Math 54, Fall '10
Quiz 7, October 13

1. (3 points) Let W be the subspace of $\mathbb{R}^3$ spanned by $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$. Write the vector $y = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ as the sum of a vector in W and a vector orthogonal to W.

   Solution. We write $y = p + o$, where $p = \mathrm{pr}_W(y)$ is the projection of y on W, and $o = y - p$ is orthogonal to W. Since $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal ($\mathbf{u}_1 \cdot \mathbf{u}_2 = 5 + 3 - 8 = 0$), we can calculate p using the formula

$$p = \left(\frac{y \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right) \cdot \mathbf{u}_1 + \left(\frac{y \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\right) \cdot \mathbf{u}_2.$$

   Since $y \cdot \mathbf{u}_1 = 0$, $y \cdot \mathbf{u}_2 = 28$ and $\mathbf{u}_2 \cdot \mathbf{u}_2 = 42$, we get

$$p = 0 \cdot \mathbf{u}_1 + \frac{28}{42} \cdot \mathbf{u}_2 = \frac{2}{3} \cdot \mathbf{u}_2 = \begin{bmatrix} 10/3 \\ 2/3 \\ 8/3 \end{bmatrix} \in W.$$

   It follows that

$$o = y - p = \begin{bmatrix} -7/3 \\ 7/3 \\ 7/3 \end{bmatrix},$$

   which is indeed orthogonal to W. □

2. (3 points) Find an orthogonal basis for the column space of the matrix

$$\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}.$$

   Solution. We use the Gram-Schmidt method. We start with the vectors $v_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix}$, which span the column space of our matrix.

We have $v_1 \cdot v_2 = -40 \neq 0$, so v_1, v_2 are not orthogonal. We then replace v_2 by $v_2 - \mathrm{pr}_{\mathrm{Span}\{v_1\}}(v_2)$, i.e. set

$$v_2' = v_2 - \left(\frac{v_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1 = v_2 + 2v_1 = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}.$$

We have $v_3 \cdot v_1 = 30$ and $v_3 \cdot v_2' = -10$, so v_3 is not orthogonal to $\mathrm{Span}\{v_1, v_2'\}$. We replace v_3 by $v_3 - \mathrm{pr}_{\mathrm{Span}\{v_1, v_2'\}}(v_3)$, i.e. set

$$v_3' = v_3 - \left(\frac{v_3 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1 - \left(\frac{v_3 \cdot v_2'}{v_2' \cdot v_2'}\right) \cdot v_2' = v_3 - \frac{3}{2}v_1 + \frac{1}{2}v_2' = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}.$$

It follows that

$$\{v_1, v_2', v_3'\} = \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

is an orthogonal basis for the column space of our matrix. □