow \mathcal{C}^0 \rightarrow \dots \rightarrow \mathcal{C}^n \rightarrow 0.$$

Changing the differentials on the $\mathcal{C}^i$ with i even to their negatives, one obtains a double complex $\mathcal{C} = \oplus \mathcal{C}^{pq}$ with $\mathcal{C}^{pq} = \mathcal{K}'(\mathcal{A}^!, \delta M^{-p})^q$ if $p \geq 0$, $q \in \mathbb{Z}$ and $\mathcal{C}^{pq} = 0$ if $p + q > 0$. We compare the two spectral sequences of $\mathcal{C}$. Since $\mathcal{K}'(\mathcal{A}^!, \delta L(x))$ is a complex with only one non-zero term $P_!(y)$ in degree zero, one of the spectral sequences has ${}''E_1^{00} \cong P_!(y)$ and ${}''E_1^{pq} = 0$ otherwise. The other spectral sequence

has $'E_1^{in} = H^n(C^i)$, and gives a filtration $P_!(y) = ''E_\infty^{00} = F^0 \supseteq \dots \supseteq F^k = 0$ in $\text{gr } \mathcal{A}^!$ with $F^i/F^{i+1} \cong 'E_\infty^{i,-i}$. The previously stated facts about $H^n(C^i)$ complete the proof since they imply $'E_1 = 'E_\infty$.

Shellability. In this and subsequent papers, there will be situations in which the structure of the order complex of a weight poset for a $\mathcal{O}$ algebra $\mathcal{A}$ is reflected in combinatorial structure of a family of modules in $\text{gr } \mathcal{A}$; because of indications of relations between this phenomenon and the combinatorial notion of shellability of posets, we call such algebras shellable. This subsection is devoted to some formal remarks on shellable algebras; we also briefly describe an example in 2.26. Throughout, we assume that $\mathcal{A}$ is an $\mathcal{O}$ K -algebra with weight poset X , though the definition could be given more generally.

2.20 We define for each $x \in X$ a partially ordered set $\tilde{I}_x$ of isomorphism classes of graded $\mathcal{A}$ -modules with properties (a), (b) below:

- (a) For any $\{F\} \in \tilde{I}_x$, one has $[F]_{v=1} = [M(x)]_{v=1}$.
- (b) For any $\{F\} \leq \{G\}$ in $\tilde{I}_x$, one has $[G] - [F] \in (v^{-1} - v)\mathcal{R}X$ and

$$\left. \frac{[G] - [F]}{v^{-1} - v} \right|_{v=1} = \sum_{k=1}^n [M(y_i)]_{v=1}$$

for some $n \in \mathbb{N}$ and $y_1, \dots, y_n \in X$ with all $y_i < x$.

Note that in (b), the multiset $\{y_1, \dots, y_n\}$ is uniquely determined by x , $\{F\}$ and $\{G\}$; we denote it by $\mathcal{X}_x(F, G)$.

The definition of $\tilde{I}_x$ is recursive. Assume $\tilde{I}_y$ is defined for all $y < x$. For $F, G \in \text{gr } \mathcal{A}$, write $\{F\} \leq \{G\}$ if there exist $y < x$, $\{H\} \in \tilde{I}_y$, $k \in \mathbb{Z}$ and an exact sequence

$$(*) \quad 0 \longrightarrow \sigma^{k+1}H \longrightarrow F \longrightarrow G \longrightarrow \sigma^{k-1}H \longrightarrow 0.$$

Note that this implies that

$$(c) \quad \frac{[G] - [F]}{v^{-1} - v} = v^k[H].$$

Write $\{F\} \leq \{G\}$ if there exist isomorphism classes

$$\{F\} = \{F^0\} \leq \dots \leq \{F^n\} = \{G\}$$

in $\text{gr } \mathcal{A}$. We let $\tilde{I}_x$ denote the interval $[\{M(x)\}, \{\delta M(x)\}]$ in the induced order. It is clear that (a), (b) above hold and, moreover, that $\tilde{I}_x$ is a graded poset (all maximal totally ordered subsets have the same finite cardinality) with order-reversing involution given by $\{F\} \mapsto \{\delta F\}$. The modules F with $\{F\}$ in $\tilde{I}_x$ will be referred to as interpolating modules for $M(x)$.

Definition 2.21. The $\mathcal{O}$ algebra $\mathcal{A}$ is shellable if for each $x \in X$, one has $\tilde{I}_x \neq \emptyset$.

In this case, for any maximal chain

$$(*) \quad \{M(x)\} = \{F^0\} \leq \dots \leq \{F^n\} = \{\delta M(x)\}$$

in $\tilde{I}_x$, define a sequence $y_1, \dots, y_n$ in X by

$$\left. \frac{[F^i] - [F^{i-1}]}{v^{-1} - v} \right|_{v=1} = [M(y_i)]_{v=1}.$$

We call a sequence $y_1, \dots, y_n$ arising in this way a shelling sequence for $M(x)$ in $\text{gr } \mathcal{A}$.

Note that shellability of $\mathcal{A}$ is equivalent to the existence of the following data (a)-(b):

(a) For each $x \in X$ a graded poset S_x with minimum element 0_x and maximum element 1_x , and a function $S_x \rightarrow \text{gr } \mathcal{A}$, denoted $F \mapsto M(x, F)$, with $M(x, 0_x) \cong M(x)$ and $M(x, 1_x) \cong \delta M(x)$.

(b) An assignment to each pair $F \leq G$ in S_x of an exact sequence

$$0 \longrightarrow \sigma^{k+1} M(y, H) \longrightarrow M(x, F) \longrightarrow M(x, G) \longrightarrow \sigma^{k-1} M(y, H) \longrightarrow 0$$

of graded $\mathcal{A}$ -modules where $y < x$ in X , $H \in S_y$ and $k \in \mathbb{Z}$. We will indicate dependence of y, H, k on x, F, G by writing $(y, H, k) = \mathcal{Y}(x, F, G)$.

Henceforward we will call such a set of data a set of shelling data for $\mathcal{A}$.

Proposition 2.22. *Shellability of $\mathcal{A}$ implies shellability of the algebra $\mathcal{A}_Y$, for any closed subset Y of X (notation as in 2.14). In fact, if $y_1, \dots, y_n$ is a shelling sequence for $M_{\mathcal{A}}(x)$ in $\text{gr } \mathcal{A}$, and Y is any closed subset of X containing x , then the subsequence of $y_1, \dots, y_n$ consisting of elements of Y is a shelling sequence for $M_{\mathcal{A}_Y}(x)$ in $\text{gr } \mathcal{A}_Y$.*

Proof. Assume that $\mathcal{A}$ is shellable. It will be sufficient to prove the assertion if Y is either an ideal or a coideal of X . Consider $x \in Y$ and any maximal chain

$$(a) \quad \{M_{\mathcal{A}}(x)\} = \{F^0\} \leq \dots \leq \{F^n\} = \{\delta M_{\mathcal{A}}(x)\}$$

in $\tilde{I}_x$. Suppose first that X is an ideal of Y . One sees immediately by induction that the modules F_i may be regarded as $\mathcal{A}_X$ -modules and that (a) above may be regarded as a maximal chain in $\tilde{I}_x$ (for $\mathcal{A}_X$).

Now suppose that Y is a coideal of X , and let $e = \sum_{y \in Y} y$. Let $J = \{m \mid 1 \leq k \leq n, y_m \in Y\}$, say $J = \{i_1, \dots, i_k\}$ where $1 \leq i_1 < \dots < i_k \leq n$. Note that one has an exact functor $\text{gr } \mathcal{A} \rightarrow \text{gr } \mathcal{A}_Y$, intertwining the respective dualities, and defined on objects by $M \mapsto eM$. For $i = 0, \dots, n$ set $G^i = eF^i$ in $\text{gr } \mathcal{A}_Y$. One has $G^i \cong G^{i-1}$ if $i \notin J$, and it follows immediately by induction that

$$\{M_{\mathcal{A}_Y}(x)\} = \{G^0\} \leq \{G^{i_1}\} \leq \dots \leq \{G^{i_k}\} = \{\delta M_{\mathcal{A}_Y}(x)\}$$

is a maximal chain in $\tilde{I}_x$ for $\mathcal{A}_Y$, which establishes shellability of $\mathcal{A}_Y$.

2.23 We wish to make some comments on the possible relevance of this concept of shellability to the determination of multiplicities in certain highest weight categories. Consider the multiplicity matrix $p = p(\mathcal{A}, v)$ defined in 2.6(b), and define the matrix $r = p^{-1}\bar{p}$. One has

$$(a) \quad \bar{p} = pr, \quad p_{x,y} = 0 \text{ if } x \not\leq y, \quad p_{y,y} = 1 \quad \text{and} \quad p_{x,y} \in v\mathbb{Z}[v] \text{ if } x < y.$$

An argument due to Gabber (see [Dy4, 2.1]) shows that p is the unique matrix satisfying (a), and that the entries of p may be determined recursively from those of r .

Suppose that $\mathcal{A}$ is shellable, with shelling data as in 2.21. For each $x \in X$ and $G \in S_x$ with $G \neq 0_x$, choose some $G' \triangleleft G$ in S_x . For $x \leq y$ in X and $F \in S_x$, let $\mathcal{C}_F(x, y)$ denote the set of sequences

$$(a) \quad C : (y, F) = (y_0, F_0), \dots, (Y_N, F_N) = (x, 0_x) \quad y_i \in X, F_i \in S_{y_i}$$

such that for $0 \leq i < N$ one has $F_i \neq 0_{y_i}$ and $\mathcal{Y}(y_i, (F_i)', F_i) = (y_{i+1}, F_{i+1}, k_{i+1})$ where $k_{i+1} \in \mathbb{Z}$; set $l(C) = N$ and $k(C) = \sum_{i=1}^N k_i$. Now define

$$(b) \quad r_{x,y}(F) = \sum_{C \in \mathcal{C}_F(x,y)} v^{k(C)} (v^{-1} - v)^{l(C)}.$$

This is independent of the choices of the G' since by 2.19(c), one has in $\text{Gr}_*(\mathcal{A})$ that

$$(c) \quad [M(y, F)] = \sum_{x \leq y} r_F(x, y) [M(x)].$$

In particular, $r_{x,y}(1_y)$ coincides with $r_{x,y}$ as defined above. Hence if shelling data for $\mathcal{A}$ can be explicitly described (directly or recursively), then the multiplicity matrix p is recursively computable. Moreover, a conjecture on the multiplicity matrix arising in this way from conjectural shelling data could be proved by establishing the existence of the corresponding family of interpolating modules (together with the exact sequences and dualities relating them).

For $x \in X$, $G = \delta M(x)$ and $F = M(x)$, the equation 2.20(b) becomes a ‘‘Jantzen sum’’ formula

$$(d) \quad \sum_{n \geq 0} n \dim yM(x)_n = \sum_{i=1}^n \dim yM(y_i), \text{ for some } y_1, \dots, y_n < x.$$

(e) The multiset $\Gamma_x = \{y_1, \dots, y_n\}$ above depends only on x , and one may define a directed graph Ω with vertex set X and edge-set E determined by $(y, x) \in E$ iff the multiplicity of y in Γ_x is non-zero, for $x, y \in X$ with $y < x$. It follows immediately from (d) by induction that $L(y)$ is a composition factor of $M(x)$ iff there is a directed path $y = y_0, \dots, y_n = x$, $(y_{i-1}, y_i) \in E$ in Ω from y to x . It is expected that for the regular algebras satisfying Conjecture 1.6 (and the singular algebras constructed from them), any non-zero homomorphism of (ungraded) Verma modules is an embedding, and that there is an embedding of $M(y)$ in $M(x)$ iff $L(y)$ is a composition factor of $M(x)$; I don’t know if these properties are general consequences of shellability.

2.24 In general, one cannot describe the structure of the posets $\tilde{I}_x$ for a shellable algebra entirely in terms of the directed graph Ω defined above. However, for the algebras in conjecture 1.6, it is expected that the map $F \mapsto \mathcal{X}_x(\{M(x)\}, F)$, $F \in \tilde{I}_x$, should be an isomorphism of $\tilde{I}(x)$ onto a subposet of the inclusion-ordered poset of subsets of $E_x = \{y \in X \mid (y, x) \in E\}$. Under this identification, maximal chains in $\tilde{I}_x$ would correspond to certain orderings of E_x which should admit a recursive characterization analogous to the definition [BW, 3.1] of a recursive atom ordering in combinatorics. We illustrate by giving an easy part of the expected connection in the face lattice case.

