Math 30530 — Introduction to Probability

Fall 2009 final exam

December 17, 2009, 1.45pm-3.45pm

Name: ________________________________ **Instructor:** David Galvin

This examination contains 7 problems on 8 pages. It is open-book, open-notes. You may use a calculator. **Show all your work** on the paper provided. The honor code is in effect for this examination.

Scores

Question	Score	Out of
1		10
2		10
3		10
4		10
5		10
6		10
7		10
Total		70

GOOD LUCK !!!

1. My wife participates in a football pool at work with seven other people. Each week during the Colts' 16 week regular season, each person pays in \$2 (for a total of \$16), and the \$16 dollars goes to one person in a competition based on the final score of that week's Colts game. Assume that competitions from week to week are independent, and that each week each person has a $1/8$ chance of winning.

   (a) Let X be the number of times my wife wins during the 16 week season. What is the expectation and variance of X?

   (b) What is $P(X \geq 2)$? (Since she puts in a total of \$32, this is the probability that she at least breaks even in the football pool.)

   (c) Let Y be the number of people in the pool who at least break even over the whole 16 weeks. Compute $E(Y)$.

2. I have nine balls, numbered 1 through 9. Balls 1 through 3 are red, and the remaining six balls are green. I also have three boxes labeled A, B and C. I put the balls into the three boxes so that each box gets exactly 3 balls. (The order within each box doesn't matter.)

   (a) In how many ways can I do this?

   (b) What is the probability that all three red balls go into the same box?

   (c) What is the probability that there ends up being one red ball in each of the three boxes?.

3. Ray Rice of the Baltimore Ravens averages 4 yards per carry with a standard deviation of 1.2.

   (a) Use an appropriate inequality to put a lower bound on the probability that in three successive carries his net gain is between 8 and 16 yards (assume that different carries are independent of each other).

   (b) Given the extra information that Rice's yards per carry is normally distributed, compute the exact probability that in three successive carries his net gain is between 8 and 16 yards.

4. (a) X and Y are two continuous random variables which have a joint density $f(x, y)$ that is non-zero only on the square $[0, 1] \times [0, 1]$. Write down an integral whose value is the probability that $X^2 \leq Y \leq X$.

(b) An introverted professor X rarely turns her face away from the blackboard. The moment when she first faces her students is equally likely to occur at any point during her hour-long lecture. X's student, Y, is very busy. He's always at least 10 minutes late, though he always manages to get to class (at a completely random moment) before the lecture is halfway through. How likely is it that when X faces the class for the first time, she'll see Y eagerly taking notes?

5. (a) I toss a coin, and let X be the number of heads I get and Y the number of tails (so both X and Y take values either 0 or 1).

   i. Compute the covariance of X and Y.

   ii. Are X and Y independent? Briefly justify.

(b) For general random variables X and Y, show that $\mathrm{Cov}(X+Y, X-Y) = 0$ if and only if $\mathrm{Var}(X) = \mathrm{Var}(Y)$

6. (a) I want use an exponential random variable to model the time I have to wait (in minutes) until the first person asks a question during today's exam. Describe in words what I should choose for the parameter λ.

(b) Calls arrive at a telephone exchange at a rate of 10 per minute. Using an appropriate random variable to model the number of calls arriving at the exchange, compute the probability that more than two calls arrive in a space of 6 seconds.

(c) If X is an exponential random variable with $P(X \geq 2) = P(X < 2)$, compute its parameter λ.

7. At a given moment, the velocity v of a particular particle (measured in meters per second) is a random variable with the following density:

$$f(x) = \begin{cases} \frac{1}{10}(4x+1) & \text{if } 0 \leq x \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Calculate the expectation of the velocity.

(b) Calculate the probability that the particle has a velocity greater than the mean velocity.

(c) Compute the distribution function of the particle's kinetic energy $K = mv^2/2$. (Here m, the mass of the particle, is a certain constant.)