

# COMMON REAL SECANTS TO PAIRS OF REAL TWISTED CUBIC CURVES

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ABSTRACT. It is well established that a general pair of twisted cubic curves in complex projective space has ten common secant lines. As an initial investigation, we show that the monodromy group of the ten common secant lines over the complex numbers is the full symmetric group, demonstrating that the common secant lines have no special structure over the complex numbers. We then investigate a novel question in real algebraic geometry: describe the possible collections of ten common real secant lines to a pair of real projective twisted cubic curves. In addition to distinguishing between real and nonreal secant lines, we introduce a refinement of this classification which takes intersection points into account yielding totally real, partially real, and minimally real secant lines. Using computational algebraic geometry as well as combinatorics, we show that for each  $k$  between 0 and 10, there exist pairs of real twisted cubic curves with exactly  $k$  common totally real secant lines. We also obtain examples of real twisted cubics whose sets of common real secants cover a wide range of possibilities within our admissible classification of common real secant lines.

## 1. INTRODUCTION

[Twisted cubic curves](#) are smooth rational curves of degree three in  $\mathbb{P}^3(\mathbb{C})$ . This class of curves have been studied for nearly two centuries [14] and they are of particular interest to algebraic geometers as they admit simple parameterizations while exhibiting subtle geometric behavior. As stated in [8, Ex. 1.10], “This is everybody’s first example of a concrete variety that is not a hypersurface, linear space, or finite set of points.”

As a curve in  $\mathbb{P}^3(\mathbb{C})$ , a twisted cubic curve has codimension two but is not a complete intersection as it cannot be defined by the vanishing of two polynomial equations since three are required. For example, the [standard twisted cubic](#) in  $\mathbb{P}^3(\mathbb{C})$  is defined to be the curve

$$(1) \quad C_0 = \{[s^3, s^2t, st^2, t^3] \mid [s, t] \in \mathbb{P}^1(\mathbb{C})\} \subseteq \mathbb{P}^3(\mathbb{C})$$

and the ideal of polynomials vanishing on  $C_0$  is

$$(2) \quad I(C_0) = \langle x_1^2 - x_0x_2, x_1x_2 - x_0x_3, x_1x_3 - x_2^2 \rangle \subseteq \mathbb{C}[x_0, x_1, x_2, x_3].$$

The dimension of the complex parameter space  $\mathcal{H}$  consisting of all twisted cubic curves in  $\mathbb{P}^3(\mathbb{C})$  is 12 [8, Lecture 12].

Given a twisted cubic curve  $C \subset \mathbb{P}^3(\mathbb{C})$ , a line  $\ell \subset \mathbb{P}^3(\mathbb{C})$  is a [secant line](#) of  $C$  if  $C$  and  $\ell$  intersect in at least two distinct points, i.e.,  $\#(C \cap \ell) \geq 2$ . In fact, this must be an equality since a twisted cubic curve can intersect a line in at most two distinct points [2, §3.1]. Figure 1 is an illustration of the standard twisted cubic  $C_0$  and a secant line  $\ell$ .

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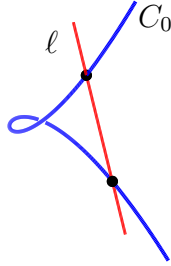


FIGURE 1. Illustration of the standard twisted cubic curve  $C_0$  together with a secant line  $\ell$ .

Since the closure of the set of secant lines to a twisted cubic curve  $C \subset \mathbb{P}^3(\mathbb{C})$  is a surface in the four-dimensional Grassmannian of lines in  $\mathbb{P}^3(\mathbb{C})$ , one expects that, given two general twisted cubic curves, the set of lines that are secants to both, i.e., [common secant lines](#), is finite. Indeed, it is a classical enumerative problem to count the number of such common secant lines to two general twisted cubic curves, e.g., see [5, Ch. 3, Keynote Question (c)] and [6, Ex. 14.7.7(c)].

**Problem 1.1.** Given two general twisted cubic curves in  $\mathbb{P}^3(\mathbb{C})$ , how many secant lines do they have in common?

Schubert Calculus can be used to obtain the following classical result, which is a consequence of [5, Prop. 3.14]. Another approach using dynamic projection appears in [5, §3.5.2].

**Theorem 1.2.** *Two general twisted cubic curves in  $\mathbb{P}^3(\mathbb{C})$  have ten common secant lines.*

Two natural follow-up questions arise from this theorem and, to the best of our knowledge, neither of which has previously been explored: the monodromy group and a real analogue of Problem 1.1.

**Problem 1.3.** What is the monodromy group of the ten common secant lines to two twisted cubic curves in  $\mathbb{P}^3(\mathbb{C})$ ?

From Theorem 1.2, the monodromy group is a subgroup of  $S_{10}$ , the symmetric group on ten elements, since it permutes the ten common secants. Our first result is that it is indeed the full symmetric group.

**Theorem 1.4.** *The monodromy group of the ten common secant lines to two twisted cubic curves in  $\mathbb{P}^3(\mathbb{C})$  is  $S_{10}$ .*

We prove Theorem 1.4 via computational algebraic geometry. First, we utilize the software systems `Bertini` [3] and `Mathematica` to heuristically determine two monodromy loops whose corresponding monodromy actions appear to generate  $S_{10}$ . Once identified, we use the certified path-tracking software `CertifiedHomotopyTracking.jl` [12] to prove this is indeed the case.

We now consider formulating the real analogue of Problem 1.1. A curve in  $\mathbb{P}^3(\mathbb{C})$  is said to be [real](#) if it has a set of defining equations with real coefficients. This defines both a real twisted cubic curve and a real line in  $\mathbb{P}^3(\mathbb{C})$ . For example, the standard twisted cubic curve  $C_0$  in (1) is real as observed from the defining equations in (2).

Suppose that  $C_1$  and  $C_2$  are general twisted cubic curves that are real and  $\ell$  is a real line that is a common secant line, i.e., a **common real secant line**. By considering the set of intersections  $C_1 \cap \ell$  and  $C_2 \cap \ell$ , there are only three possibilities:

- $\ell$  is **totally real** if both  $C_1 \cap \ell$  and  $C_2 \cap \ell$  consist of two real points;
- $\ell$  is **partially real** if one of  $C_1 \cap \ell$  or  $C_2 \cap \ell$  consists of two real points and the other consists of a pair of nonreal complex conjugate points; and
- $\ell$  is **minimally real** if both  $C_1 \cap \ell$  and  $C_2 \cap \ell$  consist of a pair of nonreal complex conjugate points.

Thus, a common real secant line must be either totally real, partially real, or minimally real. A common secant line that is not real is called a **common nonreal secant line**. This classification yields the following real analogue to Problem 1.1.

**Problem 1.5.** Given two general twisted cubic curves in  $\mathbb{P}^3(\mathbb{C})$  that are real, how many totally real, partially real, minimally real, and nonreal common secant lines are possible?

As an initial answer to Problem 1.5, we show that there is no constraint on the number of totally real lines.

**Theorem 1.6.** *For each  $k \in \{0, 1, \dots, 10\}$ , there exists real twisted cubic curves  $C_1^{(k)}, C_2^{(k)}$  having 10 common secant lines with exactly  $k$  common totally real secant lines.*

We prove Theorem 1.6 by exhibiting specific curves  $C_1^{(k)}, C_2^{(k)}$  for each  $k$  (see Section 4.3).

By Theorem 1.2, for a generic pair of twisted cubics that are real, the sum of the numbers of totally real, partially real, minimally real, and nonreal common secant lines must be 10. Thus, we can represent these counts using the 3-tuple  $(n_t, n_p, n_m)$  where  $n_t, n_p,$  and  $n_m$  are the number of totally real, partially real, and minimally real common secant lines, respectively. The number of common real secant lines will be denoted by  $n_{\mathbb{R}} := n_t + n_p + n_m$  with the number of common nonreal secant lines being  $10 - n_{\mathbb{R}}$ . In particular,  $n_t, n_p, n_m \in \{0, 1, \dots, 10\}$  such that  $n_{\mathbb{R}} = n_t + n_p + n_m$  is an even number between 0 and 10 as common nonreal secant lines must arise in complex conjugate pairs. By a combinatorial argument (Proposition 2.5), there are a total of 161 distinct 3-tuples  $(n_t, n_p, n_m)$  satisfying these conditions, which we call **admissible 3-tuples**. An admissible 3-tuple  $(n_t, n_p, n_m)$  is **realizable** if there exists real twisted cubic curves  $C_1, C_2$  having 10 common secant lines with exactly  $n_t, n_p,$  and  $n_m$  totally real, partially real, and minimally real common secant lines, respectively. We say  $C_1, C_2$  **realize**  $(n_t, n_p, n_m)$  and  $(n_t, n_p, n_m)$  is **realized** by  $C_1$  and  $C_2$ .

