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MATH 22580 – Linear Algebra and Differential Equations Tutorial #4

Tutorial Worksheet

Topic: Vector Spaces, Bases, and Linear Transformations

P1. Determine whether the following statements are **true or false**. Justify your answer.

(a) A set containing a single vector is linearly independent.

(b) The set of vectors $\{\mathbf{v}, k\mathbf{v}\}$ is linearly dependent for every scalar k.

(c) Every linearly dependent set contains the zero vector.

(d) The span of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is the column space of the matrix whose columns are $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

(e) The column space of a matrix A is the set of vectors $\mathbf{x}$ which are solutions to $A\mathbf{x} = \mathbf{b}$.

(f) The system $A\mathbf{x} = \mathbf{b}$ is inconsistent if and only if $\mathbf{b}$ is not in the column space of A.

Solution:

(a) **False.** A set containing a single vector $\{\mathbf{v}\}$ is linearly independent *unless* that vector is the zero vector. The set $\{\mathbf{0}\}$ is linearly dependent because the vector equation $c\mathbf{0} = \mathbf{0}$ has non-trivial solutions (for example, any scalar $c \neq 0$).

(b) **True.** A set of two vectors is linearly dependent if one is a scalar multiple of the other (this is not the case if the set contains more than two vectors, beware! That is, three vectors can be linearly dependent and none of them be a scalar multiple of the other).

(c) **False.** The set of vectors $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$ is linearly dependent because the second vector is twice the first, yet neither is the zero vector. A linearly dependent set does not require the presence of the zero vector.

(d) **True.** This is the definition of the column space.

(e) **False.** The column space of A is the set of all vectors $\mathbf{b}$ for which the system $A\mathbf{x} = \mathbf{b}$ has a solution. Moreover, if you fix the vector $\mathbf{b}$, then the set of vectors $\mathbf{x}$ that satisfy $A\mathbf{x} = \mathbf{b}$ is **not** a subspace unless $\mathbf{b} = 0$.

(f) **True.** The matrix equation $A\mathbf{x} = \mathbf{b}$ is simply asking: "Can the vector $\mathbf{b}$ be formed by a linear combination of the columns of A?" Because the column space of A contains *all* such linear combinations, the system will only be consistent if $\mathbf{b} \in \text{Col}(A)$. Conversely, if $\mathbf{b} \notin \text{Col}(A)$, the system is definitively inconsistent.

P2. Consider the vector space $\mathscr{P}_2$ and the linear transformation $T : \mathscr{P}_2 \to \mathscr{P}_2$ given by

$$T(f(x)) = f'(x)$$

Two bases of $\mathscr{P}_2$ are $\mathcal{B} = \{1 + x, x + x^2, x^2\}$ and $\mathcal{C} = \{(x-1)^2, x-1, 1\}$.

(a) Compute the standard matrix $[T]$ of T with respect to the standard basis $\{1, x, x^2\}$ of $\mathscr{P}_2$

(b) Compute the matrix $[T]_{\mathcal{B}}$ of the transformation T with respect to $\mathcal{B}$.

(c) Compute the matrix $[T]_{\mathcal{C}}$ with respect to $\mathcal{C}$.

(d) Compute $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$.

Solution:

(a) The standard matrix is found by applying the transformation $T(f(x)) = f'(x)$ to each basis vector in the standard basis $\mathcal{E} = \{1, x, x^2\}$ and writing the results as coordinate vectors relative to $\mathcal{E}$:

$$T(1) = 0 = 0(1) + 0(x) + 0(x^2) \implies [T(1)]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T(x) = 1 = 1(1) + 0(x) + 0(x^2) \implies [T(x)]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T(x^2) = 2x = 0(1) + 2(x) + 0(x^2) \implies [T(x^2)]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

Placing these column vectors into a matrix yields:

$$\boxed{[T] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}}$$

(b) We apply T to each basis vector in $\mathcal{B}$ and express the output as a linear combination of the vectors in $\mathcal{B}$:

$$T(1+x) = 1 = 1(1+x) - 1(x+x^2) + 1(x^2) \implies \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}_{\mathcal{B}}$$

$$T(x+x^2) = 1 + 2x = 1(1+x) + 1(x+x^2) - 1(x^2) \implies \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}_{\mathcal{B}}$$

$$T(x^2) = 2x = 0(1+x) + 2(x+x^2) - 2(x^2) \implies \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}_{\mathcal{B}}$$

