Worksheet 4 - Solutions

Claudiu Raicu

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1. a) Suppose that $\mathbf{u}$ is in the first quadrant. Then $\mathbf{u} = (a, b)$ with $a, b \geq 0$. It follows that $c\mathbf{u} = (ca, cb)$. The numbers ca, cb have the same sign (which is also the sign of c), so $c\mathbf{u}$ is either in the first quadrant (if $c \geq 0$), or in the third quadrant (if $c \leq 0$). An analogous reasoning shows that $c\mathbf{u}$ is in the first or third quadrant when $\mathbf{u}$ is in the third quadrant. Note that geometrically this observation is obvious (is it?).

   b) Take $\mathbf{u} = (1, 2)$ and $\mathbf{v} = (-2, -1)$. We have $\mathbf{u} + \mathbf{v} = (-1, 1)$, which is in the second quadrant, i.e. not in W. It follows that W is not a vector space, because it is not closed under addition.

2. As formulated, the question is incomplete: it does not mention the vector space structure on the line L given by $x + y = 1$.

   We can only say that L is not a vector subspace of $\mathbb{R}^2$, and this happens for all the possible reasons:

   - it doesn't contain the 0 vector $(0, 0)$;
   - it is not closed under addition, since $(1, 0)$, $(0, 1) \in L$, but

   $$(1, 0) + (0, 1) = (1, 1) \notin L;$$

   - it is not closed under scalar multiplication, because $(1, 0) \in L$, but

   $$2 \cdot (1, 0) = (2, 0) \notin L.$$

3. a) Yes (check this!).

   b) No, $2(a + t^2) = 2a + 2t^2$ is not equal to $b + t^2$ for any $b \in \mathbb{R}$.

   c) No, because it is not closed under multiplication: the polynomial t^2 has integer coefficients, but $\frac{1}{2} \cdot t^2$ doesn't.

   d) Yes (check this!).

   e) Yes. It is closed under addition because if f, g satisfy $4f(x) = xf'(x)$ and $4g(x) = xg'(x)$, we can add these relations together to obtain

   $$4(f + g)(x) = x(f + g)'(x),$$

   so $f + g$ satisfies the desired differential equation. To the same to show that the set of polynomials satisfying $4f(x) = xf'(x)$ is closed under multiplication by scalars.

   Note. This is just the space of polynomials of the form cx^4 with $c \in \mathbb{R}$ (remember from your calculus class?).

4. H is a subspace: it is closed under addition because if $A, B \in H$, then $FA = 0$ and $FB = 0$, so $F(A+B) = FA + FB = 0$, i.e. $A + B \in H$. Similarly, H is closed under multiplication by scalars.

5. a) Let's first see that $H + K$ is closed under addition: take $w_1, w_2 \in H + K$, and write them as

$$w_1 = u_1 + v_1, \quad w_2 = u_2 + v_2, \text{ with } u_1, u_2 \in H,\ v_1, v_2 \in K.$$

It follows that

$$w_1 + w_2 = u_1 + v_1 + u_2 + v_2 = (u_1 + u_2) + (v_1 + v_2).$$

We have $u = u_1 + u_2 \in H$ and $v = v_1 + v_2 \in K$, because H and K are subspaces of V, so

$$w_1 + w_2 = u + v \in H + K.$$

To show that $H + K$ is closed under multiplication by scalars, we proceed similarly. Take

$$w = u + v \text{ with } u \in H \text{ and } v \in K,$$

and let c be a scalar. It follows that $cu \in H$ and $cv \in K$, because H and K are closed under scalar multiplication (being subspaces of V), so

$$cw = c(u + v) = cu + cv \in H + K.$$

b) H and K are both subsets of $H+K$: if $h \in H$, then $h = h + 0 \in H + K$, so $H \subset H + K$ (do the same for K!). They are closed under addition and multiplication by scalars, since they're subspaces of V, so they are subspaces of $H + K$.

6. Row reducing A, we obtain

$$A \sim \begin{bmatrix} ① & 0 & -7 & 6 \\ 0 & ① & 4 & -2 \end{bmatrix},$$

so x_3, x_4 are free variables, and the null space of A is given by $x_1 = 7x_3 - 6x_4$, $x_2 = -4x_3 + 2x_4$. In vector notation, we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7x_3 - 6x_4 \\ -4x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

Row reducing B, we obtain

$$B \sim \begin{bmatrix} ① & -6 & 0 & 0 \\ 0 & 0 & ① & 0 \end{bmatrix},$$

so x_2, x_4 are free variables, and the null space of B is given by $x_1 = 6x_2$, $x_3 = 0$. In vector notation, we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6x_2 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

C is already in reduced echelon form:

$$C = \begin{bmatrix} \textcircled{1} & -2 & 0 & 4 & 0 \\ 0 & 0 & \textcircled{1} & -9 & 0 \\ 0 & 0 & 0 & 0 & \textcircled{1} \end{bmatrix},$$

so x_2, x_4 are free variables, and the null space of C is given by $x_1 = 2x_2 - 4x_4$, $x_3 = 9x_4$, $x_5 = 0$. In vector notation, we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 - 4x_4 \\ x_2 \\ 9x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix}.$$

7. $\mathbf{p}$ is a polynomial in the kernel of T if and only if $\mathbf{p}(0) = 0$. If we write $\mathbf{p} = a_0 + a_1 x + a_2 x^2$, the condition $\mathbf{p}(0) = 0$ translates into $a_0 = 0$. It follows that

$$\text{Ker}(T) = \{a_1 x + a_2 x^2 : a_1, a_2 \in \mathbb{R}\}.$$

We can therefore take $\mathbf{p}_1 = x$ and $\mathbf{p}_2 = x^2$ to be polynomials that span the kernel of T.

