Name: February 26, 2026

MATH 22580 – Linear Algebra and Differential Equations Tutorial # 5

Tutorial Worksheet

Topic: Linear Transformations and Determinants

P1. Find the determinant of the matrix below and state whether it is invertible.

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & \sqrt[444]{666667} & 7777\pi^{333} \\ 1 & 1 & 0 & 2 + \frac{1}{2+\frac{1}{2+\frac{1}{2+\dots}}} & 999^{\arctan 555} \\ 1 & 1 & 1 & 1 & (\sinh 1)^{888} \\ 0 & 0 & 0 & 0 & -1013 \end{bmatrix}$$

Solution:

Perform the following row operations:

$$R_2 \leftarrow R_2 + 2R_1, \qquad R_3 \leftarrow R_3 - R_1, \qquad R_4 \leftarrow R_4 - R_1.$$

This yields a row–equivalent matrix

$$B = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & \sqrt[444]{666667} & 7777\pi^{333} \\ 0 & 0 & -1 & 2 + \frac{1}{2+\frac{1}{2+\frac{1}{2+\dots}}} & 999^{\arctan 555} \\ 0 & 0 & 0 & 1 & (\sinh 1)^{888} \\ 0 & 0 & 0 & 0 & -1013 \end{bmatrix}$$

Since the elementary row operation of adding to a row a multiple of another row do not change the determinant, we have

$$\det(A) = \det(B).$$

Caution: all other elementary row operations **do** change the determinant. Swapping two rows changes the determinant by a factor of (-1) and multiplying a row by a constant c changes the determinant by multiplying it by c.

Going back to the problem, because B is upper triangular,

$$\det(A) = 1 \cdot 2 \cdot (-1) \cdot 1 \cdot (-1013) = 2026.$$

Since $\det(A) \neq 0$, A is invertible.

P2. Let A, B, C be 3×3 matrices with $\det A = 4$, $\det B = -1$, and $\det C = 9$. Compute following,

$$\det\left(2A(B^T)^2C^{-1}\right)$$

Solution:

$$\det\left(2A(B^T)^2C^{-1}\right) = 2^3 \det(A)\det(B)^2\det(C)^{-1} = 8 \cdot 4 \cdot (-1)^2 \cdot \frac{1}{9} = \frac{32}{9}.$$

P3. Consider the vector space $\mathcal{P}_2$ of polynomials of degree at most 2 and the following linear transformation:

$$T : \mathcal{P}_2 \to \mathbb{R}^3, \qquad T\left(p(x)\right) = \begin{bmatrix} p(1) \\ p'(1) \\ p''(1) \end{bmatrix}$$

(a) Use determinants to show that $\mathcal{B} = \{1 + x^2, 2 - x, (1 + x)^2\}$ is a basis of $\mathcal{P}_2$.

(b) Find the matrix of T relative to the basis $\mathcal{B}$ of $\mathcal{P}_2$ from part (b) and the standard basis of $\mathbb{R}^3$.

(c) Suppose that $p(x)$ is a polynomial whose $\mathcal{B}$ coordinate vector is $[p(x)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Find $p(x)$ and $T(p(x))$.

Solution:

(a) Relative to the standard basis $\mathcal{C} = \{1, x, x^2\}$ of $\mathcal{P}_2$, we have

$$[1 + x^2]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad [2 - x]_{\mathcal{C}} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \qquad [(1 + x)^2]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Thus it suffices to check that

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

is a basis of $\mathbb{R}^3$. Compute

$$\det \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix} = 4 \neq 0.$$

Hence $\mathcal{B}$ is a basis of $\mathcal{P}_2$.

(b) We compute the action of T on the basis $\mathcal{B}$:

$$T(1 + x^2) = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \qquad T(2 - x) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \qquad T((1 + x)^2) = \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix}.$$

Thus the matrix of T is

$$[T] = \begin{bmatrix} 2 & 1 & 4 \\ 2 & -1 & 4 \\ 2 & 0 & 2 \end{bmatrix}.$$

(c) We have

$$\begin{aligned} p(x) &= 1(1+x^2) - 1(2-x) + 1(1+x)^2 \\ &= 3x + 2x^2. \end{aligned}$$

Using the matrix from part (b)

$$T(p(x)) = [T]\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 4 \end{bmatrix}.$$