Our main result is the following theorem.

**Theorem 1.7.** *Given an admissible 3-tuple  $(n_t, n_p, n_m)$  with  $n_{\mathbb{R}} = n_t + n_p + n_m$  such that*

$$n_{\mathbb{R}} \in \{0, 2, 4, 6\} \quad \text{or} \quad n_{\mathbb{R}} = 8 \text{ and } n_t \neq 1 \quad \text{or} \quad n_{\mathbb{R}} = 10 \text{ and } n_t \in \{5, 6, 7, 8, 9, 10\},$$

*there exists real twisted cubic curves  $C_1, C_2$  that realize  $(n_t, n_p, n_m)$ .*

**Remark 1.8.** For  $n_{\mathbb{R}} \in \{8, 10\}$ , we have obtained partial results (see Appendix A). In particular, it remains an open problem whether the admissible 3-tuples of the form  $(0, n_p, 10 - n_p)$  for  $0 \leq n_p \leq 10$  and  $(1, n_p, 9 - n_p)$  for  $0 \leq n_p \leq 9$  are realizable.

We prove Theorem 1.7 by utilizing computational algebraic geometry to exhibit examples of real twisted cubic curves realizing such admissible 3-tuples. We explore  $\mathcal{H}(\mathbb{R})$ , the parameter space of real twisted cubic curves, sampling specific parameter points and using

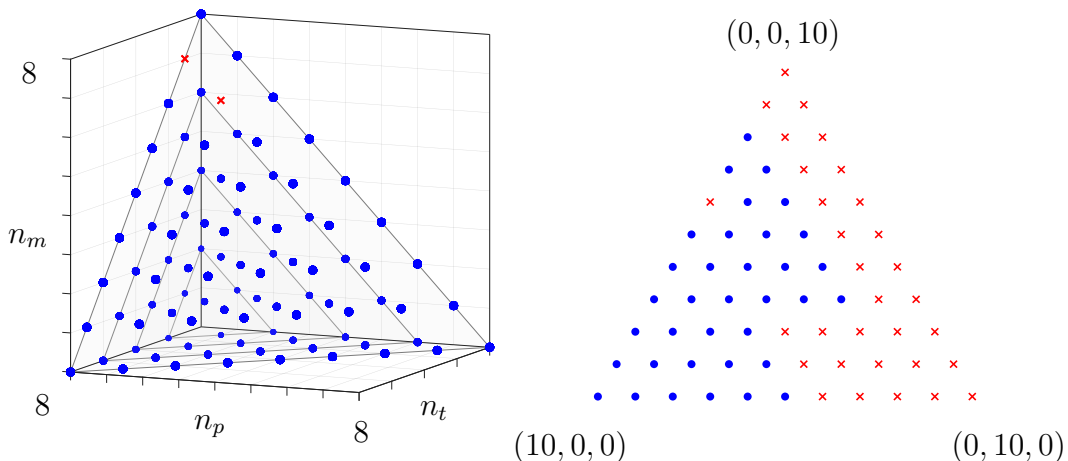


FIGURE 2. Realized 3-tuples are plotted with  $\bullet$ . Admissible but not yet realized 3-tuples are plotted with  $\times$ . The figure on the left shows the 3-tuples with  $n_{\mathbb{R}} \in \{0, 2, 4, 6, 8\}$ . The figure on the right shows the 3-tuples with  $n_{\mathbb{R}} = 10$ .

**Bertini** to heuristically determine the number of totally real, partially real, and minimally real common secant lines. Then, we utilize `alphaCertified` [11] to prove that those counts are indeed correct. Our code and sample parameter points are contained in a `GitHub` repository [1]. Figure 2 shows the admissible and realized 3-tuples with Table 4 in Appendix A listing the 3-tuples explicitly.

The rest of the paper is organized as follows. Section 2 considers the formulation of a common secant to two twisted cubic curves as a parameterized polynomial system, which will be used to solve Problem 1.3 and to investigate Problem 1.5. Section 3 proves Theorem 1.4 by computing two elements contained in the monodromy group that generate  $S_{10}$ . Section 4 proves Theorems 1.6 and 1.7, building on necessary conditions from Section 2 (Proposition 2.5) with computational algebraic geometry exhibiting an example for each admissible 3-tuple realized. Concluding remarks and open problems are provided in Section 5.

## 2. SECANTS OF TWISTED CUBICS

Our proofs of Theorems 1.4, 1.6, and 1.7 will use a formulation of the problem which is intrinsic to the geometry of the parameter space  $\mathcal{H}$  of twisted cubic curves [8, Example 12.9]. Using a parametrization of the standard twisted cubic  $C_0$ , we describe common secant lines via rank conditions. To that end, we regard vectors in  $\mathbb{C}^4$  (and, by extension, points in  $\mathbb{P}^3(\mathbb{C})$ ) as column vectors. Let  $C_0$  denote the standard twisted cubic as in (1) and  $v(a) := (1, a, a^2, a^3)^T \in \mathbb{C}^4$ . When  $s \neq 0$ , we have  $[s, t] = [1, a] \in \mathbb{P}^1(\mathbb{C})$  where  $a = t/s$ . Hence,

$$C_0 = \overline{\{[v(a)] \mid a \in \mathbb{C}\}} = \{[v(a)] \mid a \in \mathbb{C}\} \cup \{[0, 0, 0, 1]\}.$$

It is known that every twisted cubic in  $\mathbb{P}^3(\mathbb{C})$  is the image of  $C_0$  under a projective linear transformation [8, Example 1.14]. Let  $\mathrm{GL}_4(\mathbb{C})$  be the group of invertible  $4 \times 4$  matrices with entries in  $\mathbb{C}$ . Given a matrix  $M \in \mathrm{GL}_4(\mathbb{C})$ , the action of  $M$  on  $\mathbb{P}^3(\mathbb{C})$  given by

$$M \cdot [x] := [Mx], \quad 0 \neq x \in \mathbb{C}^4$$

is well-defined and scaling  $M$  does not change the action. Now, let

$$C_1 := \overline{\{[Mv(s)] \in \mathbb{P}^3(\mathbb{C}) \mid s \in \mathbb{C}\}}.$$

In particular,  $C_1$  only depends on the equivalence class  $[M] \in \text{PGL}_4(\mathbb{C}) := \text{GL}_4(\mathbb{C})/\mathbb{C}^\times$  where  $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ . The group  $\text{PGL}_4(\mathbb{C})$ , which describes all projective linear transformations of  $\mathbb{P}^3(\mathbb{C})$ , is also identified with the Zariski open subset of  $\mathbb{P}(\mathbb{C}^{4 \times 4}) \cong \mathbb{P}^{15}(\mathbb{C})$  consisting of points  $[M]$  with  $\det(M) \neq 0$ . Hence, we can define a map  $\text{PGL}_4(\mathbb{C}) \rightarrow \mathcal{H}$  sending  $[M] \in \text{PGL}_4(\mathbb{C})$  to  $\overline{\{[Mv(s)] \in \mathbb{P}^3(\mathbb{C}) \mid s \in \mathbb{C}\}}$ . Such a map is surjective and the fiber over any  $C \in \mathcal{H}$  consists of those projective linear transformations sending  $C$  to itself, which form a subgroup isomorphic to  $\text{PGL}_2(\mathbb{C})$ , e.g., see [8, Example 10.9]. Hence,  $\mathcal{H} \cong \text{PGL}_4(\mathbb{C})/\text{PGL}_2(\mathbb{C})$  which has dimension  $(4^2 - 1) - (2^2 - 1) = 12$ .

**Remark 2.1.** When working with  $[M] \in \text{PGL}_4(\mathbb{C})$  we choose representatives  $M \in \text{GL}_4(\mathbb{C})$ .