Proposition 2.25. *Let X be the face lattice of a polyhedral cone, with minimal element 0_X . Suppose that there is a shellable $\mathcal{O}$ algebra $\mathcal{A}$ such that multiplicities of irreducibles in Verma modules are described by Stanley's polynomials i.e. one has an equality $[M(x)] = P(x, \emptyset)$ in $\text{Gr } * \mathcal{A}$ for all $x \in X$, where $P(x, \emptyset)$ is defined in 1.5. Then any shelling sequence for $M(x)$ is a recursive coatom order of the interval $[0_X, x]$.*

Proof. Suppose that $y_1, \dots, y_n$ is a shelling sequence for $M(x)$ associated to a maximal chain

$$(a) \quad \{M(x)\} = \{F^0\} < \dots < \{F^n\} = \{\delta M(x)\}$$

in $\tilde{I}(x)$. Then the multiset $\{y_1, \dots, y_n\} = \mathcal{X}_x(M(x), \delta M(x))$ is determined by 2.20(b). Comparing with 1.5, one sees that it coincides with the multiset $\{y \in X \mid (y, x) \in E\}$ (the argument thus far applies also to Bruhat intervals). In the face lattice case, which we now consider exclusively, this gives that $y_1, \dots, y_n$ is an ordering of the complete set of coatoms of $[0_X, x]$ (without repetition). We prove the following claim:

(*) for each $i = 1, \dots, n$, there is a shelling sequence $z_1, \dots, z_m$ for $M(y_i)$ such that for some $0 \leq j \leq m$, the set $\{z_1, \dots, z_j\}$ is precisely the set of coatoms of $[0_X, y_i]$ which are covered by at least one of the elements $y_1, \dots, y_{i-1}$; moreover, j is non-zero if $i \geq 2$.

The proposition follows trivially from (*) by induction and the definition of a recursive coatom order. To prove (*), we use some simple facts from [St, 2]. Specifically, we need only that the assumption $[M(y)] = P(y, \emptyset)$ implies (b) below:

(b) for any $y < z$ in X , $p_{y,z}(\mathcal{A}, v)$ is a monic polynomial (in v) without constant term, of degree $l(y, z)$, and is in fact equal to $v^{l(y,z)}$ if $l(y, z) \leq 2$.

Now applying 2.22 (and its proof) to the length one closed subintervals of X shows that all the integers k occurring in exact sequences 2.20(*) with $\{F\} < \{G\}$ in $\tilde{I}(x)$ are equal to zero. It follows that for any $y < x$ with $l(y, x) \leq 2$ and $k \in \mathbb{Z}$, one has that $\dim y(F^i)_k$ is equal to 1 if

$$k = l(y, x) - 2\#\{j \mid 1 \leq j \leq i \text{ and } y_j \in [y, x]\}$$

and to zero otherwise. Now if $1 \leq i \leq n$, one has immediately from 2.20(c) that

$$(c) \quad [F^i] - [F^{i-1}] = (v^{-1} - v)[H]$$

for some $\{H\} \in \tilde{I}(y_i)$. From 2.20, there is a shelling sequence $z_1, \dots, z_m$ for $M(y_i)$ and $0 \leq j \leq m$ such that $\mathcal{X}_{y_i}(M(y_i), H) = \{z_1, \dots, z_j\}$; for $1 \leq k \leq m$, $q \in \mathbb{Z}$ one has that $\dim z_k H_q$ is equal to $\delta_{q,1}$ (resp., $\delta_{q,-1}$) if $k > j$ (resp., if $k \leq j$). Evaluating the coefficient in $\mathcal{R}$ of $z_k = [L(z_k)]$ on both sides of (c) for $1 \leq k \leq m$ shows immediately, by the above remarks on $\dim y(F^i)_k$, that $k \leq j$ iff z_k is covered by at least one of $y_1, \dots, y_{i-1}$.

It remains to show that $j > 0$ if $i \geq 2$. First consider (c) in the special case $i = 1$. Evaluating coefficients of z_k as before gives $j = 0$ in this case, and it follows that $H \cong M(y_1)$. Using (b), one now sees inductively that for any $1 \leq i \leq m$, the coefficient in $\mathcal{R}$ of 0_X in $[F^i]$ is a sum of powers v^k with integral $k < l(0_X, x)$, and for $i > 1$ that H in (c) cannot be isomorphic to $M(y_i)$. This completes the proof.

In fact, Stanley's polynomials [St, 5.5(a)] are defined for more general Eulerian lattices X , and the above argument shows that they could not be described in terms of characters of interpolating modules for some shellable $\mathcal{O}$ algebra with weight poset X unless X^{op} is an EL-shellable poset [BW] (of course, this does not exclude the possibility of other interpretations, for instance involving some other class of modules). We mention also that it is not clear whether other parts of the conjectures [St, 4.2] (asserting essentially that the sequences of coefficients of the Stanley polynomials for face lattices are M -vectors) can be approached through the ideas of this paper.

We won't describe in this paper the (conjecturally) appropriate analogue of recursive atom orderings for Bruhat intervals, but mention it is obtained by combining the definition of recursive atom orderings with that of shelling data for dihedral groups as in 3.20.

The algebras constructed in this and subsequent papers to establish special cases of 1.6 form a very special class of $\mathcal{O}$ algebras; we wish to record a rather different example (see 3.34(b) for some motivation for this example).

Example 2.26. Consider the K -algebra $\mathcal{A}_0 \cong K^{n+1}$ (a product of fields) and give the set $X = \{e_0, \dots, e_n\}$ of primitive idempotents of $\mathcal{A}_0$ the ordering $e_n < \dots < e_0$. For each $i = 1, \dots, n$ fix a finite-dimensional vector space V_i over K . Let V be a $\mathcal{A}_0$ -bimodule such that for $0 \leq i, j \leq n$ one has a given isomorphism $\phi_{i,j}: V_{|i-j|} \rightarrow e_i V e_j$ if i, j are distinct, and $e_i V e_j = 0$ if $i = j$. Now let W denote the $\mathcal{A}_0$ -sub-bimodule of $T_{\mathcal{A}_0}^2(V) = \bigoplus_{i=0}^n V e_i \otimes_K e_i V$ spanned by elements of the following types (a), (b):

$$(a) \quad \phi_{j,i}(y) \otimes \phi_{i,m}(x) - \phi_{j,k}(x) \otimes \phi_{k,m}(y)$$

where $i + k = j + m$, $x \in V_{|i-m|}$, $y \in V_{|j-i|}$, and either $0 \leq j < i, k < m \leq n$ or $0 \leq m < i, k < j \leq n$.

$$(b) \quad \phi_{k,m}(y) \otimes \phi_{m,i}(x) - \phi_{k,j}(x) \otimes \phi_{j,i}(y)$$

where $j = i + k - m$, $n \geq m > i, k \geq 0$, $x \in V_{|i-m|}$, $y \in V_{|j-i|}$ and $\phi_{k,j}(x) \otimes \phi_{j,i}(y)$ is interpreted as 0 if $j < 0$.

One can check using 2.15(b) that $\mathcal{A} := T_{\mathcal{A}_0}(V)/\langle W \rangle$ is a shellable $\mathcal{O}$ algebra. The multiplicities of irreducibles as composition factors of Verma modules may be described as follows. Let S denote the tensor product over K of symmetric algebras $S := S(V_1) \otimes \dots \otimes S(V_n)$; this has a natural grading with graded components denoted S_j such that each V_i has degree 1, and another grading with graded components $S_{(j)}$ such that each V_i has degree i . Then $\dim e_j M(e_i)_k = \dim(S_{(k)} \cap S_{j-i})$ for any integers $0 \leq i \leq j \leq n$ and k . Similarly, one has $\dim e_i M!(e_j)_k = \dim(\Lambda_{(k)} \cap \Lambda_{j-i})$ where the terms on the right are defined by replacing S (symmetric algebra) by Λ (exterior algebra) in the above. For any $0 \leq i < n$, a sequence $a_1, \dots, a_N$ of elements of $\{e_{i+1}, \dots, e_n\}$ is a shelling sequence of $M(e_i)$ if for each $i < j \leq n$, it contains e_j with multiplicity $\sum \dim V_d$ where the sum is over positive integers d dividing $j - i$. In general, $\mathcal{A}^!$ is not shellable.

The reader wishing to examine some examples of shellability in detail may find it helpful to construct specific interpolating modules for some shelling sequences in the specially simple cases $\dim V_i = \delta_{i,1}$, or $n = 2$; the first of these examples also arises in 3.27 as a singular algebra associated to a dihedral group.

Remark 2.27. An $\mathcal{O}$ algebra $\mathcal{A}$ with weight poset Y has a “Kazhdan-Lusztig theory” in the sense of [CPS, 3] iff there exists a function $l : Y \rightarrow \mathbb{Z}$ such that for $x, y \in Y$ with $l(x)$ of the same parity as $l(y)$, one has $x\mathcal{A}_1y = 0$. In particular, the algebras in 2.26 do not have Kazhdan-Lusztig theories in general, though $\mathcal{O}$ algebras satisfying 1.6, and “singular” algebras constructed from them, would. Conversely, results described in [loc cit, Appendix to 3] would imply that if a (basic) quasi-hereditary algebra with anti-involution has a graded Kazhdan-Lusztig theory, it is a $\mathcal{O}$ algebra.

3. THE MULTIPLICITY FREE CASE

In this section, we establish the basic properties of a family of algebras which satisfy conjecture 1.6 in the simplest case of an interval isomorphic to a product of dihedral Bruhat intervals. We begin by describing the general data from which the construction in this and subsequent papers will begin, then immediately impose a special assumption 3.5 which will ensure that the Verma modules for the algebra are multiplicity free.

3.1 Throughout this section, unless otherwise stated, X denotes a finite poset such that the order complex of any non-empty open subinterval of X is a combinatorial sphere. This implies that for $x \leq y$ in X , all maximal chains $x = x_0 < \dots < x_n = y$ from x to y have the same length $n = l(x, y)$. Let us say that another maximal chain $x = x'_0 < \dots < x'_n = y$ is adjacent to the first if there is a unique j with $1 \leq j \leq n - 1$ such that $x_j \neq x'_j$. Then one has

- (a) any closed length 2 subinterval $[x, y]$ of X has exactly two atoms
- (b) if C, D are maximal chains from x to y , where $x \leq y$ in X , there exists a sequence $C = C_0, \dots, C_n = D$ of maximal chains from x to y such that C_i is adjacent to C_{i+1} for $i = 1, \dots, n - 1$.
- (c) for any $x \leq z$ in X , one has $\sum_{y \in [x, z]} (-1)^{l(y, z)} = \delta_{x, z}$

3.2 Let E_0 denote the edge set of the Hasse diagram of X . We assume that X is endowed with a labelling function $\ell : E_0 \rightarrow V \setminus 0$, where V is a finite-dimensional vector space over a field K , satisfying the following conditions (a),(b):

- (a) For any length two subinterval $[x, z]$ of X with atoms y_1, y_2 , one has, writing $\alpha_i = \ell(x, y_i)$ and $\beta_i = \ell(y_i, z)$, that α_1, α_2 are linearly independent and that for $i = 1, 2$ one has $\beta_i = \sum_{j=1}^2 a_{ij}\alpha_j$ for some $(a_{ij}) \in \text{GL}_2(K)$ with $a_{12} = a_{21} \neq 0$
- (b) Let I be a closed length 3 subinterval of X . If $K\langle \ell(x, y) \mid x, y \in I, x < y \rangle$ is two-dimensional, then I has exactly two atoms (equivalently, exactly two coatoms).

Remarks 3.3. (a) Any two labellings as above which for each subinterval give the the same matrices (a_{ij}) will yield isomorphic algebras. The fact that the matrices (a_{ij}) are obtained from a labelling should be regarded as giving subtle global relations between them for the various length 2 subintervals.

(b) For any closed interval $I = [x, y]$ of X , set $\ell(I) = K\langle \ell(v, w) \mid x \leq v < w \leq y \rangle$. Then $\ell(I) = K\langle \ell(y', y) \mid x \leq y' < y \rangle = K\langle \ell(x, x') \mid x < x' \leq y \rangle$ and for any maximal chain $x = x_0 < \dots < x_n = y$, one has $\ell(I) = K\langle \ell(x_{i-1}, x_i) \mid 1 \leq i \leq n \rangle$.

