

AN EXCEPTIONAL SPHERICAL FIVE-BAR APPLIED TO A DEPLOYABLE CONTROL SURFACE

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ABSTRACT

A spherical five-bar linkage generally has two degrees-of-freedom. However, some configurations arise where the mechanism is able to move with an additional degree-of-freedom, even with both input angles fixed. These configurations are termed *exceptional*. They appear in a configuration surface as embedded straight lines in the direction of its outputs. Without prior knowledge, these lines can be discovered by structured slicing of the configuration surface using the tools of polynomial homotopy continuation. This search procedure is conducted for spherical five-bars of equal-link-length. Cases for adjacent and nonadjacent input angles are analyzed. An equal-link-length spherical five-bar with 90° links is used for the design of a deployable control surface for an aircraft. Taking advantage of the exceptional sets in the mechanism, one motor is able to lock, deploy, and articulate the control surface.

Keywords: Exceptional sets, deployable control surfaces, polynomial homotopy continuation, spherical linkages

1. INTRODUCTION

Various strategies for stowing and deploying the wings and other control surfaces of deployable aircraft have been employed, including rigid hinges, compliant morphing, inflation, and origami-like folding [1]. Since these aircraft must fit into specific volumes when stowed, any control surfaces used for steering and stability must also be folded and deployed. Traditionally, aircraft are controlled with a combination of ailerons on the main wings and a tail with both elevators and rudders. The requirement to both deploy and actuate these control surfaces while minimizing weight and complexity presents a unique design challenge. The Firefly deployable UAV [2] uses an unconventional X-tail mechanism with angled hinges that both deploy the surfaces and facilitate flight control control. Compliant control surfaces have also been developed using composite structures such as hinges and deployable booms [3].

In this work, we present a deployable tail for an aircraft using special folding and locking kinematics. Certain mechanisms have the ability to move with unexpected degrees-of-freedom (DOF), often described as *kinematotropic*, *metamorphic*, or *foldable*. Examples include mechanisms which are overconstrained according to the Grübler-Kutzbach criterion, yet still exhibit mo-

bility, such as the Bennett [4] and Bricard [5, 6] linkages. Foldable mechanisms often display this behavior, such as tessellated origami structures [7, 8]. Mechanisms that are able to change their degrees-of-freedom are termed *kinematotropic*. Analysis of these mechanisms and their mobility has been previously studied at length by Dai et al. [9, 10], Müller et al. [11, 12], Fanghella et al. [13], and Zeng et al. [14]. Use cases for these types of mechanisms include robotic hands [15], aerospace manipulators [16], and switchable walking/wheeled robots [17].

Our design capitalizes on exceptional configurations of a spherical five-bar linkage. Specifically, we take advantage of configurations where the mechanism moves with one degree-of-freedom even though its inputs are locked. From an input/output point of view, this might be considered a case of a multiple-mode or variable-DOF mechanism [18, 19], but considered as a single-loop mechanism, the five-bar has a single two DOF configuration surface. The exceptional sets are lines embedded in this surface that happen to be aligned with an output direction. To find these sets of exceptional mobility, we use the algebro-geometric ideas first described in [20] and applied to kinematics in [21]. This gives a polynomial system that we solve using numerical algebraic geometry, based on polynomial continuation, as implemented in the Bertini software package [22].

2. EXCEPTIONAL SET THEORY

Following [21], let $\mathbf{f}(\boldsymbol{\theta}; \mathbf{p})$ be the loop-closure equations of a mechanism with configuration space variables, $\boldsymbol{\theta} \in \Theta$, and parameters, $\mathbf{p} \in P$, where Θ is the configuration space and P is the parameter space. While generally $\mathbf{p}$ could include link lengths of the mechanism, in this paper, the parameters will be two angles chosen as inputs. For a given parameter point $\mathbf{p}^* \in P$, the solution set of $\mathbf{f}(\boldsymbol{\theta}; \mathbf{p}^*) = 0$ is denoted $V(\mathbf{f}(\boldsymbol{\theta}; \mathbf{p}^*))$, while $V(\mathbf{f}(\boldsymbol{\theta}; \mathbf{p}))$ is the whole solution set as $\mathbf{p}$ is allowed to vary, that is the whole configuration surface. In the application of this paper where $\mathbf{p}^*$ is a given pair of input angles, $V(\mathbf{f}(\boldsymbol{\theta}; \mathbf{p}^*))$ is the solution of the forward kinematics problem.

For any $\mathbf{p}^* \in P$, $V(\mathbf{f}(\boldsymbol{\theta}; \mathbf{p}^*))$ is called the *fiber* over $\mathbf{p}^*$ of $V(\mathbf{f}(\boldsymbol{\theta}; \mathbf{p}))$. The fiber is an algebraic set that can be decomposed into one or more irreducible components. The number of such components and their dimensions stay the same¹ as $\mathbf{p}^*$ varies

¹Including complex components

generically in P , but there may exist exceptional values of $\mathbf{p}^*$ where this decomposition changes, and in particular, there may be parameter values where an irreducible component in the fiber has a larger dimension than the fibers for the general neighbors of $\mathbf{p}^*$. Letting D_m be the union of all the fibers of dimension m , we may say that if there exist two of these, say D_m and $D_{m'}$, both not empty, with $m' > m$, then $D_{m'}$ is termed a set of *exceptional dimension*. In application to kinematics, where m becomes the mobility of the mechanism with fixed parameters $\mathbf{p}^*$, $D_{m'}$ is a set of *exceptional mobility*. In context to this paper where $\mathbf{p}$ comprises of two input angles, generic configurations belong to D_0 and exceptional configurations belong to D_1 .

If irreducible algebraic set $X \subset D_m$ is an irreducible component of $V(\mathbf{f}(\boldsymbol{\theta}; \mathbf{p}))$, then it will be revealed by computing an irreducible decomposition of $V(\mathbf{f}(\boldsymbol{\theta}; \mathbf{p}))$. However, when D_m and $D_{m'}$ both exist, with $m' > m$, it often happens that an irreducible component $X' \subset D_{m'}$ is embedded inside D_m . An example of this is illustrated in Fig. 1 where $\mathbf{p} = \{\theta_A, \theta_B\}$ are input parameters and $\theta = \theta_C$ is a coordinate of the configuration space. The fiber over general $\{\theta_A, \theta_B\}$, consists of isolated points, and the closure² of the union of all such points forms D_0 , the two-dimensional surface shown in Fig. 1. However, there exist several points in P where the fiber becomes a one-dimensional line. These fibers, which form D_1 , are embedded inside D_0 . Consequently, an irreducible decomposition of $V(\mathbf{f}(\boldsymbol{\theta}; \mathbf{p}))$ reveals D_0 , but D_1 is hidden.

