Basic Combinatorics

Math 40210, Section 01 — Fall 2012

Homework 9 — Solutions

- **2.6.2 1**:

$$\begin{aligned}\binom{\alpha-1}{k}+\binom{\alpha-1}{k-1} &= \frac{(\alpha-1)(\alpha-2)\dots(\alpha-k)}{k!}+\frac{(\alpha-1)(\alpha-2)\dots(\alpha-k+1)}{(k-1)!}\\ &= \left(\frac{(\alpha-1)(\alpha-2)\dots(\alpha-k+1)}{(k-1)!}\right)\left(\frac{\alpha-k}{k}+1\right)\\ &= \left(\frac{(\alpha-1)(\alpha-2)\dots(\alpha-k+1)}{(k-1)!}\right)\left(\frac{\alpha}{k}\right)\\ &= \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-k+1)}{k!}\\ &= \binom{\alpha}{k}.\end{aligned}$$

- **2.6.2 6**: In this question we use the basic fact, derived in class and in the textbook, that the number of ways to place m identical objects into n distinguishable bins is the same as the number of ways to select m objects from a set of n different types of objects with repetition allowed, and in both cases the answer is $\binom{n+m-1}{m}$.

- **2.6.2 6a**: Here we are placing $m = 50$ identical objects (the burgers) into $n = 20$ distinguishable bins (the guests), so there are $\binom{20+50-1}{50} = \binom{69}{50}$ ways.

- **2.6.2 6b**: One each guest has received one burger each, there are $m = 30$ left over, which must be distributed among $n = 20$ guests, so there are $\binom{20+30-1}{30} = \binom{49}{30}$ ways.

- **2.6.2 6c**: In the first part, there are 51 possibilities for the number k of burgers left over. If k are left over, then we are dealing with the problem $m = 50 - k$ and $n = 20$, leading to a count of $\binom{20+50-k-1}{50-k} = \binom{69-k}{50-k}$. So the total count is

$$\sum_{k=0}^{50}\binom{69-k}{50-k}.$$

In the second part, again we start by giving each guest one vertex each. Then there are 31 possibilities for the number k of burgers left over when the balance is distributed. If k are left over, then we are dealing with the problem $m = 30 - k$ and $n = 20$, leading to a count of $\binom{20+30-k-1}{30-k} = \binom{49-k}{30-k}$. So the total count is

$$\sum_{k=0}^{30} \binom{49-k}{30-k}.$$

There's a "trick" way to do both of these parts: introduce a phantom 21st guest to receive the unused burgers. In the first part we are now dealing with the $m = 50$, $n = 21$ problem, so there are $\binom{21+50-1}{50} = \binom{70}{50}$ ways. In the second part we are dealing with the $m = 30$, $n = 21$ problem, so there are $\binom{21+30-1}{30} = \binom{50}{30}$ ways.

General comment: We have just given a combinatorial proof of the following identity: for all m, n,

$$\sum_{k=0}^{m} \binom{n+m-k-1}{m-k} = \binom{n+m}{m}.$$

This is identical to

$$\sum_{k=0}^{m} \binom{n+k-1}{k} = \binom{n+m}{m},$$

which is a more standard way to present this identity.

- **2.6.4 1(d)**: Experiment suggests $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$. We prove this by induction on n. Base case is $n = 1$, which is trivial. We now try to deduce the $n+1$ case from the n case. The trick is to find exactly the right terms to apply the Fibonacci recurrence to:

$$\begin{aligned} F_{(n+1)+1}F_{(n+1)-1} - F_{n+1}^2 &= F_{n+2}F_n - F_{n+1}^2 \\ &= (F_{n+1} + F_n)\, F_n - (F_n + F_{n-1})\, F_{n+1} \\ &= F_{n+1}F_n + F_n^2 - F_nF_{n+1} - F_{n-1}F_{n+1} \\ &= -\left(F_{n+1}F_{n-1} - F_n^2\right) \\ &= -(-1)^n \quad \text{(induction hypothesis)} \\ &= (-1)^{n+1}, \end{aligned}$$

as required.

- **2.6.4 3**: Form generating function:

$$\begin{aligned} A(x) &= 5x + a_2x^2 + a_3x^3 + \ldots \\ &= 5x + (a_1 + 6a_0)x^2 + (a_2 + 6a_1)x^3 + \ldots \\ &= 5x + x(a_1x + a_2x^2 + \ldots) + 6x^2(a_0 + a_1x + \ldots) \\ &= 5x + xA(x) + 6x^2A_x. \end{aligned}$$

So

$$A(x) = \frac{5x}{1 - x - 6x^2} = \frac{5x}{(1+2x)(1-3x)} = \frac{1}{1-3x} - \frac{1}{1+2x}$$

and $a_n = 3^n - (-1)^n2^n$.

