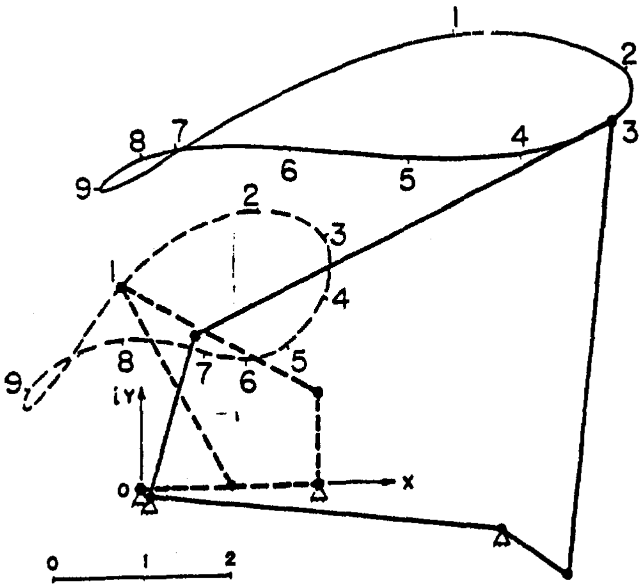

Fig. 2 Example 1, minus-one geared five-bar. The synthesized linkage and its generated path (solid lines); the starting linkage and its locus (broken lines).

of an ellipse. The resultant coupler curve deviates so much over the remaining three quarters, that one hardly can talk about an approximation to an ellipse. A better method is to prescribe as many intersections as possible with the desired curve in order to get a better approximation [4] and then select the solution out from the infinite many possible solutions. The result I have obtained is an approximation to the desired arbitrary ellipse so that it is almost impossible to separate the two curves by the eye. Even better approximations than those I have obtained by the first try are possible. (The method is not limited to specific curves.)

From my own experience with coupler curves I have found what I call the invariance of coupler curves, i.e. by changing the dimensions of the links but maintaining certain initial synthesis requirements the resultant coupler curves change very little [5]. This leads me to think that this phenomenon is causing the many multiple solutions found in Wampler et al.

Certainly there is not always a solution when the nine points are chosen arbitrarily and this was the problem originally stated by H. Alt [1]. When the points are separated along the entire curve and not too close to each other there seems to be only one solution if there is a solution (plus, of course, the Robert's cognates and the geared double cranks with a transmission ratio of +1:1).

The number of precision points can be increased considerably when the problem at hand calls for symmetrical coupler curves. Using geometrical methods a solution is obtained on a PC in no time. I have obtained 12 precision points [3] by a four-bar linkage. The number of precision points I have achieved for the centric slider-crank is 10 and for the eccentric slider-crank 8 [5]. Geometrical methods often allow one to see whether there is a solution to the problem and within what range, and the solutions are found more easily. The writing of equations after equations do not allow this kind of a deeper insight into the problems.

Summary

Obviously the conclusion to what is the general requirement to be imposed on the precision points is, that the points must lie on a curve the shape of which is characteristic for the coupler curves for the desired type of linkage.

References

1 Alt, H., "Ueber die Erzeugung gegebener ebener Kurven mit Hilfe des Gelenkvierseits," ZAMM, Vol. 3, 1923, pp. 13–19.

2 Roth, B., et al., "Synthesis of Path-Generating Mechanisms by Numerical Methods," *Transactions of the ASME,* August 1963, pp. 298–306.

3 Jensen, Preben W., "Synthesis of Four-Bar Linkages With a Coupler Point Passing Through 12 Points," *Mechanisms and Machine Theory,* Vol. 19, No. 1, 1984, pp. 149–156.

4 Jensen, Preben W., "The Polode Synthesis Method," *Engineering Research 6,* VDI-Verlag, Duesseldorf, Germany, 1992, pp. 152–163.

5 Jensen, Preben W., *Analysis, Synthesis, and Design of Mechanisms and Mechanical Devices,* to be published.

Authors' Closure

C. Wampler,[3] A. Morgan,[4] and A. Sommese[5] We appreciate the interest that Prof. Jensen has taken in our work and the opportunity to clarify the contribution made by our paper (Wampler et al., 1992). At our request, Prof. Jensen sent copies of several of his articles, so we have also gained the chance to become more familiar with his ingenious methods.