To find D_1 , we take advantage of the fact that if we slice $V(\mathbf{f}(\boldsymbol{\theta}; \mathbf{p}))$ horizontally, we are sure to intersect all of its fibers. Fig. 1 shows three such slices as curves lying inside D_0 , each corresponding to $V(\mathbf{f}(\boldsymbol{\theta}; \mathbf{p}), \theta_C - \theta_C^*)$ for different values of θ_C^* . Denoting the projection onto parameter space as $\pi : (\boldsymbol{\theta}; \mathbf{p}) \mapsto (\mathbf{p})$, we see that for each θ_C^* , the projection $\pi(V(\mathbf{f}(\boldsymbol{\theta}; \mathbf{p}), \theta_C - \theta_C^*))$ is a curve in the parameter space. For the case at hand, any two of these curves intersect in a finite number of points that includes the exceptional cases $\pi(D_1)$. Intersecting the projections of three or more slices isolates just the exceptional points in parameter space.

The theory in [20, 21] describes how to treat exceptional sets of arbitrary dimensions, but the simplified illustration of Fig. 1 suffices for this paper. In Fig. 1, the parameter space (θ_A, θ_B) is wholly shown, but only one of three output angles $(\theta_C, \theta_D, \theta_E)$ is shown. Therefore Fig. 1 does not provide a complete view of $V(\mathbf{f}(\boldsymbol{\theta}; \mathbf{p}))$. Choosing θ_C as the output angle, exceptional sets are found where θ_C moves while the inputs are constant by solving the so-called fiber product system:

$$\mathbf{G}_2^{(0,1)}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2; \mathbf{p}) = \left\{ \begin{array}{l} \mathbf{f}(\boldsymbol{\theta}_1; \mathbf{p}) \\ \theta_C - \theta_{C1}^* \\ \mathbf{f}(\boldsymbol{\theta}_2; \mathbf{p}) \\ \theta_C - \theta_{C2}^* \end{array} \right\} = 0 \quad (1)$$

where we solve for both copies of variables and parameters at once, that is $\boldsymbol{\theta}_1 = \{\theta_{C1}, \theta_{D1}, \theta_{E1}\}$, $\boldsymbol{\theta}_2 = \{\theta_{C2}, \theta_{D2}, \theta_{E2}\}$, and $\mathbf{p} = \{\theta_A, \theta_B\}$. The notation $\mathbf{G}_2^{(0,1)}$ follows [21]. The subscript 2 indicates two copies of $\mathbf{f}$. The superscript 1 indicates one slice of the variables per copy. The superscript 0 indicates zero slices

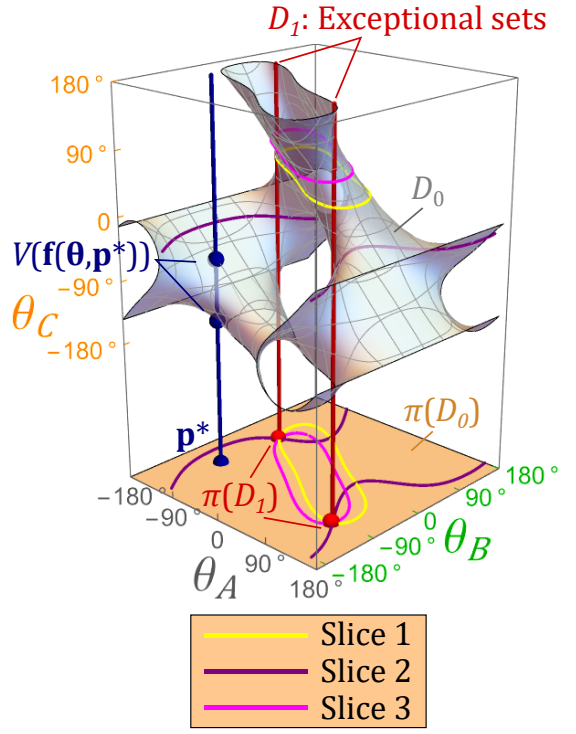


FIGURE 1: A configuration space is sliced to find exceptional sets. Shown is the configuration subspace for θ_C in the equal-link-length spherical five-bar when $\alpha = 45^\circ$ and inputs are adjacent.

of the parameters. However, in Sec. 4 this process is simplified by breaking the configuration space of the mechanism into three separate subspaces and directly substituting in linear slicing equations.

Notice in particular that in forming a fiber product for $\mathbf{f}(\boldsymbol{\theta}; \mathbf{p})$, we have multiple copies of $\boldsymbol{\theta}$ but only one copy of $\mathbf{p}$. This corresponds to taking the intersection of projections onto P . For a single-loop spherical mechanism, the loop closure equations $\mathbf{f}(\boldsymbol{\theta}; \mathbf{p}) = 0$ consist of three independent equations that equate the orientation at the beginning and end of the loop, so we see that (1) is a set of 6 equations in 6 unknowns. However, to solve this as a polynomial system, we need to replace each unknown angle, say θ , with its sine-cosine pair, $s_\theta = \sin \theta$, $c_\theta = \cos \theta$, and append the identity $c_\theta^2 + s_\theta^2 - 1 = 0$. To facilitate computation, we eliminate the angles θ_D and θ_E , at which point f becomes a single equation in the variables θ_C and parameters $\{\theta_A, \theta_B\}$.

As illustrated in Fig. 1, although the intersection of the projected slices will include the exceptional set, there may be additional intersections that are not exceptional. Consequently, the solutions to (1) must be checked to determine which ones are truly exceptional. One way to accomplish this is to consider a third general slice $\theta_C - \theta_{C3}^* = 0$. The projections of all three slices will meet only in exceptional parameter points. A more efficient approach is to check the Jacobian matrix of $\mathbf{f}$. If a point of $V(\mathbf{G}_2^{(0,1)})$ is nonsingular while the Jacobian $[\frac{\partial \mathbf{f}}{\partial \boldsymbol{\theta}}]$ is less than full rank, then the device can move with $\mathbf{p}$ held fixed, implying that the point is in D_1 rather than in $D_0 \setminus D_1$ (the complement of D_1 inside D_0). To study the five-bar loop thoroughly, we consider variations of (1) where different angles are assigned to $\boldsymbol{\theta}$ and $\mathbf{p}$.

²Closure means we include any point that is the limit of a path inside the set

3. KINEMATIC MODEL

The spherical five-bar linkage is a single-loop two DOF mechanism with five links and five revolute joints. The axes of all joints intersect at a common point, which creates spherical motion for the mechanism's links. The kinematic model for the spherical five-bar is shown in Fig. 2.