- **2.6.4 5(d)**: Induction on n, base case $n = 0$ trivial. For the induction step:

$$\begin{aligned}\sum_{k=0}^{n+1} F_k^2 &= \left(\sum_{k=0}^{n} F_k^2\right) + F_{n+1}^2 \\ &= F_n F_{n+1} + F_{n+1}^2 \quad \text{(induction hypothesis)} \\ &= (F_n + F_{n+1})\, F_{n+1} \\ &= F_{n+2} F_{n+1} \\ &= F_{(n+1)+1} F_{n+1},\end{aligned}$$

  as required.

- **2.6.4 8(a)**: We prove this by induction on n, the base cases $n = 1$ and 2 being trivial. For the induction step:

$$\begin{aligned}L_{n+1} &= L_n + L_{n-1} \\ &= (F_{n+1} + F_{n-1}) + (F_n + F_{n-2}) \quad \text{(inductive hypothesis)} \\ &= (F_{n+1} + F_n) + (F_{n-1} + F_{n-2}) \\ &= F_{n+2} + F_n,\end{aligned}$$

  as required. Notice that base cases $n = 1, 2$ need to be verified here, since the induction hypothesis is applied both to L_n **and** L_{n-1}.

- **2.6.4 10**: Let h_n be the number of hopscotch boards with n squares. It's clear that $h_0 = 1$ and $h_1 = 1$. For $n \geq 2$, there are h_{n-1} boards that begin with a single-square position (once that square had been put down, it can be completed to a legitimate board by the addition of any $(n-1)$-square board), and there are h_{n-2} boards that begin with a two-square position. So $h_n = h_{n-1} + h_{n-2}$ for $n \geq 2$. The h's are thus just a "shifted" Fibonacci sequence: $h_n = F_{n+1}$.

- **2.6.4 11**: From the preceding problem, F_n is the number of hopscotch boards with $n-1$ squares. How many such Hopscotch boards have exactly k two-square positions? The k two-square positions account for $2k$ of the squares, leaving $n-1-2k$ single-square positions, so $(n-1-2k+k = n-k-1$ positions in all. To construct an $(n-1)$-square Hopscotch board with exactly k two-square positions, we just select which k of the $n-k-1$ positions are the two-square ones, so $\binom{n-k-1}{k}$ choices in all. Summing over all possible k we get the total number of $(n-1)$-square Hopscotch boards:

$$h_{n-1} = F_n = \sum_k \binom{n-k-1}{k}.$$

  (Practically, k goes from 0 to the last k with $2k \leq n-1$, but for any other k the binomial coefficient is automatically zero, so we might as well sum over all k).

- **2.6.5 2(a)**: Form generating function:

$$\begin{aligned} A(x) &= a_0 + a_1 x + a_2 x^2 + \ldots \\ &= a_0 + (a_0 + c)x + (a_1 + c)x^2 + \ldots \\ &= a_0 + x(a_0 + a_1 x + \ldots) + cx(1 + x + x^2 \ldots) \\ &= a_0 + xA(x) + \frac{cx}{1-x}. \end{aligned}$$

So

$$A(x) = \frac{a_0}{1-x} + \frac{cx}{(1-x)^2}.$$

The coefficient of x^n in $a_0/(1-x)$ is a_0 times the coefficient of x^n in $1/(1-x)$, which is a_0 times 1 or a_0. The coefficient of x^n in $cx/(1-x)^2$ is c times the coefficient of x^n in $x/(1-x)^2$, which is c times n or cn (this is equation (2.44) of the book, on page 183). So

$$a_n = a_0 + cn.$$

- **2.6.5 2(e)**: Form generating function:

$$\begin{aligned} A(x) &= a_0 + a_1 x + a_2 x^2 + \ldots \\ &= a_0 + (ba_0 + c)x + (ba_1 + 2c)x^2 + (ba_2 + 3c)x^3 + \ldots \\ &= a_0 + bx(a_0 + a_1 x + \ldots) + cx(1 + 2x + 3x^2 + \ldots) \\ &= a_0 + bxA(x) + \frac{cx}{(1-x)^2}. \end{aligned}$$

