Math 60850, Spring 2016

Midsemester exam

Due February 29 at 3pm

Instructions: Each problem should be started on a **separate page**. Write on **one side only**. Put your answers together in numerical order, add a cover page with your name and the information that this is the Math 60850 Midsemester exam for Spring 2016, and staple it all together.

You must think about the questions on your own. Collaboration with colleagues is **not** acceptable. Any resources that you use (textbooks, published papers) must be cited.

1. $(5+5)$ Let P and Q be two probability measures defined on the same sample space Ω and σ-algebra $\mathcal{F}$.

   (a) Suppose that $P(A) = Q(A)$ for all $A \in \mathcal{F}$ with $P(A) \le 1/2$. Prove that $P(A) = Q(A)$ for all $A \in \mathcal{F}$.

   (b) Give an example where $P(A) = Q(A)$ for all $A \in \mathcal{F}$ with $P(A) < 1/2$, but such that $P(A) \neq Q(A)$ for some $A \in \mathcal{F}$.

2. (2×5) Let X be a real-valued random variable defined on a probability triple $(\Omega, \mathcal{F}, P)$. Fill in the following blanks to make (mathematically) correct sentences.

   (a) $\mathcal{F}$ is a collection of subsets of ________.

   (b) $P(A)$ is a well-defined element of ________ provided that A is an element of ________.

   (c) $\{X < 5\}$ is shorthand notation for the particular subset of ________ which is defined by: ________ .

   (d) If S is a subset of ________, then $\{X \in S\}$ is a subset of ________.

   (e) If S is a ________ subset of ________, then $\{X \in S\}$ must be an element of ________.

3. $(5+5)$ Let $A_1, A_2, \ldots, B_1, B_2, \ldots$ be events in a σ-algebra $\mathcal{F}$ on Ω.

   (a) Prove that
   $$(\limsup A_n) \cap (\limsup B_n) \supseteq \limsup (A_n \cap B_n).$$

   (b) Give an example where the above inclusion is strict, and another example where it holds with equality.

4. $(5+5)$

   (a) Let X be a random variable with $P(X > 0) > 0$. Prove that there is $\delta > 0$ such that $P(X > \delta) > 0$.

   (b) Let $X_1, X_2, \ldots$ be defined jointly on some probability space $(\Omega, \mathcal{F}, P)$, with $E(X_i) = 0$ and $E(X_i^2) = 1$ for all i. Prove that $P(X_n > n \ i.o.) = 0$.

5. $(5+5)$ Let $X_1, X_2, \ldots$ be a sequence of independent random variables, all defined on the same probability space, with $P(X_n = 1) = p_n$ and $P(X_n = 0) = 1 - p_n$ (so each X_n takes on only two values, 0 and 1).

   (a) Show that $X_n \to 0$ in probability if and only if $p_n \to 0$.

   (b) Show that $X_n \to 0$ almost surely if and only if $\sum_{n=1}^{\infty} p_n < \infty$.

6. $(5+5)$

   (a) Let $(\Omega, \mathcal{F}, P)$ be the model of independently tossing a coin infinitely often, as discussed in Section 2.6 (so Ω is the set of all countable 0-1 strings). Let H_i be the event that the ith toss is a head (so H_i is the set of all countable 0-1 strings whose ith entry is 1). Show that

   $$H_1 \notin \sigma(H_2, H_3, \ldots)$$

   (the right-hand side being the σ-algebra generated by $H_2, H_3, \ldots$).

   (b) Either prove that the following statement is true, or give a counter-example: if $A_1, A_2, \ldots$ are countably many independent events in a probability triple $(\Omega, \mathcal{F}, P)$ then

   $$A_1 \notin \sigma(A_2, A_3, \ldots).$$