Unit vectors $\{\mathbf{s}_{A0}, \mathbf{s}_{B0}, \mathbf{s}_{C0}, \mathbf{s}_{D0}, \mathbf{s}_{E0}\}$ represent the five joints axes in a reference configuration. In this paper, $\mathbf{s}_{A0}$ and $\mathbf{s}_{B0}$ are considered ground pivots, but in general, any two adjacent joints can be grounded. Forming a closed loop starting from joint A, the angles $\{\theta_A, \theta_B, \theta_C, \theta_D, \theta_E\}$ parameterize relative rotations between consecutive links. By convention, each angle indicates a right-handed relative rotation of the succeeding link with respect to the previous link about their connecting joint axis. For example, the angle θ_B is the relative rotation of link BC about $\mathbf{s}_B$ with respect to link AB, θ_C is the relative rotation of link CD about $\mathbf{s}_C$ with respect to link BC, and so forth, in the order of the closed loop. By this convention, a right-handed rotation of the link AE about $\mathbf{s}_{A0}$ with respect to the ground link AB is $-\theta_A$.

In axis-angle form, a rotation matrix about $\mathbf{s}$ by angle θ takes the form,

$$[\mathcal{R}(\mathbf{s}, \theta)] := [I] + \sin \theta [\tilde{\mathbf{s}}] + (1 - \cos \theta) [\tilde{\mathbf{s}}]^2, \quad (2)$$

$$\text{where } \mathbf{s} = \begin{Bmatrix} s_x \\ s_y \\ s_z \end{Bmatrix}, \quad \text{and } [\tilde{\mathbf{s}}] = \begin{bmatrix} 0 & -s_z & s_y \\ s_z & 0 & -s_x \\ -s_y & s_x & 0 \end{bmatrix}.$$

Here, $[I]$ is a 3×3 identity matrix. Using Eq. (2), the kinematic constraints can be written for the closed loop in the spherical five-bar,

$$[\mathcal{R}(\mathbf{s}_{A0}, \theta_A)][\mathcal{R}(\mathbf{s}_{B0}, \theta_B)][\mathcal{R}(\mathbf{s}_{C0}, \theta_C)][\mathcal{R}(\mathbf{s}_{D0}, \theta_D)][\mathcal{R}(\mathbf{s}_{E0}, \theta_E)] = [I]$$

This equation defines the constraints between angles $\theta_A, \theta_B, \theta_C, \theta_D$, and θ_E . To set up the search for exceptional sets, two angles are designated as inputs, initially θ_A and θ_B , and the rest are called outputs. Applying the theory of Sec. 2, we initially set $\mathbf{p} = \{\theta_A, \theta_B\}$ as the parameters, and $\boldsymbol{\theta} = \{\theta_C, \theta_D, \theta_E\}$ as the variables.

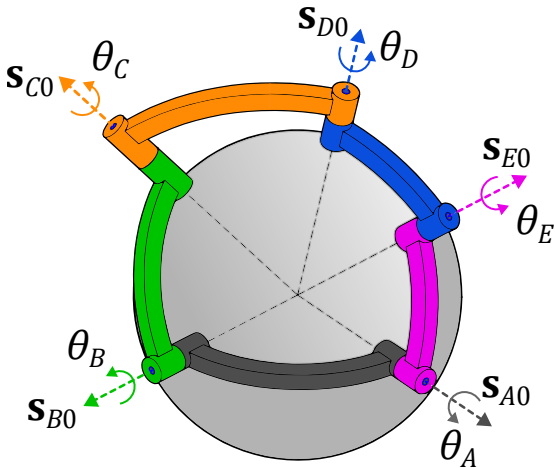


FIGURE 2: Kinematic diagram for the general spherical five-bar linkage

In the proceeding, we consider each output variable individually, forming three configuration subspaces that are analyzed. Because of the symmetry of the mechanism being analyzed, we only consider the cases for input angles $\mathbf{p} = \{\theta_A, \theta_B\}$ and $\mathbf{p} = \{\theta_A, \theta_C\}$. The first we call the *adjacent* case and the second we call *nonadjacent*. Any other selection of input angles simply amounts to a relabeling of these two cases.

For the adjacent case, that is when $\mathbf{p} = \{\theta_A, \theta_B\}$ are selected as inputs, the remaining three angles, $\theta_C, \theta_D, \theta_E$ can be isolated into three respective configuration subspaces,

$$\begin{aligned} h_C(\theta_C; \mathbf{p}) &= \mathbf{s}_{E0} \cdot [\mathcal{R}(\mathbf{s}_{A0}, \theta_A)][\mathcal{R}(\mathbf{s}_{B0}, \theta_B)][\mathcal{R}(\mathbf{s}_{C0}, \theta_C)]\mathbf{s}_{D0} \\ &\quad - \mathbf{s}_{E0} \cdot \mathbf{s}_{D0} = 0 \end{aligned} \quad (3)$$

$$\begin{aligned} h_D(\theta_D; \mathbf{p}) &= \mathbf{s}_{E0} \cdot [\mathcal{R}(\mathbf{s}_{A0}, \theta_A)][\mathcal{R}(\mathbf{s}_{B0}, \theta_B)]\mathbf{s}_{C0} \\ &\quad - \mathbf{s}_{E0} \cdot [\mathcal{R}(\mathbf{s}_{D0}, \theta_D)]^T \mathbf{s}_{C0} = 0 \end{aligned} \quad (4)$$

$$\begin{aligned} h_E(\theta_E; \mathbf{p}) &= \mathbf{s}_{D0} \cdot [\mathcal{R}(\mathbf{s}_{E0}, \theta_E)][\mathcal{R}(\mathbf{s}_{A0}, \theta_A)][\mathcal{R}(\mathbf{s}_{B0}, \theta_B)]\mathbf{s}_{C0} \\ &\quad - \mathbf{s}_{D0} \cdot \mathbf{s}_{C0} = 0 \end{aligned} \quad (5)$$

Eqs. (3), (4), and (5) are each one equation in three variables. Each defines a two dimensional surface of their respective subspaces. Note that Eqs. (3) and (5) are simply a relabeling of variables. Indeed, this similarity is due to the symmetry of the mechanism, so going forward, we exclude the analysis with θ_E as the output. Eqs. (3) and (4) can be rewritten as polynomials by replacing trigonometric functions $\cos \theta$ and $\sin \theta$ for each angle with two variables, c_θ and s_θ , and appending a circle constraint. With this substitution, Eq. (2) becomes

$$[\mathcal{R}(\mathbf{s}, c_\theta, s_\theta)] = [I] + s_\theta [\tilde{\mathbf{s}}] + (1 - c_\theta) [\tilde{\mathbf{s}}]^2 \quad (6)$$

The three polynomial systems to analyze then become

$$\begin{aligned} \mathbf{f}_C(c_{\theta_C}, s_{\theta_C}; \mathbf{p}) &= \begin{Bmatrix} h_C(c_{\theta_C}, s_{\theta_C}; \mathbf{p}) \\ c_{\theta_A}^2 + s_{\theta_A}^2 - 1 \\ c_{\theta_B}^2 + s_{\theta_B}^2 - 1 \\ c_{\theta_C}^2 + s_{\theta_C}^2 - 1 \end{Bmatrix} & \mathbf{f}_D(c_{\theta_D}, s_{\theta_D}; \mathbf{p}) &= \begin{Bmatrix} h_D(c_{\theta_D}, s_{\theta_D}; \mathbf{p}) \\ c_{\theta_A}^2 + s_{\theta_A}^2 - 1 \\ c_{\theta_B}^2 + s_{\theta_B}^2 - 1 \\ c_{\theta_D}^2 + s_{\theta_D}^2 - 1 \end{Bmatrix} \end{aligned} \quad (7)$$

where now $\mathbf{p}$ is redefined as $\mathbf{p} = \{c_{\theta_A}, s_{\theta_A}, c_{\theta_B}, s_{\theta_B}\}$. Making this substitution, each configuration subspace becomes a system of four equations (one kinematic constraint and three circle constraints) in six variables, coinciding with two DOF. By turning the equations into polynomials their varieties $V(\mathbf{f}_C)$, and $V(\mathbf{f}_D)$ are bounded so that slicing down to isolated points will not yield a countably infinite set.