Incidence conditions between  $C_0$  and  $C_1$  may be expressed in terms of the linear dependence of vectors in  $\mathbb{C}^4$ . In fact, a line meeting  $\{[v(a)] \mid a \in \mathbb{C}\}$  and  $\{[Mv(s)] \in \mathbb{P}^3(\mathbb{C}) \mid s \in \mathbb{C}\}$  in two points each corresponds to a rank condition on the  $4 \times 4$  matrix whose columns are

$$v(t_1), \quad v(t_2), \quad Mv(s_1), \quad Mv(s_2)$$

where  $t_1 \neq t_2$  and  $s_1 \neq s_2$ . In particular, when  $t_1 \neq t_2$ , the vectors  $v(t_1), v(t_2)$  are linearly independent and

$$(3) \quad \ell(t_1, t_2) = \text{Span}(v(t_1), v(t_2)) \subset \mathbb{P}^3(\mathbb{C})$$

is a secant line to  $C_0$ . The line  $\ell(t_1, t_2)$  intersects  $\{[Mv(s)] \in \mathbb{P}^3(\mathbb{C}) \mid s \in \mathbb{C}\}$  in two distinct points if and only if there exist  $s_1 \neq s_2$  such that

$$Mv(s_1), Mv(s_2) \in \ell(t_1, t_2).$$

Equivalently, for each  $i = 1, 2$ , the vectors  $v(t_1), v(t_2), Mv(s_i)^T \in \mathbb{C}^4$  are linearly dependent:

$$(4) \quad \text{rank} \begin{pmatrix} v(t_1) & v(t_2) & Mv(s_i) \end{pmatrix} \leq 2, \quad i = 1, 2.$$

Since  $t_1 \neq t_2$  implies that  $\text{rank} \begin{pmatrix} v(t_1) & v(t_2) \end{pmatrix} = 2$  with the leading  $2 \times 2$  block being invertible, Condition (4) with  $t_1 \neq t_2$  is equivalent to the vanishing of the  $3 \times 3$  minors of  $\begin{pmatrix} v(t_1) & v(t_2) & Mv(s_i) \end{pmatrix}$  using rows 1, 2, 3 and 1, 2, 4. For convenience, write  $M = (m_{ij})_{1 \leq i, j \leq 4}$  and define

$$(5) \quad p_i(s; M) := m_{i1} + m_{i2}s + m_{i3}s^2 + m_{i4}s^3, \quad i = 1, 2, 3, 4.$$

Then, computing these two minors and factoring out the common factor  $t_1 - t_2$ , we obtain

$$\begin{aligned} f_1(t_1, t_2, s_i; M) &= p_3(s_i; M) - (t_1 + t_2)p_2(s_i; M) + t_1t_2p_1(s_i; M), \\ f_2(t_1, t_2, s_i; M) &= p_4(s_i; M) - (t_1^2 + 2t_1t_2 + t_2^2)p_2(s_i; M) + t_1t_2p_1(s_i; M), \end{aligned} \quad i = 1, 2.$$

In totality, this yields a parametrized system of four polynomial equations in the four unknowns  $(t_1, t_2, s_1, s_2) \in \mathbb{C}^4$  with parameters being the entries of  $M$ :

$$(6) \quad F(t_1, t_2, s_1, s_2; M) = \begin{bmatrix} f_1(t_1, t_2, s_1; M) \\ f_1(t_1, t_2, s_2; M) \\ f_2(t_1, t_2, s_1; M) \\ f_2(t_1, t_2, s_2; M) \end{bmatrix} = \begin{bmatrix} p_3(s_1; M) - (t_1 + t_2)p_2(s_1; M) + t_1t_2p_1(s_1; M) \\ p_3(s_2; M) - (t_1 + t_2)p_2(s_2; M) + t_1t_2p_1(s_2; M) \\ p_4(s_1; M) - (t_1^2 + 2t_1t_2 + t_2^2)p_2(s_1; M) + t_1t_2p_1(s_1; M) \\ p_4(s_2; M) - (t_1^2 + 2t_1t_2 + t_2^2)p_2(s_2; M) + t_1t_2p_1(s_2; M) \end{bmatrix}$$

where we only consider solutions with  $t_1 \neq t_2$  and  $s_1 \neq s_2$ . Since the labeling of  $t_1$  and  $t_2$  as well as the labeling of  $s_1$  and  $s_2$  are both arbitrary, there is a natural  $(S_2 \times S_2)$ -action on the set of solutions where the first  $S_2$  permutes  $t_1$  and  $t_2$  while the second  $S_2$  permutes  $s_1$  and  $s_2$ . Hence, a common secant line corresponds to a single orbit of size four in the open subset  $t_1 \neq t_2, s_1 \neq s_2$  of the solution set, namely

$$(7) \quad (t_1, t_2, s_1, s_2), \quad (t_2, t_1, s_1, s_2), \quad (t_1, t_2, s_2, s_1), \quad \text{and} \quad (t_2, t_1, s_2, s_1).$$

**Proposition 2.2.** *For a general  $[M] \in \text{PGL}_4(\mathbb{C})$ , the system (6) has 40 isolated nonsingular solutions  $(t_1, t_2, s_1, s_2) \in \mathbb{C}^4$  and all such nonsingular solutions satisfy  $t_1 \neq t_2$  and  $s_1 \neq s_2$ . These nonsingular solutions are grouped into ten orbits of size four under the  $(S_2 \times S_2)$ -action in accordance with Theorem 1.2.*

**Proof.** From the structure of  $F$  in (6), if  $s_1 = s_2$ , then  $F$  only consists of two distinct polynomials in three variables. Hence, any solution with  $s_1 = s_2$  must lie on a positive-dimensional component and thus is singular. Similarly, it is easy to observe that  $\partial F / \partial t_1(t_1, t_1, s_1, s_2; M) = \partial F / \partial t_2(t_1, t_1, s_1, s_2; M)$  so that any solution with  $t_1 = t_2$  must also be singular.

For the standard twisted cubic  $C_0$  and a general  $C_1 \in \mathcal{H}$ , the 10 common secant lines are not contained in the plane defined by  $x_0 = 0$  in  $\mathbb{P}^3(\mathbb{C})$ . Hence, for general  $[M] \in \text{PGL}_4(\mathbb{C})$ , there are 10 common secant lines between  $\{[v(a)] \mid a \in \mathbb{C}\}$  and  $\{[Mv(s)] \in \mathbb{P}^3(\mathbb{C}) \mid s \in \mathbb{C}\}$ . Due to (7), these correspond with 40 solutions of  $F(t_1, t_2, s_1, s_2; M) = 0$  with  $t_1 \neq t_2$  and  $s_1 \neq s_2$ . To show generic nonsingularity, it suffices to show nonsingularity for a chosen  $M$ . For example, using `Macaulay2` [7] with the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 3 & 1 \end{pmatrix},$$

the ideal  $J = \langle F \rangle : \langle (t_1 - t_2)(s_1 - s_2) \rangle^\infty \subset \mathbb{Q}[t_1, t_2, s_1, s_2]$  has  $\deg J = \deg \sqrt{J} = 40$  showing that all 40 solutions satisfying  $t_1 \neq t_2$  and  $s_1 \neq s_2$  are nonsingular.  $\square$

For parameterized polynomial systems such as  $F$  in (6), the monodromy group associated with the generically nonsingular isolated solutions is an invariant that encodes information about the structure of the solutions, e.g., see [10]. Due to the relationship described in (7), Theorem 1.4 posits that the monodromy group associated with the 40 generically nonsingular isolated solutions of  $F$  is  $S_{10} \times S_2 \times S_2$ . We will prove Theorem 1.4 in Section 3.

Now, suppose that the entries of  $M$  are real so that  $C_1$  is a real twisted cubic curve. Given  $(t_1, t_2, s_1, s_2) \in \mathbb{C}^4$  with  $t_1 \neq t_2$ ,  $s_1 \neq s_2$ , and  $F(t_1, t_2, s_1, s_2; M) = 0$ , we now need to determine if the corresponding common secant line  $\ell := \ell(t_1, t_2)$  in (3) is totally real, partially real, minimally real, or nonreal. The following describes this process.

First, one determines if  $\ell$  is a real line. A real secant line to  $C_0$  must intersect  $C_0$  in either two real points, implying  $(t_1, t_2) \in \mathbb{R}^2$ , or a pair of nonreal complex conjugate points, implying  $t_2 = \text{conj}(t_1)$  where  $\text{conj}(a)$  is the complex conjugate of  $a$ . Conversely, if  $(t_1, t_2) \in \mathbb{R}^2$ , then  $\ell$  is trivially observed to be a real line and, if  $t_2 = \text{conj}(t_1)$ , then  $\ell$  is a line that interpolates between two nonreal complex conjugate points and is therefore real.

Similarly,  $\ell$  is real if and only if the intersection points of  $\ell$  with  $C_1$ , i.e.  $Mv(s_i)$ ,  $i = 1, 2$ , are either real or nonreal complex conjugates. Since  $M$  is real and invertible, this is equivalent to

$(s_1, s_2) \in \mathbb{R}^2$  or  $s_2 = \text{conj}(s_1)$ , respectively. This yields the following approach to classifying common secant lines.

