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MATH 22580 – Linear Algebra and Differential Equations Tutorial # 8

Tutorial Worksheet

Topic: QR factorization, ordinary differential equations, separation of variables

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P1. Find the QR factorization of the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

Solution:

Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$. We apply the Gram-Schmidt process to the columns of A to find the orthogonal vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$:

Step 1: $\mathbf{u}_1 = \mathbf{a}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. The norm is $\|\mathbf{u}_1\| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$. Thus, $\mathbf{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Step 2: $\mathbf{u}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. The norm is $\|\mathbf{u}_2\| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$. Thus, $\mathbf{q}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Step 3: $\mathbf{u}_3 = \mathbf{a}_3 - \frac{\mathbf{a}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 - \frac{\mathbf{a}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{0}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$. The norm is $\|\mathbf{u}_3\| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}$. Thus, $\mathbf{q}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

The matrix Q is formed by the orthonormal vectors:

$$Q = \begin{bmatrix} 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix}$$

The upper triangular matrix R is given by $R = Q^T A$:

$$R = \begin{bmatrix} \mathbf{q}_1 \cdot \mathbf{a}_1 & \mathbf{q}_1 \cdot \mathbf{a}_2 & \mathbf{q}_1 \cdot \mathbf{a}_3 \\ 0 & \mathbf{q}_2 \cdot \mathbf{a}_2 & \mathbf{q}_2 \cdot \mathbf{a}_3 \\ 0 & 0 & \mathbf{q}_3 \cdot \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{6} \end{bmatrix}$$

P2. For each of the following ordinary differential equations, determine:

(a) the order, and
(b) if the equation is linear or nonlinear.

Recall that a linear differential equation has the following form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

(a) $(1-x)y'' - 4xy' + 5y = \cos(y)$

Solution:
Order: 2 **Linearity:** Nonlinear (due to the $\cos(y)$ term)

(b) $x^5 y^{(4)} - y^3 y'' + 6y = 0$

Solution:
Order: 4 **Linearity:** Nonlinear (due to the $y^3 y''$ term)

(c) $\dfrac{d^2y}{dx^2} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$

Solution:
Order: 2 **Linearity:** Nonlinear (because dy/dx is squared and inside a radical)

(d) $(\sin x)y''' - (\cos x)y' = 2y$

Solution:
Order: 3 **Linearity:** Linear

P3. For each of the following, verify that the indicated function $y = \phi(x)$ is an explicit solution of the given first-order differential equation.

(a) $(y - x)y' = y - x + 2; \quad y = x + 2\sqrt{x+1}$

(b) $y' = 4xy^2; \quad y = 1/(1 - 2x^2)$

Solution:

(a) First, find the derivative y':

$$y' = 1 + \frac{1}{\sqrt{x+1}}$$

Substitute y and y' into the left-hand side (LHS) of the ODE:

$$\begin{aligned} \text{LHS} &= \left(x + 2\sqrt{x+1} - x\right)\left(1 + \frac{1}{\sqrt{x+1}}\right) \\ &= 2\sqrt{x+1}\left(1 + \frac{1}{\sqrt{x+1}}\right) \\ &= 2\sqrt{x+1} + 2 \end{aligned}$$

Substitute y into the right-hand side (RHS) of the ODE:

$$\begin{aligned} \text{RHS} &= (x + 2\sqrt{x+1}) - x + 2 \\ &= 2\sqrt{x+1} + 2 \end{aligned}$$

Since LHS = RHS, the solution is verified.

(b) First, find the derivative y' using the chain rule:

$$y' = -(1 - 2x^2)^{-2}(-4x) = \frac{4x}{(1 - 2x^2)^2}$$

Substitute y into the right-hand side (RHS) of the ODE:

$$\text{RHS} = 4x\left(\frac{1}{1 - 2x^2}\right)^2 = \frac{4x}{(1 - 2x^2)^2}$$

Since $y' = \text{RHS}$, the solution is verified.

P4. For each of the following, solve the given differential equation by separation of variables.

(a) $\dfrac{dy}{dx} = \cos(2x)$

(b) $x\dfrac{dy}{dx} = y - xy$

Solution:

(a) Separate the variables and integrate both sides:

$$\begin{aligned} dy &= \cos(2x)\,dx \\ \int dy &= \int \cos(2x)\,dx \\ y &= \frac{1}{2}\sin(2x) + C \end{aligned}$$

(b) Factor out y on the right side, then separate variables:

$$\begin{aligned} x\frac{dy}{dx} &= y(1-x) \\ \frac{1}{y}\,dy &= \frac{1-x}{x}\,dx \\ \int \frac{1}{y}\,dy &= \int \left(\frac{1}{x} - 1\right)\,dx \\ \ln|y| &= \ln|x| - x + C_1 \\ |y| &= e^{\ln|x| - x + C_1} = e^{C_1}|x|e^{-x} \end{aligned}$$

Since $\pm e^{C_1}$ is a nonzero constant and noting $y = 0$ is a trivial solution, the general solution is:

$$y = Cxe^{-x},$$

where C is an artibrary constant.

P5. For each of the following, solve the given initial-value problem.

(a) $\dfrac{dy}{dx} = \cos(2x), \quad y(\pi/4) = 1$

(b) $x\dfrac{dy}{dx} = 2y, \quad y(1) = 2.$

Solution:

(a) From 4(a), the general solution is $y = \frac{1}{2}\sin(2x) + C$. Apply the initial condition $y(\pi/4) = 1$:

$$\begin{aligned} 1 &= \frac{1}{2}\sin\left(2\left(\frac{\pi}{4}\right)\right) + C \\ 1 &= \frac{1}{2}\sin\left(\frac{\pi}{2}\right) + C \\ 1 &= \frac{1}{2}(1) + C \implies C = \frac{1}{2} \end{aligned}$$

The particular solution is:

$$y = \frac{1}{2}\sin(2x) + \frac{1}{2}$$

(b) Separate variables and integrate:

$$\begin{aligned} \frac{1}{y}\,dy &= \frac{2}{x}\,dx \\ \int \frac{1}{y}\,dy &= \int \frac{2}{x}\,dx \\ \ln|y| &= 2\ln|x| + C_1 \\ \ln|y| &= \ln(x^2) + C_1 \\ y &= Cx^2 \end{aligned}$$

Apply the initial condition $y(1) = 2$:

$$2 = C(1)^2 \implies C = 2$$

The particular solution is:

$$y = 2x^2$$