The range of T can be described as the set of vectors

$$\left\{ \begin{bmatrix} c \\ c \end{bmatrix} : c \in \mathbb{R} \right\} = \left\{ c \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} : c \in \mathbb{R} \right\}.$$

8. a) To check that T is linear it suffices to check that T preserves sums and multiplication by scalars (which in particular implies that $T(0) = 0$).

Sums:

$$T(A+B) = (A+B)+(A+B)^T = A+B+A^T+B^T = (A+A^T)+(B+B^T) = T(A)+T(B).$$

Multiplication by scalars:

$$T(c \cdot A) = (c \cdot A)^T = c \cdot A^T = c \cdot T(A).$$

b) Take $A = \frac{1}{2}B$. Then $A^T = \frac{1}{2}B^T = \frac{1}{2}B$, so

$$T(A) = A + A^T = \frac{1}{2}B + \frac{1}{2}B = B.$$

c) Part b) shows that any B with $B = B^T$ is in the range of T. It remains to show that every element B in the range of T has the property that $B^T = B$. Let $B = T(A)$ be such an element. We have

$$B^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A = B.$$

d) A matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is in the kernel of T if and only if

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = A + A^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix},$$

if and only if $2a = b + c = 2d = 0$, i.e. $a = d = 0$ and $c = -b$. So the kernel of T is the set

$$\ker(T) = \left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} : b \in \mathbb{R} \right\}.$$

A basis for $\ker(T)$ consists of the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

9. The solutions to $x + 2y + z = 0$ are precisely the null space of the matrix

$$\begin{bmatrix} \textcircled{1} & 2 & 1 \end{bmatrix}.$$

This has two nonpivot columns, corresponding to the free variables y and z. The set of solutions is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y - z \\ y \\ z \end{bmatrix} = y \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

so a basis for the set of vectors in the plane $x + 2y + z = 0$ is

$$\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

10. A basis for the null space of A is

$$\left\{ \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

A basis for the null space of B is

$$\left\{ \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

A basis for the null space of C is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

11. Check that $4\mathbf{v}_1 + 5\mathbf{v}_2 = 3\mathbf{v}_3$! This shows that

$$\mathbf{v}_3 = \frac{4}{3} \cdot \mathbf{v}_1 + \frac{5}{3} \cdot \mathbf{v}_2 \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}.$$

It follows that $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a spanning set for H. $\mathcal{B}$ is also a linearly independent set because $\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiples of each other (check this!), so $\mathcal{B}$ is a basis for H.

Check that $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_3\}$ is also a basis for H, as well as $\mathcal{D} = \{\mathbf{v}_2, \mathbf{v}_3\}$!

12. If $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ wasn't a basis, the Spanning Set Theorem would imply that there is a subset of $\mathcal{B}$ which is a basis, i.e. $\mathbb{R}^4$ would have a basis consisting of fewer than 4 elements. But the Basis Theorem states that any two bases have the same number of elements, and we already know that the standard basis $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{R}^4$ has 4 elements, so we would get a contradiction. See also the next exercise!

13. The elements of S are the columns of an $n \times k$ matrix A. If S was a basis of $\mathbb{R}^n$, it would also be a spanning set for $\mathbb{R}^n$, i.e. the columns of A would span $\mathbb{R}^n$. This can only happen if A has a pivot in every row, but A can have at most k pivots, since it only has k columns. As $k < n$ and A has n rows, this is impossible.

14. Since $\mathbf{p_1}$ and $\mathbf{p_2}$ are not multiples of each other (check this!), $\{\mathbf{p_1}, \mathbf{p_2}\}$ is a linearly independent set. It is not a basis, because the polynomial $t \in \mathbb{P}_2$ is not a linear combination of $\mathbf{p_1}$ and $\mathbf{p_2}$ (check this!).

15. We have

$$\mathbf{p_1} + \mathbf{p_2} = (1 + t) + (1 - t) = 2 = \mathbf{p_3}.$$

It follows that $\mathbf{p_3} \in \text{Span}\{\mathbf{p_1}, \mathbf{p_2}\}$. Since $\mathbf{p_1}$ and $\mathbf{p_2}$ are not multiples of each other, they must be independent. It follows as in exercise 11 that $\{\mathbf{p_1}, \mathbf{p_2}\}$ is a basis for $\text{Span}\{\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}\}$.

You can check that $\{\mathbf{p_1}, \mathbf{p_3}\}$ and $\{\mathbf{p_2}, \mathbf{p_3}\}$ are also bases for $\text{Span}\{\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}\}$.