When nonadjacent angles $\mathbf{p} = \{\theta_A, \theta_C\}$ are selected as inputs,

the equations are similar. Eqs. (3)-(5) update to

$$\begin{aligned} h_B(\theta_B; \mathbf{p}) &= \mathbf{s}_{E0} \cdot [\mathcal{R}(\mathbf{s}_{A0}, \theta_A)] [\mathcal{R}(\mathbf{s}_{B0}, \theta_B)] [\mathcal{R}(\mathbf{s}_{C0}, \theta_C)] \mathbf{s}_{D0} \\ &\quad - \mathbf{s}_{E0} \cdot \mathbf{s}_{D0} = 0 \end{aligned} \quad (8)$$

$$\begin{aligned} h_D(\theta_D; \mathbf{p}) &= \mathbf{s}_{B0} \cdot [\mathcal{R}(\mathbf{s}_{C0}, \theta_C)] [\mathcal{R}(\mathbf{s}_{D0}, \theta_D)] \mathbf{s}_{E0} \\ &\quad - \mathbf{s}_{B0} \cdot [\mathcal{R}(\mathbf{s}_{A0}, \theta_A)]^T \mathbf{s}_{E0} = 0 \end{aligned} \quad (9)$$

$$\begin{aligned} h(\theta_E; \mathbf{p}) &= \mathbf{s}_{D0} \cdot [\mathcal{R}(\mathbf{s}_{E0}, \theta_E)] [\mathcal{R}(\mathbf{s}_{A0}, \theta_A)] \mathbf{s}_{B0} \\ &\quad - \mathbf{s}_{D0} \cdot [\mathcal{R}(\mathbf{s}_{C0}, \theta_C)]^T \mathbf{s}_{B0} = 0 \end{aligned} \quad (10)$$

Similar to earlier, note that Eqs. (9) and (10) are simply a relabeling of variables. This similarity is due to the symmetry of the mechanism, so going forward, we exclude the analysis with θ_E as the output. By transforming Eqs. (8) and (9) into polynomials we form the nonadjacent version of Eqs. (7),

$$\begin{aligned} \mathbf{f}_B(c_{\theta_B}, s_{\theta_B}; \mathbf{p}) &= \begin{Bmatrix} h_B(c_{\theta_B}, s_{\theta_B}; \mathbf{p}) \\ c_{\theta_A}^2 + s_{\theta_A}^2 - 1 \\ c_{\theta_C}^2 + s_{\theta_C}^2 - 1 \\ c_{\theta_B}^2 + s_{\theta_B}^2 - 1 \end{Bmatrix} & \mathbf{f}_D(c_{\theta_D}, s_{\theta_D}; \mathbf{p}) &= \begin{Bmatrix} h_D(c_{\theta_D}, s_{\theta_D}; \mathbf{p}) \\ c_{\theta_A}^2 + s_{\theta_A}^2 - 1 \\ c_{\theta_C}^2 + s_{\theta_C}^2 - 1 \\ c_{\theta_D}^2 + s_{\theta_D}^2 - 1 \end{Bmatrix} \end{aligned} \quad (11)$$

where $\mathbf{p}$ is redefined as $\mathbf{p} = \{c_{\theta_A}, s_{\theta_A}, c_{\theta_C}, s_{\theta_C}\}$

3.1 Equal-Link-Length Spherical Five-bar

The characteristic length of each spherical link is defined by the angle between its two joint axes, since the kinematics are not changed by scaling in the radial direction. Spherical five-bars with five equal links can be defined with the symmetric reference configuration shown in Fig. 3. Its corresponding equations are

$$\begin{aligned} \mathbf{s}_{A0} &= \{1, 0, 0\}^T \\ \mathbf{s}_{B0} &= \{\cos \alpha, \sin -\alpha, 0\}^T \\ \mathbf{s}_{C0} &= [\mathcal{R}(\mathbf{s}_{E0}, \gamma)] \mathbf{s}_{A0} \\ \mathbf{s}_{D0} &= [\mathcal{R}(\mathbf{s}_{B0}, \gamma)] \mathbf{s}_{A0} \\ \mathbf{s}_{E0} &= \{\cos \alpha, \sin \alpha, 0\}^T \end{aligned} \quad (12)$$

$$\text{where } \gamma = \cos^{-1} \left(\frac{\cos \alpha / 2 - \cos^2 \alpha}{\sin^2 \alpha} \right)$$

Eq. (12) can be used without loss of generality for equal-link-length spherical five-bars up to $\alpha = 144^\circ$. For $144^\circ < \alpha < 180^\circ$, the mechanism cannot be assembled. Spherical link lengths $\alpha > 180^\circ$ are equivalent to links with length $360^\circ - \alpha$, so only the range $0^\circ < \alpha < 144^\circ$ needs to be considered to fully capture the set of all possible equal-link-length spherical five-bars.

4. EXCEPTIONAL SETS OF THE EQUAL-LINK-LENGTH SPHERICAL FIVE-BAR

Spherical five-bars with five equal link lengths are able to fold onto themselves when two joint axes become colinear. In these configurations, the mechanism gains mobility, and some links are able to rotate independently of the two inputs. We apply the search procedure of Sec. 2 to find all exceptional configurations

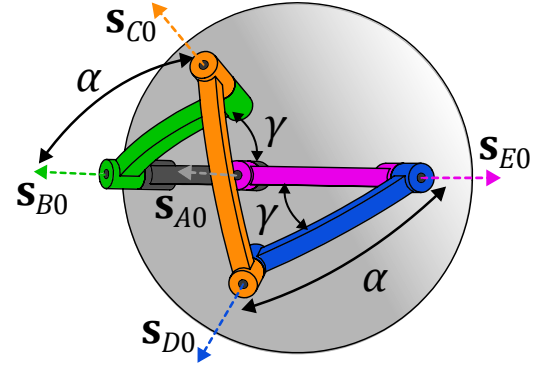


FIGURE 3: Reference for the spherical five-bar linkage with equal-link-length α

while varying the sole structural parameter α of the equal-link-length spherical five-bar. Following the model of Sec. 3, we analyze the cases for *adjacent* and *nonadjacent* input angles in Sections 4.1 and 4.2, respectively.