Examples 3.4. (i) By [Dy3, 1.4], face lattices of polyhedral cones with the labelling 1.2 (or the labelling [loc cit, 2.10]) satisfy all conditions in 3.1, 3.2 (one takes $K = \mathbb{R}$ or, by complexifying V , $K = \mathbb{C}$). The same is true of Bruhat intervals with the labelling in 1.3, and more generally, of twisted Bruhat posets with either labelling [loc cit, 3.5 or 3.15]. Examples of this type satisfying an additional condition 3.15(b)

(which is redundant for intervals and may be so in general) will be called standard labelled posets. Each such labelled poset X has the following property:

(a) the real numbers a_{12} occurring in 3.2(a), for the various length 2 subintervals of X , are either all positive or all negative.

The proof of this claim easily reduces to the special cases of dihedral groups and two-dimensional cones, where it follows by simple computations.

(ii) If a Coxeter group has a crystallographic root system, the edge-labels for a poset may be taken in this root system and one can carry out our constructions over $K = \mathbb{Q}$. Since the edge-labels lie in a lattice, suitably labelled posets can also be obtained over fields K of (sufficiently large) prime characteristic p by "reducing the labels" modulo p .

Assumption 3.5. Throughout the whole of Section 3, we will assume unless otherwise stated that every length 3 subinterval of X has at most 3 atoms. For the standard labelled posets of 3.4, it will be seen in 3.19 that this is equivalent to the assumption that every closed subinterval of X is isomorphic as poset to a product of closed dihedral Bruhat intervals (note that this is an isomorphism only as poset, not as labelled poset).

The following sections 3.6–3.8 define a quadratic algebra $\mathcal{A}_X$ which will eventually be shown to be a $\mathcal{O}$ algebra satisfying Conjecture 1.6 whenever (X, ℓ) is a standard labelled poset. We mention that essentially the same conclusions apply to algebras constructed from the labellings over $\mathbb{Q}$ or fields of sufficiently large prime characteristic p in 3.4(ii) (some of the results or their proofs may require that p be larger than needed merely for existence of the algebra).

3.6 We first define an associative K -algebra $\mathcal{A}'_X$ with a K -basis consisting of all symbols $x_n \dots x_0$ where $n \geq 0$, each $x_i \in X$ and for all $i > 0$, either $x_{i-1} \leq x_i$ or $x_i \leq x_{i-1}$. Multiplication is defined on this basis by setting $(y_m \dots y_0)(x_n \dots x_0)$ equal to $y_m \dots y_1 x_n \dots x_0$ if $y_0 = x_n$ and equal to 0 otherwise, then extended to $\mathcal{A}'_X$ by linearity.

The identity element of $\mathcal{A}'_X$ is $\sum_{x \in X} x$. Let $\mathcal{A}'_n$ denote the subspace of $\mathcal{A}'_X$ spanned by all basis elements $x_n \dots x_0$. It is clear that $\mathcal{A}'$ may be identified with the tensor algebra of $\mathcal{A}'_1$ over $\mathcal{A}'_0$, and that there is a K -algebra anti-involution ω of $\mathcal{A}'_X$ defined on the basis elements by $\omega: x_n \dots x_0 \mapsto x_0 \dots x_n$.

3.7 Suppose that $x, y, z, w \in X$ satisfy $x \leq y \leq w$, $x \leq z \leq w$ (we allow $y = z$). Let z' denote the atom of $[x, w]$ unequal to z and use 3.2(a) to write

$$(a) \quad \ell(y, w) = c_{zxy;w} \ell(x, z) + d_{zxy;w} \ell(x, z')$$

where $c_{zxy;w}, d_{zxy;w} \in K$. Note that by the symmetry of (a_{ij}) in 3.2(a), one has

$$(b) \quad c_{zxy;w} = c_{yxz;w}.$$

Now for $x, y, z \in X$ satisfy $x \leq y$ and $x \leq z$, define

$$(c) \quad r_{zxy} = zxy - \sum c_{zxy;w} zwy \in \mathcal{A}'_X$$

where the sum is over $w \in X$ satisfying $y \leq w$ and $z \leq w$. Note that by (b),

$$(d) \quad \omega(r_{zxy}) = r_{yxz}.$$

Definition 3.8. The algebra $\mathcal{A}_X$ is the quotient of $\mathcal{A}'_X$ by the two-sided ideal I generated by elements of the following types (a), (b):

- (a) r_{zxy} for $x, y, z \in X$ with $x \leq y$ and $x \leq z$.
- (b) $xy_1z - xy_2z, zy_1x - zy_2x$ for $y_1 \neq y_2, x, z \in X$ with $x \leq y_i \leq z$ ($i = 1, 2$)

3.9 It is clear that $\mathcal{A}_X$ is a quadratic algebra in the sense of 2.8. By 3.1(b) and the relations 3.8(b), for any $x \leq y \in X$, there are well-defined elements xy and yx in $\mathcal{A}_X$ such that for any maximal chain $x = x_0, \dots, x_n = y$ in X one has $xy = x_0 \dots x_n + I$ and $yx = x_n \dots x_0 + I$. We will abbreviate a product $(y_1y_2)(y_2y_3) \dots (y_{m-1}y_m)$ of such elements in $\mathcal{A}_X$ by $y_1y_2 \dots y_m$, and write x for xx , for any $x \in X$.

Part (b) of the following lemma is immediate from 3.7(d). We will prove part (a) by constructing a representation of A_X in 3.11–3.13 below (alternative proofs could be given).

Lemma 3.10. (a) *The elements xyz with $x \leq y, z \leq y$ in X form a basis of A_X over K .*

(b) *There is a K -algebra anti-involution ω of $\mathcal{A}_X$ satisfying $\omega(xyz) = zyx$ for $x, y, z \in X$ with $x \leq y, z \leq y$.*

3.11 For each $y \in X$, let W_y be a K -vector space with basis $\{\alpha_{yz}\}_{z \geq y}$. For any $x \leq y$ in X , define a linear map $\theta_x^y: W_y \rightarrow W_x$ by setting

$$(a) \quad \theta_x^y(\alpha_{yw}) = c_{yxy;w} \alpha_{xy} + d_{yxy;w} \alpha_{xy'}$$

where $w \geq y$ and y' is the atom of $[x, w]$ unequal to y , and extending by linearity. For any $x \leq y$, choose a maximal chain $x = x_0, \dots, x_n = y$ from x to y and set

$$\theta_x^y = \theta_{x_0}^{x_1} \circ \dots \circ \theta_{x_{n-1}}^{x_n}: W_y \rightarrow W_x.$$

Note that if $x \leq y \leq z$ and $\theta_x^y(\alpha_{yz}) = \sum_{u \geq x} a_u \alpha_{xu}$ where the $a_u \in K$, then $\ell(y, z) = \sum_{u \geq x} a_u \ell(x, u)$ and $a_u = 0$ unless $u \leq z$. The labels $\ell(x, u)$, for fixed x , are not necessarily linearly independent, but θ_x^y is independent of the choice of the maximal chain anyway. In the special case $\ell(x, y) = 2$, this is a consequence of 3.2(b) and the assumption 3.5; in general, it then follows from 3.1(b).

3.12 For each $y \in X$, let V_y be a K -vector space with basis $\{\alpha_z\}_{z \geq y}$. For $x \leq y$, let $i_x^y: V_y \rightarrow V_x$ denote the inclusion, and define $p_y^x: V_x \rightarrow V_y$ by setting

$$(a) \quad p_y^x(\alpha_z) = \sum_{z' \geq z} c_{z'y} \alpha_{z'}$$

where for $z' \geq z$, $c_{z'y} \in K$ is determined by

$$(b) \quad \theta_x^z(\alpha_{zz'}) = \sum_{y' \geq x} c_{z'y'} \alpha_{xy'}.$$

It follows immediately from 3.11 that for $x, y, z \in X$ satisfying, $x \leq y$ and $x \leq z$, one has

$$(c) \quad p_z^x \circ i_x^y = \sum c_{zxy:w} i_z^w \circ p_w^y$$

where the sum is over $w \in X$ satisfying $y \leq w$ and $z \leq w$.

Now suppose that $y_1 \neq y_2$ and $x \leq y_j \leq z$ for $j = 1, 2$. Obviously,

$$(d) \quad i_x^{y_j} \circ i_{y_j}^z \text{ is independent of } j,$$

and we will show by induction that

$$(e) \quad p_z^{y_j} \circ p_{y_j}^x \text{ is independent of } j.$$

Assume (e) holds with z replaced by any $z' < z$. Since $p_z^{y_i} \circ p_{y_i}^x(\alpha_x) = \alpha_z$, it will be sufficient to show that for $w \geq x$, the composite $p_z^{y_j} \circ p_{y_j}^x \circ i_x^w$ is independent of j . Using (c) twice for each $j = 1, 2$, one obtains formulae

$$(f) \quad p_z^{y_j} \circ p_{y_j}^x \circ i_x^w = \sum_{v \geq z} \sum_{w \leq u \leq v} d_{vuj} i_z^v \circ p_v^u \circ p_u^w$$

for certain $d_{vuj} \in K$. But by 3.7(b), one has similarly that

$$(g) \quad p_w^x \circ i_x^{y_j} \circ i_{y_j}^z = \sum_{v \geq z} \sum_{w \leq u \leq v} d_{vuj} i_w^u \circ i_u^v \circ p_v^z.$$

Now by (d), the LHS of (g) is independent of j , and hence, by evaluating the RHS at α_z , one has that for $v \geq w$, $\sum_{w \leq u \leq v} d_{vuj}$ is independent of j . By induction, this implies that the RHS of (f) is independent of j as required.

Now form the direct sum $U = \bigoplus_{x \in X} V_x$, and for $x \in X$, let $\pi_x: U \rightarrow V_x$ and $j_x: V_x \rightarrow U$ denote the corresponding projections and inclusions.

Lemma 3.13. *There exists a unique homomorphism of K -algebras $\psi: \mathcal{A}_X \rightarrow \text{End}_K(U)$ such that $\psi(x) = j_x \circ \pi_x$ for $x \in X$ and*

$$\psi(yx) = j_y \circ p_y^x \circ \pi_x, \quad \psi(xy) = j_x \circ i_x^y \circ \pi_y$$

if $x \leq y$ in X . Further, ψ is injective.

Proof of 3.10 and 3.13. The existence of ψ is immediate from 3.12(c)–(e). It follows from the relations 3.8(a) that the elements xyz in 3.10(a) span $\mathcal{A}_X$ over K . Moreover, for $\alpha_z \in V_z \subseteq U$ one has $\psi(xyz)(\alpha_z) = \alpha_y \in V_x \subseteq U$, and hence the elements $\psi(xyz)$ with $z \leq y$, $x \leq y$ are linearly independent over K . Both 3.10 and 3.13 follow.

Proposition 3.14. *The algebra $\mathcal{A} = \mathcal{A}_X$ defined in 3.8 is a qh algebra. One has*

$$(*) \quad p_{x,y}(\mathcal{A}, v) = v^{l(x,y)} \text{ for any } x \leq y \text{ in } X.$$

Proof. All conditions of 2.2 are clearly satisfied by $\mathcal{A}$ except perhaps for (d). But by 3.10 and the relations 3.8, one has

$$(\dagger) \quad M(x) = \mathcal{A}x / \mathcal{A}\mathcal{A}_1^+ x \text{ has a } K\text{-basis } \{B_y\}_{y \leq x} \text{ where } B_y = (yx)v_x.$$

In fact, for $x, u, z \in X$ with $z \geq x$, $z \geq u$ one has $(uzx)B_y = B_u$ if $x = z = u$ and $(uzx)B_y = 0$ otherwise.

3.15 In order to establish the analogue of Beilinson-Ginsburg duality, we will have to impose the following two additional assumptions (a)–(b) (one can show that (a) follows from 3.1 and 3.2 if X is an interval and the labels of edges in some maximal chain are linearly independent, but I do not know if (a)–(b) are consequences of the conditions in 3.1–3.2 more generally). First, let $E' = \{(x, y) \mid (y, x) \in E_0\}$ denote the edge set of X^{op} .