While there is an element of truth underlying some of Prof. Jensen's comments, there are also some misleading statements and some misdirected criticisms. We will attempt to sort these out and to address in brief how our methods can be applied to several of the problems mentioned in his commentary.

Prof. Jensen contends that, contrary to the claim of our title, our solution is not complete. Lest there be any confusion on the point, we do not claim to offer a comprehensive method for designing four-bar linkages for path generation. Rather, we offer a complete solution to the long-standing nine-point problem; that is, given nine points in the plane, find all four-bar linkages whose coupler curves pass through the points. We use the adjective "complete" to distinguish our method from previous approaches, which only find a partial list of solutions, such as the methods published by Roth and Freudenstein (1963) and Tsai and Lu (1990). The situation is analogous to the well-known Burmester problem of finding all centerpoint-circlepoint pairs that guide a body through five given positions in the plane. A complete solution to the Burmester problem generates a list of four centerpoint-circlepoint pairs, 0, 2, or 4 of which may be real. In the nine-point problem, our numerical evidence strongly indicates that taken over the complex numbers, a complete solution list consists of 1442 triples (Roberts cognates), some subset of which will be real. In the same manner that solution methods for the Burmester problem are useful in designing four-bars for body guidance, our solution method for the nine-point problem can aid in the design of four-bars for path generation.

Prof. Jensen's chief criticism of our methodology lies in a supposed inadequacy in our example Problem 1, drawn from Roth and Freudenstein (1963). He complains that the points have not been selected "arbitrarily." We cannot answer Prof. Jensen's questions concerning the etiology of this example, but we must stress that it has no bearing whatsoever on the validity of our numerical experiment. As we stated on page 156, our initial solution was computed using random, complex precision points. These were generated using a random number generator, which is as arbitrary as one could imagine! Roth and Freudenstein's problem is merely an example application which, being the only previously published solved example, we felt obliged to treat.

Prof. Jensen's contention that a sequence of points in a zigzag pattern will lead to no solutions at all is valid, as no four-bar coupler curve can cross a line more than six times. By finding a complete solution list in which no suitable linkages appear, our method will confirm this impossibility and the designer will

[3,4] General Motors R&D Center, Warren, MI
[5] University of Notre Dame, Notre Dame, IN

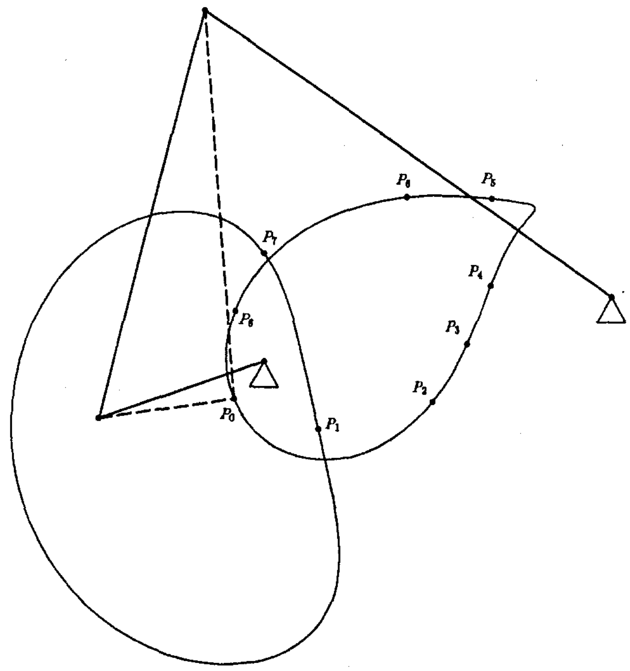

Fig. 1 A solution to Problem 2 of Wampler et al. (1992). No four-bar coupler curve traverses this set of precision points in order on a single circuit.

