

Tutorial Worksheet*Topic: Autonomous, First Order Linear, Separable, and Exact Differential Equations***P1.** Find all stable equilibrium solutions of

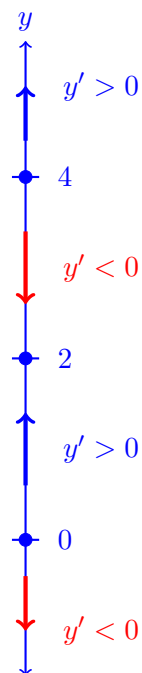
$$\frac{dy}{dt} = 8y - 6y^2 + y^3.$$

Solution:To find the equilibrium solutions, we set $\frac{dy}{dt} = 0$:

$$y^3 - 6y^2 + 8y = 0 \implies y(y^2 - 6y + 8) = 0 \implies y(y - 2)(y - 4) = 0$$

The equilibrium solutions are $y = 0$, $y = 2$, and $y = 4$.To determine stability, we can analyze the sign of $\frac{dy}{dt}$ in the intervals between these equilibria:

$$\begin{array}{ll} y > 4 & \implies y' > 0 \\ 2 < y < 4 & \implies y' < 0 \\ 0 < y < 2 & \implies y' > 0 \\ y < 0 & \implies y' < 0 \end{array}$$

Since solutions approach $y = 2$ from both below and above, the only stable equilibrium solution is $y = 2$.

P2. Consider

$$y' = (y^3 - y)(y^2 - 16), \quad y(0) = -3.$$

Find $\lim_{t \rightarrow \infty} y(t)$.

Solution:

First, find the equilibrium solutions by setting $y' = 0$:

$$(y^3 - y)(y^2 - 16) = y(y - 1)(y + 1)(y - 4)(y + 4) = 0$$

The roots are $y = -4, -1, 0, 1, 4$.

The initial condition is $y(0) = -3$. This value lies in the interval $(-4, -1)$. We test a point in this interval, such as $y = -2$, to find the sign of y' :

$$y' = y(y - 1)(y + 1)(y - 4)(y + 4) \quad \longrightarrow \quad (-)(-)(-)(-)(+) = +$$

Since $y' > 0$ for $-4 < y < -1$, the solution $y(t)$ is strictly increasing. It will asymptotically approach the next equilibrium solution from below. Therefore, $\lim_{t \rightarrow \infty} y(t) = -1$.

P3. (a) Solve the initial value problem

$$\frac{dy}{dx} = \frac{2 \cos^2 y}{x^2}, \quad y(1) = \frac{\pi}{4}.$$

Solution:

The equation is separable. Dividing by $\cos^2 y$ and multiplying by dx :

$$\frac{1}{\cos^2 y} dy = \frac{2}{x^2} dx \implies \sec^2 y dy = 2x^{-2} dx$$

Integrating both sides:

$$\int \sec^2 y dy = \int 2x^{-2} dx \implies \tan y = -\frac{2}{x} + C$$

Apply the initial condition $y(1) = \frac{\pi}{4}$:

$$\tan\left(\frac{\pi}{4}\right) = -\frac{2}{1} + C \implies 1 = -2 + C \implies C = 3$$

Thus, $\tan y = 3 - \frac{2}{x}$, which gives the explicit solution:

$$y(x) = \arctan\left(3 - \frac{2}{x}\right)$$

(b) Find the interval of definition.

Solution:

The function inside the arctan must be defined, so $x \neq 0$. The interval of definition must be a continuous, connected interval containing the initial $x_0 = 1$. Therefore, the interval is $(0, \infty)$.

(c) Compute $y(2)$.

Solution:

Substitute $x = 2$ into the solution:

$$y(2) = \arctan\left(3 - \frac{2}{2}\right) = \arctan(2)$$

P4. Solve the initial value problem

$$y' = x^3y + e^{x^4/4} \sin x, \quad y(0) = 1.$$

Solution:

Rewrite the equation in standard linear form:

$$y' - x^3y = e^{x^4/4} \sin x$$

Find the integrating factor $\mu(x)$:

$$\mu(x) = e^{\int -x^3 dx} = e^{-x^4/4}$$

Multiply the entire differential equation by $\mu(x)$:

$$e^{-x^4/4}y' - x^3e^{-x^4/4}y = \sin x \implies \frac{d}{dx} \left[e^{-x^4/4}y \right] = \sin x$$

Integrate both sides with respect to x :

$$e^{-x^4/4}y = -\cos x + C \implies y(x) = e^{x^4/4}(C - \cos x)$$

Apply the initial condition $y(0) = 1$:

$$1 = e^0(C - \cos 0) \implies 1 = C - 1 \implies C = 2$$

The specific solution is:

$$y(x) = e^{x^4/4}(2 - \cos x)$$

P5. Find the general solution y to

$$(y - \cos x) + (x + \sin y) \frac{dy}{dx} = 0.$$

Solution:

Let $M(x, y) = y - \cos x$ and $N(x, y) = x + \sin y$. We check for exactness:

$$\frac{\partial M}{\partial y} = 1 \quad \text{and} \quad \frac{\partial N}{\partial x} = 1$$

Since $M_y = N_x$, the equation is exact. We seek a potential function $f(x, y)$ such that $f_x = M$ and $f_y = N$.

$$f(x, y) = \int M \, dx = \int (y - \cos x) \, dx = xy - \sin x + g(y)$$

Take the partial derivative with respect to y and set it equal to N :

$$\frac{\partial f}{\partial y} = x + g'(y) = x + \sin y \implies g'(y) = \sin y$$

Integrating gives $g(y) = -\cos y$. The potential function is $f(x, y) = xy - \sin x - \cos y$. The general solution is implicitly given by:

$$xy - \sin x - \cos y = C$$

P6. Consider

$$2xy^2 + (4x^2y + 3)y' = 0.$$

(a) Is it exact? If not, find an integrating factor.

Solution:

Let $M = 2xy^2$ and $N = 4x^2y + 3$.

$$\frac{\partial M}{\partial y} = 4xy \quad \text{and} \quad \frac{\partial N}{\partial x} = 8xy$$

Since $4xy \neq 8xy$, it is **not exact**.

We look for an integrating factor depending only on y :

$$\frac{N_x - M_y}{M} = \frac{8xy - 4xy}{2xy^2} = \frac{4xy}{2xy^2} = \frac{2}{y}$$

Since this is a function of y alone, the integrating factor is:

$$\mu(y) = e^{\int \frac{2}{y} dy} = e^{2 \ln |y|} = y^2$$

(b) Solve with $y(1) = -1$.

Solution:

Multiply the original equation by the integrating factor $\mu(y) = y^2$:

$$2xy^4 + (4x^2y^3 + 3y^2)y' = 0$$

This is now exact ($\tilde{M}_y = 8xy^3 = \tilde{N}_x$). Find the potential function $f(x, y)$:

$$f(x, y) = \int 2xy^4 dx = x^2y^4 + h(y)$$

$$\frac{\partial f}{\partial y} = 4x^2y^3 + h'(y) = 4x^2y^3 + 3y^2 \implies h'(y) = 3y^2 \implies h(y) = y^3$$

The general solution is $x^2y^4 + y^3 = C$.

Apply the initial condition $y(1) = -1$:

$$(1)^2(-1)^4 + (-1)^3 = C \implies 1 - 1 = C \implies C = 0$$

The particular solution is implicitly $x^2y^4 + y^3 = 0$.

Since $y = 0$ is not a solution that passes through $(1, -1)$, we can divide by y^3 :

$$x^2y + 1 = 0 \implies y = -\frac{1}{x^2}$$