(a) There is a “dual” labelling $\ell': E' \rightarrow V'$ satisfying 3.2(a)–(b) (with X replaced by X^{op}), and related to ℓ as follows. For $x < z$ in X as in 3.2, define y_i , (a_{ij}) as in 3.2 and $(b_{ij}) \in \text{GL}_2(F)$ by $\alpha'_i = \sum_{j=1}^2 b_{ij}\beta'_j$ where $\alpha'_i = \ell'(y_i, x)$ and $\beta'_i = \ell'(z, y_i)$. Then for $1 \leq i, j \leq 2$, one has $a_{ij} = -b_{ij}$ if $i = j$ and $a_{ij} = b_{ij}$ if $i \neq j$.

Secondly, we require an “orientability” assumption which is redundant by 3.1 if X is an interval.

(b) There is a function $\epsilon: E_0 \rightarrow \{\pm 1\}$ such that if $e_1, \dots, e_4$ are the four edges of the Hasse diagram of a length 2 subinterval of X , one has $\prod_{i=1}^4 \epsilon(e_i) = -1$.

Results in [Dy1, Dy2] imply that (b) holds at least for all those twisted Bruhat orders associated to chains of parabolic subgroups [Dy1, 2.11] (in particular, ordinary Bruhat order and its reverse) and those twisted orders arising in the multiplicity conjecture [Dy1, 6]. We recall that a standard labelled poset was defined to be one of the labelled posets as in 3.4(i) satisfying the condition (b) above; the condition (a) then holds by [Dy3, 2.10 and 3.15].

For the following proposition, note that for any closed subset Y of X one has an algebra $\mathcal{A}_Y$ defined in the same way as $\mathcal{A}_X$, but using the Hasse diagram of Y labelled by the restriction of the labelling of the Hasse diagram of X .

Proposition 3.16. (a) *Let X be a labelled poset satisfying all conditions 3.1–3.2, 3.5 and 3.15(a)–(b). Then the quadratic dual $\mathcal{A}^!$ of $\mathcal{A} = \mathcal{A}_X$ is isomorphic to $\mathcal{A}_Y$ where Y denotes X^{op} with the dual labelling ℓ' in 3.15(a). Moreover, $\mathcal{A}$ is an $\mathcal{O}$ algebra.*

(b) *For any closed subset Y of X , the algebra $\mathcal{A}_Y$ may be identified with the algebra $\mathcal{A}_Y$ defined as in 2.14 by truncation of the weight poset of $\mathcal{A}$.*

Proof. Now $\mathcal{A}_X = \mathcal{A}'_X / I$ where $\mathcal{A}'_X = T_{\mathcal{A}'_0}(\mathcal{A}'_1)$ is as in 3.6, and I is the two-sided ideal of $\mathcal{A}'_X$ generated by the subspace W of $T^2(\mathcal{A}'_1)$ spanned by elements as in 3.8(a)–(b). Similarly, $\mathcal{A}_Y = \mathcal{A}'_Y / J$ where J is the two-sided ideal of $\mathcal{A}'_Y$ generated by elements of the following types (a)–(b):

$$(a) \quad r'_{zxy} \text{ for } x, y, z \in X \text{ with } x \succ y \text{ and } x \succ z$$

$$(b) \quad xy_1z - xy_2z, zy_1x - zy_2x \text{ for } y_1 \neq y_2, x, z \in X \\ \text{with } x \leq y_i \leq z \text{ (} i = 1, 2 \text{)}.$$

Here, for $w < x$ with $l(w, x) = 2$ and $x \succ y \succ w$, $x \succ z \succ w$, we let z' denote the atom of $[w, x]$ unequal to z and write

$$\ell'(y, w) = c'_{zxy;w} \ell'(x, z) + d'_{zxy;w} \ell'(x, z')$$

with $c'_{zxy;w}, d'_{zxy;w} \in K$. Then for $x, y, z \in X$ with $x \succ y$ and $x \succ z$, one has

$$r'_{zxy} = zxy - \sum_{27} c'_{zxy;w} zwy$$

where the sum is over $w \in X$ satisfying $y \succ w$ and $z \succ w$. Set $V = \mathcal{A}'_1$; then $\mathcal{A}'_X = T(V)$ has K -basis consisting of all elements $x_0 \dots x_n$ as in 3.6, and these symbols with $n = 1$ are a basis of V . Let $(x_1 x_0)^\#$ denote the element of the dual basis of the vector space $V^\#$ corresponding to $x_0 x_1$. Then $T_{\mathcal{A}_0}(V^\#)$ has a K -basis consisting of symbols $(x_0 \dots x_n)^\#$ with multiplication

$$(x_0 \dots x_n)^\# (x_n \dots x_{n+k})^\# = (x_0 \dots x_{n+k})^\#.$$

Now there is an algebra isomorphism $\theta: T(V) \rightarrow T(V^\#)$ which is the identity on $\mathcal{A}'_0$ and such that for $x \prec y$, one has $\theta(xy) = \epsilon(x, y)(xy)^\#$ and $\theta(yx) = \epsilon(x, y)(yx)^\#$, where ϵ is as in 3.14(b). It follows that $\mathcal{A}_Y \cong T(V^\#)/\langle U \rangle$ where U is the subspace of $T^2(V^\#)$ spanned by elements of the following forms:

$$(c) \quad xy_1 z^\# + xy_2 z^\#, zy_1 x^\# + zy_2 x^\#$$

for $y_1 \neq y_2, x, z \in X$ with $x \prec y_i \prec z$ ($i = 1, 2$)

$$(d) \quad zxy^\# - (2\delta_{z,y} - 1) \sum c'_{zxy,w} zwy^\# \in \mathcal{A}'_X \text{ for } x, y, z \text{ as in (a).}$$

Now $f(r) = 0$ for all $f \in U \subseteq T^2(V^\#)$, $r \in W \subseteq T^2(V)$; in fact, one easily sees that it is sufficient to check this if X is a length 2 interval, when it follows immediately from 3.15(a). But for $x < y$ in X with $l(x, y) = 2$, one has $\dim xWy = \dim yWx = 1$ and for other x, y one has $\dim xWy = \#\{z \in X \mid z \prec x, z \prec y\}$. Using the analogous fact for U , it follows by a dimension count that U is the orthogonal complement of W , so $\mathcal{A}_Y \cong \mathcal{A}'_X$ as required.

Finally for (a), it remains to verify that $\mathcal{A}$ is a $\mathcal{O}$ algebra. This can be proved by directly checking that the complexes $\mathcal{K}(M(x)^t, \delta M_l(y))$ in 2.15(b) are acyclic if $x < y$. We omit the details here and supply them in a more complex situation (for singular algebras) in 3.27. Part (b) is trivial from 3.8.

Shellability. 3.17 The purpose of this section is to complete the proof of conjecture 1.6 in the multiplicity free case by establishing the existence of the interpolating modules $M(x, F)$. In fact, we will explicitly determine the posets $\tilde{I}_x$ of 2.20 for $x \in X$.

Throughout this subsection, unless otherwise stated, X denotes a standard labelled poset satisfying 3.5 and $\mathcal{A}$ denotes the K -algebra $\mathcal{A}_X$ defined in 3.8; for definiteness, we take $K = \mathbb{R}$, though this is not essential. We begin by describing in detail the structure of dihedral Bruhat intervals and their products.

3.18 Consider the set $\Lambda = \mathbb{Z} \times \{\pm 1\}$, endowed with the order such that $(m, \epsilon) < (m', \epsilon')$ iff $m < m'$ in $\mathbb{Z}$. We call any (non-empty) poset isomorphic to an open, closed or half-open interval in Λ a dihedral interval; any Bruhat interval in a dihedral group is of this type.

Now give Λ the structure of a directed graph, with an edge $((m, \epsilon), (m', \epsilon'))$ iff $m < m'$ and $m' - m$ is odd. Any directed graph isomorphic to a full subgraph of Λ with vertex set a (non-empty) interval in Λ will be called a dihedral graph.

We recall that there is a directed graph Ω associated to X . For face lattices of polyhedral cones, the edgeset E of Ω is just the set E_0 of edges of the Hasse diagram of X . For twisted Bruhat posets, $(x, y) \in E$ iff $x < y$ and xy^{-1} is a reflection in the Coxeter group W . By [Dy1, 2.7] (and the fact that face lattices are lattices), one has

(a) the graph Ω depends only on the isomorphism type of the poset X .

Let E' denote the edge set of the undirected graph underlying Ω . Consider x, y, z in X with $x \neq z$ such that $\{x, y\}, \{y, z\} \in E'$. Let $\Omega_{x,y,z}$ denote the full subgraph of Ω on vertex set A , where A is the smallest subset of X such that (b)–(c) below hold:

(b) $A \supseteq \{x, y, z\}$ and

(c) for $z \in X$ and distinct $p, q \in A$ with $\{z, q\}, \{z, p\} \in E'$, one has $z \in A$.

Then by [loc cit, 2.6] for twisted Bruhat intervals and trivially for face lattices,

(d) $\Omega_{x,y,z}$ is a dihedral graph.

We call such $\Omega_{x,y,z}$ the dihedral subgraphs of X . The above facts (a),(d) are valid for standard posets not necessarily satisfying 3.5. In this paper, we will use the following simple consequence of (d).

(d') if $z \leq x_i \leq z_j$ (resp. $z \geq x_i \geq z_j$) in X for all $i = 1, 2$ and $j = 1, 2, 3$, where x_1, x_2 are distinct, then the z_j are not all distinct.

Now consider the special case in which $X = [u, w]$ is isomorphic as poset to a product of closed dihedral intervals. Then by (a), the graph Ω is the product of the dihedral graphs of these intervals; for $x, y \in X$, one has

(e) $(x, y) \in E$ iff $x < y$, $l(x, y)$ is odd and $[x, y]$ is dihedral.

(f) Let us describe the dihedral subgraphs of X containing w as a vertex. Identify X with a product $X = I_1 \times \dots \times I_m$, ($m > 0$) where I_i is a dihedral interval of length n_i satisfying $n_i = 1$ or $n_i \geq 3$; we regard I_i as a subinterval of X with w as maximal element. If (a_i, w) is an edge of the graph of I_i , then for $i \neq j$, one has a dihedral subgraph of Ω with vertex set $\{w, a_i, a_j, \text{glb}(a_i, a_j)\}$; this subgraph is isomorphic to the graph of a closed length 2 dihedral interval. The remaining dihedral subgraphs of Ω containing w are the full subgraphs on the vertex sets I_i , where $n_i \geq 3$.

The following result generalizes the well-known fact that for $d \geq 3$ a convex d -polytope which is both simple and simplicial is a simplex.

Proposition 3.19. *Let X be a standard labelled poset. Then the following two conditions are equivalent*

(a) *every length 3 subinterval of X has at most 3 atoms*

(b) *every closed subinterval of X is isomorphic to a product of dihedral intervals.*

Proof. We first make some more remarks on products of dihedral intervals. Let $I = [x, u]$, $x \neq u$, be a poset isomorphic to a product of closed dihedral intervals. Then one has

(c) there exist unique $x_i < u$ in I , $1 \leq i \leq m$, such that $[x_i, u]$ is a closed dihedral interval of length n_i ($n_i = 1$ or $n_i \geq 3$) and such that

$$(y_1, \dots, y_m) \mapsto \text{glb}\{y_1, \dots, y_m\}$$

is a (well-defined) isomorphism $[x_1, u] \times \dots \times [x_m, u] \rightarrow I$.

(d) if $x < y_i$ (resp., $y_i > x$) in I for $i = 1, 2$, there exists $z \in I$ with $y_i < z$ (resp., $z < y_i$) for $i = 1, 2$.

We will use the following observation several times in the proof below. If I_1 and I_2 are closed intervals of the same length satisfying the condition 3.1, then any injective order-preserving map $\psi: I_1 \rightarrow I_2$ is an isomorphism of posets. This follows because ψ induces an isomorphism of the order complex $\Delta(I'_1)$ onto a homogeneous, boundaryless subcomplex of $\Delta(I'_2)$, which must be all of $\Delta(I'_2)$ since the latter is a combinatorial sphere (here, I'_i denotes I_i with its top and bottom elements deleted).

We now give a proof of 3.19; it will use that X is a finite poset satisfying the condition 3.1 and the condition on the dihedral subgraphs in 3.18(d). We assume that X is an interval $X = [x, y]$ and proceed by induction on the length $n = l(x, y)$ of X . The proposition is clear if $n \leq 3$, so we assume $n \geq 4$. It will be convenient to let I_n denote a fixed dihedral interval of length n , and use notation $(p, q], [p, q)$, to denote half-open intervals as usual.