(The last part above is obtained either by noticing that the derivative of $1/(1-x)$ is $1/(1-x)^2$, and that the derivative of $1+x+x^2+\ldots$, the power series of $1/(1-x)$, is $1+2x+3x^2+\ldots$, so this must be the power series of $1/(1-x)^2$; or by using equation (2.44) of the text on page 183). So

$$A(x) = \frac{a_0}{1-bx} + \frac{cx}{(1-bx)(1-x)^2}.$$

We use partial fractions for the second term. Since $b \neq 1$ we write

$$\frac{cx}{(1-bx)(1-x)^2} = \frac{A}{1-bx} + \frac{B}{1-x} + \frac{C}{(1-x)^2}$$

and solve to get

$$A = \frac{bc}{(1-b)^2}, \quad B = \frac{-c}{1-b}, \quad C = \frac{1}{1-b},$$

so

$$A(x) = \frac{a_0}{1-bx} + \left(\frac{bc}{(1-b)^2}\right)\frac{1}{1-bx} - \left(\frac{c}{1-b}\right)\frac{1}{1-x} + \left(\frac{c}{1-b}\right)\frac{1}{(1-x)^2}$$

and

$$a_n = a_0 b^n + \left(\frac{bc}{(1-b)^2}\right) b^n + \left(\frac{c}{1-b}\right) + (n+1)\left(\frac{c}{1-b}\right),$$

(the last part using the equation before (2.44) of the book, on page 183).

- **2.6.5 7(a)**: $t_0 = 1$, $t_1 = 2$, $t_2 = 4$, $t_3 = 7$ (in this last case, only 111 is left out). For a recurrence: consider how the sequence starts - with a 0, with 10, or with with 110 (it can't start with anything other than these three possibilities). This lets us say

$$t_n = t_{n_1} + t_{n-2} + t_{n-3}.$$

  We can use this for $n \geq 3$, since it already gives $t_3 = 7$.

- **2.6.5 7(b)**: Here's the generating function of the t's:

$$\begin{aligned} T(x) &= t_0 + t_1 x + t_2 x^2 + t_3 x^3 + t_4 x^4 + \dots \\ &= 1 + 2x + 4x^2 + (t_2 + t_1 + t_0)x^3 + (t_3 + t_2 + t_1)x^4 + \dots \\ &= 1 + 2x + 4x^2 + x(T(x) - 2x - 1) + x^2(T(x) - 1) + x^3 T(x) \end{aligned}$$

  so

$$T(x) = \frac{1 + x + x^2}{1 - x - x^2 - x^3}.$$

- **2.6.5 7(c)**: Here's the generating function of the $t^\star$'s:

$$\begin{aligned} T^\star(x) &= t_0^\star + t_1^\star x + t_2^\star x^2 + t_3^\star x^3 + \dots \\ &= x^2 + t_0 x^3 + t_1 x^4 + \dots \\ &= x^2 + x^3 T(x) \\ &= x^2 + \frac{x^3 + x^4 + x^5}{1 - x - x^2 - x^3} \\ &= \frac{x^2}{1 - x - x^2 - x^3} \end{aligned}$$

- **2.6.6 3(a)**: $p_3 = 1$, $p_4 = 2$, $p_5 = 5$, $p_6 = 14$. Pictures of p_3 through p_5 are easy to come up with; for p_6, see a picture at `http://en.wikipedia.org/wiki/Catalan_number` in the section "Applications in Combinatorics".

- **2.6.6 3(b)**: For p_7, if the vertices are labeled cyclicly 1 through n, and vertex 1 is not an endpoint of one of the the triangulation edges, then there must be an edge from 7 to 2, and there are p_6 ways to complete the triangulation.

  If 1 is in an edge, and the earliest (in numerical order) vertex that it's joined to by one of the triangulation edges is 3 then there is one way to complete the triangulation on the 123 side, and p_6 ways on the 345671 side.

  If 1 is in an edge, and the earliest vertex that it's joined to by one of the triangulation edges is 4 then there is one way to complete the triangulation on the 1234 side, and p_5 ways on the 45671 side.

  If 1 is in an edge, and the earliest vertex that it's joined to by one of the triangulation edges is 5 then there are two ways to complete the triangulation on each side of the 15 edge, independently.

If 1 is in an edge, and the earliest (in numerical order) vertex that it's joined to by one of the triangulation edges is 6 then there is one way to complete the triangulation on the 671 side, and p_5 ways on the 123456 side (26 must be an edge, since 1 can't be joined to 3, 4 or 5).

This gives a total of $p_6 + p_6 + p_5 + 2 * 2 + p_5 = 42$. S0 $p_7 = 42$.

- **2.6.6 3(c)**: The last part suggests a general strategy for counting p_n. We look at the earliest vertex (in numerical order) that 1 is joined to by an edge of the triangulation. If that vertex is 3 (the smallest possible) then there is 1 way to complete to triangulation on the 123 side (it's already completed!) and p_{n-1} ways on the other side.