4.1 Adjacent inputs

With adjacent angles θ_A and θ_B designated as inputs, we consider configuration subspaces $(\theta_A, \theta_B, \theta_C)$, $(\theta_A, \theta_B, \theta_D)$, and $(\theta_A, \theta_B, \theta_E)$. As was already mentioned in Sec. 3, the last subspace is structurally similar to the first so it is omitted. The goal of this section is to find the exceptional sets of $V(\mathbf{f}_C)$ and $V(\mathbf{f}_D)$ of Eq. (7). The parameters for this section are $\mathbf{p} = \{c_{\theta_A}, s_{\theta_A}, c_{\theta_B}, s_{\theta_B}\}$. Note that a generic choice of $\mathbf{p} = \mathbf{p}^*$ in accordance with circle constraints identifies isolated points of D_0 as illustrated in Fig. 1. Following Sec. 2, D_1 can be discovered by computing the zero sets of two copies of the systems from Eq. (7), that is,

$$\begin{aligned} V \left(\begin{Bmatrix} \mathbf{f}_C(c_{\theta_{C1}}^*, s_{\theta_{C1}}^*; \mathbf{p}) \\ \mathbf{f}_C(c_{\theta_{C2}}^*, s_{\theta_{C2}}^*; \mathbf{p}) \end{Bmatrix} \right) &= V \left(\begin{Bmatrix} h_C(c_{\theta_{C1}}^*, s_{\theta_{C1}}^*; \mathbf{p}) \\ h_C(c_{\theta_{C2}}^*, s_{\theta_{C2}}^*; \mathbf{p}) \\ c_{\theta_A}^2 + s_{\theta_A}^2 - 1 \\ c_{\theta_B}^2 + s_{\theta_B}^2 - 1 \end{Bmatrix} \right) \\ V \left(\begin{Bmatrix} \mathbf{f}_D(c_{\theta_{D1}}^*, s_{\theta_{D1}}^*; \mathbf{p}) \\ \mathbf{f}_D(c_{\theta_{D2}}^*, s_{\theta_{D2}}^*; \mathbf{p}) \end{Bmatrix} \right) &= V \left(\begin{Bmatrix} h_D(c_{\theta_{D1}}^*, s_{\theta_{D1}}^*; \mathbf{p}) \\ h_D(c_{\theta_{D2}}^*, s_{\theta_{D2}}^*; \mathbf{p}) \\ c_{\theta_A}^2 + s_{\theta_A}^2 - 1 \\ c_{\theta_B}^2 + s_{\theta_B}^2 - 1 \end{Bmatrix} \right) \end{aligned} \quad (13)$$

The left and right sides of Eq. (13) correspond to the fiber product systems of $(\theta_A, \theta_B, \theta_C)$ and $(\theta_A, \theta_B, \theta_D)$ subspaces, respectively. Starting with $(\theta_A, \theta_B, \theta_C)$, slicing was performed by choosing two random real values θ_{C1}^* & θ_{C2}^* , and plugging in for the variables $c_{\theta_C}^*$ & $s_{\theta_C}^*$. The same thing was done for the $(\theta_A, \theta_B, \theta_D)$ subspace, but instead random real values were generated for θ_{D1}^* & θ_{D2}^* . Note this slicing approach differs from the more general procedure given in [21] whereby instead random linear slices in the variable space would be generated. For this paper, the nongeneral route is sufficient and produces better visualizations. In reference to the notation of [21], the systems of Eq. (13) are equivalent to $\mathbf{G}_2^{(0,1)}(\theta_1, \theta_2; \mathbf{p})$.

Systems were solved using Bertini [22] for varying values of α from 1° to 144° by 1° increments. Computational results are displayed in Table 1. The systems of Eq. (13) yield only isolated points in $\mathbf{p}$, that is the intersection of the projections

TABLE 1: Number of exceptional sets for spherical five-bars of equal-link-length α *Exceptional sets are singular **Exceptional sets are complex

Output angle	Adjacent inputs (θ_A, θ_B)		Nonadjacent inputs (θ_A, θ_C)	
	θ_C/θ_E	θ_D	θ_B	θ_D/θ_E
Finite solutions	4	0	8	4
α	Number of exceptional sets			
$0^\circ - 90^\circ$	2	0	4	2
90°	4	0	8	4
$90^\circ - 120^\circ$	2	0	4	2
120°	1*	0	2*	1*
$120^\circ - 144^\circ$	2**	0	4**	2**

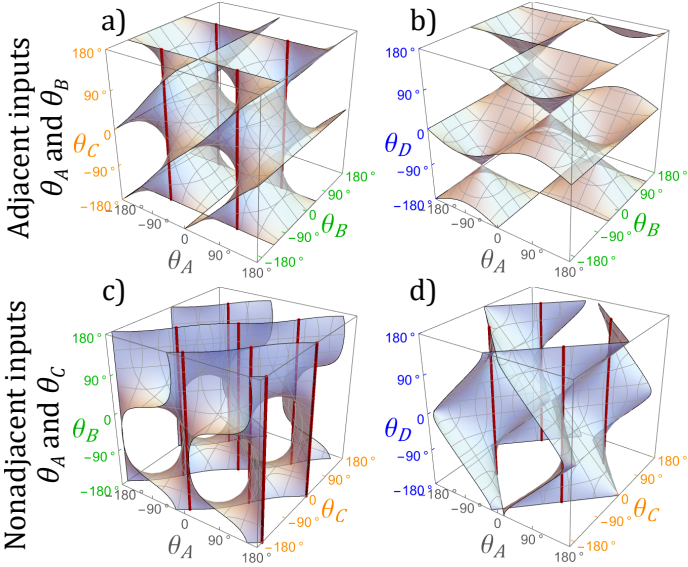


FIGURE 4: Exceptional sets are found in the outputs of the spherical five-bar for the cases where inputs are adjacent (a-b) and nonadjacent (c-d). The exceptional sets for $\alpha = 90^\circ$ are shown.

as shown in Fig. 1. A subset of these points identifies D_1 . To pick out the projections $\pi(D_1)$ from other extraneous points, a third slice could be taken. Alternatively, the intersections can be verified using the Jacobian. Since we eliminated (θ_D, θ_E) , this just means checking whether $\frac{\partial h_C}{\partial \theta_C}$ is zero. At the points (θ_A, θ_B) that pass this test, θ_C can move independently of the inputs. This repeated slicing can be visualized in Fig. 1, which depicts the two exceptional sets in θ_C when $\alpha = 45^\circ$.