**Proposition 2.3.** *If the entries of  $M$  are real and  $(t_1, t_2, s_1, s_2) \in \mathbb{C}^4$  is a solution to  $F(t_1, t_2, s_1, s_2; M) = 0$  such that  $t_1 \neq t_2, s_1 \neq s_2$ , then the corresponding common secant line  $\ell(t_1, t_2)$  to  $C_0$  and  $C_1$  as in (3) is classified as follows:*

- $\ell$  is **totally real** if and only if  $(t_1, t_2, s_1, s_2) \in \mathbb{R}^4$ ;
- $\ell$  is **partially real** if and only if either  $(t_1, t_2) \in \mathbb{R}^2$  and  $(s_1, s_2) \in \mathbb{C}^2 \setminus \mathbb{R}^2$  such that  $s_2 = \text{conj}(s_1)$ , or  $(s_1, s_2) \in \mathbb{R}^2$  and  $(t_1, t_2) \in \mathbb{C}^2 \setminus \mathbb{R}^2$  such that  $t_2 = \text{conj}(t_1)$ ;
- $\ell$  is **minimally real** if and only if  $(t_1, t_2, s_1, s_2) \in \mathbb{C}^4 \setminus \mathbb{R}^4$  such that  $t_2 = \text{conj}(t_1)$  and  $s_2 = \text{conj}(s_1)$ ;
- $\ell$  is **nonreal** if and only if  $(t_1, t_2) \in \mathbb{C}^2 \setminus \mathbb{R}^2$  and  $t_2 \neq \text{conj}(t_1)$ .

**Example 2.4** (10 totally real lines). Take the following parameters as exact:

$$(8) \quad M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1.4351 & 0.6797 & 3.9435 & 7.7238 \\ -1.3085 & 4.6694 & 13.1949 & 5.1509 \\ 1.1573 & 1.2007 & 5.7272 & 8.6591 \end{pmatrix}.$$

Then, there are 40 isolated solutions in  $\mathbb{C}^4$  to  $F(t_1, t_2, s_1, s_2; M) = 0$  with  $t_1 \neq t_2$  and  $s_1 \neq s_2$  which arise in 10 groups of four as in (7). We utilized `alphaCertified` to certify that all 40 are real. Rounded to four decimal places, one representative from each group is:

$$\begin{aligned} x_1 &:= (0.9750, 0.1977, 0.4144, 2.4773), \\ x_2 &:= (0.1052, 1.0123, 0.4063, -0.7032), \\ x_3 &:= (0.1089, 1.0266, 2.7003, -0.7031), \\ x_4 &:= (1.0006, -1.0064, 0.2684, -0.8378), \\ x_5 &:= (1.0311, -1.0077, -0.8386, -2.0733), \\ x_6 &:= (0.9846, -1.0399, -1.8183, 0.2619), \\ x_7 &:= (-0.0610, -0.8453, -0.4690, -0.0085), \\ x_8 &:= (-1.0078, 0.1115, -0.8657, -0.5436), \\ x_9 &:= (-0.6075, -0.3172, -0.4000, -0.0465), \\ x_{10} &:= (-1.0093, 0.1324, -0.8664, 0.0316). \end{aligned}$$

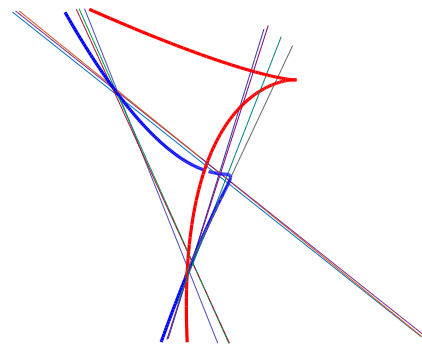


FIGURE 3. Ten totally real common real secant lines between  $C_0$  and  $C_1$ .

Each of these corresponds to a totally real common real secant line to  $C_0$  and  $C_1$ , as illustrated in Figure 3. The 3-tuple  $(n_t, n_p, n_m) = (10, 0, 0)$  is realized by these two curves.  $\diamond$

We can build off of the description in Proposition 2.3 to determine which of the admissible 3-tuples  $(n_t, n_p, n_m)$  are realized by taking into account the behavior of solutions to systems of polynomial equations with real coefficients, as common nonreal secant lines must arise in complex conjugate pairs.

**Proposition 2.5.** *Let  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$  be the set of nonnegative integers. The total number of admissible 3-tuples is*

$$\# \left( \bigcup_{n_{\mathbb{R}} \in \{0, 2, 4, 6, 8, 10\}} \{(n_t, n_p, n_m) \in \mathbb{N}_0^3 \mid n_t + n_p + n_m = n_{\mathbb{R}}\} \right) = \sum_{n_{\mathbb{R}} \in \{0, 2, 4, 6, 8, 10\}} \binom{n_{\mathbb{R}} + 2}{2} = 161.$$

**Proof.** To find the total number of admissible 3-tuples, we use a version of the Stars and Bars Theorem [15, Section 1.2] from combinatorics which posits that there are exactly  $\binom{m+2}{2}$  3-tuples of nonnegative integers whose sum is  $m$ . Since  $n_{\mathbb{R}}$  must be even, adding the 6 values corresponding to even sums from 0 to 10 yields the result.  $\square$

### 3. MONODROMY

The monodromy group of ten common secant lines to twisted cubic curves is a subgroup of  $S_{10}$ . This monodromy group encodes which permutations of the ten common secant lines are possible as the pair of twisted cubic curves move in a closed loop such that the common secant lines vary smoothly. The following proves Theorem 1.4 by finding two closed loops for which the corresponding monodromy action generates  $S_{10}$  using  $F$  in (6). To simplify notation, set  $x := (t_1, t_2, s_1, s_2) \in \mathbb{C}^4$  so that (6) can be written as  $F(x; M) = 0$ .

We utilize the following procedure which has been utilized for other problems before, e.g., see [9, 13]. We select a sufficiently general base point  $[M_0] \in \text{PGL}_4(\mathbb{C})$  for which  $F(x; M_0) = 0$  has 40 nonsingular isolated solutions which arise in 10 groups of four, as summarized in Proposition 2.2. This is sufficient since  $\mathcal{H}$  is path-connected. For each of the 10 groups, we select a representative and assign an ordering  $x_1, \dots, x_{10}$ . We then follow a two-step process to prove Theorem 1.4: a heuristic search followed by certification.

For the proof of our result, we will take  $M_0$  to be the parameters listed in Example 2.4 and  $x_1, \dots, x_{10}$  to be the points as listed there as well.

**3.1. Heuristic searching for triangular loops.** Suppose that  $\gamma : [0, 1] \rightarrow \text{PGL}_4(\mathbb{C})$  is a loop with  $\gamma(0) = \gamma(1) = [M_0]$  such that  $F(x; \gamma(a)) = 0$  has 40 nonsingular isolated solutions for every  $a \in [0, 1]$ . Then, for every  $i = 1, \dots, 10$ , there is a path  $X_i : [0, 1] \rightarrow \mathbb{C}^4$  such that  $X_i(a)$  is a nonsingular solution of  $F(x; \gamma(a)) = 0$  for every  $a \in [0, 1]$  with  $X_i(0) = x_i$ . Since  $\gamma$  is a loop, there exists a unique  $j_i \in \{1, \dots, 10\}$  such that  $X_i(1)$  is in the same  $(S_2 \times S_2)$ -orbit of  $x_{j_i}$ . Let  $\sigma_\gamma \in S_{10}$  be the permutation this describes, i.e.,  $\sigma_\gamma(i) = j_i$ . The monodromy group consists of all such permutations  $\sigma_\gamma$ .

For our heuristic search, we generate loops  $\gamma$  as the edges of a triangle where one of the vertices is  $M_0$ , the aforementioned fixed base point. We select the other two vertices randomly and track the corresponding solution paths  $X_i(a)$  using numerical path tracking implemented in `Bertini` to determine what we purport to be the permutation  $\sigma_\gamma$ .