Fix $z \in X$ with $l(x, z) = 1$. Define a map

$$\theta' : \{w \in X \mid z \triangleleft w \leq y\} \rightarrow \{u \in X \mid x \triangleleft u \leq y\}$$

by requiring that $w' = \theta'(w)$ is the atom of $[x, w]$ unequal to z . We consider the cases θ' injective and θ' not injective separately.

First suppose that θ' is injective. We will show that $X \cong I_1 \times [z, y]$. By the inductive hypothesis, for $z \leq t < y$ one has

$$(e) \quad [x, t] \cong [z, t] \times I_1.$$

Hence θ extends to a (order-preserving) map $\theta: [z, y] \rightarrow X$ such that $w' = \theta(w)$ is the unique coatom of $[x, w]$ with $z \not\leq w'$. We claim that θ is injective. Suppose to the contrary that $w_1 \neq w_2 \in [z, y]$ with $w'_1 = w'_2$. Then $l(z, w_1) = l(z, w_2) = k \geq 2$, say. Assume without loss of generality that the w_i are chosen to make k as small as possible. Now for any $z \leq p \triangleleft w_1$, one has $p' \triangleleft w'_1 = w'_2$ so $p \triangleleft w_2$ by (e) and minimality of k . Since $[z, w_1]$ has at least two coatoms (which are also coatoms of $[z, w_2]$) and also $w'_1 \triangleleft w_i$ for $i = 1, 2$, one has a contradiction to 3.18(d'). It follows that we have an order-preserving injection $\psi': [x, z] \times [z, y] \rightarrow [x, y]$ such that $\psi'(z, p) = p$ and $\psi'(x, p) = p'$ for $z \leq p < y$. Now consider $z \triangleleft w < y$. By induction, $[w', y] \cong [w, y] \times [w', w]$ so there is a unique coatom y_w of $[w', y]$ such that $w \not\leq y_w$. For any coatom q of $[w, y]$, one sees that y_w is determined by the requirement that the coatoms of $[q', y]$ are q and y_w ; hence $y' := y_w$ is independent of w . It follows that ψ' extends to an order-preserving injection $\psi: [x, z] \times [z, y] \rightarrow [x, y]$ with $\psi(z, y) = y$, $\psi(x, y) = y'$. By an earlier remark, ψ is an isomorphism.

We now consider the case θ' not injective. Let $z \triangleleft v_i < y$, $i = 1, 2$ with $v_1 \neq v_2$ but $v'_1 = v'_2 = u$. Suppose first that $l(x, y) = 4$. Then there exists $w \in X$ with $v_i \triangleleft w \triangleleft y$ for $i = 1, 2$. One sees easily that either $X \cong I_4$ or that the argument for the preceding case applies to the reverse poset X^{op} with w taking the role of z . Henceforward, we suppose $l(x, y) \geq 5$. We now claim

(f) if $z \triangleleft u_i \leq y$, $i = 1, 2$, with $u_1 \neq u_2$ and $u'_1 = u'_2$, then $\{v_1, v_2\} = \{u_1, u_2\}$.

For if $u'_i = u$, then $u'_i \in \{v_1, v_2\}$ by considering the dihedral subgraph $\Omega_{z, v_1, u}$. In particular, u_1, u_2, v_1, v_2 are all distinct. By induction applied to $[z, y]$ and (d), one may choose $p \in [z, y]$ with $l(z, p) = 4$ such that $p > u_i$, $p > v_i$ for $i = 1, 2$. By the length 4 case considered earlier, one has (since $[z, p]$ has at least 4 coatoms) that $[z, p] \cong I_1^4$. Choose $q \triangleleft p$ with $v_i < q$ for $i = 1, 2$; then $[z, q] \cong I_1^3$ and so by the length 4 case again, $[x, q] \cong I_1 \times I_3$. In particular, every coatom of $[x, q]$ is a coatom of $[z, q]$. This implies that the first case applies to $[x, p]^{\text{op}}$ with q playing the role of z , and one obtains $[x, p] \cong I_1 \times [x, q]$. From this, it follows that one cannot have $u'_1 = u'_2$ and $v'_1 = v'_2$, and the claim (f) is proved.

We claim that there does not exist $v_3 \triangleright z$, $v_3 \neq v_2, v_1$, and $q \in [z, y]$ with $l(z, q) \geq 3$ such that the atoms of $[z, q]$ are precisely v_1 and v_3 . For if such v_3, q existed, one could choose them with $[z, q] \cong I_3$. By inductive hypothesis applied

to $[z, y]$, one could choose $n > q$ such that $n > v_2$. Now $[z, n] \cong I_1 \times I_3$ and so $[v_2, n] \cong I_3$. Since $[u, n]$ has at least three coatoms (those of $[v_2, n]$ together with q), one gets $[u, n] \cong I_1 \times I_3$. Let q_1, q_2 denote the coatoms of $[v_1, q]$ and v_4 be the atom of $[u, n]$ unequal to v_1 and v_2 (one has $v_4 \neq v_3$ also else $\Omega_{z,x,u}$ is not dihedral). But then $v_i \leq q_j \leq q$ for $i = 1, 3, 4$ and $j = 1, 2$ contrary to 3.18(d'), which proves the claim. By symmetry, the analogous claim with v_1 and v_2 interchanged also holds.

Now let p denote a maximal element of $[z, y]$ with only v_1, v_2 as atoms (there is a unique such p by induction applied to $[z, y]$). One has that p is also the maximum element of $[u, y]$ with only v_1, v_2 as atoms, and $[x, p]$ is a dihedral interval. By the inductive hypothesis applied to $[z, y]$, the claims of the preceeding paragraph and (c), the set of elements q of $[z, y]$ such that neither v_1 nor v_2 is an atom of $[z, q]$ is an interval $[z, w]$, and the map $(r, s) \mapsto \text{lub}(r, s)$, for $r \in [z, p]$ and $s \in [z, w]$, defines an isomorphism $\psi_1: [z, p] \times [z, w] \rightarrow [z, y]$ of posets. Similarly, one has an isomorphism of posets $\psi_2: [z, p] \times [z, t] \rightarrow [z, y]$, $(r, s) \mapsto \text{lub}(r, s)$, where t is the maximum element of $[u, y]$ such that neither v_1 nor v_2 is an atom of $[u, t]$. We now claim that

$$(g) \quad [u, y] \cap [z, y] = [v_1, y] \cup [v_2, y].$$

Indeed, the containment of the right side in the left is clear. Suppose $q \in [u, y] \cap [z, y]$ but $q \notin [v_1, y] \cup [v_2, y]$. Then $q \neq y$, $q \geq z$ and $q \geq u$. By induction applied to $[x, q]$, there exists $s \in [x, q]$ with $s > z$, $s > u$. One has $q \in [z, w]$, so $s \neq v_i$ for $i = 1, 2$ which contradicts that $\Omega_{z,v_1,u}$ is a dihedral graph. This establishes (g).

For $i = 1, 2$ let $m_i = \psi_1(v_i, w) \in [z, y]$. Then m_i is the unique maximal element of $[v_i, y]$ such that $(v_i, m_i) \cap (v_i, p) = \emptyset$, so $m_i = \psi_2(v_i, t)$ also. Using the inductive hypothesis and (f), (c), one sees that there is an isomorphism $\phi_i: [x, v_i] \times [x, q] \rightarrow [x, m_i]$, $\phi_i: (r, s) \mapsto \text{lub}(r, s)$, where q is the maximal element of $[x, m_1]$ (equivalently, $[x, m_2]$) with neither z nor u as an atom. One must have that ϕ_i maps $\{z\} \times [x, q]$ isomorphically onto $[z, w]$ and $\{u\} \times [x, q]$ isomorphically onto $[u, t]$. Moreover, for $s \in [x, q]$ and $i = 1, 2$ one has

$$\psi_1(v_i, \phi_i(z, s)) = \phi_i(v_i, s) = \psi_2(v_i, \phi_i(u, s)).$$

Using this, one sees that there is an order-preserving map $\psi: [x, p] \times [x, q] \rightarrow [x, y]$ such that $\psi(r, s) = \phi_1(r, s) = \phi_2(r, s)$ if $r \in \{x, z, u\}$ and

$$\psi(r, s) = \psi_1(r, \phi_1(z, s)) = \psi_2(r, \phi_1(u, s))$$

otherwise. By (g), ψ is injective, so it is an isomorphism, and this completes the proof.

3.20 We describe shelling data $\mathcal{V}$ for dihedral intervals, as a preliminary to the general description for the multiplicity-free case in section 3.21.

Let Γ be a dihedral graph with dihedral interval I as vertex set and edge set E_Γ . For $x, y \in I$ with $x \leq y$, let $l_I(x, y)$ denote the length of a maximal chain in I from x to y . For $z \in I$, set $\Gamma_z = \{y \mid (y, z) \in E_\Gamma\}$. Let S_z^Γ denote the set of subsets A of Γ_z satisfying (a)–(b) below:

- (a) if $y \in A$ and $l_I(y, z) = n > 1$, there exists $y' \in A$ with $l_I(y', z) = n - 2$
- (b) if $y \in \Gamma_z \setminus A$ and $l_I(y, z) = n > 1$, there exists $y' \in \Gamma_z \setminus A$ with $l_I(y', z) = n - 2$.

It is easily checked that, ordered by inclusion, S_z^Γ is a graded poset. For $y, z \in I$ and $F, G \in S_z^\Gamma$ with $G \setminus F = \{y\}$, define $\mathcal{V}(z, F, G) \in S_y^\Gamma$ by

(c) $\mathcal{V}(z, F, G) = \Gamma_y$ if $l(y', z) = l(y, z)$ for some $y' \in F$ and $\mathcal{V}(z, F, G) = \emptyset$ otherwise.

3.21 For the rest of this subsection, we consider a fixed standard labelled poset X in which every closed subinterval is isomorphic to a product of dihedral intervals. We let Ω denote the corresponding graph with edge set E , and $\mathcal{A} = \mathcal{A}_X$ denote the algebra defined in 3.8.

For $x \in X$, let $\Omega_x = \{y \mid (y, x) \in E\}$. Let S_x denote the set of subsets A of Ω_x with the property that for any dihedral subgraph Γ of X containing x as vertex, one has $A \cap \Gamma_x \in S_x^\Gamma$. It is easily verified that, ordered by inclusion, S_x is a graded poset with minimum (resp., maximum) element $\emptyset$ (resp., Ω_x); one has $l(A, B) = \sharp(B) - \sharp(A)$ for $A \subseteq B$ in S_x .

For $y \leq x$ in X and $F \in S_x$, define

$$(a) \quad n(y, x, F) = l(y, x) - 2\sharp(F \cap [y, x]) \in \mathbb{Z},$$

and

$$(b) \quad P(x, F) = \sum_{y \leq x} v^{n(y, x, F)} y \in \mathcal{R}X.$$

We can now define a function $\mathcal{Y}$ as in 2.21 as part of proposed shelling data for $\mathcal{A}$. Suppose that $y \leq x$ and $F, G \in S_x$ satisfy $G \setminus F = \{y\}$. Define $\mathcal{Y}(x, F, G) = (y, H, 0)$ where $H \subseteq \Omega_y$ is defined as follows. Let $z \in \Omega_y$ and consider the dihedral subgraph $\Gamma = \Omega_{x, y, z}$ of Ω . Then $z \in H$ iff $z \in \mathcal{V}(x, F \cap \Gamma_x, G \cap \Gamma_x)$ (see 3.20(c)). One may check that $H \in S_y$; indeed, it is sufficient to check this in the case that X is an interval, when it is easily seen from the description of the dihedral subgraphs in 3.18(f). By reducing to the case of a dihedral interval, one readily verifies the claims (c)–(e) below.

$$(c) \quad P(x, G) = P(x, F) + (v^{-1} - v)P(y, H) \quad (x, y, F, G \text{ as above}).$$

(d) For $x \leq z$ in X and $F \in S_z$, the set of $w \in [x, z]$ such that $n(w, z, F) = n(x, z, F) - l(x, w)$ is a closed interval $[x, u]$. A similar statement holds with “ $-$ ” replaced by “ $+$.”

(e) for $x < y \leq z$ in X , and $F \in S_z$, one has $n(x, z, F) = n(y, z, F) \pm 1$.