Alternatively,

$$T(p(t)) = \begin{bmatrix} p(1) \\ p'(1) \\ p''(1) \end{bmatrix} = \begin{bmatrix} 3+2 \\ 3+4 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 4 \end{bmatrix}.$$

P4. Consider the vector space $\mathcal{P}_2$ of polynomials of degree at most 2 in the and the linear transformation below:

$$T : \mathcal{P}_2 \to \mathcal{P}_2, \qquad T\big(p(x)\big) = \frac{d}{dx}\big(p(x+2)\big)$$

(a) Let $\mathcal{B} = \{1, x, x^2\}$. Write down $[T]_{\mathcal{B}}$

(b) Find bases for kernel and range of T.

Solution:

(a) First, observe that if $p(x) = a_0 + a_1x + a_2x^2$, then

$$\begin{aligned} T(a_0 + a_1x + a_2x^2) &= \frac{d}{dx}\Big(a_0 + a_1(x+2) + a_2(x+2)^2\Big) \\ &= a_1 + 4a_2 + 2a_2x \end{aligned}$$

and so T can be expressed more explicitly as

$$T(a_0 + a_1x + a_2x^2) = (a_1 + 4a_2) + 2a_2x.$$

Now, with respect to $\mathcal{B} = \{1, x, x^2\}$,

$$[T]_{\mathcal{B}} = \big[\, [T(1)]_{\mathcal{B}} \;\; [T(x)]_{\mathcal{B}} \;\; [T(x^2)]_{\mathcal{B}} \,\big] = \begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) Row-reducing the matrix from part (a),

$$A \longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

From this, a basis for the null space of A is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Hence a basis for $\ker(T)$ is

$$\{\, 1 + 0x + 0x^2 \,\} = \{\, 1 \,\}.$$

A basis for the column space of A is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} \right\}.$$

Therefore, a basis for the range of T is

$$\{\, 1 + 0x + 0x^2,\; 4 + 2x + 0x^2 \,\} = \{\, 1,\; 4 + 2x \,\}.$$

P5. Consider following system of linear equations,

$$\begin{cases} x_1 - 2x_2 + 3x_3 = 1 \\ -x_1 + 3x_3 = 2 \\ 2x_1 - x_2 = 2 \end{cases}$$

Use Cramer's Rule to find x_3.

Solution:

To compute x_3 by Cramer's rule we need to compute

$$x_3 = \frac{\det(A_3(\vec{b}))}{\det(A)},$$

where

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 0 & 3 \\ 2 & -1 & 0 \end{bmatrix} \qquad \text{and} \qquad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix},$$

and $A_3(\vec{b})$ is the matrix of A with the 3rd column replaced with $\vec{b}$:

$$A_3(\vec{b}) = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 2 \\ 2 & -1 & 2 \end{bmatrix}.$$

To compute $\det(A)$ first do one step in the row reduction: clear the rest of column 1.

$$\begin{vmatrix} 1 & -2 & 3 \\ -1 & 0 & 3 \\ 2 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 \\ 0 & -2 & 6 \\ 0 & 3 & -6 \end{vmatrix}.$$

Then do expansion down the first column:

$$\begin{vmatrix} 1 & -2 & 3 \\ 0 & -2 & 6 \\ 0 & 3 & -6 \end{vmatrix} = 1 \begin{vmatrix} -2 & 6 \\ 3 & -6 \end{vmatrix} = 1\big((-2)(-6) - 3 \cdot 6\big) = 12 - 18 = -6.$$

To compute $\det(A_3(\vec{b}))$ first do one step in the row reduction: clear the rest of column 1.

$$\begin{vmatrix} 1 & -2 & 1 \\ -1 & 0 & 2 \\ 2 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 1 \\ 0 & -2 & 3 \\ 0 & 3 & 0 \end{vmatrix}.$$

Then do expansion down the first column:

$$\begin{vmatrix} 1 & -2 & 1 \\ 0 & -2 & 3 \\ 0 & 3 & 0 \end{vmatrix} = 1 \begin{vmatrix} -2 & 3 \\ 3 & 0 \end{vmatrix} = 1\big((-2) \cdot 0 - 3 \cdot 3\big) = -9.$$

Hence

$$x_3 = \frac{-9}{-6} = \frac{3}{2}.$$

Just for completeness, the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 5 \\ 6 \\ 3 \end{bmatrix}.$$