The number of exceptional sets are listed in Table 1 for notable values of α . For θ_C (and θ_E by symmetry), there are two real exceptional sets when $\alpha \in (0^\circ, 90^\circ)$ and $\alpha \in (90^\circ, 120^\circ)$. At $\alpha = 90^\circ$, there are four real exceptional sets. In fact, the number of exceptional sets doubles at $\alpha = 90^\circ$ for all cases in Table 1. The locations of exceptional sets when $\alpha = 90^\circ$ are depicted in the corresponding configuration subspaces in Fig. 4.

At $\alpha = 120^\circ$, the two real exceptional sets collapse into one singular value. Above $\alpha = 120^\circ$, the two exceptional sets become complex, losing any physical utility. Fig. 5 shows the exceptional sets in θ_C in their configuration subspaces for α in the ranges

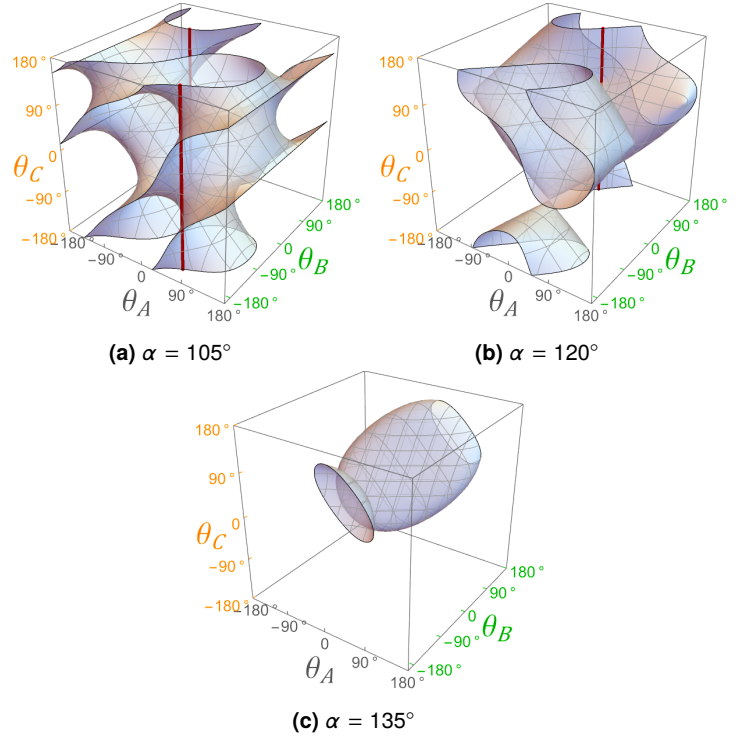


FIGURE 5: Exceptional sets for θ_C with adjacent inputs θ_A and θ_B

$(90^\circ, 120^\circ)$, 120° , and $(120^\circ, 144^\circ)$.

There are no exceptional sets in θ_D when inputs are adjacent.

4.2 Nonadjacent inputs

With nonadjacent angles θ_A and θ_C designated as inputs, we consider configuration subspaces $(\theta_A, \theta_C, \theta_B)$, $(\theta_A, \theta_C, \theta_D)$, and $(\theta_A, \theta_C, \theta_E)$. As was already mentioned in Sec. 3, the last subspace is structurally similar to the second so it is omitted. The goal of this section is to find the exceptional sets of $V(\mathbf{f}_B)$ and $V(\mathbf{f}_D)$ of Eq. (11). The parameters for this section are $\mathbf{p} = \{c_{\theta_A}, s_{\theta_A}, c_{\theta_C}, s_{\theta_C}\}$. Note that a generic choice of $\mathbf{p} = \mathbf{p}^*$ in accordance with circle constraints identifies isolated points of D_0 . Following Sec. 2, D_1 can be discovered by computing the zero sets of two copies of the systems from Eq. (11), that is,

$$\begin{aligned} V\left(\begin{Bmatrix} \mathbf{f}_B(c_{\theta_{B1}}^*, s_{\theta_{B1}}^*; \mathbf{p}) \\ \mathbf{f}_B(c_{\theta_{B2}}^*, s_{\theta_{B2}}^*; \mathbf{p}) \end{Bmatrix}\right) &= V\left(\begin{Bmatrix} h_B(c_{\theta_{B1}}^*, s_{\theta_{B1}}^*; \mathbf{p}) \\ h_B(c_{\theta_{B2}}^*, s_{\theta_{B2}}^*; \mathbf{p}) \\ c_{\theta_A}^2 + s_{\theta_A}^2 - 1 \\ c_{\theta_C}^2 + s_{\theta_C}^2 - 1 \end{Bmatrix}\right) \\ V\left(\begin{Bmatrix} \mathbf{f}_D(c_{\theta_{D1}}^*, s_{\theta_{D1}}^*; \mathbf{p}) \\ \mathbf{f}_D(c_{\theta_{D2}}^*, s_{\theta_{D2}}^*; \mathbf{p}) \end{Bmatrix}\right) &= V\left(\begin{Bmatrix} h_D(c_{\theta_{D1}}^*, s_{\theta_{D1}}^*; \mathbf{p}) \\ h_D(c_{\theta_{D2}}^*, s_{\theta_{D2}}^*; \mathbf{p}) \\ c_{\theta_A}^2 + s_{\theta_A}^2 - 1 \\ c_{\theta_C}^2 + s_{\theta_C}^2 - 1 \end{Bmatrix}\right) \end{aligned} \quad (14)$$

The left and right sides of Eq. (14) correspond to the fiber product systems of $(\theta_A, \theta_C, \theta_B)$ and $(\theta_A, \theta_C, \theta_D)$ subspaces, respectively. In the same way as Sec. 4.1, slicing was performed by choosing random real values for θ_B^* and θ_D^* .

Eqs. (14) were solved using Bertini [22], which tracked 16 paths and found 8 finite solutions. The exceptional sets among the 8 solutions were verified the same way as Sec. 4.1. The exceptional sets in θ_B for nonadjacent inputs follow a similar

pattern to the exceptional sets in θ_C for adjacent inputs, but with twice as many exceptional sets. For $\alpha \in (0^\circ, 90^\circ)$ and $(90^\circ, 120^\circ)$, there are 4 real exceptional sets. At $\alpha = 90^\circ$, the number of real exceptional sets doubles to 8. There are two sets of real, singular exceptional sets for $\alpha = 120^\circ$, which become 4 complex sets for $\alpha \in (120^\circ, 144^\circ)$.

For θ_D (and θ_E by symmetry), the number of exceptional sets follows the same pattern as θ_B , with half as many exceptional sets in each α range. The exceptional set quantities for both nonadjacent cases are additionally listed in Table 1.

4.3 Analyzing the exceptional sets for general equal-link-length spherical five-bars

After discovering the exceptional sets systematically, we analyze the physical realization of these sets in the spherical five-bar linkage. Aside from the special cases at $\alpha = 90^\circ$ and $\alpha = 120^\circ$, the number of exceptional sets in the four input/output cases in Table 1 remains constant for different α . In Fig. 6, we show the structure of a spherical five-bar linkage with $\alpha = 45^\circ$ configured in each of these exceptional sets.

