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MATH 22580 – Linear Algebra and Differential Equations Tutorial #2

Tutorial Worksheet

Linear Transformations, Linear Independence, Subspaces, and Bases

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P1. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformations defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} y+z \\ x \end{bmatrix} \text{ and } S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3y \\ x-2y \end{bmatrix},$$

respectively.

(a) Find the standard matrix for $S \circ T$, i.e., the matrix A (of appropriate dimensions) such that $(S \circ T)(\mathbf{u}) = A\mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^3$.

Solution: Represented via their standard matrices,

$$[T] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad [S] = \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix}.$$

The standard matrix representing the composition $S \circ T$ is the product of the matrices

$$[S] \cdot [T] = \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 1 & 1 \end{bmatrix}$$

(b) Does the composition $T \circ S$ make sense? If so, find its standard matrix.

Solution: The composition $T \circ S$ **does not make sense** because we cannot perform the product $[T] \cdot [S]$ for dimension reasons: T has 3 columns while S has 2 rows.

More clearly, $T \circ S$ means first apply S and then apply T. Since $S : \mathbb{R}^2 \to \mathbb{R}^2$, the transformation S receives a vector in $\mathbb{R}^2$ and returns a vector in $\mathbb{R}^2$, which will become the input of T. However, $T : \mathbb{R}^3 \to \mathbb{R}^2$ receives vectors in $\mathbb{R}^3$, not in $\mathbb{R}^2$, so the composition $T \circ S$ does not make sense.

P2. Find three linearly dependent vectors in $\mathbb{R}^3$ such that any two of them are linearly independent.

Solution: We need to find 3 vectors generating a plane a such that no two of them generate the same line. For example

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \begin{bmatrix}1\\1\\0\end{bmatrix}.$$

P3. Determine if the following vectors are linearly independent.

(a) $\begin{bmatrix}2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$

(b) $\begin{bmatrix}12\\5\\2\end{bmatrix}, \begin{bmatrix}5\\8\\1\end{bmatrix}, \begin{bmatrix}7\\-3\\1\end{bmatrix}$

Solution: (a) We have

$$\begin{bmatrix}2 & 0 & 0\\0 & 0 & -1\\0 & 1 & 0\end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix}2 & 0 & 0\\0 & 1 & 0\\0 & 0 & -1\end{bmatrix}$$

Since the matrix has 3 pivots (it has rank 3), the three vectors are linearly independent.

(b) Note that

$$\begin{bmatrix}12\\5\\2\end{bmatrix} - \begin{bmatrix}5\\8\\1\end{bmatrix} = \begin{bmatrix}7\\-3\\1\end{bmatrix}$$

So the vectors are linearly dependent. You can also check this by writing the matrix with the 3 vectors as columns and row reducing. The matrix will have only 2 pivots, so has rank 2.

P4. Show that the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

form a basis for $\mathbb{R}^3$.

Solution: To show that this set of 3 vectors form a basis for $\mathbb{R}^3$, we need to check that the matrix with those 3 vectors as columns has rank 3. Hence, we do row reduction:

$$\begin{bmatrix} 3 & 1 & 2 \\ -1 & -3 & 0 \\ 1 & -1 & 2 \end{bmatrix} \xrightarrow[R_2 \leftarrow 3R_2 + R_1]{R_3 \leftarrow 3R_3 - R_1} \begin{bmatrix} 3 & 1 & 2 \\ 0 & -8 & 2 \\ 0 & -4 & 4 \end{bmatrix} \xrightarrow{R_3 \leftarrow 2R_3 - R2} \begin{bmatrix} 3 & 1 & 2 \\ 0 & -8 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

This matrix has 3 pivots, so it has rank 3, showing that the 3 vectors are linearly independent. Any 3 linearly independent vectors in $\mathbb{R}^3$ form a basis for $\mathbb{R}^3$.

P5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - 5z \\ -z \\ -2x + 2z \end{bmatrix}.$$

(a) Find the standard matrix of T

(b) Find the column space of the standard matrix.

(c) Find the null space of the standard matrix.

Solution: (a) The standard matrix is

$$[T] = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 0 & -1 \\ -2 & 0 & 2 \end{bmatrix}.$$

(b) The column space of $[T]$ is the space generated by its columns, so we could be lazy and say that

$$\text{col}([T]) = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -1 \\ 2 \end{bmatrix}\right\}.$$

Of course the second vector (the zero vector) adds nothing to the span, so actually

$$\text{col}([T]) = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -5 \\ -1 \\ 2 \end{bmatrix}\right\}.$$

We can see these two vectors are linearly independent (because in the second entries one is zero while the other is not), so these vectors form a basis for the column space.

(c) We can clearly see that the null space is the set of all vectors of the form $\begin{bmatrix} 0 \\ y \\ 0. \end{bmatrix}$ To verify this, we need to find x, y and z such that

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - 5z \\ -z \\ -2x + 2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0. \end{bmatrix}$$

Therefore, $-z = 0$, which gives $z = 0$. Since $x - 5z = 0$ and $z = 0$, we have $x = 0$. This gives the null space:

$$\text{Null}([T]) = \text{Span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}.$$

P6. Which of the following sets of vectors $\mathbf{v} = (a_1, a_2, a_3)$ in $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$?

(a) All $\mathbf{v}$ such that a_1 is even;

(b) All $\mathbf{v}$ such that $a_1 > a_2$;

(c) All $\mathbf{v}$ such that $a_2 \geq 0$;

(d) All $\mathbf{v}$ such that $a_1 + 3a_3 = 2a_2$.

Solution: By definition (see Section 3.5 in Poole's book), a subspace of $\mathbb{R}^3$ is any collection S of vectors in $\mathbb{R}^3$ such that:

(1) The zero vector $\mathbf{0}$ is in S.

(2) S is closed under linear combinations, i.e., if $\mathbf{v}$ and $\mathbf{w}$ are in S and c, d are scalars, then $c\mathbf{v} + d\mathbf{w}$ is in S.

(a) Not a subspace. Take the vector $\mathbf{v} = (2, 4, 4)$. Scaling it, $\frac{1}{2}\mathbf{v} = (1, 2, 2)$ and the first entry is no longer even. The set is not closed under scalar multiplication.

(b) Not a subspace. The zero vector $(0, 0, 0)$ is not in the set with $a_1 > a_2$ because $0 > 0$ is not true.

(c) Not a subspace. The vector $\mathbf{v} = (0, 1, 0)$ is in the set because $1 \geq 0$, but $-\mathbf{v} = (0, -1, 0)$ is not in the set as $-1 \not\geq 0$. The set is not closed under scalar multiplication.

(d) Yes, it is a subspace. Since $a_1 = 2a_2 - 3a_3$, we let $a_2 = s$ and $a_3 = t$ so that the subspace is the set of vectors of the form

$$\left\{ \begin{bmatrix} 2s - 3t \\ s \\ t \end{bmatrix} \right\} = \text{Span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Because it is a span, it is a subspace.