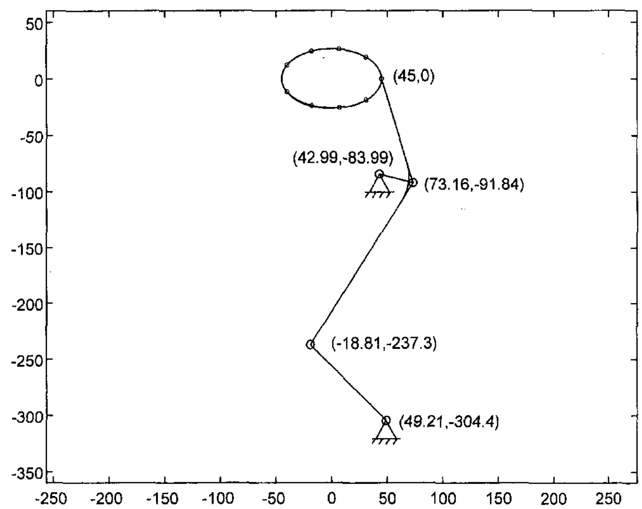

Fig. 2 A solution to a nine-point problem for generating an ellipse

know that his precision point specification is unattainable. This can happen for points in a more reasonable pattern, and in fact, we showed just such an example in our Problem 2. Note that the solution list for that problem is not empty, but every linkage in the list either traversed the points out of order or else some points fell on each circuit of a bicursal (Grashof) coupler curve. As shown in Fig. 4 of our paper (reproduced here as Fig. 1), there is a solution which passes through 7 of the 9 precision points on one circuit, which suggests how the remaining points might be adjusted to obtain a feasible linkage.

Prof. Jensen next turns his attention to the generation of elliptical coupler curves. His statement that a precision-point specification will not yield a curve having the least possible deviation is correct, although the same can be said of his own polode synthesis method. A close approximation generated by either approach could be refined iteratively using a direct measure of deviation as an optimization criterion. No matter what approach is used, a designer must expect a trade-off between minimizing deviations and optimizing transmission angles. These comments apply to any path approximation problem, not just ellipses.

The criticism that our method has only approximated one quarter of an ellipse is unfair: in placing all nine precision points in one quadrant in our example Problem 4, we presumed a scenario where that portion of the ellipse should be approximated quite accurately while the return path was relatively free. To approximate an entire ellipse, one should of course distribute the precision points all the way around. Figures 2 and 3 show the best approximations that result from such a specification for an ellipse with major axis 90 and minor axis 52, as was considered in Jensen (1992). The mechanism shown in Fig. 2 is close to Jensen's solution, whereas the symmetric four-bar shown in Fig. 3 is quite different. For the latter, two precision points fall on the second circuit (light line), but the main circuit is still a good approximation to the ellipse. Unfortunately, the transmission angle of this linkage is unacceptable. In addition to Cardan (elliptical trammel) mechanisms, our method found 100 other real mechanisms, which vary greatly in accuracy of approximation and transmission angle. The computation took 12 minutes of CPU time on an IBM RS6000 workstation.

Finally, Prof. Jensen brings up the interesting issue of symmetric coupler curves. This too can be explored using our method. First, one must determine the number of conditions imposed by the symmetry requirement, which reduces the number of precision points that can be independently specified. If the precision points and their mirror images are specified, the line of symmetry is determined. As can be seen in Fig. 4, this places four conditions on the mechanism, so only five independent precision points can be given. While we would prefer to call this a five-point problem with symmetry, Jensen (1984) prefers to count both the independent precision points and their mirror images, in which case one may logically pose a "10-point" problem. Jensen's claim of 12 precision points is misleading, because he lets the line of symmetry float, so that the mirror images of the 6 independent points are not really determined until after the design is finished. (His "10-point" solution in the same paper suffers the same shortcoming.)

Consider a nine-point problem composed of five independent points along with four mirror-image points. (It does not matter which four of the five are mirrored.) Among the 1442 solutions to the nine-point problem will be some symmetric linkages, which will also generate the fifth mirror point. In fact, any nonsingular solution to the symmetrical five-point problem will also be a solution to the symmetrically-posed nine-point problem. We have carried out this computation, using randomly selected precision points and their mirror images. After 6.5 minutes of CPU time, the list of 1442 solution triples was

