MATH 22580 – Linear Algebra and Differential Equations Tutorial #1

Tutorial Worksheet
Spans, Matrix Operations, and Linear Transformations

P1. Determine whether the vector $\mathbf{w}$ can be written as a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. If yes, find scalars c_1, c_2, c_3 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{w}$.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

Solution:

We form the augmented matrix corresponding to $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{w}$:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

Write in Reduced Echelon Form:

$$\xrightarrow{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & -2 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

Conclusion: The last row yields the equation $0 = -3$, which is a contradiction. Therefore, the system is inconsistent, and $\mathbf{w}$ **cannot** be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$.

P2. Determine whether the vector $\mathbf{w}$ can be written as a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$. If yes, find scalars c_1, c_2, c_3 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{w}$.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 3 \\ 0 \\ 3 \\ 1 \end{bmatrix}.$$

Solution:

We set up the augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 3 \\ 0 & -2 & 2 & 3 & 0 \\ 0 & 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

We can solve this system using back substitution directly from the bottom rows:

Row 3: $-c_3 = 3 \implies c_3 = -3$.

Row 4: $c_3 + c_4 = 1$. Substitute $c_3 = -3$:

$$-3 + c_4 = 1 \implies c_4 = 4$$

Row 2: $-2c_2 + 2c_3 + 3c_4 = 0$. Substitute $c_3 = -3, c_4 = 4$:

$$-2c_2 + 2(-3) + 3(4) = 0 \implies c_2 = 3$$

Row 1: $c_1 + c_2 + 2c_3 + c_4 = 3$. Substitute all found values:

$$c_1 + 3 + 2(-3) + 4 = 3 \implies c_1 = 2$$

Answer: Yes, $\mathbf{w}$ is a linear combination with scalars:

$$c_1 = 2, \quad c_2 = 3, \quad c_3 = -3, \quad c_4 = 4.$$

P3. For each one of the following matrices, determine whether the matrix is invertible, and find its inverse if it exists.

(a) $\begin{bmatrix} 3 & 2 \\ 7 & 3 \end{bmatrix}$

Solution:

A 2×2 matri $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if $ad - bc \neq 0$ and the inverse is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Therefore, for the matrix in this problem $ad - bc = 9 - 14 = -5$ is not zero, so the matrix is invertible and the inverse is

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 3 & -2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 3/5 & -2/5 \\ -7/5 & 3/5 \end{bmatrix}$$

(b) $\begin{bmatrix} 5 & 10 \\ -2 & -4 \end{bmatrix}$

Solution:

Since $ad - bc = -20 - (-20) = 0$, the matrix is not invertible.

P4. Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Compute (if possible) the following:

(a) AB

Solution:

To multiply a matrix of size $m \times n$ and a matrix of size $p \times q$ we need $n = p$ (number of columns of the first one must equal the number of rows of the second one), and the resulting size is $m \times q$. In this case, A has size 3×2 whereas B is 3×3 so the product AB **cannot be computed**.

(b) BA

Solution:

$$\begin{bmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 5 & 9 \\ 8 & 2 \end{bmatrix}$$

(c) $A^T B$

Solution:

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 8 \\ 4 & -7 & 2 \end{bmatrix}$$

P5. (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation. If $T(\mathbf{u}) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $T(\mathbf{v}) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $T(\mathbf{w}) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Find $T(\mathbf{x})$, where $\mathbf{x} = 2\mathbf{u} + 5\mathbf{v} + 4\mathbf{w}$.

Solution:

$$\begin{aligned} T(\mathbf{x}) &= T(2\mathbf{u} + 5\mathbf{v} + 4\mathbf{w}) \\ &= 2T(\mathbf{u}) + 5T(\mathbf{v}) + 4T(\mathbf{w}) \\ &= 2\begin{bmatrix} 2 \\ 4 \end{bmatrix} + 5\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 8 \end{bmatrix} + \begin{bmatrix} 15 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 8 \end{bmatrix} \\ &= \begin{bmatrix} 19 \\ 21 \end{bmatrix}. \end{aligned}$$

(b) Continuing from part (a), if we know $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for any $\mathbf{x}$ in $\mathbb{R}^3$.

Solution:

We are looking for a 2×3 matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ such that

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \qquad \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

If you do the three matix multiplications you see that

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 4 & 1 & 2 \end{bmatrix}$$

P6. For each of the following linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, find the standard matrix for the linear transformation T, i.e., find a 2×2 matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

(a) $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 2x_2 \\ 2x_1 \end{bmatrix}$, for all $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$.

Solution:
$$A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

(b) $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 \\ x_1 \end{bmatrix}$, for all $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$.

Solution:
$$A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

P7. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined as
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - 2y \\ x + y \\ y \end{bmatrix}.$$

Is T a linear transformation? If yes, find the standard matrix for T. If not, show which property fails.

Hint: Recall that a map $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if the following two properties hold:

(a) $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$ for all $\mathbf{v}_1, \mathbf{v}_2$ in $\mathbb{R}^n$.
(b) $T(c \cdot \mathbf{v}) = c \cdot T(\mathbf{v})$ for all vectors $\mathbf{v}$ in $\mathbb{R}^n$ and scalars c.

Solution:

Note that
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

so T is a linear transformation and its standard matrix is $T = \begin{bmatrix} 1 & -2 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$

P8. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as
$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ x + 1 \end{bmatrix}.$$

Is T a linear transformation? If yes, find the standard matrix for T. If not, show which property fails.

Solution:

One key property of linear transformations (which follows from (b)) is that $T(0) = 0$. For this map,
$$T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

so it fails property (b). You can also find two vectors for which property (a) fails .