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MATH 22580 – Linear Algebra and Differential Equations Tutorial # 10

Tutorial Worksheet

Topic: second order homogeneous equations with constant coefficients, method of undetermined coefficients, Wronskians, reduction of order, variation of parameters

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P1. Find the solution of following initial value problem.

$$\begin{cases} y'' + 3y' + 2y = 0, \\ y(0) = 0, \quad y'(0) = -1. \end{cases}$$

Solution:

We want to solve the second-order linear homogeneous differential equation

$$y'' + 3y' + 2y = 0$$

subject to the initial conditions

$$y(0) = 0, \qquad y'(0) = -1.$$

Since this is a linear differential equation with constant coefficients, we use the standard method of solving the associated characteristic equation.

Step 1: Form the characteristic equation.

We look for solutions of the form

$$y(t) = e^{rt},$$

where r is a constant to be determined. Then

$$y'(t) = re^{rt}, \qquad y''(t) = r^2e^{rt}.$$

Substituting these into the differential equation gives

$$r^2e^{rt} + 3re^{rt} + 2e^{rt} = 0.$$

Since $e^{rt} \neq 0$ for all t, we may divide through by e^{rt} and obtain

$$r^2 + 3r + 2 = 0.$$

Thus the characteristic equation is

$$r^2 + 3r + 2 = 0.$$

Step 2: Solve the characteristic equation.

We factor the quadratic:

$$r^2 + 3r + 2 = (r + 1)(r + 2).$$

Therefore the roots are

$$r = -1, \qquad r = -2.$$

Because the characteristic equation has two distinct real roots, the general solution of the differential equation is

$$y(t) = C_1 e^{-t} + C_2 e^{-2t},$$

where C_1 and C_2 are constants.

Step 3: Apply the initial condition $y(0) = 0$.

Substituting $t = 0$ into the general solution, we get

$$y(0) = C_1 e^0 + C_2 e^0 = C_1 + C_2.$$

Since $y(0) = 0$, it follows that

$$C_1 + C_2 = 0.$$

Hence

$$C_2 = -C_1.$$

Step 4: Compute $y'(t)$ and apply the initial condition $y'(0) = -1$.

Differentiate the general solution:

$$y'(t) = \frac{d}{dt}\left(C_1 e^{-t} + C_2 e^{-2t}\right) = -C_1 e^{-t} - 2C_2 e^{-2t}.$$

Now substitute $t = 0$:

$$y'(0) = -C_1 - 2C_2.$$

Since $y'(0) = -1$, we obtain

$$-C_1 - 2C_2 = -1.$$

Using the relation $C_2 = -C_1$, we substitute:

$$-C_1 - 2(-C_1) = -1.$$

Simplifying,

$$-C_1 + 2C_1 = -1,$$

so

$$C_1 = -1.$$

Therefore

$$C_2 = 1.$$

Step 5: Write the final solution.

Substituting these constants into the general solution gives

$$y(t) = -e^{-t} + e^{-2t}.$$

Rewriting,

$$\boxed{y(t) = e^{-2t} - e^{-t}.}$$

Verification.

We check that this function satisfies both the differential equation and the initial conditions.

First,

$$y(t) = e^{-2t} - e^{-t}.$$

Then

$$y'(t) = -2e^{-2t} + e^{-t},$$

and

$$y''(t) = 4e^{-2t} - e^{-t}.$$

Now compute

$$y'' + 3y' + 2y = \left(4e^{-2t} - e^{-t}\right) + 3\left(-2e^{-2t} + e^{-t}\right) + 2\left(e^{-2t} - e^{-t}\right).$$

Expanding,

$$y'' + 3y' + 2y = 4e^{-2t} - e^{-t} - 6e^{-2t} + 3e^{-t} + 2e^{-2t} - 2e^{-t}.$$

Combine like terms:

$$(4 - 6 + 2)e^{-2t} + (-1 + 3 - 2)e^{-t} = 0 + 0 = 0.$$

So the differential equation is satisfied.

Also,

$$y(0) = e^0 - e^0 = 1 - 1 = 0,$$

and

$$y'(0) = -2e^0 + e^0 = -2 + 1 = -1.$$

Thus the initial conditions are satisfied as well.

Therefore the solution is

$$\boxed{y(t) = e^{-2t} - e^{-t}.}$$

P2. Use the method of variation of parameters to find the general solution of the differential equation

$$y'' + y = \frac{1}{\cos t}.$$

Solution:

First, rewrite the non-homogeneous term for convenience:

$$y'' + y = \sec t$$

Step 1: Find the complementary solution, $y_c(t)$.
The characteristic equation for the associated homogeneous equation $y'' + y = 0$ is:

$$r^2 + 1 = 0 \implies r = \pm i$$

Thus, our fundamental solutions are $y_1(t) = \cos t$ and $y_2(t) = \sin t$. The complementary solution is:

$$y_c(t) = C_1 \cos t + C_2 \sin t$$

Step 2: Calculate the Wronskian, W.

$$W(y_1, y_2) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = (\cos t)(\cos t) - (\sin t)(-\sin t) = \cos^2 t + \sin^2 t = 1$$

Step 3: Use the Variation of Parameters formulas to find $u_1(t)$ and $u_2(t)$.
Given $f(t) = \sec t$ and $W = 1$:

$$\begin{aligned} u_1(t) &= -\int \frac{y_2(t)f(t)}{W}\,dt \\ &= -\int \frac{\sin t \cdot \sec t}{1}\,dt \\ &= -\int \tan t\,dt \\ &= \ln|\cos t| \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{y_1(t)f(t)}{W}\,dt \\ &= \int \frac{\cos t \cdot \sec t}{1}\,dt \\ &= \int 1\,dt \\ &= t \end{aligned}$$