Finally, to establish connections with 1.5 and thereby Kazhdan-Lusztig-Stanley polynomials, we need the following fact (f):

(f) Consider $z, w \in X$, $F, G \in I_z$ and $H \in I_w$ as in 1.5. Set $F' = \{u \leq z \mid \ell(u, z) \in F\}$ and define G', H' similarly. Then $F', G' \in S_z$, $H' \in S_w$ and $\mathcal{Y}(z, F', G') = (w, H', 0)$.

In the case of a twisted Bruhat poset it is sufficient to check (f) if X is dihedral, when it is immediate from [Dy3, 3.4]; in the face lattice case, it is sufficient to verify (f) for a length 2 interval, when it follows from [loc cit 2.3(*)].

Proposition 3.22. *Fix $z \in X$ and $F \in S_z$. There is a unique up to isomorphism graded $\mathcal{A}$ -module $M(z, F)$ with K -basis b_x , ($x \in X$, $x \leq z$), such that (a)–(b) below hold:*

$$(a) \quad b_x \in xM(z, F)_{n(x, z, F)}$$

(b) For $x < z$, there exists y with $x < y \leq z$ such that there is a non-semisimple subquotient of (the ungraded module underlying) $M(z, F)$ with composition factors $L(x)$ and $L(y)$ i.e.

$$(*) \quad \begin{aligned} (xy)b_y &\in K^*b_x \text{ if } n(x, z, F) > n(y, z, F) \text{ and} \\ (yx)b_x &\in K^*b_y \text{ if } n(x, z, F) < n(y, z, F). \end{aligned}$$

Moreover, $(*)$ holds for all $x < y \leq z$.

Proof. The proposition will be proved by induction on $\sharp(X)$. Note that if $M(z, F)$ satisfies the conditions (a)–(b) and Y is a coideal of X containing z , then $eM(z, F)$ may be identified with the module $M(z, F \cap Y)$ for $\mathcal{A}_Y = e\mathcal{A}e$, where $e = \sum_{y \in Y} y$. It will therefore be enough to show that in case $Y = X \setminus \{x\}$, where x is a minimal element of X , that there is an essentially unique $\mathcal{A}$ -module structure on $M(z, F)$ inducing the inductively given $\mathcal{A}_Y$ -module structure on $M' = M(z, F \cap Y) = eM(z, F)$ and satisfying $(*)$ for some (resp. all) $y \in X$ with $x < y \leq z$.

This is trivial if $x \not\leq z$ or $x = z$, so assume that $x < z$. By 3.21(d), there exist $u, w \in [x, z]$ such that $\{y \in [x, z] \mid n(y, z, F) = n(x, z, F) - l(x, y)\} = [x, w]$ and $\{y \in [x, z] \mid n(y, z, F) = n(x, z, F) + l(x, y)\} = [x, u]$. By the relations 3.8(b), one may assume without loss of generality that for $x < y \leq w$ one has $(yw)b_w = b_y$ in M' , and that for $u \leq y < x$ one has $(uy)b_y = b_u$. We now consider separately the three cases $x = u$, $x = w$ and $x \neq u, x \neq w$.

If $x = u$, then by 3.21(e) one has $y \leq w$ for all y with $x < y \leq z$. For any (graded) $\mathcal{A}$ -module structure on $M(z, F)$ satisfying the condition $(*)$ and inducing the $\mathcal{A}_Y$ -module structure on M' , one must have $(xw)b_w = cb_x$ for some $c \in K^*$, and then one necessarily has $(xy)b_y = cb_y$ and $(yx)b_x = 0$ for all y satisfying $x < y \leq z$. Also, for any $c \in K^*$, it is easily seen that there is a unique $\mathcal{A}$ -module structure on $M(z, F)$ satisfying these conditions and inducing the given $\mathcal{A}_Y$ -module structure on M' , and that all these modules for varying $c \in K^*$ are isomorphic (see the argument in the paragraph below). This completes the inductive step for the case $x = u$. The case $x = w$ is similar; one has $c \in K^*$ with $(yx)b_x = cb_y$ for all y with $x < y \leq z$.

We now consider the remaining case $x \neq u, x \neq w$. Then $uxw = (ux)(xw) \in e\mathcal{A}e = \mathcal{A}_Y$. Hence one has in M' that $(uxw)b_w = cb_u$ for some $c \in K$. We will show later that $c \neq 0$, but first complete the argument for this case assuming this condition holds. For any $\mathcal{A}$ -module structure on $M(z, F)$ inducing the $\mathcal{A}_Y$ -module structure on M' , there exist $c_1, c_2 \in K$ with $c_1c_2 = c$ such that for any $x < p \leq w$ and $x < q \leq u$ one has

$$(c) \quad \begin{aligned} (xp)b_p &= c_1 b_x \text{ and } (px)b_x = 0 \\ (xq)b_q &= 0 \text{ and } (qx)b_x = c_2 b_q \end{aligned}$$

This makes it clear that $M(z, F)$ is uniquely determined if it exists. But given c_1, c_2 with $c_1c_2 = c$, one may extend the $\mathcal{A}_Y$ -module structure on M' to an $\mathcal{A}$ -module structure on $M(z, F)$ satisfying (c) (and $ab_r = 0$ if $a \in \mathcal{A}_t$ where $r, t \in X$ are distinct). To show this, one has to show only that the proposed module structure respects the quadratic relations 3.8 involving x . The relations 3.8(b) are easily seen to be respected; for instance, if $x < y_i < p \leq z$ where $y_1 \neq y_2$, then $(xy_i p)b_p = c_1 b_p$ if $p \leq w$ and $(xy_i p)b_p = 0$ otherwise. Now consider the relations 3.8(a); suppose

$x \leq p \leq z$, $x \leq q \leq z$. Recall that $r_{qxp} = qxp - \sum c_{qxp;s}qsp$ where the sum is over $s \geq p$, $s \geq q$. If $q \not\leq u$ or $p \not\leq w$, then one has $(qs)(sp)b_p = 0$ for $s = x$ or $s \geq p$, $s \geq q$ and so r_{qxp} is respected. Suppose that $q \leq u$ and $p \leq w$. One has

$$(uq)(\sum c_{qxp;s}qsp)(pw) = (uq)(qxp)(pw) = (ux)(xw)$$

so by definition of c , one has $(uq)(\sum c_{qxp;s}qsp)(pw)b_w = cb_u$. This implies that

$$\sum c_{qxp;s}qsp b_p = cb_q = (qx)(xp)b_p$$

as required to show that r_{qxp} is respected.

To complete the proof, it remains to show that $c \neq 0$. Fix arbitrary $p, q \in X$ with $x \leq p \leq w$ and $x \leq q \leq u$. Let Ω' denote the graph associated to the poset $[x, z]$ and consider the dihedral subgraph $\Omega'' = \Omega'_{p,x,q}$ as in 3.18. The vertex set of Ω'' is an interval $[x, y]$. By reducing to the case when $[x, z]$ is a dihedral interval, one may easily check that $l(x, y) = 2m$ for some $m \in \mathbb{Z}$, that $n(y, z, F) = n(x, z, F)$ and that one may denote the elements s of $[x, y]$ with $l(s, y) = i$ ($1 \leq i < 2m$) by x_i, x'_i in such a way that

$$n(x_i, z, F) = n(x_i, z, F) = n(x, z, F) \text{ if } i \text{ is even and}$$

$$n(x_i, z, F) + 1 = n(x_i, z, F) - 1 = n(x, z, F) \text{ if } i \text{ is odd.}$$

Set $x_0 = x'_0 = y$ and $x_{2m} = x'_{2m} = x$. Note that the coefficients $c_{fgh;k}$ of 3.7(a) with $f \neq h$ are either all positive or all negative by 3.4(a). Using this and the relations 3.8(b), one sees by induction on i that

$$(x_{2i+1}x_{2i}x'_{2i+1})b_{x'_{2i+1}} \in \mathbb{R}_{>0}(x_{2i+1}x'_{2i}x'_{2i+1})b_{x'_{2i+1}}$$

in M' for $i = 1, \dots, m-1$. Finally, since $p = x'_{2m-1}$ and $q = x_{2m-1}$, one gets $cb_q = (qxp)b_p = \sum c_{qxp;s}(qsp)b_p \neq 0$ (sum over $s \in I$ with $s \geq p$ and $s \geq q$).

Proposition 3.23. (a) For $x \in X$ and $F \leq G \in S_x$, there is a (unique up to isomorphism) exact sequence as in 2.21(b) where $(y, H, k) = \mathcal{Y}(x, F, G)$. Together with the posets S_x and the modules $M(x, F)$, these exact sequences constitute a set of shelling data for $\mathcal{A}$ in the sense of 2.21.

(b) The shelling data in (a) may be identified with the full shelling data described in 2.20. That is, if $\{F\} \in \tilde{I}_x$, $x \in X$, then $F \cong M(x, A)$ for a unique $A \in S_x$ and any exact sequence as in 2.20(*), with $\{F\} \leq \{G\} \in \tilde{I}_x$, is equivalent to (a unique) one as in 2.21(b).

Proof. (a) First, note that $n(x, z, F) = -n(x, z, \Omega_z \setminus F)$; from 3.23, it is clear that $M(z, \emptyset) \cong M(z)$ and $M(z, \Omega_z \setminus F) \cong \delta M(z, F)$.

Now consider $F \leq G \in S_x$, $x \in X$, and set $(y, H, K) = \mathcal{Y}(x, F, G)$. From 3.21(c),(d), one has that if $p \leq y$ and $p \leq q \leq x$ but $q \not\leq y$, then $n(p, x, F) = n(q, x, F) + 1$. It follows that $\sum_{p \leq y} yM(x, F)$ is a submodule of $M(x, F)$, and by 3.21(c) and 3.22, one sees this submodule is isomorphic to $\sigma M(y, H)$. Hence one obtains an exact sequence

$$0 \longrightarrow \sigma M(y, H) \longrightarrow M(x, F) \longrightarrow N \longrightarrow 0$$

for some $N \in \text{gr } \mathcal{A}$. By duality, one also has an exact sequence

$$0 \longrightarrow N' \longrightarrow M(x, G) \longrightarrow \sigma^{-1}M(y, H) \longrightarrow 0$$

for some $N' \in \text{gr } \mathcal{A}$. Set $B = \{q \in X \mid q \not\leq y\}$. Now N and N' are both annihilated by $f = \sum_{p \leq y} p$, and clearly as graded $(1-f)\mathcal{A}(1-f) = \mathcal{A}_B$ modules, one has $N \cong N' \cong M_{\mathcal{A}_B}(x, F \setminus B)$. It follows that $N \cong N'$ in $\text{gr } \mathcal{A}$. The uniqueness assertion is clear.

(b) Let $x \in X$ such that (b) of the proposition holds with x replaced by any $x' < x$. Consider a maximal chain in $\tilde{I}_x$ as in 2.21(*), with corresponding shelling sequence $y_1, \dots, y_n$. By 2.23(e), one has

$$(c) \quad \{y_1, \dots, y_n\} = \Omega_x$$

as multisets. Note that one necessarily has

$$(d) \quad [F_i] = \sum_{p \leq x} v^{l(p, x) - 2\#\{[p, x] \cap \{y_1, \dots, y_i\}\}} p \in \mathcal{R}X$$

and $\#\{i \mid 1 \leq i \leq n, y_i \in [z, x]\} = l(z, x)$ for any $z \leq x$.

Consider for a moment the special case when $X = [u, x]$ is a closed dihedral interval of length $n = 2m + 1$. One can show by induction on m that one has

$$(d) \quad \{y_1, \dots, y_i\} \in S_x, \quad i = 0, \dots, n.$$

Indeed, using 2.22, one has only to show that $u = y_{m+1}$. But if $u = y_j$ for some $1 \leq j \leq m$ then one may check from (d) that $[F_{j+1}] - [F_j] \neq (v^{k-1} - v^{k+1})[H]$ for any $\{H\} \in \tilde{I}_{y_{j+1}}$, $k \in \mathbb{Z}$ contrary to 2.20(c); similarly $u \neq y_j$ for any $m+2 \leq j \leq 2m+1$.

Still in this special dihedral case, fix $w \in X$ with $u < w \leq x$. Note that if $\{H\} \in \tilde{I}_y$ for some $w < y < x$, then (by induction) H has a non-semisimple subquotient with composition factors $L(u), L(w)$. It follows from this and (d) by induction on i that

(e) F_i has a non-semisimple subquotient with composition factors $L(u), L(w)$ for $i = 0, \dots, m$, and the same is true for $i = m+1, \dots, 2m+1$ by symmetry.