Since  $S_{10}$  can be generated by two permutations, we heuristically searched for two loops  $\gamma_1$  and  $\gamma_2$  such that the purported permutations  $\sigma_{\gamma_1}$  and  $\sigma_{\gamma_2}$  generate  $S_{10}$ . Each edge of the triangle loop  $\gamma_k$  is parametrized by a straight-line homotopy in parameter space as illustrated in Figure 4. If an edge joins  $M_\alpha^{(k)}$  to  $M_\beta^{(k)}$ , we track along the straight line  $aM_\alpha^{(k)} + (1-a)M_\beta^{(k)}$  for  $a \in [0, 1]$ . In order for the path to be well-defined in  $\text{PGL}_4(\mathbb{C})$ , we need to ensure that  $\det(aM_i^{(k)} + (1-a)M_j^{(k)}) \neq 0$  for all  $a \in [0, 1]$ . Since  $\det(aM_i^{(k)} + (1-a)M_j^{(k)})$  is a polynomial in  $a$ , one just needs to check that the roots do not lie in  $[0, 1]$ . Tables 1 and 2 show the roots for the corresponding path  $\gamma_1$  and  $\gamma_2$ , respectively, where the paths  $\gamma_1$  and  $\gamma_2$  are defined by the following matrices:

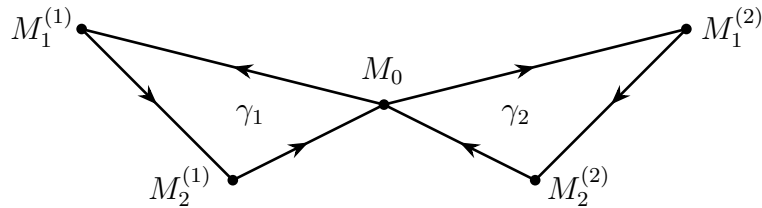
$$M_1^{(1)} = \begin{pmatrix} 4.1182-3.6621i & -5.6621+6.1182i & 3.1182-4.8843i & -6.1182+5.6621i \\ 1.1182-2.1182i & -10.6621-3.1182i & 5.1182+0.8843i & 8.1182-0.8843i \\ -1.1182-1.1182i & 18.1182+0.8843i & 25.1182-9.1182i & 3.1182+0.8843i \\ 12.1182+8.1182i & 3.1182-0.8843i & 8.1182-2.1182i & 20.1182-6.1182i \end{pmatrix},$$

$$M_2^{(1)} = \begin{pmatrix} -1.8843+2.6621i & 4.6621-7.1182i & -5.5529+1.4412i & 5.6621-2.1182i \\ -7.8843+6.4412i & 2.1182-7.8843i & -10.6621+5.1182i & 20.4412+4.6621i \\ 7.5529-3.1182i & -6.1182-4.6621i & 10.6621+5.4412i & -7.8843+9.1182i \\ 2.4412-5.1182i & 4.6621-10.6621i & -3.1182+7.8843i & -0.6621-6.4412i \end{pmatrix},$$

$$M_1^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -0.9132+13.7745i & -14.8821-1.3348i & 20.4413+2.1187i & -6.7724-14.2031i \\ 15.9982-3.1146i & 5.4418+12.9926i & 26.1184+2.7741i & 3.8825+16.1183i \\ -2.1187+10.4419i & 15.9923-1.8824i & -7.4412+0.9926i & 9.1184-11.7745i \end{pmatrix},$$

$$M_2^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -11.4413+1.1184i & 14.8826-0.7742i & -8.9921+11.4410i & 6.1183-13.9926i \\ -2.8824-15.4417i & 4.1186+9.8821i & 24.9928+3.1184i & -6.1187+11.7742i \\ -10.4412+13.8826i & 12.1184-9.4413i & 4.8821+1.1184i & 19.7745-0.8826i \end{pmatrix}.$$

Edge	Determinant	Roots of det = 0
$M_0-M_1^{(1)}$	$\det(aM_0 + (1-a)M_1^{(1)})$	$a_1 \approx 0.9942 + 0.0016i$
		$a_2 \approx 1.0449 + 0.0260i$
		$a_3 \approx 1.3711 + 0.0331i$
		$a_4 \approx 2.6126 + 0.7410i$
$M_1^{(1)}-M_2^{(1)}$	$\det(aM_1^{(1)} + (1-a)M_2^{(1)})$	$a_1 \approx -0.5899 + 1.5925i$
		$a_2 \approx 0.0428 - 0.4212i$
		$a_3 \approx 0.3658 + 0.3845i$
		$a_4 \approx 0.4319 - 0.0771i$
$M_2^{(1)}-M_0$	$\det(aM_2^{(1)} + (1-a)M_0)$	$a_1 \approx -0.0044 + 0.0110i$
		$a_2 \approx 0.0214 - 0.5554i$
		$a_3 \approx 0.0348 - 0.0091i$
		$a_4 \approx 0.8297 + 0.7387i$

TABLE 1. Roots of determinant polynomials along the edges of loop  $\gamma_1$ .FIGURE 4. Illustration of loops  $\gamma_1$  and  $\gamma_2$ 

In our heuristic process, we found that the loops  $\gamma_1$  and  $\gamma_2$  appeared to yield

(9)  $\sigma_{\gamma_1} = (1\ 7\ 5\ 9\ 8\ 6\ 3\ 4\ 2)(10)$  and  $\sigma_{\gamma_2} = (1\ 8\ 9\ 3\ 10\ 5\ 7\ 2)(4\ 6)$ .

Edge	Determinant	Roots of det = 0
$M_0-M_1^{(2)}$	$\det(aM_0 + (1-a)M_1^{(2)})$	$a_1 \approx 0.8047 + 0.4024i$ $a_2 \approx 0.9982 - 0.0000i$ $a_3 \approx 1.6446 - 0.6637i$
$M_1^{(2)}-M_2^{(2)}$	$\det(aM_1^{(1)} + (1-a)M_2^{(2)})$	$a_1 \approx -2.1843 + 4.3611i$ $a_2 \approx 0.4278 + 0.2305i$ $a_3 \approx 0.6036 - 1.0162i$
$M_2^{(2)}-M_0$	$\det(aM_2^{(2)} + (1-a)M_0)$	$a_1 \approx -3.5780 + 0.1699i$ $a_2 \approx -0.0684 - 0.0127i$ $a_3 \approx -0.0099 + 0.0070i$

TABLE 2. Roots of determinant polynomials along the edges of loop  $\gamma_2$ .

It remains to certify that the numerical path tracking indeed produces these permutations.

**3.2. Certification.** Although path tracking in `Bertini` is robust, it is heuristic and thus one needs to utilize certification to prove that  $\gamma_1$  and  $\gamma_2$  yield smooth solution paths and the purported permutations  $\sigma_{\gamma_1}$  and  $\sigma_{\gamma_2}$  in (9), respectively.

**Proof.** [Proof of Theorem 1.4]

For  $j = 1, 2$  and  $i = 1, \dots, 10$ , we utilized the certified homotopy tracking software `CertifiedHomotopyTracking.jl` [12] arising from the Krawczyk homotopy method from [4] to prove that the path  $X_i^{(j)}(a)$  defined by  $F(X_i^{(j)}(a); \gamma_j(a)) = 0$  for  $a \in [0, 1]$  with  $X_i^{(j)}(0) = x_i$  has  $X_i^{(j)}(a)$  being a nonsingular isolated solution of  $F(x; \gamma_j(a)) = 0$  and  $X_i^{(j)}(1) = x_{\sigma_{\gamma_j}(i)}$  where  $\sigma_{\gamma_j}$  in (9). All scripts, input files, and certification certificates are available at the public GitHub repository [1]. Finally, we execute the following code in `Mathematica`:

```

s1 = Cycles[{{1,7,5,9,8,6,3,4,2},{10}}];
s2 = Cycles[{{1,3,4,7},{2,5},{6,8,10,9}}];
GroupOrder[PermutationGroup[{s1,s2}]]

```

The output is 3,628,800. Thus,  $\langle \sigma_{\gamma_1}, \sigma_{\gamma_2} \rangle$  is a subgroup of  $S_{10}$  of size 3,628,800 = 10!. Since  $\#S_{10} = 10!$ , this proves  $\langle \sigma_{\gamma_1}, \sigma_{\gamma_2} \rangle = S_{10}$ .  $\square$

#### 4. REALIZABLE 3-TUPLES

We sample the parameter space  $\mathcal{H}(\mathbb{R}) := \text{PGL}_4(\mathbb{R})/\text{PGL}_2(\mathbb{R})$  to investigate how the number of real and nonreal solutions to (6) varies.

**4.1. Uniform sampling.** To explore  $\mathcal{H}(\mathbb{R})$ , we sample uniformly in the real projective space  $\mathbb{P}^{15}(\mathbb{R})$ . Since  $\mathbb{P}^{15}(\mathbb{R})$  can be realized as the quotient  $\mathbb{S}^{15}/\{\pm 1\}$  where  $\mathbb{S}^{15} \subset \mathbb{R}^{16}$  is the unit sphere, with antipodal points identified, we first sample  $\mathbb{S}^{15}$  uniformly and then select a representative from each antipodal pair. Uniform sampling on  $\mathbb{S}^{15}$  is obtained by drawing vectors from the standard Gaussian distribution in  $\mathbb{R}^{16}$  and normalizing them to unit length. To pass to projective space, we fix a representative for each line through the origin by requiring the last coordinate be nonnegative.

A Python driver script automates parameter sampling, tracking parameter homotopies in Bertini, and the certified classification of real versus complex solutions. Each sample produces a 3-tuple  $(n_t, n_p, n_m)$  representing the number of various real secant lines. Algorithm 1 summarizes this approach.