Thus, the matrix is:

$$\boxed{[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 2 \\ 1 & -1 & -2 \end{bmatrix}}$$

(c) We apply T to each basis vector in $\mathcal{C}$ and express the output as a linear combination of the vectors in $\mathcal{C}$:

$$T((x-1)^2) = 2(x-1) = 0(x-1)^2 + 2(x-1) + 0(1) \implies \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}_{\mathcal{C}}$$

$$T(x-1) = 1 = 0(x-1)^2 + 0(x-1) + 1(1) \implies \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{\mathcal{C}}$$

$$T(1) = 0 = 0(x-1)^2 + 0(x-1) + 0(1) \implies \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{C}}$$

Thus, the matrix is:

$$\boxed{[T]_{\mathcal{C}} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}$$

(d) Note that $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$ is the matrix that receives a vector in $\mathcal{B}$ coordinates, applies T, and then returns the answer in $\mathcal{C}$ coordinates. To obtain it, we evaluate T on the basis vectors of $\mathcal{B}$, but express the results in terms of $\mathcal{C}$:

$$T(1+x) = 1 = 0(x-1)^2 + 0(x-1) + 1(1) \implies \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{\mathcal{C}}$$

$$T(x+x^2) = 1 + 2x = 2(x-1) + 3 = 0(x-1)^2 + 2(x-1) + 3(1) \implies \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}_{\mathcal{C}}$$

$$T(x^2) = 2x = 2(x-1) + 2 = 0(x-1)^2 + 2(x-1) + 2(1) \implies \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}_{\mathcal{C}}$$

Placing these coordinate vectors as columns gives the transformation matrix relative to $\mathcal{B}$ and $\mathcal{C}$:

$$\boxed{[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 1 & 3 & 2 \end{bmatrix}}$$

Another way: If you compute the change-of-basis matrix $\mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}}$, then the transformation $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$ can be computed in the following ways:

1. First change the input vector from $\mathcal{B}$ coordinates to $\mathcal{C}$ coordinates and then apply T in $\mathcal{C}$ coordinates:

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} = [T]_{\mathcal{C}} \cdot \mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}}$$

2. First apply the transformation to the input vector in $\mathcal{B}$ coordinates and then change to $\mathcal{C}$ coordinates:

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}} \cdot [T]_{\mathcal{B}}.$$

P3. Let $T : \mathbb{R}^n \to \mathbb{R}^m$, $S : \mathbb{R}^m \to \mathbb{R}^k$ be linear transformations. Determine whether the following statements are **true or false**.

(a) If $n > m$, T cannot be one-to-one.

(b) If $n > m$, T must be onto.

(c) If $n < m$, T must be one-to-one.

(d) If $n < m$, T cannot be onto.

(e) If $\ker(T) = \{0\}$ and $\ker(S \circ T) = \{0\}$, then $\ker(S) = \{0\}$.

(f) If $\ker(S) = \{0\}$ and $\ker(S \circ T) = \{0\}$, then $\ker(T) = \{0\}$.

Solution:

(a) **True.** By the Rank-Nullity Theorem, $\dim(\ker(T)) + \text{rank}(T) = n$. Since the codomain is $\mathbb{R}^m$, the maximum possible rank (dimension of the image) is m. If $n > m$, then $\dim(\ker(T)) = n - \text{rank}(T) \geq n - m > 0$. Because the dimension of the kernel is strictly greater than zero, the kernel contains non-zero vectors, meaning T cannot be one-to-one.

(b) **False.** A linear transformation from a higher-dimensional space to a lower-dimensional space is not *forced* to cover the entire codomain. For example, let T be the zero transformation, $T(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^n$. The image is just the zero vector, which does not fill $\mathbb{R}^m$, so it is not onto.

(c) **False.** Mapping from a smaller space to a larger space does not guarantee a one-to-one mapping. The zero transformation is again a perfect counterexample: if $T(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^n$, then $\ker(T) = \mathbb{R}^n$. Since $n \geq 1$, $\ker(T) \neq \{\mathbf{0}\}$, so it is not one-to-one.

(d) **True.** The dimension of the image of T cannot exceed the dimension of its domain, n. Since $n < m$, the image of T is a subspace of dimension at most n, which is strictly less than the dimension of the codomain m. Therefore, the image cannot possibly be the entirety of $\mathbb{R}^m$, meaning T cannot be onto.

