
Final exam.

This exam is for Math 60610, Spring 2017.

This exam contains 5 problems on 11 pages (including the front cover).

For full credit complete any **4** of the problems.

This is a take-home exam.

For full credit **show all your work** on the paper provided.

Feel free to ask for hints/discuss approaches with me.

Please return the exam to my mailbox by Thursday May 11 at 6.15pm.

Scores

Question	Score	Out of
1		7 + 3
2		8 + 2
3		3 + 7
4		3 + 7
5		4 + 6
Total		40

MAY THE ODDS BE EVER IN YOUR FAVOR!

1. For a graph G denote by $t(G)$ the sum of the number of triangles in G and the number of triangles in $\overline{G}$, the complement of G (the graph on the same vertex set as G , obtained from G by declaring uv to be an edge of $\overline{G}$ if and only if uv is not an edge of G).

- Show that if G has n vertices and m edges then

$$t(G) \geq \binom{n}{3} + \frac{2m^2}{n} - m(n-1).$$

- Show that for large n , for every 2-colouring of the edges of K_n at least nearly a quarter of all triangles will be monochromatic. Specifically, with $\Delta(\chi)$ the number of monochromatic triangles in a colouring χ , show that for all $\varepsilon > 0$ and all large enough $n = n(\varepsilon)$, for every 2-colouring χ of the edges of K_n we have

$$\frac{\Delta(\chi)}{\binom{n}{3}} \geq \frac{1}{4} - \varepsilon.$$

2. Let T be a tree with k edges. This problem shows that $\lim_{n \rightarrow \infty} \text{ex}(n, T)/n$ (if it exists) lies between $(k-1)/2$ and $k-1$.
- Let G be a graph with n vertices and more than $(k-1)n$ edges. Show that G contains T as a subgraph.

- Show that there are infinitely many n for which there is a graph G with n vertices and no more than $(k-1)n/2$ edges for which G does not contain T as a subgraph.

3. Let $n = 2m + 1$ be an odd integer (with $m \geq 0$). Let $\mathcal{F}$ be an antichain (Sperner family) on $\{1, \dots, n\}$ of largest possible size, that is, size $\binom{n}{\lfloor n/2 \rfloor}$.
- Show that $\mathcal{F}$ consists only of elements of size m or $m + 1$.

- Show that in fact there are only two possibilities for $\mathcal{F}$: it is either the set of all subsets of $\{1, \dots, n\}$ of size m , or the set of all subsets of $\{1, \dots, n\}$ of size $m + 1$.

4. Let $\mathcal{F}$ be a non-empty collection of subsets of the set $\{1, \dots, n\}$ with the property that if $A, B \in \mathcal{F}$ then $A \cup B \in \mathcal{F}$.
- Show that if $\mathcal{F}$ contains a set $\{x, y\}$ of size 2 then one of x, y is an element of at least half the members of $\mathcal{F}$.

- Show that if $n = 12$ then there is an ℓ , $1 \leq \ell \leq 12$, such that ℓ is an element of at least half the members of $\mathcal{F}$.

5. For $s \geq 1$ an integer, an s -distance set in $\mathbb{R}^n$ is a set of vectors $v_1, \dots, v_m$ such that there is some set S of size s of positive reals with $d(v_i, v_j)$ (Euclidean distance) in S for all $i \neq j$.
- Improve the proof of Theorem 59.3 of the lecture notes, to show that the maximum size of a 1-distance set in $\mathbb{R}^n$ is $n + 1$.

- Show that the maximum size of a 2-distance set in $\mathbb{R}^n$ is $\binom{n+2}{2}$.