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MATH 22580 – Linear Algebra and Differential Equations Tutorial #0

Row Reduction Practice

P1. For each augmented matrix $[A \,|\, \mathbf{b}]$ below, perform the following steps:

(1) **Gaussian Elimination:** Use elementary row operations to transform the matrix into Row Echelon Form (REF). (Note: You do not need to reach Reduced Row Echelon Form unless you find it helpful).

(2) **Consistency Check:** Determine if the system is consistent (has at least one solution) or inconsistent (has no solutions).

(3) If the system is consistent, identify:

i. The Rank of the coefficient matrix A.

ii. The number of Free Variables.

(a) $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right]$

Consistent? Yes / No Rank:__________ # of free vars: __________

Solution:

i. $R_2 \leftarrow R_2 - 3R_1$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right]$$

ii. $R_3 \leftarrow R_3 - 2R_2$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right]$$

Results:

- **Consistent?** Yes (No row of form $[0 \dots 0|b]$ where $b \neq 0$).
- **Rank:** 3 (There are 3 pivots in the coefficient part).
- **# of free variables:** 0 (3 variables - 3 pivots = 0).

(b) $\left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right]$

Consistent? Yes / No Rank:__________ # of free vars: __________

Solution:

i. $R_2 \leftarrow R_2 + R_1$ and $R_3 \leftarrow R_3 - 2R_1$:

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 10 & 10 & 0 \end{array}\right]$$

ii. $R_3 \leftarrow R_3 + 10R_2$:

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Results:

- **Consistent?** Yes (and in fact, the system has infinitely many solutions because of the free variable).
- **Rank:** 2.
- **# of free variables:** 1 (3 variables - 2 pivots = 1).

(c) $\left[\begin{array}{cccc|c} 0 & 2 & 0 & 0 & 0 \\ 1 & 4 & 2 & 1 & 0 \\ 0 & 0 & 3 & 1 & 0 \end{array}\right]$

Consistent? Yes / No Rank:__________ # of free vars: __________

Solution:

i. Swap $R_1 \leftrightarrow R_2$ to bring a pivot to the first column:

$$\left[\begin{array}{cccc|c} 1 & 4 & 2 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \end{array}\right]$$

The matrix is now in Row Echelon Form.

Results:

- **Consistent?** Yes (infinitely many solutions).
- **Rank:** 3.
- **# of free variables:** 1 (4 variables - 3 pivots = 1).

(d) $\left[\begin{array}{cccc|c} 1 & 0 & 3 & 4 & -1 \\ 0 & 1 & -2 & 2 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right]$

Consistent? Yes / No Rank:__________ # of free vars: __________

Solution:

i. $R_4 \leftarrow R_4 - 2R_3$:

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 4 & -1 \\ 0 & 1 & -2 & 2 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Results:

- **Consistent?** Yes (infinitely many solutions).
- **Rank:** 3.
- **# of free variables:** 1 (4 variables - 3 pivots = 1).

(e) $\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 2 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 2 & 3 & -2 \\ 1 & 1 & 1 & 2 & 0 \end{array}\right]$

Consistent? Yes / No   Rank:__________   # of free vars: __________

Solution:

i. $R_4 \leftarrow R_4 - R_1$:

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 2 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 2 & 3 & -2 \\ 0 & 1 & 2 & 3 & -2 \end{array}\right]$$

ii. $R_4 \leftarrow R_4 - R_2$:

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 2 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 2 & 3 & -2 \\ 0 & 0 & 0 & 0 & -3 \end{array}\right]$$

The last row implies $0 = -3$, which is a contradiction.

Results:

- **Consistent?** No (System has no solution).
- **Rank:** 3.
- **# of free variables:** N/A (System is inconsistent).

(f) $\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 3 \\ 0 & -2 & 2 & 3 & 0 \\ 0 & 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$

Consistent? Yes / No   Rank:__________   # of free vars: __________

Solution:

i. $R_4 \leftarrow R_4 + R_3$:

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 3 \\ 0 & -2 & 2 & 3 & 0 \\ 0 & 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

Results:

- **Consistent?** Yes (unique solution).
- **Rank:** 4.
- **# of free variables:** 0 (4 variables - 4 pivots = 0).

P2. Use back substitution to solve the system of equations in the variables x_1, x_2, x_3, x_4 corresponding to the matrix (f) in Problem 1.

Solution:

Using the Row Echelon Form from P1(f):

$$\begin{aligned} x_1 + x_2 + 2x_3 + x_4 &= 3 \\ -2x_2 + 2x_3 + 3x_4 &= 0 \\ -x_3 &= 3 \\ x_4 &= 2 \end{aligned}$$

Solving backwards:

(a) From row 4: $\mathbf{x_4} = \mathbf{2}$.

(b) From row 3: $-x_3 = 3 \implies \mathbf{x_3} = \mathbf{-3}$.

(c) From row 2: $-2x_2 + 2(-3) + 3(2) = 0 \implies -2x_2 - 6 + 6 = 0 \implies -2x_2 = 0 \implies \mathbf{x_2} = \mathbf{0}$.

(d) From row 1: $x_1 + 0 + 2(-3) + 2 = 3 \implies x_1 - 6 + 2 = 3 \implies x_1 - 4 = 3 \implies \mathbf{x_1} = \mathbf{7}$.

Solution to the system: $\mathbf{x} = \begin{bmatrix} 7 \\ 0 \\ -3 \\ 2 \end{bmatrix}$

P3. Do Gauss-Jordan elimination (writing the matrix in **reduced** row echelon form) to solve the system of equations corresponding to the matrix (f) in Problem 1.

Solution:

Starting from the REF obtained in P1(f):

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 3 \\ 0 & -2 & 2 & 3 & 0 \\ 0 & 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

1. Eliminate column 4 (using R_4):
   - $R_1 \leftarrow R_1 - R_4$
   - $R_2 \leftarrow R_2 - 3R_4$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & -2 & 2 & 0 & -6 \\ 0 & 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

2. Simplify rows (making pivots equal to 1):
   - $R_2 \leftarrow -\frac{1}{2}R_2$
   - $R_3 \leftarrow -R_3$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

3. Eliminate column 3 (using R_3):

   - $R_1 \leftarrow R_1 - 2R_3$
   - $R_2 \leftarrow R_2 + R_3$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

4. Eliminate column 2 (using R_2):

   - $R_1 \leftarrow R_1 - R_2$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

Final Reduced Row Echelon Form confirms the solution: $x_1 = 7, x_2 = 0, x_3 = -3, x_4 = 2.$