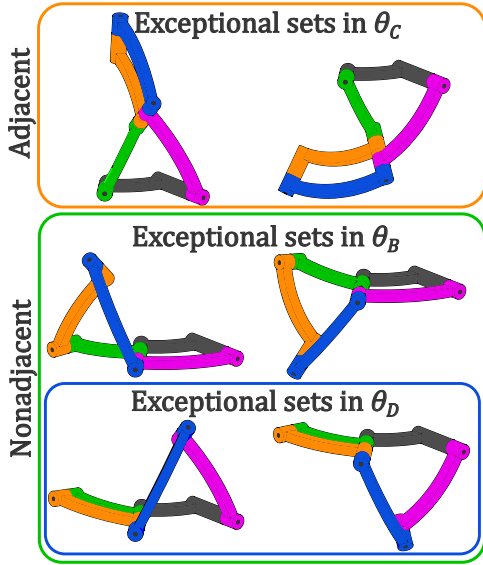


FIGURE 6: The spherical five-bar with equal-link-length $\alpha = 45^\circ$ configured in each of its exceptional sets.

In each exceptional set, the mechanism collapses into a spherical triangle and a great circle segment. Due to the exceptionality, either the triangle or great circle segment can rotate freely in each of these configurations. The exceptional sets come in pairs, which correspond to "flipping" the spherical triangles. For the non-adjacent case, the two exceptional sets in θ_D occur simultaneously with two of the exceptional sets in θ_B .

The two unique cases of exceptional sets in Table 1 occur when $\alpha = 90^\circ$ and $\alpha = 120^\circ$. Fig. 7 shows a spherical five-bar with $\alpha = 90^\circ$ in the θ_C exceptional sets with adjacent inputs.

Fig. 7 provides physical intuition for why the number of exceptional sets doubles at $\alpha = 90^\circ$. The s_{D0} (green-orange) and s_{E0} (magenta-blue) joints must coalign for the θ_C exceptional set to occur, which is generally possible in two ways, as seen in Fig. 6. However, at $\alpha = 90^\circ$, the two joints are able to align on

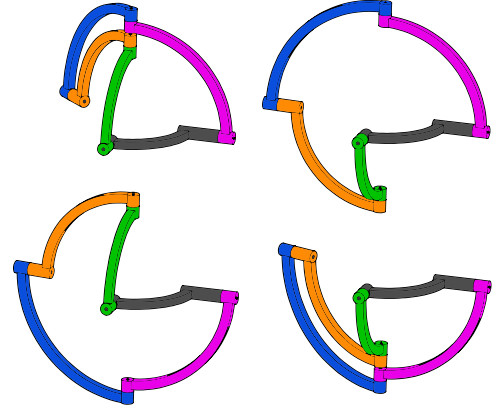


FIGURE 7: When $\alpha = 90^\circ$, the number of exceptional sets doubles. The exceptional sets in θ_C are shown for adjacent inputs.

the same side of the hemisphere or on different sides, since the total angle of any two links sums to 180° . This allows for two additional exceptional sets.

The pairs of exceptional sets found for $\alpha \in (0^\circ, 120^\circ)$ collapse into singular sets at $\alpha = 120^\circ$. Beyond this link-length, the exceptional sets become complex. An exceptional set for $\alpha = 120^\circ$ is shown in Fig. 8.

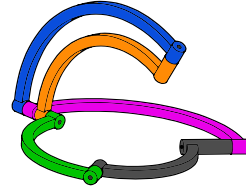


FIGURE 8: When $\alpha = 120^\circ$, the exceptional sets become singular. The singular exceptional set in θ_C is shown for adjacent inputs.

When this configuration is reached, the spherical triangle formed by three of the links collapses into a great circle. As α approaches 120° from below, the two ways for the triangle to assemble merge into one singularity. When α is greater than 120° , three equal links are no longer able to assemble in a triangle, so there are no real exceptional sets.

5. APPLICATION OF EXCEPTIONAL SPHERICAL FIVE-BARS TO DEPLOYABLE CONTROL SURFACES

In deployable aircraft design, all wings and control surfaces must begin stowed along the fuselage for compact storage, then deploy into a flight configuration on command. Additionally, the control surfaces must articulate while in flight for steering and stability. Each control surface must be actuated independently in a twisting manner and coordinated to control the roll, pitch, and yaw of the vehicle. Specific design requirements include:

1. All actuators should be fixed to the fuselage.
2. Each surface should remain passively locked while stowed.
3. The motor for each control surface should trigger deployment.
4. Each surface should deploy passively.
5. Each surface should passively lock into place when deployed.
6. Only one motor should be assigned to each surface.

In this work, we present a design for a deployable control surface using an equal-link-length spherical five-bar linkage. By taking advantage of exceptional sets in the joint space, we achieve all of the design requirements listed above.

5.1 Deployable control surface design

An exceptional spherical five-bar with $\alpha = 90^\circ$ was selected for the tail deployment mechanism. A physical prototype was constructed, as shown in Fig. 9 with the joint axes and key components labeled. The mechanism uses adjacent joints θ_A and θ_B as inputs. The first input, θ_A , is actuated with a servo motor, which is connected with a worm drive to conveniently align the motor with the body of the fuselage. The second input, θ_B is actuated by a spring through a planar four-bar linkage. Both the motor and the spring are mounted to the fuselage, which accomplishes design requirement 1.

5.2 Tail deployment and operation

The exceptional sets of the equal-link-length spherical five-bar are utilized for the deployable control surface in three connected operating stages: a stowed and locked configuration (1), tail deployment (2), and flight control (3). Fig. 10 shows the path taken by the mechanism through these stages in the $(\theta_A, \theta_B, \theta_C)$ configuration subspace. Fig. 11 shows the physical prototype in each of the operating stages.

1 - Stowed and locked configuration

While stowed, the control surface is in an input singularity [23, 24]. In this configuration, the spring applies torque to the input θ_B , but it cannot produce any output motion and stays locked

unless the motor input, θ_A is moved. The plane tangent to the configuration surface at (1) in Fig. 10 depicts this singularity. The worm drive and gearbox of the motor make it nonbackdrivable, so no energy is exerted to keep the control surface stowed. This accomplishes design requirement 2.

The stowed configuration prevents the control surface from deploying, but it also happens to be on an exceptional set in the tail twist angle, θ_C . As a result, the tail is able to twist freely when stowed, which could pull it out of the stowed singularity. To prevent this, the wing rests in a slot while stowed to prevent wing twist and premature deployment.

2 - Tail deployment

When the tail is signaled to deploy, the motor begins rotating. In the configuration space, the mechanism begins traveling along the black dashed line in Fig. 10. This horizontal line is an exceptional set wherein θ_A can move while (θ_B, θ_C) are held constant. As the motor rotates, the spring-driven input gains mechanical advantage and eventually swings the control surface out as the motor nears the deployed configuration. Design requirements 3 and 4 are achieved by using the motor to trigger deployment and letting the spring passively deploy the control surface.

