Name: CARL FRIEDRICH GAUSS March 19, 2026

MATH 22580 – Linear Algebra and Differential Equations Tutorial #6

Tutorial Worksheet

Topic: Eigenvalues and Orthogonality

P1. Consider the matrix

$$A = \begin{bmatrix} 2 & 3 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

Find all eigenvalues of the matrix A. If possible, diagonalize A, otherwise justify why you cannot diagonalize.

Solution:

Eigenvalues: Because A is an upper triangular matrix, its eigenvalues are the entries on its main diagonal. Thus, the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = -3$.

Diagonalization: Since A is a 3×3 matrix with 3 distinct real eigenvalues, it is guaranteed to be diagonalizable. We find the eigenvectors by finding the null space of the matrix $A - \lambda I$ for each eigenvalue λ:

For $\lambda_1 = 2$:

$$A - 2I = \begin{bmatrix} 0 & 3 & -5 \\ 0 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

This yields $x_2 = 0$ and $x_3 = 0$, with x_1 being a free variable. A corresponding eigenvector is $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

For $\lambda_2 = 1$:

$$A - I = \begin{bmatrix} 1 & 3 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

We see that x_2 is a free variable. Then $x_3 = 0$ and $x_1 + 3x_2 = 0 \implies x_1 = -3x_2$. A corresponding eigenvector is $v_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$.

For $\lambda_3 = -3$:

$$A + 3I = \begin{bmatrix} 5 & 3 & -5 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This implies $4x_2 = 0 \implies x_2 = 0$, and $5x_1 - 5x_3 = 0 \implies x_1 = x_3$. A corresponding eigenvector is $v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Result: The matrix can be diagonalized as $A = PDP^{-1}$ where:

$$P = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

P2. Consider the matrix

$$A = \begin{bmatrix} 1 & 8 & 7 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix}.$$

Find all eigenvalues of the matrix A.

Solution:

To get the eigenvalues we calculate the determinant of $(A - \lambda I)$ and set it to 0:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 8 & 7 \\ 0 & 4-\lambda & 0 \\ 2 & 0 & 2-\lambda \end{vmatrix} = 0$$

Expanding along the second row gives:

$$\begin{aligned} (4-\lambda)\begin{vmatrix} 1-\lambda & 7 \\ 2 & 2-\lambda \end{vmatrix} &= (4-\lambda)\left((1-\lambda)(2-\lambda) - 14\right) \\ &= (4-\lambda)(\lambda^2 - 3\lambda + 2 - 14) \\ &= (4-\lambda)(\lambda^2 - 3\lambda - 12) = 0 \end{aligned}$$

One root is $\lambda_1 = 4$. For the remaining roots, we use the quadratic formula:

$$\lambda = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-12)}}{2} = \frac{3 \pm \sqrt{9+48}}{2} = \frac{3 \pm \sqrt{57}}{2}$$

The three eigenvalues are $\lambda_1 = 4$, $\lambda_2 = \dfrac{3+\sqrt{57}}{2}$, and $\lambda_3 = \dfrac{3-\sqrt{57}}{2}$.

P3. Let $\mathbf{u} = \begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}0\\3\\-1\end{bmatrix}$.

(a) Find $\mathbf{u} \cdot \mathbf{v}$.

Solution:

$1 \cdot 0 + 1 \cdot 3 + 0 \cdot (-1) = 3$.

(b) Find a unit vector in the direction of $\mathbf{u}$.

Solution:

Since $|\mathbf{u}| = \sqrt{2}$, a unit vector in the direction of $\mathbf{u}$ is $\frac{1}{\sqrt{2}} \begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}$.

(c) Find a unit vector in the direction of $\mathbf{v}$.

Solution:

$|\mathbf{v}| = \sqrt{9+1} = \sqrt{10}$, so a unit vector in the direction of $\mathbf{v}$ is $\dfrac{\mathbf{v}}{|\mathbf{v}|} = \begin{bmatrix}0\\\frac{3}{\sqrt{10}}\\-\frac{1}{\sqrt{10}}\end{bmatrix}$.

(d) Find $\operatorname{proj}_{\mathbf{u}} \mathbf{v}$ (the orthogonal projection of $\mathbf{v}$ onto $\mathbf{u}$).

Solution:

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} = \frac{3}{2} \mathbf{u} = \begin{bmatrix}3/2\\3/2\\0\end{bmatrix}$$

(e) Find $\operatorname{proj}_{\mathbf{v}} \mathbf{u}$ (the orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$).

Solution:

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{3}{10} \mathbf{v} = \begin{bmatrix}0\\9/10\\-3/10\end{bmatrix}$$

P4. Determine whether the following set of vectors forms an orthogonal set

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Solution:

Just by looking at it we can tell none of the vectors are orthogonal to the other vectors because they all have positive entries (so there is no cancellation in the dot product). But to be explicit:

$$\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = 0 + 6 + 1 = 7$$

which is not zero, so the first two vectors are not orthogonal. Hence, the set is not an orthogonal set.

P5. Determine whether the following set of vectors forms an orthogonal basis for $\mathbb{R}^3$

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

Solution:

The first and third vectors are not orthogonal, so the set cannot be an orthonormal basis:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 1 + 2 + 1 = 4 \neq 0.$$

P6. Find the orthogonal complement $W^\perp$ of the subspace W of $\mathbb{R}^3$ where

$$W = \left\{ \begin{bmatrix} x+2y \\ 2x \\ x-y \end{bmatrix} : x, y \in \mathbb{R} \right\}.$$

Verify that $\dim W + \dim W^\perp = \dim \mathbb{R}^3$.

Hint: Recall that $\text{Null}(A)^\perp = \text{Col}(A^T)$ *and* $\text{Col}(A)^\perp = \text{Null}(A^T)$.

Solution:

Note that W can be written as

$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \right\}$$

Hence, if we define the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 1 & -1 \end{bmatrix}$$

we have that $W = \text{Col}(A)$ and so using the hint, $W^\perp = \text{Col}(A)^\perp = \text{Null}(A^T)$. To find this null space we set up the augmented matrix and row-reduce:

$$\begin{bmatrix} 1 & 2 & 1 & | & 0 \\ 2 & 0 & -1 & | & 0 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 & | & 0 \\ 0 & -4 & -3 & | & 0 \end{bmatrix}$$

This system gives us two equations in which z is a free variable (no pivot in the third column):

$$-4y - 3z = 0 \implies y = -\frac{3}{4}z$$

$$x + 2\left(-\frac{3}{4}z\right) + z = 0 \implies x - \frac{3}{2}z + z = 0 \implies x = \frac{1}{2}z$$

Therefore

$$W^\perp = \text{Col}(A)^\perp = \text{Null}(A^T) = \text{Span}\left\{ \begin{bmatrix} 1/2 \\ -3/4 \\ 1 \end{bmatrix} \right\} = \text{Span}\left\{ \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \right\}$$

(we write the last span by multiplying the vector by 4 to clear denominators).

Verification: $\dim W = 2$ (since the two column vectors forming W are linearly independent), $\dim W^\perp = 1$, and $\dim \mathbb{R}^3 = 3$. Adding them yields $2 + 1 = 3$, confirming the formula.