

Tutorial Worksheet
Basis for a Subspace, Coordinates, Change of Basis

P1. Given the matrix $A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 7 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

(a) Find a basis for $\text{Row}(A)$, $\text{Col}(A)$, and $\text{Null}(A)$.

(b) What are the dimensions of $\text{Row}(A)$, $\text{Col}(A)$, and $\text{Null}(A)$?

Hint: Row reduce A .

Solution:

We first row reduce A to REF:

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 7 \\ 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

This is the REF of A .

(a)

A basis for the row space $\text{Row}(A)$ is given by the nonzero rows of the REF:

$$\mathcal{B}_{\text{Row}} = \{ [1 \ 2 \ 0 \ 3], [0 \ 1 \ 0 \ 0], [0 \ 0 \ 1 \ 1] \}.$$

A basis for the column space $\text{Col}(A)$ is given by the pivot columns of the *original* matrix A (pivot columns are the columns which contain pivot points in the REF, here they are the first 3 columns, counting from left to right):

$$\mathcal{B}_{\text{Col}} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

$\text{Null}(A)$ is the space of solutions \vec{v} to the equation $A\vec{v} = 0$. Let $\vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. From the REF,

$$x_1 + x_2 + 3x_4 = 0, \quad x_2 = 0, \quad x_3 + x_4 = 0.$$

Use back substitution to solve this. Then x_4 is a free variable, $x_3 = -x_4$, $x_2 = 0$, $x_1 = -3x_4$. So

$$\vec{v} = x_4 \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Hence a basis for the null space is

$$\mathcal{B}_{\text{Null}} = \left\{ \begin{bmatrix} -3 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

(b) The dimension of a space is equal to the number of vectors in a basis of the space. And we always have $\dim(\text{Row}(A)) = \dim(\text{Col}(A))$ (this number is called the *rank* of A), $\dim(\text{Null}(A)) = \text{number of columns} - \text{rank of } A$. So

$$\dim(\text{Row}(A)) = \dim(\text{Col}(A)) = \text{rank}(A) = 3, \quad \dim(\text{Null}(A)) = 4 - 3 = 1.$$

P2. Let \mathcal{B} denote the basis of \mathbb{R}^3 given by

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

and let \mathbf{v} denote the vector $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ (with respect to the standard basis). Find the coordinates $[\mathbf{v}]_{\mathcal{B}}$ of \mathbf{v} with respect to \mathcal{B} .

Solution:

We want scalars c_1, c_2, c_3 such that

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

This is the matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Put it in the form of the augmented matrix,

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{array} \right].$$

This is already in row echelon form, so we solve by back substitution:

$$c_3 = 4, \quad c_2 + c_3 = 3 \Rightarrow c_2 = -1, \quad c_1 + c_2 + c_3 = 2 \Rightarrow c_1 = -1.$$

Therefore,

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}.$$

P3. Consider the basis $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ for \mathbb{R}^2 . If $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, find \mathbf{x} in standard basis coordinates.

Solution:

The coordinate vector $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ means that

$$\mathbf{x} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Compute:

$$\mathbf{x} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

Thus, the vector \mathbf{x} in standard basis coordinates is

$$\mathbf{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

P4. Suppose that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ are two bases for \mathbb{R}^2 . Also suppose that the change-of-coordinate matrix from \mathcal{B} to \mathcal{C} is given as:

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

For $\mathbf{v} = 2\mathbf{b}_1 - \mathbf{b}_2$, what is $[\mathbf{v}]_{\mathcal{C}}$?

Solution:

$\mathbf{v} = 2\mathbf{b}_1 - \mathbf{b}_2$ means that the coordinates $[v]_{\mathcal{B}}$ of \mathbf{v} with respect to \mathcal{B} is

$$[v]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Using the change-of-coordinate matrix,

$$[v]_{\mathcal{C}} = P_{\mathcal{C} \leftarrow \mathcal{B}}[v]_{\mathcal{B}} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}.$$

P5. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be two bases for \mathbb{R}^2 , where:

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{c}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Find the change-of-basis matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$.

Solution:

The change-of-coordinate matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$ is defined by the property that, for any vector v , multiplying $P_{\mathcal{C} \leftarrow \mathcal{B}}$ with $[v]_{\mathcal{B}}$, we get $[v]_{\mathcal{C}}$. So

$$[v]_{\mathcal{C}} = P_{\mathcal{C} \leftarrow \mathcal{B}}[v]_{\mathcal{B}}.$$

Its columns are the coordinate vectors of the basis vectors in \mathcal{B} , expressed in the basis \mathcal{C} :

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} [b_1]_{\mathcal{C}} & [b_2]_{\mathcal{C}} \end{bmatrix}.$$

To compute $[b_1]_{\mathcal{C}}$, write b_1 as a linear combination of c_1 and c_2 :

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

This gives the system

$$\begin{cases} x - y = 1, \\ x + y = 0, \end{cases}$$

which has solution $x = \frac{1}{2}$, $y = -\frac{1}{2}$. Hence

$$[b_1]_{\mathcal{C}} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}.$$

Next, write b_2 in terms of c_1 and c_2 :

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

which gives

$$\begin{cases} x - y = 0, \\ x + y = 1. \end{cases}$$

This system has solution $x = \frac{1}{2}$, $y = \frac{1}{2}$, so

$$[b_2]_{\mathcal{C}} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Therefore,

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Solution:

Another way to solve it:

Form the augmented matrix $[c_1 \ c_2 \mid b_1 \ b_2]$ and row reduce, until the left side becomes the identity matrix

$$\begin{aligned} \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array} \right] &\xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array} \right] &\xrightarrow{R_2 \leftarrow \frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{array} \right] \\ & & & \xrightarrow{R_1 \leftarrow R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{array} \right]. \end{aligned}$$

Then the answer is just the augmented part on the right side

$$P_{C \leftarrow B} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

P6. If you have not done it already, solve Problem 2 again by using the change-of-basis matrix $P_{\mathcal{B} \leftarrow \text{std}}$.

Solution:

Here $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ and $v = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ (standard coordinates). The change-of-coordinate matrix $P_{\mathcal{B} \leftarrow \text{std}}$ is defined by

$$[v]_{\mathcal{B}} = P_{\mathcal{B} \leftarrow \text{std}}[v]_{\text{std}}.$$

To find $P_{\mathcal{B} \leftarrow \text{std}}$, first form the matrix that converts \mathcal{B} -coordinates to standard coordinates:

$$P_{\text{std} \leftarrow \mathcal{B}} = [b_1 \ b_2 \ b_3] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

To go the other direction (from standard coordinates to \mathcal{B} -coordinates), we just invert the above matrix:

$$P_{\mathcal{B} \leftarrow \text{std}} = (P_{\text{std} \leftarrow \mathcal{B}})^{-1}.$$

Compute the inverse by row reduction:

$$\begin{aligned} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] &\xrightarrow{R_2 \leftarrow R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] &\xrightarrow{R_1 \leftarrow R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \\ &&&&&&&&\xrightarrow{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]. \end{aligned}$$

So

$$P_{\mathcal{B} \leftarrow \text{std}} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Finally,

$$[v]_{\mathcal{B}} = P_{\mathcal{B} \leftarrow \text{std}}[v]_{\text{std}} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}.$$