Name: SIR ISAAC NEWTON March 26, 2026

MATH 22580 – Linear Algebra and Differential Equations Tutorial # 7

Tutorial Worksheet

Topic: Orthogonality, the Gram-Schmidt Process, and Least Squares Solutions

---

P1. Determine whether the following statements are true or false and justify your answers.

(a) Any orthogonal set of vectors is linearly independent.

(b) Any linearly independent set of vectors is orthogonal.

(c) Every nontrivial subspace of $\mathbb{R}^n$ has an orthonormal basis.

(d) For a vector $\mathbf{x}$ in $\mathbb{R}^n$ and a subspace W of $\mathbb{R}^n$, the orthogonal projection of $\mathbf{x}$ onto W, $\text{proj}_W \mathbf{x}$ is orthogonal to every vector in W.

(e) If $A\mathbf{x} = \mathbf{b}$ is a consistent linear system, then $A^TA\mathbf{x} = A^T\mathbf{b}$ is also consistent.

(f) If $A\mathbf{x} = \mathbf{b}$ is an inconsistent linear system, then $A^TA\mathbf{x} = A^T\mathbf{b}$ is also inconsistent.

Solution:

(a) **False.** An orthogonal set can contain the zero vector (since the zero vector is orthogonal to every vector). Any set containing the zero vector is linearly dependent. (However, any orthogonal set of *non-zero* vectors is linearly independent).

(b) **False.** For example, the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are linearly independent but they are not orthogonal.

(c) **True.** Every nontrivial subspace has a basis and so we can apply the Gram-Schmidt process to that basis and obtain an orthogonal basis. We can then normalize those vectors to produce an orthonormal basis.

(d) **False.** The vector $\text{proj}_W\mathbf{x}$ is *in* the subspace W and not orthogonal with itself unless it is the zero vector.

(e) **True.** If $A\mathbf{x} = \mathbf{b}$ is consistent, there is some $\mathbf{x}$ that satisfies the equation. Multiplying both sides by A^T gives $A^TA\mathbf{x} = A^T\mathbf{b}$, meaning the exact same $\mathbf{x}$ is a solution to the normal equations.

(f) **False.** The normal equations $A^TA\mathbf{x} = A^T\mathbf{b}$ are *always* consistent, regardless of whether $A\mathbf{x} = \mathbf{b}$ is consistent or not. They yield the least squares solution(s).

P2. Let $\mathbf{v} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$, and let W be the subspace of $\mathbb{R}^3$ spanned by $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$.

(a) Find the orthogonal projection of $\mathbf{v}$ onto W.

(b) Find the distance from $\mathbf{v}$ to W.

(c) Find the distance from $\mathbf{x} = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$ to W.

Hint: How does $\mathbf{x}$ relate to $\mathbf{u}_1$ and $\mathbf{u}_2$?

Solution:

(a) **Solution #1:** Since $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal, we use the projection formula:

$$\text{proj}_W \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{v} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2$$

Calculating the necessary dot products:

$$\begin{aligned} \mathbf{v} \cdot \mathbf{u}_1 &= (0)(1) + (-1)(0) + (2)(1) = 2 \\ \mathbf{u}_1 \cdot \mathbf{u}_1 &= 1^2 + 0^2 + 1^2 = 2 \\ \mathbf{v} \cdot \mathbf{u}_2 &= (0)(-2) + (-1)(1) + (2)(2) = 3 \\ \mathbf{u}_2 \cdot \mathbf{u}_2 &= (-2)^2 + 1^2 + 2^2 = 9 \end{aligned}$$

Substituting these into the formula:

$$\text{proj}_W \mathbf{v} = \frac{2}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{3}{9}\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 5/3 \end{bmatrix}$$

Solution #2: Alternatively, we can use the projection matrix $P = A(A^TA)^{-1}A^T$ where A is the matrix whose columns are the basis vectors of W:

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$$

Here is a step-by-step calculation of P:

$$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 9 \end{bmatrix}$$

$$(A^TA)^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/9 \end{bmatrix}$$

$$(A^TA)^{-1}A^T = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/9 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ -2/9 & 1/9 & 2/9 \end{bmatrix}$$

$$P = A(A^TA)^{-1}A^T = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1/2 & 0 & 1/2 \\ -2/9 & 1/9 & 2/9 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 17 & -4 & 1 \\ -4 & 2 & 4 \\ 1 & 4 & 17 \end{bmatrix}$$

We compute the projection by multiplying the projection matrix P by $\mathbf{v}$:

$$\text{proj}_W\mathbf{v} = P\mathbf{v} = \frac{1}{18}\begin{bmatrix} 17 & -4 & 1 \\ -4 & 2 & 4 \\ 1 & 4 & 17 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 6 \\ 6 \\ 30 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 5/3 \end{bmatrix}$$

(b) The distance from $\mathbf{v}$ to W is the magnitude of the orthogonal component (the error vector), $\|\mathbf{v} - \text{proj}_W\mathbf{v}\|$.

$$\mathbf{v} - \text{proj}_W\mathbf{v} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \\ 5/3 \end{bmatrix} = \begin{bmatrix} -1/3 \\ -4/3 \\ 1/3 \end{bmatrix}$$

$$\text{Distance} = \|\mathbf{v} - \text{proj}_W\mathbf{v}\| = \sqrt{(-1/3)^2 + (-4/3)^2 + (1/3)^2} = \sqrt{\frac{1}{9} + \frac{16}{9} + \frac{1}{9}} = \sqrt{\frac{18}{9}} = \sqrt{2}$$

(c) Since $\mathbf{x} = \mathbf{u}_2 - \mathbf{u}_1$, the vector $\mathbf{x}$ is already in W, and so the distance from $\mathbf{x}$ to W is zero.