Now return to the case of arbitrary X . By 2.22, the statement (d) still holds. To complete the proof, we verify that for $i = 1, \dots, n+1$, one has

$$(f) \quad F_{i-1} \cong M(x, A_{i-1})$$

where $A_i = \{y_1, \dots, y_i\}$. Recall that (f) holds for $i = 1$; assume inductively that it is true for some $i \leq n$. Then one has an exact sequence

$$0 \longrightarrow \sigma^{k+1}M(y_i, H) \longrightarrow M(x, A_{i-1}) \longrightarrow F_i \longrightarrow \sigma^{k-1}M(y_i, H) \longrightarrow 0$$

for some $H \in \Omega_{y_i}$ and $k \in \mathbb{Z}$. Since this exact sequence implies that

$$[M(x, A_{i-1})] - v^{k+1}[M(y_i, H)] \in \sum_{z \in X} \mathbb{N}[v, v^{-1}]z \subseteq \mathcal{R}X,$$

one must have $(y_i, H, k) = \mathcal{Y}(x, A_{i-1}, A_i)$. Now choose $w \in X$ with $y_i < w \leq x$, and set $Y = [y_i, x]$, $e = \sum_{y \in Y} y$. By the proof of 2.22, one has in $\text{gr } \mathcal{A}_Y$ that $\{eF_i\} \in \tilde{I}_x$, where $\tilde{I}_x$ is defined as in 2.20 but for $\mathcal{A}_Y$. Applying (e) above to eF_i in $\text{gr } \mathcal{A}_Y$, one finds that $F_i \in \text{gr } \mathcal{A}$ has a non-semisimple subquotient with composition factors $L(y_i), L(w)$. From 3.22, one now sees immediately that $F_i \cong M(x, A_i)$ as required to complete the proof.

Finally, we record the following consequence of the results of this subsection.

Proposition 3.24. *Let $I = [x, w]$ be a (non-empty) Bruhat interval in a Coxeter group. Then the following conditions (a)–(c) are equivalent:*

- (a) *the Kazhdan-Lusztig polynomial $P_{u,y} = 1$ for all $u \leq y$ in I with $l(u, y) = 3$*
- (b) *the Kazhdan-Lusztig polynomial $P_{u,y} = 1$ for all $u \leq y$ in I*
- (c) *I is isomorphic to a product of dihedral Bruhat intervals.*

Proof. It is well known that for $u \leq y$ with $l(u, y) = 3$, one has $P_{u,y} = 1$ iff $[u, y]$ has at most three atoms. Hence (a) implies (c) by 3.19. That (c) implies (b) follows from 3.21(c),(f), 1.5 and [Dy3, 1.8(a)].

Remarks 3.25. (a) A similar statement holds for the inverse Kazhdan-Lusztig polynomials $Q_{u,y}$. More generally, the analogous result holds for twisted Bruhat intervals.

(b) It may be interesting to study the complexes $\mathcal{K}(M^t, N)$ where M, N are interpolating modules for $\mathcal{A}$ and $\mathcal{A}^\dagger$ respectively. Exactness of certain of these complexes would provide a natural explanation for a combinatorial identity in [Dy4, 1.9] in the Coxeter group case.

(c) Let $\mathcal{A}$ be a $\mathcal{O}$ algebra satisfying 1.6 where X is an interval in a twisted Bruhat order. It is hoped that it will eventually be possible to show that $p(\mathcal{A}, v)$ is the expected matrix of (renormalized) Kazhdan-Lusztig polynomials associated to X , without necessarily establishing existence of the interpolating modules in 1.6, by applying Hecke algebra methods based on the existence of translation functors to a suitable sequence of $\mathcal{O}$ algebras. We mention that if X is not an interval in untwisted Bruhat order or its reverse, it might not be possible to choose the weight posets of these algebras as subposets of X (see [Dy3, 2]) and in this case there is no reason to expect that $\text{gr } \mathcal{A}$ should arise by truncating the weight poset of a highest weight category with a Kazhdan-Lusztig conjecture in the sense of [CPS2, 6].

Singular and Parabolic Algebras. Throughout this subsection, (W, S) denotes a fixed Coxeter system with length function l' and we fix $J \subseteq S$. Let $\leq$ denote a fixed twisted Bruhat order on W (e.g. Bruhat order or its reverse). Fix a standard poset X in W as in 3.4, satisfying 3.5, with labelling as in 3.3 or [Dy3, 3.5] (see 3.30) and let $\mathcal{A}$ denote the K -algebra associated to the labelled poset (X, ℓ) in 3.8 (we take $K = \mathbb{R}$ or $K = \mathbb{C}$). Let W_J denote the standard parabolic subgroup of W generated by $J \subseteq S$, l' denote the length function on (W_J, J) and let Y denote the subposet of “shortest left coset representatives” $Y := \{x \in X \mid sx > x \text{ for all } s \in J\}$ (we assume that Y is non-empty). In this subsection, we study a certain “singular” quasi-hereditary algebra $\mathcal{B}$ with weight poset Y that can be constructed from $\mathcal{A}$, and its “parabolic” quadratic dual $\mathcal{B}^\dagger$.

3.26 For $x, y \in X$, we set $[x, y] = \{z \in Y \mid x \leq z \leq y\}$ and $[x, y]_X = \{z \in X \mid x \leq z \leq y\}$, and say that $[x, y]_X$ is full if $[x, y] = [x, y]_X$. Let $X' = \{x \in X \mid x \geq y \text{ for some } y \in Y\}$. We first record some facts concerning $\leq$ and Y (see [Dy1, 1.9, 3.9]).

(a) If $x, y \in W$ and $s \in S$ with $sx \geq x$ and $sy \geq y$, then the following conditions (i)–(iii) are equivalent: (i) $x \leq y$ (ii) $x \leq sy$ (iii) $sx \leq sy$. Moreover, any element $x \in X'$ can be uniquely written $x = x_J \pi(x)$ where $x_J \in W_J$ and $\pi(x) \in Y$. One has $l(\pi(x), x) = l'(x_J)$.

(b) The poset Y is graded; in fact, if $x < y$ in Y with $n = l(x, y) \geq 2$, then $\Delta(\{z \in Y \mid x < z < y\})$ is either a combinatorial ball or a combinatorial sphere (of dimension $n - 2$). The complex is a sphere iff the interval $[x, y]_X$ is full.

Proposition 3.27. *Let $\mathcal{B}$ denote the subalgebra of $\mathcal{A}$ (with identity $e = \sum_{y \in Y} y$ different from that of $\mathcal{A}$) generated by Y and $\cup_{x,y \in Y} x\mathcal{A}_1y$. Then*

- (a) $\mathcal{B}$ is an $\mathcal{O}$ algebra and its quadratic dual algebra is $\mathcal{B}^\dagger \cong \mathcal{A}^\dagger / \langle X \setminus Y \rangle$
- (b) for $x, y \in Y$, one has $p_{x,y}(\mathcal{B}, v) = v^{l(x,y)}$ if $x \leq y$, and $p_{x,y}(\mathcal{B}, v) = 0$ otherwise
- (c) for $x, y \in Y$, one has $p_{y,x}(\mathcal{B}^\dagger, v) = v^{l(x,y)}$ if $x \leq y$ and $[x, y]$ is full in X , and $p_{y,x}(\mathcal{B}^\dagger, v) = 0$ otherwise.

Proof. We have $\mathcal{A} \cong T_{\mathcal{A}_0}(V)/\langle R \rangle$ where $V = \mathcal{A}_1$ and R is the $\mathcal{A}_0$ -sub-bimodule of $T^2(V)$ spanned as K -vector space by the elements of type 3.8(a)–(b). Set $\mathcal{A}'_0 = \sum_{y \in Y} Ky$ and $V' = \mathcal{A}'_0 V \mathcal{A}'_0$, and identify $T(V') := T_{\mathcal{A}'_0}(V')$ with the subalgebra (with different identity) of $T_{\mathcal{A}_0}(V)$ generated by $\mathcal{A}'_0$ and V' . Let $\mathcal{B}' = T_{\mathcal{A}'_0}(V')/\langle R' \rangle$ where $R' = R \cap T^2(V')$. We will show first that the natural surjective K -algebra homomorphism $\theta: \mathcal{B}' \rightarrow \mathcal{B}$ (given by $a + \langle R' \rangle \mapsto a + \langle R \rangle$, $a \in T(V')$) is an isomorphism.

Using 3.26(a), one sees that for $x, y, z, w \in X$ as in 3.7 with $x, y, z \in Y$, one has $c_{zxy;w} = 0$ unless $w \in Y$. This implies that

- (d) R' is spanned as K -vector space by elements of the type 3.8(a) with $x, y, z \in Y$ and elements of type 3.8(b) with $x, y_1, y_2, z \in Y$.

Using 3.10(a), one finds that

- (e) the elements xyz of $\mathcal{A}$ with $x \leq y, z \leq y$ in Y form a K -basis of $\mathcal{B}$.

Note that 3.26(b) implies that

- (f) the poset Y satisfies the condition 3.1(b).

It follows using (d) that for $x, y, z \in Y$ as in (e), there is an element

$$x_0 \dots x_n \dots x_m + \langle R' \rangle \in \mathcal{B}',$$

independent of the choice of maximal chains $x = x_0 \leq x_1 \leq \dots \leq x_n = y$ and $z = x_m \leq x_{m-1} \leq \dots \leq x_n = y$ in Y , and that these elements (for $x, y, z \in Y$ as in (e)), span $\mathcal{B}'$. By (e), one has $\dim \mathcal{B}' \leq \dim \mathcal{B}$ and since θ is surjective, it must be an isomorphism. Now from (e) and 3.14(†), one sees that for $y \in Y$, the $\mathcal{B}$ -module $eM(y)$ may be identified with $\mathcal{B}y/\mathcal{B}\mathcal{B}_1^+y$. It follows that $\mathcal{B}$ satisfies the hypotheses of 2.3, and $p(\mathcal{B}, v)$ is as stated in (b).

Now set $\mathcal{C} = \mathcal{A}^\dagger / \langle X \setminus Y \rangle$, where $\mathcal{A}^\dagger = T(V^\#)/\langle U \rangle$ as in the proof of 3.16. Note one has $V = V' \oplus V''$ where $V'' = \oplus xVy$ (sum over $(x, y) \in X \times X$ with $x \notin Y$ or $y \notin Y$). Use this decomposition to identify $T_{\mathcal{A}'_0}(V'^\#)$ with the subalgebra (with different identity) of $T_{\mathcal{A}_0}(V^\#)$ generated by $\mathcal{A}'_0$ and $(V')^\#$. There is a projection $T^2(V^\#) \rightarrow T^2((V')^\#)$ with kernel $V^\# \otimes (V'')^\# + (V'')^\# \otimes V^\#$ and we let $U' \subseteq T^2((V')^\#)$ denote the image of U under this projection. Using 3.26(a) again, one checks that

- (g) U' is spanned by elements of the form 3.16(d) with $x, y, z \in Y$ and the sum only over $w \in Y$, by elements of type 3.16(c) with $x, y_1, y_2, z \in Y$ and the elements $xyz^\#, zyx^\#$ with $x \leq y \leq z$ in Y such that $[x, z]_X$ is not full.

Define $\mathcal{C}' = T((V')^\#)/\langle U' \rangle$. An argument similar to the one above for θ shows that the natural surjective homomorphism $\mathcal{C}' \rightarrow \mathcal{C}$ is an isomorphism; one uses that

- (h) the ideal $\langle X \setminus Y \rangle$ of $\mathcal{A}^\dagger$ has K -basis consisting of the elements $xyz_\#$ with $x \geq y, z \geq y$ in X such that either $[y, x]_X$ is not full or $[y, z]_X$ is not full.

Moreover, one has that $R' \subseteq T^2(V')$ is orthogonal to $U' \subseteq T^2(V')^\#$ and it follows by dimension count as in 3.16 that $\mathcal{C}' \cong \mathcal{B}^\dagger$. Next, one verifies using (h)

and 3.14(†) that for $y \in Y$, one has in $\text{gr } C$ that $\mathcal{C}y/\mathcal{C}C_1^+y \cong M_!(y)/\mathcal{C}(X \setminus Y)M_!(y)$ where $\mathcal{C}(X \setminus Y)M_!(y)$ has K -basis consisting of the elements $(xy)^\#v_y$, with $x \in X$ and $[x, y]_X$ not full (here, $0 \neq v_y \in yM_!(y)$ is fixed and $C_1^+ = \sum zC_1w$, the sum over $z < w$ in Y). It follows that $\mathcal{B}^!$ satisfies the conditions of 2.3 and that $p(\mathcal{B}^!, v)$ is as in (c).