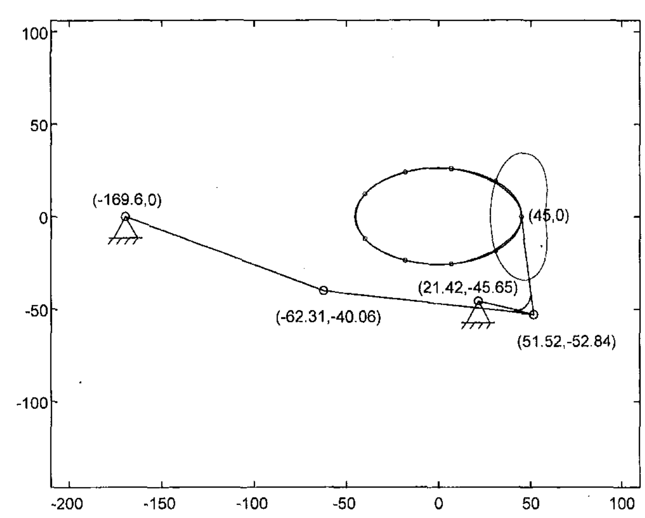

Fig. 3 Another solution for approximating an ellipse

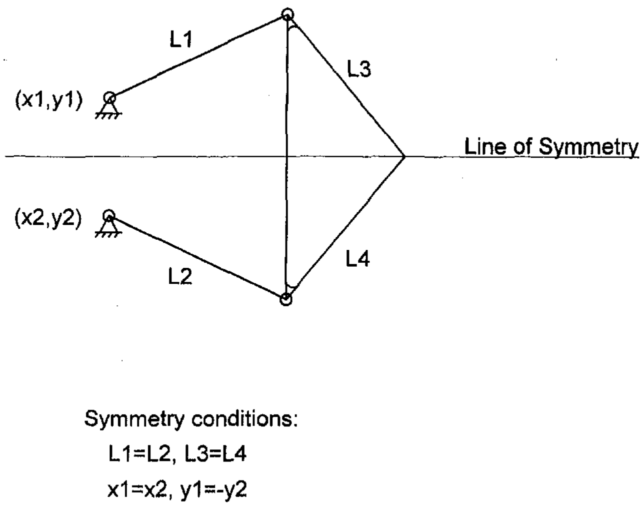

Fig. 4 Conditions for a coupler curve to have a prescribed line of symmetry. Two cognates of the illustrated linkage are also possible.

examined to find that 78 were symmetric. Subsequent symmetrical five-point problems can be solved using just these symmetric solutions as startpoints in a 78-path parameter continuation. To do so, the points must be consistently numbered such that the lone un-mirrored point is last, so that a linear interpolation between the start and target point sets always displays the requisite symmetry. Such runs typically take less than two minutes of CPU time. This method can be applied to any symmetrical five-point problem, including the special example considered in Jensen (1984), in which two points lie on a common tangent line. As given in the table below, we approximated the double tangency condition by two pairs of closely spaced points and then added a fifth precision point near the line of symmetry to control the size of the return loop of the coupler curve. After 1.8 minutes of CPU time, our method returned 11 real mechanisms, one of which traverses the points in a simple loop and resembles Jensen's result, see Fig. 5.

Prof. Jensen's concluding remark is a tautology: to obtain a reasonable four-bar from a precision-point synthesis method, the precision points must be prescribed so as to lie on the coupler curve of a reasonable four-bar. However, the comment reflects a genuine difficulty. By prescribing the maximum number of precision points, one has the potential for maximal control of the coupler curve, but all control of the transmission properties of the linkage or of the region occupied by the links is relinquished. A natural answer to this difficulty is to prescribe fewer points. For example, one could assign only eight precision points and then plot the corresponding centerpoint and circlepoint curves, perhaps color-coded according to transmission angle or other criterion. This is analogous to the centerpoint-circlepoint curves commonly drawn for the variant of the Burmester problem in which only four positions of the coupler link are given. Polynomial continuation could readily address this eight-point problem, but it would be improper to investigate it in this forum.

