Affine Toric Equivalence Relations are Effective

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Motivating Question

Under what circumstances do quotients by finite equivalence relations exist?

Outline of talk:

Definition of Equivalence Relations

Given a scheme X over a base S, a **scheme theoretic equivalence relation** on X over S is an S-scheme R together with a morphism

$$f : R \to X \times_S X$$

over S such that for any S-scheme T, the set map

$$f(T) : R(T) \to X(T) \times X(T)$$

is injective and its image is the graph of an equivalence relation on $X(T)$ (here $Z(T)$ denotes the set of S-maps from T to Z).

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R is said to be **finite** if the two projections

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The Affine Case

If k is a field and $X = \mathbb{A}_k^n$ is the n-dimensional affine space over k, then $\mathcal{O}_X \simeq k[\boldsymbol{x}]$, where $\boldsymbol{x} = (x_1, \cdots, x_n)$. An equivalence relation $R \subset X \times_k X$ corresponds to an ideal $I(\boldsymbol{x}, \boldsymbol{y}) \subset k[\boldsymbol{x}, \boldsymbol{y}]$

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R is finite if and only if I satisfies

4. (*finiteness*)
$$k[\boldsymbol{x}, \boldsymbol{y}]/I(\boldsymbol{x}, \boldsymbol{y}) \text{ is finite over } k[\boldsymbol{x}]$$

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Answer: No. Example: to come. Also, Hironaka's.
"Theorem" If X, Y and $f : X \to Y$ are "nice", and if it happens that the effective equivalence relation $R = X \times_Y X$ defined by f is finite, then the quotient X/R exists.

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Remarks:

- The theorem holds without any finiteness assumptions.
- If R is finite, the quotient exists and is also an affine toric variety.
- The theorem is false in the nonaffine case: an equivalence relation on $X = \mathbb{P}^2$ identifying the points of a (torus-invariant) line L can't be effective; if it were, then the map $X \to Y$ defining it would have to contract L and therefore be constant.

Definition of the Amitsur Complex

Given a commutative ring A and an A-algebra B, we consider the **Amitsur complex**

$$C(A,B): \; B \to B \otimes_A B \to \cdots \to B^{\otimes_A m} \to \cdots$$

with differentials given by the formula

$$d(b_1 \otimes b_2 \otimes \cdots \otimes b_m) = \sum_{i=1}^{m+1} (-1)^i b_1 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_m.$$

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It is well known that if B is a faithfully flat or augmented A-algebra, then $C(A,B)$ is exact. In these cases, the kernel of the first differential is A.

Exactness of the Amitsur Complex

It turns out that exactness holds also when A, B are monoid rings and the map $A \to B$ is defined on the monoid level:

Theorem (–, 2009)

Let k be any commutative ring, let τ and σ be commutative monoids, and let $\varphi : \tau \to \sigma$ be a map of monoids. If $A = k[\tau]$, $B = k[\sigma]$, and B is considered as an A-algebra via the map $A \to B$ induced by φ, then the Amitsur complex $C(A, B)$ is exact.

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As opposed to the faithfully flat and augmented cases, the kernel of the first differential

$$d : B \to B \otimes_A B, \quad b \mapsto b \otimes 1 - 1 \otimes b$$

is usually larger than A.

A 1–Dimensional Zig–zag

If we consider

$$A = k[t^3, t^5] \subset B = k[t]$$

then $t^7 \in B$ is not an element of A, but it goes to zero under the first differential in the Amitsur complex.

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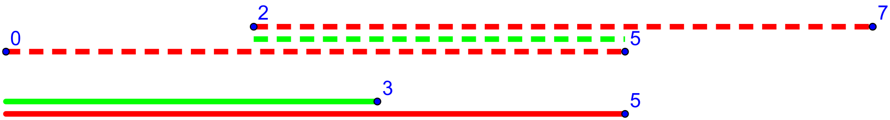
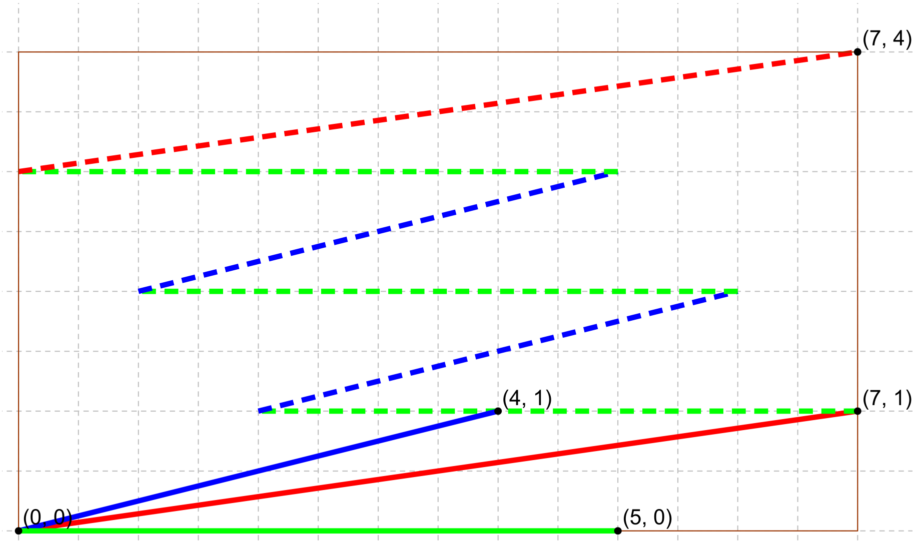


A 2–Dimensional Zig–zag



A Noneffective Affine Equivalence Relation

If k is any ring, $A = k[f_1, \cdots, f_m] \subset B = k[\boldsymbol{x}]$, and $f(\boldsymbol{x}, \boldsymbol{y})$ is a 1–cocycle in the Amitsur complex $C(A, B)$, i.e.

$$f(\boldsymbol{y}, \boldsymbol{z}) - f(\boldsymbol{x}, \boldsymbol{z}) + f(\boldsymbol{x}, \boldsymbol{y}) = 0 \in k[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]/(f_i(\boldsymbol{x}) - f_i(\boldsymbol{y}), f_i(\boldsymbol{x}) - f_i(\boldsymbol{z})),$$

then the ideal

$$I(\boldsymbol{x}, \boldsymbol{y}) = (f(\boldsymbol{x}, \boldsymbol{y}), f_i(\boldsymbol{x}) - f_i(\boldsymbol{y}) : i = 1, \cdots, m) \subset k[\boldsymbol{x}, \boldsymbol{y}]$$

defines an equivalence relation on Spec(B). When the f_i's are homogeneous, noneffectivity of this equivalence relation amounts to f not being a coboundary.

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Example

$$f_1(\boldsymbol{x}) = x_1^2,\ f_2(\boldsymbol{x}) = x_1 x_2 - x_2^2,\ f_3(\boldsymbol{x}) = x_2^3,$$
$$f(\boldsymbol{x}, \boldsymbol{y}) = (x_1 y_2 - x_2 y_1) y_2^3.$$

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- Given a finite equivalence relation on an affine variety, is there a method of producing invariant sections?
- Are finite toric equivalence relations effective?
- Is there a geometric way of explaining the noneffective equivalence relations coming from the nonvanishing of the first cohomology of the Amitsur complex?