---

**Algorithm 1** Uniformly sampling parameter points in  $\mathcal{H}(\mathbb{R})$  for the system  $F$  (6)

---

**Input:** The number  $N$  of parameters to be sampled.

**Output:** A list of 3-tuples  $(n_t, n_p, n_m)$  giving the count of real secant lines obtained from the  $N$  parameters uniformly sampled.

1. Generate  $N$  sample parameter points  $M_1, \dots, M_N$  uniformly in  $\mathbb{P}^{15}(\mathbb{R})$ .

**For each**  $i \in \{1, \dots, N\}$  **do:**

2. Solve  $F(x; M_i) = 0$ . Generically there are 40 nonsingular isolated solutions.

3. Identify real and complex-conjugate pairs among the solutions and pick 10 representatives  $(t_1, t_2, s_1, s_2) \in \mathbb{C}^4$  corresponding to their  $(S_2 \times S_2)$ -orbits.

4. Classify each representative according to Proposition 2.3.

**Return** A list of 3-tuples  $(n_t, n_p, n_m)$  giving the count of real secant lines.

---

We performed 100,000 uniform random samples of  $\mathcal{H}(\mathbb{R})$ . Figure 5 shows that the distribution of real common secants is concentrated at  $n_{\mathbb{R}} = 2$  and  $n_{\mathbb{R}} = 4$ . Configurations with ten real secant lines occurred only in 0.06% of samples.

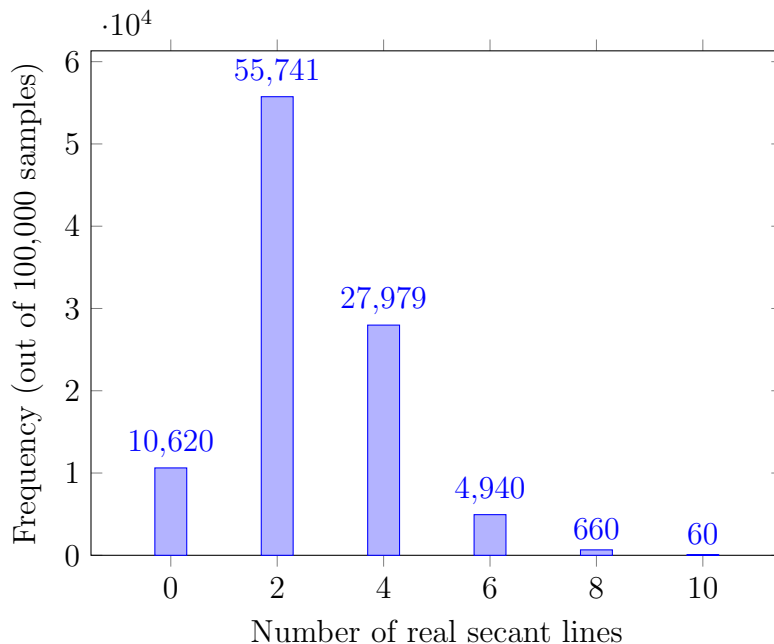


FIGURE 5. Frequency distribution of the number of real common secant lines among 100,000 uniform random samples.

As expected, we observe that the induced distribution on 3-tuples  $(n_t, n_p, n_m)$  is highly nonuniform with a small number of 3-tuples arising in the majority of samples. Many admissible cases were not realized in 100,000 samples. Table 3 shows the 10 most frequent 3-tuples  $(n_t, n_p, n_m)$  for 100,000 samples.

TABLE 3. Top 10 most frequent 3-tuples  $(n_t, n_p, n_m)$  for 100,000 samples.

$n_t$	$n_p$	$n_m$	Count
0	1	1	21,830
0	2	0	11,811
0	0	0	10,620
1	0	1	9,960
0	0	2	7,280
0	4	0	5,309
0	3	1	4,340
1	3	0	3,510
2	1	1	3,390
1	1	0	3,310

**4.2. Localizing sampling.** Rather than uniformly sampling, we aim to find additional realizable 3-tuples by generating samples nearby a given parameter point. Algorithm 2 summarizes this localized approach which we use to complete the proof of our main theorem, Theorem 1.7.

---

**Algorithm 2** Sampling parameter points near a given point in  $\mathcal{H}(\mathbb{R})$ .

---

**Input:** A sample point  $[M_0] \in \mathcal{H}(\mathbb{R})$  representing parameters of  $F(x; M_0) = 0$  as in (6), a number  $\varepsilon \in (0, 1)$ , and a number  $N$  of parameters to be sampled in  $B_\varepsilon(M_0)$ .

**Output:** A list of 3-tuples  $(n_t, n_p, n_m)$  giving the count of real secant lines obtained from the  $N$  parameters sampled in  $B_\varepsilon(M_0)$ .

1. Generate  $N$  sample parameter points  $M_1, \dots, M_N$  in  $B_\varepsilon(M_0)$ .

**For each**  $i \in N$  **do:**

2. Run a parameter homotopy from the set of 10 solution representatives of  $F(x; M_0) = 0$  to  $F(x; M_i) = 0$ .

3. Classify each representative according to Proposition 2.3.

**Return** A list of 3-tuples  $(n_t, n_p, n_m)$  giving the count of real secant lines.

---

**Proof.** [Proof of Theorem 1.7] By Proposition 2.5, there are a total of 161 admissible 3-tuples  $(n_t, n_p, n_m)$ . By sampling a large number of initial parameter points in  $\mathcal{H}(\mathbb{R})$ , and repeatedly performing Algorithm 2 on points in our sample whose 3-tuples  $(n_t, n_p, n_m)$  are located near admissible but not yet realized 3-tuples, we first used heuristic solving to determine parameters  $M \in \mathcal{H}(\mathbb{R})$  which realize an admissible 3-tuple that has yet to be realized. This process generated parameters for 128 out of the 161 admissible 3-tuples.

For each of these parameters, we proved realizability by certifying approximate solutions using `alphaCertified`. The certified data, including rational representations of the solutions, are archived in the `GitHub` repository [1].

Table 4 in the Appendix shows the lists of realized 3-tuples as well as admissible 3-tuples that have not been realized, which is also plotted in Figure 2. An inspection of the data confirms that all admissible 3-tuples satisfying the hypothesis of the theorem have been realized and certified.  $\square$

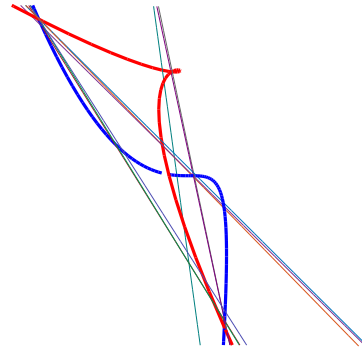
**4.3. Totally real secant lines.** The following subsection records examples constituting the proof of Theorem 1.6. Since Example 2.4 showed the existence of two real twisted cubics with 10 totally real secant lines, the remaining cases are considered below.

**Example 4.1** (9 totally real lines). Taking the following parameters as exact,

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.7519 & 0.1524 & 2.8796 & 7.6158 \\ -1.0069 & 3.7048 & 15.1121 & 7.9388 \\ 1.2271 & 0.1795 & 3.6345 & 10.6654 \end{pmatrix},$$

there are 40 isolated solutions in  $\mathbb{C}^4$  to  $F(x; M) = 0$  arising in 10 groups of four as in (7). From these, there are 9 representatives that are totally real. Truncated to four decimal places, one representative from each of the 9 groups is given below.

$$\begin{pmatrix} -0.0973, & 1.2278, & 0.2872, & 17.9477 \\ -0.0477, & 1.2050, & -0.5659, & 13.7519 \\ -0.0357, & 1.2281, & -0.5654, & 0.2871 \\ -1.1977, & 1.2280, & -0.7631, & 0.2883 \\ -1.1961, & 1.2023, & -1.4348, & -0.7619 \\ -1.1699, & 1.2279, & -1.5291, & 0.2883 \\ -1.4968, & 0.1556, & -0.3699, & 0.0476 \\ -1.1959, & -0.0525, & -0.7887, & -0.2850 \\ -1.1954, & -0.0624, & -0.7886, & -0.0035 \end{pmatrix}$$



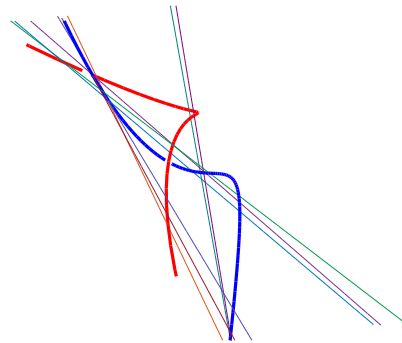
These data correspond to the 3-tuple  $(9, 1, 0)$ . ◇