  If the earliest vertex joined to 1 is k for some $n - 1 \leq k > 3$, then, since 1 cannot be joined to any of 3 through $k - 1$, it must be that to triangulate the $12 \dots k$ side we have an edge from 2 to k, leaving p_{k-1} completions on the polygon $23 \dots k$. On the other side $(k \dots 1)$ there are p_{n-k+2} triangulations (since what's left is a $(n - k + 2)$-sided polygon), and these triangulations can be done independently of the triangulations of the $12 \dots k$ side, giving $p_{k-1}p_{n-k+2}$ in all.

  Finally, if 1 is not an endpoint of one of the the triangulation edges, then there must be an edge from n to 2, and there are p_{n-1} ways to complete the triangulation.

  We get the recurrence: $p_3 = 1$ and for $n \geq 3$,

$$p_n = p_{n-1} + p_3 p_{n-2} + p_4 p_{n-3} + \dots + p_{n-2} p_3 + p_{n-1}.$$

  Defining $p_2 = 1$, this can also be written as $p_2 = 1$ and for $n \geq 3$,

$$p_n = p_2 p_{n-1} + p_3 p_{n-2} + p_4 p_{n-3} + \dots + p_{n-2} p_3 + p_{n-1} p_2.$$

  Setting $p_{n+2} = c_n$, this becomes: $c_0 = 1$ and for $n \geq 1$,

$$c_n = c_0 c_{n-1} + c_1 c_{n-2} + c_2 c_{n-3} + \dots + c_{n-2} c_1 + c_{n-1} c_0.$$

  This is the Catalan recurrence exactly, so

$$c_n = \frac{\binom{2n}{n}}{n+1}, \quad p_n = \frac{\binom{2n-4}{n-2}}{n-1}.$$

- **2.6.6 5**: If we interpret UP steps as runs scored by White Sox, and DOWN steps as runs scored by Cubs, then a mountain ridgeline is exactly a game between the teams that ends in an n-n tie and in which the Cubs never hold the lead, so r_n is exactly the nth Catalan number, as we discussed in class.

- **2.6.6 8**: The prime number p divides $k!$ exactly

$$\left[\frac{k}{p}\right] + \left[\frac{k}{p^2}\right] + \left[\frac{k}{p^3}\right] + \dots$$

  times, where $[x]$ is the greatest integer less than or equal to x. The term $[k/p]$ counts the number of multiples of p that are at most k; each of these contributes a factor of p; the term

$[k/p^2]$ counts the number of multiples of p^2 that are at most k; each of these contributes a new factor of p that wasn't counted in the first term; and so on. Notice that the sum can be thought of as an infinite one: as soon as we get to a term $[k/p^\ell]$ where p^ℓ is greater than k, we just start getting 0's.

So, for each prime p, the number of times it divides $(2k)!$ is exactly

$$\left[\frac{2k}{p}\right] + \left[\frac{2k}{p^2}\right] + \left[\frac{2k}{p^3}\right] + \dots$$

the number of times it divides $k!(k+1)!$ is exactly

$$\left(\left[\frac{k}{p}\right] + \left[\frac{k}{p^2}\right] + \left[\frac{k}{p^3}\right] + \dots\right) + \left(\left[\frac{k+1}{p}\right] + \left[\frac{k+1}{p^2}\right] + \left[\frac{k+1}{p^3}\right] + \dots\right).$$

To show that $(2k)!/(k!(k+1)!)$ is an integer, we need to show that for every prime p, the first expression is at least as big as the second.

It is enough to show that for all integers $\alpha \geq 1$ and all k and p (a prime),

$$\left[\frac{2k}{p^\alpha}\right] \geq \left[\frac{k}{p^\alpha}\right] + \left[\frac{k+1}{p^\alpha}\right]$$

Let's say $k/p^\alpha = mp^\alpha + r$ where $0 \leq r \leq p^\alpha - 1$. Then $2k/p^\alpha = 2mp^\alpha + 2r$. and $(k+1)/p^\alpha = mp^\alpha + r + 1$. We have

$$\left[\frac{2k}{p^\alpha}\right] = \begin{cases} 2m & \text{if } 2r < p^\alpha \\ 2m+1 & \text{if } 2r \geq p^\alpha, \end{cases}$$

$$\left[\frac{k+1}{p^\alpha}\right] = \begin{cases} m+1 & \text{if } r = p^\alpha - 1 \\ m & \text{othetrwise,} \end{cases}$$

and

$$\left[\frac{k}{p^\alpha}\right] = m.$$

The only way it can happen that

$$\left[\frac{2k}{p^\alpha}\right] < \left[\frac{k}{p^\alpha}\right] + \left[\frac{k+1}{p^\alpha}\right]$$

is when $r = p^\alpha - 1$ and $2r < p^\alpha$; but this can only happen if $p^\alpha < 2$, which cannot happen since p is a prime, so ≥ 2, and $\alpha \geq 1$. So we are done.