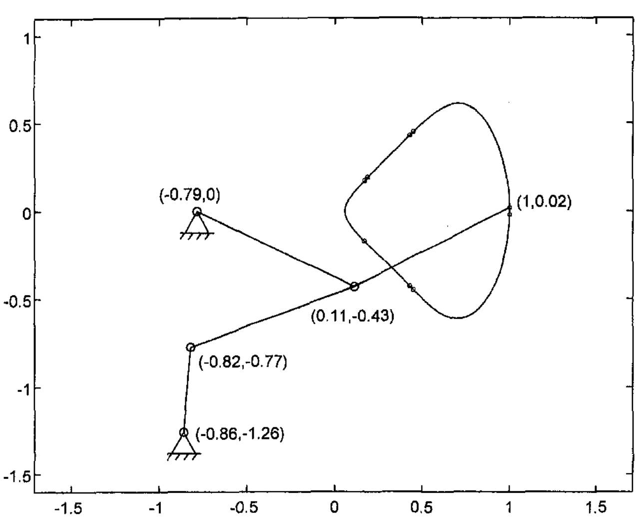

Fig. 5 Solution to a symmetrical five-point problem found using our nine-point algorithm

Table 1 Precision points for symmetric five-point example problem

	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
x	1.00	0.17	0.43	0.45	1.00	0.17	0.43	0.45	0.19
y	0.02	0.17	0.43	0.45	-0.02	-0.17	-0.43	-0.45	0.19

Table 2 Location of pivot points (see Fig. 5)

	x	y
A_0	-0.7850	0.0000
A	0.1124	-0.4316
B_0	-0.8620	-1.2586
B	-0.8236	-0.7717

We believe that the preceding examples establish the applicability and generality of our nine-point solution method. Moreover, polynomial continuation has wide application in all kinds of mechanism design, both planar and spatial, as well as problems concerning the kinematics of robots. We readily admit that geometrical methods can often yield significant insight, and we applaud Prof. Jensen's contributions in this area. However, the insight that can be derived from algebraic and numerical methods should not be discounted. For example, Roberts (1875) discovered the triple generation of four-bar curves using purely algebraic reasoning, while numerical experiments using polynomial continuation first demonstrated the existence of 16 solutions for the inverse kinematics of general six-revolute serial-link robots (Tsai and Morgan, 1985) and 40 solutions for the forward kinematics of general Gough-Stewart platform manipulators (Raghavan, 1993). Both of these latter results have now been confirmed analytically. A discussion placing polynomial continuation in context with other methods for solving kinematic equations can be found in Raghavan and Roth (1995). We feel that our results on the nine-point problem, and now on the related symmetrical five-point problem, make a significant contribution to the understanding of coupler curve generation, while also providing an example of the power of polynomial continuation.

References

Jensen, P. W., 1992, "The Polode Synthesis Method," *Eng. Research Bd. 58, No. 6, VDI-Verlag,* Duesseldorf, Germany, pp. 152–163.

Jensen, P. W., 1984, "Synthesis of Four-Bar Linkages with a Coupler Point Passing Through 12 Points," *Mech. Mach. Theory,* Vol. 19, No. 1, pp. 149–156.

Raghavan, M., 1993, "The Stewart Platform of General Geometry Has 40 Configurations," ASME JOURNAL OF MECHANICAL DESIGN, Vol. 115, pp. 277–282.

Roberts, S., 1875, "On Three-Bar Motion in Plane Space," *Proc. London Math. Society,* Vol. VII, pp. 14–23.

Roth, B., and Freudenstein, F., 1963, "Synthesis of Path-Generating Mechanisms by Numerical Methods," ASME *Journal of Engineering for Industry,* Vol. 85, Series B, No. 3, pp. 298–306.

Tsai, L.-W., and Lu, J.-J., 1990, "Coupler-Point-Curve Synthesis Using Homotopy Methods," ASME JOURNAL OF MECHANICAL DESIGN, Vol. 112, pp. 384–389.

Tsai, L.-W., and Morgan, A. P., 1985, "Solving the Kinematics of the Most General Six- and Five-Degree-of-Freedom Manipulators by Continuation Methods," ASME JOURNAL OF MECHANISMS, TRANSMISSIONS, AND AUTOMATION IN DESIGN, Vol. 107, pp. 189–200.

Wampler, C. W., Morgan, A. P., and Sommese, A. J., 1992, "Complete Solution of the Nine-Point Path Synthesis Problem for Four-Bar Linkages," ASME JOURNAL OF MECHANICAL DESIGN, Vol. 114, pp. 153–159.