**Example 4.2** (8 totally real lines). Taking the following parameters as exact,

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.7712 & 1.0684 & 4.0153 & 6.8183 \\ -0.3876 & 3.8568 & 13.8269 & 6.0562 \\ 0.7904 & 1.0239 & 4.9333 & 9.2222 \end{pmatrix},$$

there are 40 isolated solutions in  $\mathbb{C}^4$  to  $F(x; M) = 0$  arising in 10 groups of four as in (7). From these, there are 8 representatives that are totally real. Truncated to four decimal places, one representative from each of the 8 groups is given below.

$$\begin{pmatrix} 0.2845, & 0.9851, & 0.2431, & 3.5926 \\ 0.1204, & 1.0904, & -0.4944, & 4.3169 \\ -1.2920, & 0.9966, & -1.3992, & 0.1953 \\ 0.0682, & 1.0255, & -0.4963, & 0.2388 \\ -1.2405, & 1.1041, & -1.6107, & -0.6670 \\ -1.1738, & 1.0127, & -0.6361, & 0.2009 \\ -1.2561, & 0.1874, & -0.9975, & -0.0484 \\ -1.2469, & 0.1460, & -0.9924, & -0.2785 \end{pmatrix}$$



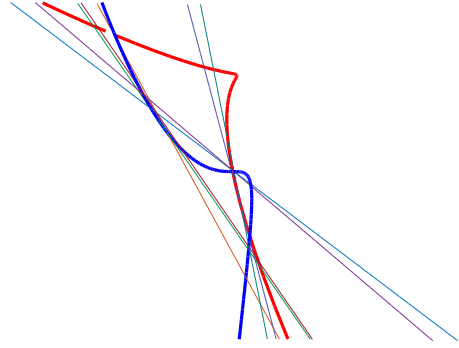
These data correspond to the 3-tuple  $(8, 0, 0)$ . ◇

**Example 4.3** (7 totally real lines). Taking the following parameters as exact,

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1.5875 & 1.0292 & 2.9179 & 8.5023 \\ -2.2410 & 3.8004 & 16.0167 & 6.6939 \\ 1.7921 & 0.5681 & 3.0176 & 10.5398 \end{pmatrix},$$

there are 40 isolated solutions in  $\mathbb{C}^4$  to  $F(x; M) = 0$  arising in 10 groups of four as in (7). From these, there are 7 representatives that are totally real. Truncated to four decimal places, one representative from each of the 7 groups is given below.

$$\begin{pmatrix} -0.0294, & 1.0400, & -0.6271, & 0.4619 \\ -0.0263, & 1.1190, & -0.6270, & 5.0823 \\ -1.2378, & 0.8883, & -1.2100, & 0.2845 \\ -1.1210, & 0.9786, & -0.7724, & 0.3145 \\ -1.1058, & 1.1389, & -2.0696, & -0.7637 \\ -1.0642, & 0.0213, & -0.6763, & 0.0904 \\ -1.1036, & -0.0243, & -0.6875, & -0.5651 \end{pmatrix}$$



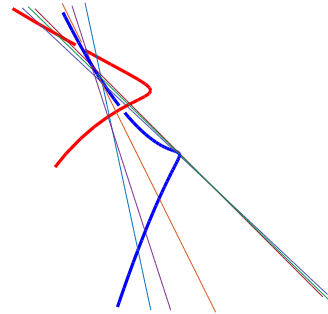
These data correspond to the 3-tuple  $(7, 3, 0)$ . ◇

**Example 4.4** (6 totally real lines). Taking the following parameters as exact,

$$(10) \quad M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1.0321 & 1.8134 & 4.5132 & 7.8117 \\ 0.3835 & 2.8268 & 14.8081 & 5.2579 \\ 1.4025 & 2.0517 & 6.1857 & 10.9519 \end{pmatrix},$$

there are 40 isolated solutions in  $\mathbb{C}^4$  to  $F(x; M) = 0$  arising in 10 groups of four as in (7). From these, there are 6 representatives that are totally real. Truncated to four decimal places, one representative from each of the 6 groups is given below.

$$\begin{pmatrix} 1.1362, & -1.2222, & 0.1631, & -2.3167 \\ 1.2304, & -1.1314, & -2.9760, & -0.4230 \\ 1.1713, & -0.9253, & 0.1881, & -0.3896 \\ 1.1766, & -0.2196, & -0.3302, & 0.2450 \\ 1.2165, & -0.0779, & 2.7098, & -0.3231 \\ 1.1161, & 0.1145, & 0.2720, & 2.1508 \end{pmatrix}$$



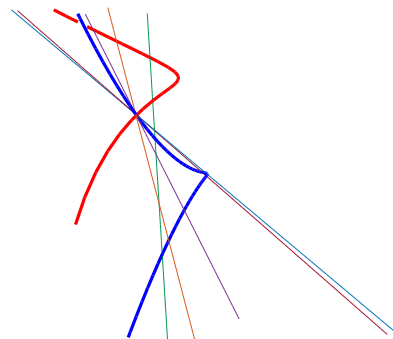
These data correspond to the 3-tuple  $(6, 0, 0)$ . ◇

**Example 4.5** (5 totally real lines). Taking the following parameters as exact,

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2.0150 & 1.9005 & 2.2737 & 9.2034 \\ -0.0362 & 2.0729 & 14.6727 & 7.2270 \\ 1.8364 & 1.7414 & 4.0076 & 10.9568 \end{pmatrix},$$

there are 40 isolated solutions in  $\mathbb{C}^4$  to  $F(x; M) = 0$  arising in 10 groups of four as in (7). From these, there are 5 representatives that are totally real. Truncated to four decimal places, one representative from each of the 5 groups is given below.

$$\begin{pmatrix} -0.0018, & 1.0256, & -0.4033, & 0.4235 \\ -0.8601, & 1.0250, & -0.4045, & 0.2321 \\ -1.0606, & 1.0271, & -1.7495, & -0.4058 \\ -1.1321, & 0.8514, & -1.3372, & 0.1018 \\ 0.1450, & 1.0265, & -0.4033, & 3.3072 \end{pmatrix}$$



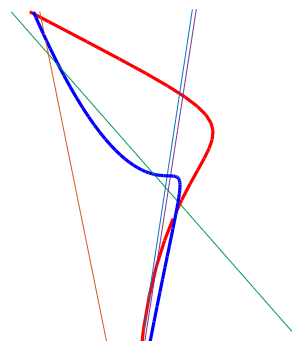
These data correspond to the 3-tuple  $(5, 1, 0)$ . ◇

**Example 4.6** (4 totally real lines). Taking the following parameters as exact,

$$(11) \quad M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.2256 & 1.2068 & 5.2039 & 7.9386 \\ -1.1289 & 3.3178 & 14.3586 & 5.7655 \\ 1.1670 & 3.1542 & 2.6854 & 9.8156 \end{pmatrix},$$

there are 40 isolated solutions in  $\mathbb{C}^4$  to  $F(x; M) = 0$  arising in 10 groups of four as in (7). From these, there are 4 representatives that are totally real. Truncated to four decimal places, one representative from each of the 4 groups is given below.

$$\begin{pmatrix} 0.0458, & -2.1130, & -0.6338, & 0.0915 \\ -0.0351, & -1.8916, & 0.0810, & -0.7441 \\ 1.3628, & -1.7109, & 0.3347, & -0.8727 \\ 1.2607, & 0.2935, & -0.4365, & 0.3849 \end{pmatrix}$$



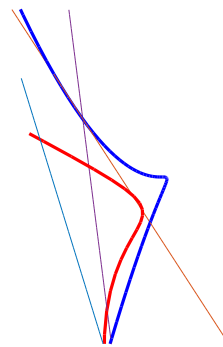
These data correspond to the 3-tuple  $(4, 1, 1)$ . ◇

**Example 4.7** (3 totally real lines). Taking the following parameters as exact,

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -0.7792 & 0.4435 & 5.0652 & 9.1369 \\ 1.2123 & 3.6701 & 12.5914 & 7.0705 \\ -0.2896 & 2.7162 & 3.8465 & 9.6116 \end{pmatrix},$$

there are 40 isolated solutions in  $\mathbb{C}^4$  to  $F(x; M) = 0$  arising in 10 groups of four as in (7). From these, there are 3 representatives that are totally real. Truncated to four decimal places, one representative from each of the 3 groups is given below.

$$\begin{pmatrix} -1.4749, & 2.0198, & 0.2309, & -0.6487 \\ -1.4330, & 1.0056, & -0.5707, & -0.8760 \\ 1.2187, & -0.9380, & -2.1231, & -0.1383 \end{pmatrix}$$