Step 4: Form the particular solution, $y_p(t)$.
Substitute $u_1(t)$ and $u_2(t)$ into the assumed form $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$:

$$y_p(t) = (\ln|\cos t|)\cos t + t \sin t$$

Step 5: Write the general solution.
The general solution is the sum of the complementary and particular solutions:

$$\boxed{y(t) = C_1 \cos t + C_2 \sin t + \cos t \ln|\cos t| + t \sin t}$$

P3. One of the solutions of the equation

$$t^2y'' - 3ty' + 3y = 0, \qquad t > 0,$$

is $y_1(t) = t$. Find a second solution $y_2(t)$ which is not a scalar multiple of y_1.

Solution:

We are given the differential equation

$$t^2y'' - 3ty' + 3y = 0, \qquad t > 0,$$

and one known solution

$$y_1(t) = t.$$

Step 1: Use reduction of order.

We look for a second solution of the form

$$y_2(t) = v(t)\,y_1(t) = v(t)\,t,$$

where $v(t)$ is to be determined.

Step 2: Compute derivatives.

$$y_2' = v't + v, \qquad y_2'' = v''t + 2v'.$$

Step 3: Substitute into the equation.

Substitute into $t^2y'' - 3ty' + 3y = 0$:

$$t^2(v''t + 2v') - 3t(v't + v) + 3(vt) = 0.$$

Expand:

$$t^3v'' + 2t^2v' - 3t^2v' - 3tv + 3tv = 0.$$

Simplify:

$$t^3v'' - t^2v' = 0.$$

Divide by t^2 (since $t > 0$):

$$tv'' - v' = 0.$$

Step 4: Solve for $v(t)$.

Let $w = v'$. Then the equation becomes

$$tw' - w = 0.$$

Rewrite:

$$\frac{w'}{w} = \frac{1}{t}.$$

Integrate:

$$\ln|w| = \ln t + C \;\Rightarrow\; w = Ct.$$

Thus

$$v' = Ct.$$

Integrate again:

$$v = \frac{C}{2}t^2 + D.$$

Step 5: Construct y_2.

$$y_2 = v(t)\,t = \left(\frac{C}{2}t^2 + D\right)t = \frac{C}{2}t^3 + Dt.$$

The term Dt is a multiple of $y_1(t) = t$, so we discard it. Taking $C = 2$, we obtain a linearly independent solution:

$$\boxed{y_2(t) = t^3.}$$

Final Answer.

A second solution linearly independent from y_1 is

$$\boxed{y_2(t) = t^3.}$$

P4. Compute the Wronskian of the functions

$$y_1(t) = t\cos t \quad \text{and} \quad y_2(t) = t\sin t.$$

Solution:

The Wronskian of two functions $y_1(t)$ and $y_2(t)$ is defined as

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'.$$

We are given

$$y_1(t) = t\cos t, \qquad y_2(t) = t\sin t.$$

Step 1: Compute derivatives.

Using the product rule,

$$y_1'(t) = \cos t - t\sin t,$$
$$y_2'(t) = \sin t + t\cos t.$$

Step 2: Compute the Wronskian.

$$W(t) = y_1 y_2' - y_2 y_1'.$$

Substitute:

$$W(t) = (t\cos t)(\sin t + t\cos t) - (t\sin t)(\cos t - t\sin t).$$

Step 3: Expand.

$$W(t) = t\cos t\sin t + t^2\cos^2 t - t\sin t\cos t + t^2\sin^2 t.$$

Step 4: Simplify.

The terms $t\cos t\sin t$ and $-t\sin t\cos t$ cancel:

$$W(t) = t^2(\cos^2 t + \sin^2 t).$$

Using $\cos^2 t + \sin^2 t = 1$, we obtain

$$\boxed{W(t) = t^2.}$$

P5. Find a suitable form for a particular solution y_p of the differential equation

$$y'' - 4y' + 4y = 4te^{2t} + t\sin(2t).$$

Solution:

We are asked to find a suitable form for a particular solution y_p using the method of undetermined coefficients.

Step 1: Solve the homogeneous equation.

Consider

$$y'' - 4y' + 4y = 0.$$

The characteristic equation is

$$r^2 - 4r + 4 = 0 \;\Rightarrow\; (r-2)^2 = 0.$$

Thus the root $r = 2$ has multiplicity 2, and the homogeneous solution is

$$y_h = (C_1 + C_2 t)e^{2t}.$$

Step 2: Handle the term $4te^{2t}$.

Since e^{2t} corresponds to a repeated root of multiplicity 2, we multiply the usual guess by t^2.

For a forcing term of the form te^{2t}, the standard guess would be

$$(At + B)e^{2t}.$$

Because of resonance, we multiply by t^2:

$$y_{p1} = t^2(At + B)e^{2t}.$$

Step 3: Handle the term $t\sin(2t)$.

For a forcing term of the form $t\sin(2t)$, we use the standard ansatz

$$y_{p2} = (Ct + D)\cos(2t) + (Et + F)\sin(2t).$$

Since $\pm 2i$ are not roots of the characteristic equation, no modification is needed.

Step 4: Combine both parts.

Thus a suitable form for the particular solution is

$$\boxed{y_p = t^2(At + B)e^{2t} + (Ct + D)\cos(2t) + (Et + F)\sin(2t).}$$