P3. Let W be the subspace of $\mathbb{R}^2$ spanned by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(a) Find the standard matrix, P, of the orthogonal projection onto W.

(b) Use P to find the orthogonal projection of $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ onto W.

Solution:

(a) For a matrix A **with independent columns**, the projection matrix in general is $P = A(A^T A)^{-1} A^T$. For a 1-dimensional subspace spanned by $\mathbf{v}$, then the matrix A consists of a single column $\mathbf{v}$. Hence, $A^T A = \mathbf{v}^T \mathbf{v}$ is the dot product $\mathbf{v} \cdot \mathbf{v}$, which is just a number. Thus, $P = \dfrac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T \mathbf{v}}$.

$$P = \frac{1}{1^2 + 2^2} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix}.$$

(b)

$$\text{proj}_W \mathbf{v} = P\mathbf{v} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/5 - 2/5 \\ 2/5 - 4/5 \end{bmatrix} = \begin{bmatrix} -1/5 \\ -2/5 \end{bmatrix}$$

P4. Use the Gram-Schmidt process to find an orthogonal basis for the column space of the matrix $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 2 & 2 \\ 1 & 0 & 1 \end{bmatrix}$.

Then, normalize the orthogonal basis.

Solution:

Let the columns be $x_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, and $x_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$. We apply the Gram-Schmidt process to find orthogonal vectors v_1, v_2, v_3:

Step 1: $v_1 = x_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Step 2: $v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$

$$x_2 \cdot v_1 = (1)(0) + (2)(-1) + (0)(1) = -2 \quad \text{and} \quad v_1 \cdot v_1 = 2$$

$$v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{-2}{2} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Step 3: $v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$

$$x_3 \cdot v_1 = (0)(0) + (2)(-1) + (1)(1) = -1 \quad \text{and} \quad x_3 \cdot v_2 = (0)(1) + (2)(1) + (1)(1) = 3$$

$$v_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \frac{-1}{2} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/2 \\ 1/2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1/2 \\ 1/2 \end{bmatrix}$$

Normalization: We divide each orthogonal vector by its magnitude to get the orthonormal basis $\{w_1, w_2, w_3\}$:

$$\|v_1\| = \sqrt{2} \implies w_1 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\|v_2\| = \sqrt{3} \implies w_2 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$\|v_3\| = \sqrt{1 + 1/4 + 1/4} = \sqrt{3/2} = \frac{\sqrt{6}}{2} \implies w_3 = \frac{2}{\sqrt{6}} \begin{bmatrix} -1 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

P5. Find a least squares solution of $A\mathbf{x} = \mathbf{b}$ for $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ -2. \end{bmatrix}$

Hints:

1) Notice that the system is inconsistent.
2) Let $\hat{\mathbf{x}}$ be a least squares solution of $A\mathbf{x} = \mathbf{b}$. Recall that $\mathbf{b} - A\hat{\mathbf{x}} = \text{perp}_{\text{Col}(A)}\,\mathbf{b}$. The least squares solution $\hat{\mathbf{x}}$ satisfies $A^T(\mathbf{b} - A\hat{\mathbf{x}}) = 0$, also expressed as $A^T\mathbf{b} = A^T A\hat{\mathbf{x}}$.

Solution:

The least squares solution $\hat{\mathbf{x}}$ satisfies the normal equations $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$. First, compute $A^T A$:

$$A^T A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} (1+4+9) & (1+0-3) \\ (1+0-3) & (1+0+1) \end{bmatrix} = \begin{bmatrix} 14 & -2 \\ -2 & 2 \end{bmatrix}$$

Next, compute $A^T\mathbf{b}$:

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2+2-6 \\ 2+0+2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

Now, solve the system $\begin{bmatrix} 14 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$:

$$\begin{aligned} 14x_1 - 2x_2 &= -2 \\ -2x_1 + 2x_2 &= 4 \end{aligned}$$

Adding the two equations together yields $12x_1 = 2$, which means $x_1 = 1/6$.
Substitute x_1 back into the second equation:

$$-2(1/6) + 2x_2 = 4 \implies -1/3 + 2x_2 = 4 \implies 2x_2 = 13/3 \implies x_2 = 13/6$$

The least squares solution is $\hat{\mathbf{x}} = \begin{bmatrix} 1/6 \\ 13/6 \end{bmatrix}$.