We now show that $\mathcal{A}$ is a $\mathcal{O}$ algebra by directly checking that the complexes $\mathcal{K}_{w,x} := \mathcal{K}_{\mathcal{A}^!}(M_!(w)^t, \delta M(x))$ in 2.15(b) are acyclic if $w < x$ in Y . Fix w and x . Let I denote the ideal $\{z \in [w, x] \mid [w, z] \text{ full}\}$ of Y , and set $I' = I \setminus \{w\}$. For $n \in \mathbb{N}$, let D_n denote a vector space with basis $\{d_z\}$ for $z \in I$ with $l(w, z) = n$. It is easy to check that $\mathcal{K}_{w,x}$ can be identified with the complex $\rightarrow D_n \rightarrow \dots \rightarrow D_1 \rightarrow D_0 \rightarrow 0$ with differential given on basis elements by

$$d_z \mapsto \sum_{y: w \leq y < z} \epsilon(y, z)d_y, \quad d_z \in D_n$$

for $n > 0$. For $z \leq x$ in Y , let I_z denote the half-open interval $(w, z]$. It is easily seen that the above complex is obtained by shifting up one degree a complex which computes the reduced homology of $\Delta(I')$ using the CW-decomposition with cells $\Delta(I_z)$, $z \in I$. Let $z_1, \dots, z_n$ denote the elements of $I(x) \setminus I$ arranged so that $l(w, z_1) \leq \dots \leq l(w, z_n)$. Now for $i = 1, \dots, n$, $\Delta(I' \cup \{z_1, \dots, z_i\})$ is obtained from $\Delta(I' \cup \{z_1, \dots, z_{i-1}\})$ by adding a cone over a ball contained in the latter. It follows that $\Delta(I')$ is homotopy equivalent to $\Delta(I_x)$ and so is contractible since I_x has a maximal element. Hence $\mathcal{K}_{w,x}$ is acyclic as required to complete the proof.

Proposition 3.28. *The singular algebra $\mathcal{B}$ of 3.27 is shellable.*

Proof. One has an exact functor $\tau: \text{gr } \mathcal{A} \rightarrow \text{gr } \mathcal{B}$ (commuting with the duality functors) given by $M \mapsto eM$ (M in $\text{gr } \mathcal{A}$). For $y \in Y$, we denote the poset $\tilde{I}_y$ for $\mathcal{B}$ by I'_y . Let $M'(y) = \tau(M(y))$ denote the Verma module for $\mathcal{B}$ with highest weight $y \in Y$. It follows immediately from 3.14(†) and 3.26(a) that for $x \in X$, one has $\tau(M(x)) = \sigma^{l'(x,j)}M'(\pi(x))$ if $x \in X'$, and $\tau(M(x)) = 0$ otherwise. It will suffice to prove the following claim: if $x \in X'$ and $\{F\} \in \tilde{I}_x$ then $\{\sigma^j\tau(F)\} \in I'_{\pi(x)}$ where $j \in \mathbb{Z}$ is chosen so $\dim \pi(x)(\sigma^j\tau(F))_0 = 1$. Assume inductively that the claim is true with x replaced by any $x' \in X$ with $x' < x$.

Consider $x \in X'$ and $F, G \in \text{gr } \mathcal{A}$ with $\{F\} < \{G\}$ in $\tilde{I}_x$, so there exist $y < x$ in X , $\{H\} \in \tilde{I}_y$ and an exact sequence

$$0 \longrightarrow \sigma^{k+1}H \longrightarrow F \longrightarrow G \longrightarrow \sigma^{k-1}H \longrightarrow 0.$$

We consider the effect of applying τ to this exact sequence. If $y \notin X'$, then $\tau(H) = 0$ and one simply obtains $\tau(F) \cong \tau(G)$. Now suppose $y \in X'$, so that by 3.26(a), one gets $\pi(y) \leq \pi(x) = z \in X'$. If $\pi(y) = z$, then for any $u \in Y$ one has $\dim u\tau(H) = \dim u\tau(F) = \dim u\tau(G)$ and applying τ to the exact sequence gives $\tau(G) \cong \sigma^{-2}\tau(F)$. In the remaining case $\pi(y) < z$, one has by induction that $\{\sigma^m\tau(H)\} \in I'_{\pi(y)}$ for some integer m ; applying the functor $\sigma^j\tau$ to the above exact sequence gives an exact sequence which shows that $\{\sigma^j\tau(F)\} < \{\sigma^j\tau(G)\}$ as isomorphism classes in $\text{gr } \mathcal{B}$. Since the above observations hold for any $\{F\} < \{G\}$ in $\tilde{I}_x$, the claim follows immediately.

Remark 3.29. I don't know whether assertions like those in 2.24 hold for the I'_x above also. The above proof can be used to show that there exists (partial) shelling

data for $\mathcal{B}$ corresponding to [Dy3, 3.11]. The parabolic algebras $\mathcal{B}^!$ are not shellable in general.

Remark 3.30. Suppose that X is a standard poset in a twisted Bruhat order (satisfying 3.5) with labelling as in [Dy3, 3.15] i.e. a labelling “dual” to those considered previously in this subsection, and let $\mathcal{A}$ denote the associated algebra as defined in 3.8. Using [Dy2, 12], one can show that all the results of this subsection have analogues for “singular” and “parabolic” algebras with weight posets given by “shortest right coset representatives” in X in the sense of [Dy3, 3.12]. In the special case of finite Coxeter systems, one obtains the same class of algebras whether one uses left or right coset representatives.

The following theorem summarizes the results of this section.

Theorem 3.31. (a) *The algebra $\mathcal{A}$ associated in 3.8 to a standard labelled poset X (satisfying 3.5) is a (shellable) $\mathcal{O}$ algebra satisfying the conjecture 1.6.*

(b) *In the Coxeter group case, the singular algebras constructed from $\mathcal{A}$ as in 3.27 (or the appropriate modification 3.30) are shellable $\mathcal{O}$ algebras.*

Category $\mathcal{O}$. If the conjecture 1.6 and the remarks 1.7 are correct, there are $\mathcal{O}$ algebras associated to twisted Bruhat intervals and posets of distinguished coset representatives in Coxeter groups. It would be natural to expect that, in the special case of crystallographic Coxeter groups, these might be closely related to more classical representation theory (in particular, this suggests the question, which can be asked independently of 1.6, of whether quasi-hereditary algebras arising in classical representation theory are “often” shellable $\mathcal{O}$ algebras or their quadratic duals). The purpose of this subsection is to indicate (3.34(a)) one possible connection between the algebras defined in this section and category $\mathcal{O}$ for Kac-Moody Lie algebras; in view of [L, 9], such questions for affine Weyl groups may be relevant to algebraic or quantum groups. We begin by using a (now standard) technique to associate a quasi-hereditary algebra to finite intervals in a weight poset of a category of representations, in this case that of category $\mathcal{O}$ for Kac-Moody Lie algebras.

3.32 Let $\mathfrak{g}$ denote a contragredient Lie algebra (e.g. Kac-Moody Lie algebra) associated to a $n \times n$ symmetrizable complex matrix and $\mathcal{O}$ denote the corresponding Bernstein-Gelfand-Gelfand category of $\mathfrak{g}$ -modules [DGK, 2,3]. Let $\mathfrak{h}$ denote the Cartan subalgebra of $\mathfrak{g}$. For $\mu \in \mathfrak{h}^\#$, let $M(\mu)$ denote the Verma module of highest weight μ and $L(\mu)$ denote the unique irreducible quotient module in $\mathcal{O}$ of $M(\mu)$. We let $\leq$ denote the locally finite partial order on $\mathfrak{h}^\#$ such that $\nu \leq \mu$ iff $L(\nu)$ is a subquotient of $M(\mu)$ (see [KK, Theorem 2]).

Fix $\lambda_1, \dots, \lambda_n$ in $\mathfrak{h}^\#$ and let $D = \{\mu \in \mathfrak{h}^\# \mid \mu \leq \lambda_i \text{ for some } 1 \leq i \leq n\}$. Let $\mathcal{O}'$ denote the full subcategory of $\mathcal{O}$ consisting of objects all of the irreducible subquotients of which are of the form $L(\mu)$ for some $\mu \in D$. From the “block” decomposition [DGK, 4.2] of $\mathcal{O}$ and [RW, 4.13], it follows that for $\mu \in D$, there is a (unique up to isomorphism) finitely-generated projective indecomposable object $P(\mu)$ with $L(\mu)$ as unique irreducible quotient (the arguments of [RW] are given only for the case $n = 1$, but extend without essential change). The following result is an immediate consequence of facts in [RW] which imply that $\mathcal{O}'$ is “nearly” a highest weight category in the sense of [CPS1, 3.1]; only the requirement that $\mathcal{O}'$ be “locally artinian” fails in general.

Proposition 3.33. *Fix a non-empty finite coideal X in D . Consider the projective object $P = \bigoplus_{\mu \in X} P(\mu)$ of $\mathcal{O}'$. Let Hom, End denote the spaces of homomorphisms,*

endomorphisms in $\mathcal{O}'$. Then

(a) $\mathcal{B} := \text{End}(P)$ is a quasi-hereditary algebra over $\mathbb{C}$ with one-dimensional simple modules $L'(\mu) = \text{Hom}(P, L(\mu))$, Verma modules $M'(\mu) = \text{Hom}(P, M(\mu))$ and projective indecomposables $P'(\mu) = \text{Hom}(P, P(\mu))$, for $\mu \in X$.

(b) Let $[A : B]$ denote the multiplicity of a simple module B as a composition factor of a module A . Then for $\mu, \nu \in X$, one has $[M(\mu) : L(\nu)] = [M'(\mu) : L'(\nu)]$.

Proof. (c.f. [CPS, 3.5(b)]) For M in $\mathcal{O}'$ and $\mu \in X$, one has by [RW, 5.3] that $[M : L(\mu)] = \dim \text{Hom}(P(\mu), M)$. Now this implies that $\dim L'(\mu) = 1$ and that $\mathcal{B}$ is finite-dimensional. The existence of a filtration for $P'(\mu)$ as in 2.1(c) follows from exactness of $\text{Hom}(P, -)$ and the existence of a similar filtration for $P(\mu)$ [RW, 4.10]. Similarly, the condition 2.1(b) for $\mathcal{B}$ and also (b) of the Proposition, follow from exactness of $\text{Hom}(P, -)$ and the definition of multiplicities of composition factors in $\mathcal{O}$ in terms of “local composition series” [DGK, 3]. Since $\mathcal{B} = \bigoplus_{\mu \in X} P'(\mu)$, the $P'(\mu)$ are clearly the full set of projective indecomposable $\mathcal{B}$ -modules.

Remarks 3.34. (a) Consider the case when $\mathfrak{g}$ is a Kac-Moody Lie algebra. Certain special intervals X in the poset $\mathfrak{h}^\#$, are isomorphic to a poset of “right coset representatives” in an associated twisted Bruhat order on the (crystallographic) Coxeter group W associated to $\mathfrak{g}$ [Dy1, 6]. One may ask whether the algebra $\mathcal{B}$ defined in 3.33 is of the type considered in 3.30 (when the poset of right coset representatives is contained in an interval of W satisfying 3.5, so that the algebra in 3.30 is known to exist).

(b) For general intervals X , the algebra defined in 3.33 would bear little relation to algebras satisfying 1.6. However, one might still ask if those in 3.33 are shellable $\mathcal{O}$ algebras. Some reason for raising this question is provided by the example 2.26; for choices of the spaces V_i occurring there as imaginary root spaces for certain affine Lie algebras, the algebras in 2.26 are shellable $\mathcal{O}$ algebras with the same multiplicity matrices as the quasi-hereditary algebras associated in 3.33 to intervals X whose top element is a “Kac-Kazhdan weight” (see [H, 4]; one does not necessarily expect the algebras to be isomorphic, the point is merely that it is at least possible for shellable $\mathcal{O}$ algebras to produce these particular multiplicity matrices). The presence of a general Jantzen sum formula for $\mathcal{O}$ with positive coefficients provides additional motivation for the above question (c.f. 2.23).

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