The spring acts on its spherical five-bar input through a planar four-bar. As the tail hits the hard stop of its deployment, the four-bar enters an output singularity. This gives the spring infinite mechanical advantage and locks the tail into its stowed configuration, satisfying design requirement 5.

3 - Flight control

With the spring input now locked in the deployed configuration, the motor can directly control the tailplane twisting in a one degree-of-freedom manner. This accomplishes the final design requirement 6. The worm drive, gearbox, and motor can be selected according to the torque and speed requirements of the expected flight conditions.

6. CONCLUSION

In this paper, we applied a procedure for finding exceptional sets to a family of spherical five-bar linkages with five equal link lengths. In these sets, the mechanism moves with one degree-of-

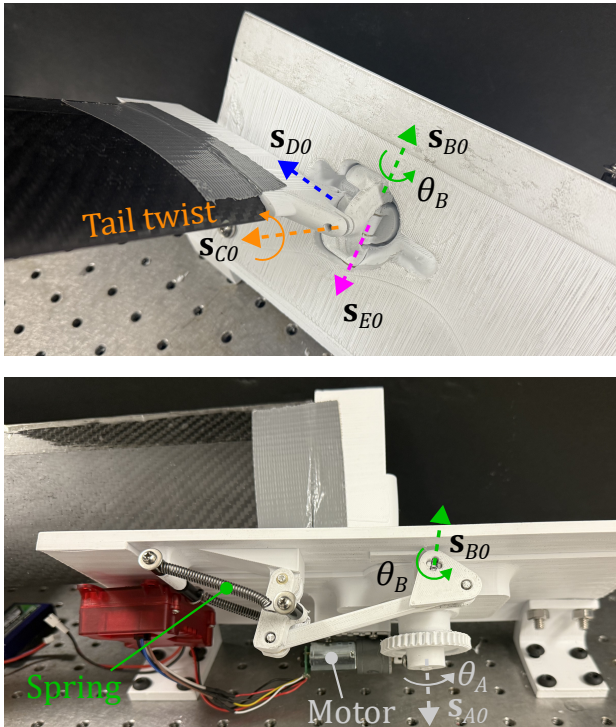


FIGURE 9: An exceptional spherical five-bar with $\alpha = 90^\circ$ is used for the deployable control surface. Kinematic joint axes and key components are labeled.

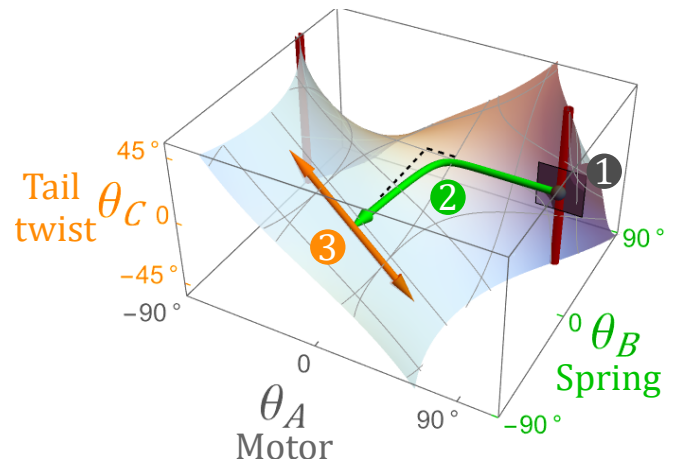


FIGURE 10: Configuration space of the deployable control surface in its three operating stages.

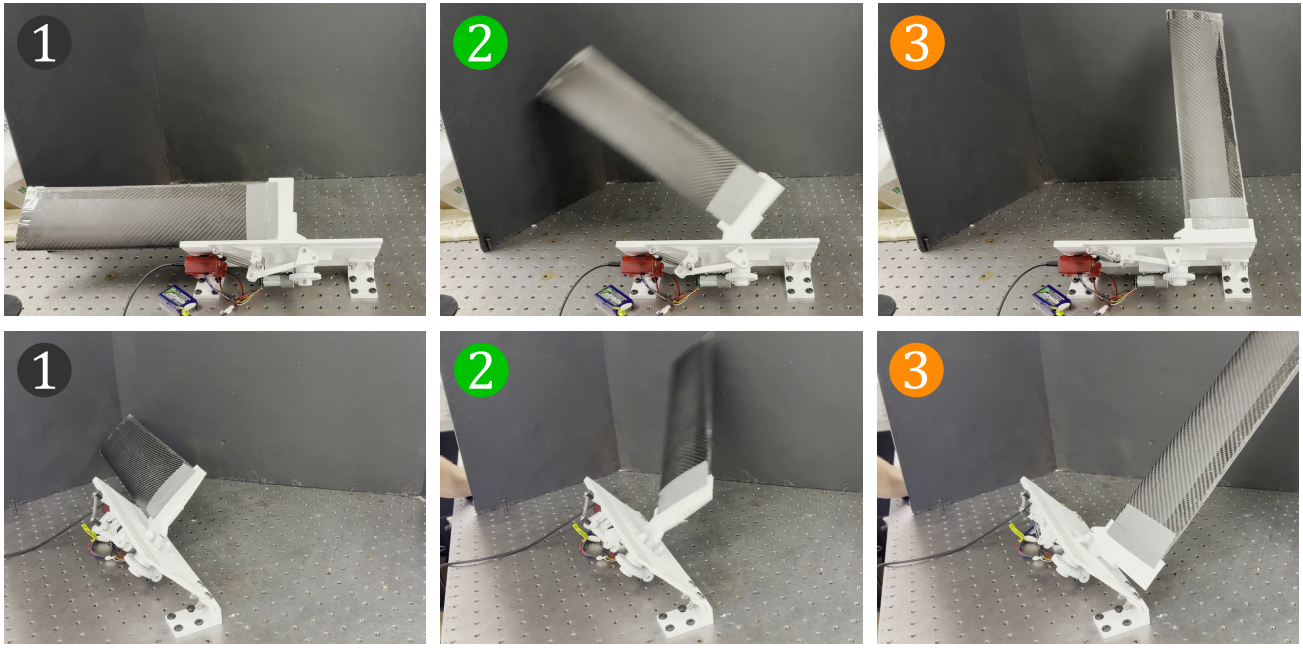


FIGURE 11: Prototype of the deployable control surface shown in its three operating stages.

freedom, even though both of its input angles are fixed. Exceptional sets are lines embedded within the configuration space, which are aligned with the mechanism's outputs. We discovered these sets through structured slicing of the configuration space and the use of the homotopy continuation solver Bertini. We found all exceptional sets for equal-link-length spherical five-bars of link length α . A spherical five-bar of equal-link-length $\alpha = 90^\circ$ was used to design a deployable control surface for an aircraft. The exceptional sets within the mechanism enable a single motor to lock, deploy, and articulate the control surface.

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