(e) **False.** Let $T : \mathbb{R}^1 \to \mathbb{R}^2$ be $T(x) = (x, 0)$ and $S : \mathbb{R}^2 \to \mathbb{R}^1$ be $S(x, y) = x$. Here, $(S \circ T)(x) = x$, so $\ker(S \circ T) = \{0\}$. However, $S(0, 1) = 0$, meaning the vector $(0, 1)$ is in the kernel of S, so $\ker(S) \neq \{\mathbf{0}\}$.

The condition $\ker(S \circ T) = \{\mathbf{0}\}$ only guarantees that S is one-to-one *when restricted to the image of* T. It says nothing about how S behaves on vectors in $\mathbb{R}^m$ that are outside the image of T.

(f) **True.** The statement would be false if you can find two transformations S and T where $\ker(S \circ T) = \{\mathbf{0}\}$ but $\ker(T) \neq \{\mathbf{0}\}$. However, if $\mathbf{x}$ is a vector in $\ker(T)$ with $\mathbf{x} \neq \mathbf{0}$, then

$$(S \circ T)(\mathbf{x}) = S(T(\mathbf{x})) = S(\mathbf{0}) = 0$$

so $\mathbf{x}$ is in the kernel of $S \circ T$, which is impossible because $\ker(S \circ T) = \{\mathbf{0}\}$.
Note: $\ker(S) = \{\mathbf{0}\}$ is actually unnecessary extra information!

P4. Let $V = \mathscr{P}_3$ and

$$S = \{x^3 - x^2 - 1,\ 2x^3 - 3x + 4\}.$$

Determine whether the polynomial $v = 5x^3 + x^2 + 3x + 2$ is in $\text{Span}(S)$. If it is, find an explicit representation of v as a linear combination of the vectors in S.

Solution: The polynomial is **not in** the span of S.

To determine whether $v = 5x^3 + x^2 + 3x + 2$ is in $\text{Span}(S)$, we must check if there exist scalars c_1 and c_2 such that:

$$c_1(x^3 - x^2 - 1) + c_2(2x^3 - 3x + 4) = 5x^3 + x^2 + 3x + 2$$

Expanding the left side and grouping the terms by powers of x, we obtain:

$$(c_1 + 2c_2)x^3 + (-c_1)x^2 + (-3c_2)x + (-c_1 + 4c_2) = 5x^3 + x^2 + 3x + 2$$

For these two polynomials to be equal, their corresponding coefficients must be strictly equal. This gives us the following system of linear equations:

$$\begin{aligned} c_1 + 2c_2 &= 5 && (x^3 \text{ coefficient}) \\ -c_1 &= 1 && (x^2 \text{ coefficient}) \\ -3c_2 &= 3 && (x \text{ coefficient}) \\ -c_1 + 4c_2 &= 2 && (x^0 \text{ constant term}) \end{aligned}$$

We can easily solve for c_1 and c_2 using the middle two equations:

- From the x^2 coefficient: $-c_1 = 1 \implies c_1 = -1$
- From the x coefficient: $-3c_2 = 3 \implies c_2 = -1$

Now, we must verify if these values satisfy the remaining equations to ensure the system is consistent. Substituting $c_1 = -1$ and $c_2 = -1$ into the first equation (x^3 coefficient):

$$c_1 + 2c_2 = -1 + 2(-1) = -3$$

However, the required coefficient for x^3 is 5. Since $-3 \neq 5$, the resulting system of equations is inconsistent. Therefore, $v = 5x^3 + x^2 + 3x + 2$ cannot be written as a linear combination of the vectors in S. This means that $v = 5x^3 + x^2 + 3x + 2$ is **not** in $\text{Span}(S)$.

P5. (a) Write down a basis for the set of 2×2 matrices M_{22}

(b) Compute the coordinate vector of the matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

in the basis that you wrote.

Solution:

(a) The simplest is the standard basis for M_{22}:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Of course this is not the only correct answer; there are infinitely many different bases for M_{22}

(b) To find the coordinate vector, we must express our target matrix as a linear combination of the basis vectors in $\mathcal{B}$:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Clearly $c_1 = 1$, $c_2 = 2$, $c_3 = 3$, and $c_4 = 4$. The coordinate vector is simply these scalars arranged as a column vector:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$