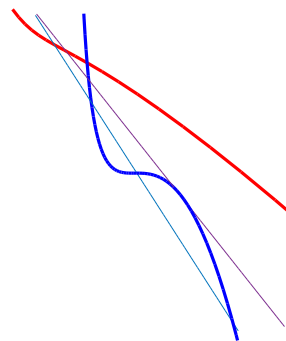
These data correspond to the 3-tuple  $(3, 0, 1)$ . ◇

**Example 4.8** (2 totally real lines). Taking the following parameters as exact,

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1.7547 & 3.4027 & 1.7429 & 7.6041 \\ -1.0823 & 4.1538 & 12.0008 & 6.6773 \\ 2.6459 & 2.3080 & 3.5738 & 8.8431 \end{pmatrix},$$

there are 40 isolated solutions in  $\mathbb{C}^4$  to  $F(x; M) = 0$  arising in 10 groups of four as in (7). From these, there are 2 representatives that are totally real. Truncated to four decimal places, one representative from each of the 2 groups is given below.

$$\begin{pmatrix} -0.0307, & 1.1047, & 0.5412, & 6.8077 \\ -0.6559, & 1.2094, & -4.1860, & 0.4012 \end{pmatrix}$$



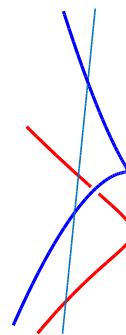
These data correspond to the 3-tuple  $(2, 0, 0)$ . ◇

**Example 4.9** (1 totally real line). Taking the following parameters as exact,

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.0983 & 4.0479 & 2.2629 & 6.9867 \\ -0.0267 & 4.2539 & 12.6067 & 5.2851 \\ -1.3234 & 2.3696 & 3.5350 & 10.0737 \end{pmatrix}$$

there are 40 isolated solutions in  $\mathbb{C}^4$  to  $F(x; M) = 0$  arising in 10 groups of four as in (7). From these, there is 1 representative that is totally real. Truncated to four decimal places, the representative is given below.

( -1.0062, 1.2979, -0.5469, -3.3602 )

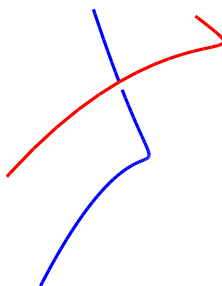


These data correspond to the 3-tuple (1, 0, 3). ◇

**Example 4.10** (0 totally real lines). Taking the following parameters as exact,

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1.6872 & 1.5909 & 2.7170 & 7.8740 \\ -1.0519 & 3.4185 & 14.1275 & 6.8633 \\ 2.7208 & 1.3832 & 4.2118 & 6.8602 \end{pmatrix}$$

there are 40 isolated solutions in  $\mathbb{C}^4$  to  $F(x; M) = 0$  arising in 10 groups of four as in (7). From these, there are no representatives that are totally real.



These data correspond to the 3-tuple (0, 0, 0). ◇

## 5. DISCUSSION AND OPEN PROBLEMS

Over the complex numbers, Theorem 1.2 provides the well-known count of ten common secants to two general twisted cubic curves. We formulated and investigated the real analogue aiming to determine the possible collections of ten common secant lines to a pair of real twisted cubics. In particular, we classified real secants as totally real, partially real, or minimally real, and used combinatorics to determine that there are a total of 161 admissible 3-tuples  $(n_t, n_p, n_m)$ . We then used Python scripts with Bertini to explore the parameter space  $\mathcal{H}(\mathbb{R})$  to realize 128 of the 161 admissible 3-tuples  $(n_t, n_p, n_m)$ . We used alphaCertified to certify the solutions associated with the realizable 3-tuples found, yielding Theorem 1.7. A particular case is formulated in Theorem 1.6, which posits that any number from 0 to 10 of totally real secant lines is possible.

We outline a few open problems naturally arising from our investigation.

**Problem 5.1.** Determine the full set of realizable 3-tuples in Table 4. Our data [1] show 128 out of 161 admissible 3-tuples have been realized.

**Problem 5.2.** Consider the problem of determining the possible collections of common secants to other rational normal curves in  $\mathbb{P}^n$ .

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During the preparation of this work the authors used ChatGPT in order to create `shell` and `Python` scripts to post process the data obtained from `Bertini` and `alphaCertified`. After using this tool, the authors reviewed and edited the content as needed and take full responsibility for the content of the published article.

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## APPENDIX A. TABLES OF 3-TUPLES

TABLE 4. Realized 3-tuples and Admissible and not Realized 3-tuples.

Realized				Admissible and not Realized		
(0, 0, 0)	(0, 0, 2)	(0, 0, 4)	(0, 0, 6)	(0, 0, 10)	(0, 1, 9)	(0, 2, 8)
(0, 0, 8)	(0, 1, 1)	(0, 1, 3)	(0, 1, 5)	(0, 3, 7)	(0, 4, 6)	(0, 5, 5)
(0, 1, 7)	(0, 2, 0)	(0, 2, 2)	(0, 2, 4)	(0, 6, 4)	(0, 7, 3)	(0, 8, 2)
(0, 2, 6)	(0, 3, 1)	(0, 3, 3)	(0, 3, 5)	(0, 9, 1)	(0, 10, 0)	(1, 0, 9)
(0, 4, 0)	(0, 4, 2)	(0, 4, 4)	(0, 5, 1)	(1, 0, 7)	(1, 1, 8)	(1, 1, 6)
(0, 5, 3)	(0, 6, 0)	(0, 6, 2)	(0, 7, 1)	(1, 2, 7)	(1, 3, 6)	(1, 4, 5)
(0, 8, 0)	(1, 0, 1)	(1, 0, 3)	(1, 0, 5)	(1, 5, 4)	(1, 6, 3)	(1, 7, 2)
(1, 1, 0)	(1, 1, 2)	(1, 1, 4)	(1, 2, 1)	(1, 8, 1)	(1, 9, 0)	(2, 6, 2)
(1, 2, 3)	(1, 2, 5)	(1, 3, 0)	(1, 3, 2)	(2, 7, 1)	(2, 8, 0)	(3, 5, 2)
(1, 3, 4)	(1, 4, 1)	(1, 4, 3)	(1, 5, 0)	(3, 6, 1)	(3, 7, 0)	(4, 0, 6)
(1, 5, 2)	(1, 6, 1)	(1, 7, 0)	(2, 0, 0)	(4, 4, 2)	(4, 5, 1)	(4, 6, 0)
(2, 0, 2)	(2, 0, 4)	(2, 0, 6)	(2, 0, 8)			
(2, 1, 1)	(2, 1, 3)	(2, 1, 5)	(2, 1, 7)			
(2, 2, 0)	(2, 2, 2)	(2, 2, 4)	(2, 2, 6)			
(2, 3, 1)	(2, 3, 3)	(2, 3, 5)	(2, 4, 0)			
(2, 4, 2)	(2, 4, 4)	(2, 5, 1)	(2, 5, 3)			
(2, 6, 0)	(3, 0, 1)	(3, 0, 3)	(3, 0, 5)			
(3, 0, 7)	(3, 1, 0)	(3, 1, 2)	(3, 1, 4)			
(3, 1, 6)	(3, 2, 1)	(3, 2, 3)	(3, 2, 5)			
(3, 3, 0)	(3, 3, 2)	(3, 3, 4)	(3, 4, 1)			
(3, 4, 3)	(3, 5, 0)	(4, 0, 0)	(4, 0, 2)			
(4, 0, 4)	(4, 1, 1)	(4, 1, 3)	(4, 1, 5)			
(4, 2, 0)	(4, 2, 2)	(4, 2, 4)	(4, 3, 1)			
(4, 3, 3)	(4, 4, 0)	(5, 0, 1)	(5, 0, 3)			
(5, 0, 5)	(5, 1, 0)	(5, 1, 2)	(5, 1, 4)			
(5, 2, 1)	(5, 2, 3)	(5, 3, 0)	(5, 3, 2)			
(5, 4, 1)	(5, 5, 0)	(6, 0, 0)	(6, 0, 2)			
(6, 0, 4)	(6, 1, 1)	(6, 1, 3)	(6, 2, 0)			
(6, 2, 2)	(6, 3, 1)	(6, 4, 0)	(7, 0, 1)			
(7, 0, 3)	(7, 1, 0)	(7, 1, 2)	(7, 2, 1)			
(7, 3, 0)	(8, 0, 0)	(8, 0, 2)	(8, 1, 1)			
(8, 2, 0)	(9, 0, 1)	(9, 1, 0)	(10, 0, 0)			

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