Math 20580 Name:______________________________

Midterm 2 Instructor:______________________________

October 26, 2023 Section:______________________________

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an $\times$ through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

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1. a ~~b~~ c d e
2. a b c d ~~e~~
3. a b c ~~d~~ e
4. a b ~~c~~ d e
5. a b ~~c~~ d e
6. a b c ~~d~~ e
7. a ~~b~~ c d e
8. ~~a~~ b c d e

Multiple Choice. ________________

9.

10.

11.

12.

Total. ________________

Part I: Multiple choice questions (7 points each)

1. Suppose that A, B, C are 2×2 matrices such that $\det(A) = 1/3$, $\det(B) = 2$ and $\det(C) = -2$. What is $\det(3A^T B^{-1} C)$?

(a) 1 (b) -3 (c) 2 (d) -1 (e) 1/3

$\det(3 I_2) \cdot \det(A^T) \cdot (\det B)^{-1} \cdot \det(C)$

$= 9 \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot (-2) = -3$

2. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

(a) 0,0,2 (b) 0,1,2 (c) 2,2,2 (d) 2,4,6 (e) 0,2,4

$$0 = \begin{vmatrix} 2-\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix}$$

$= (2-\lambda)\left[(2-\lambda)^2 - 2^2\right]$

$= (2-\lambda)(4 - 4\lambda + \lambda^2 - 4)$

$= (2-\lambda) \cdot \lambda(\lambda - 4)$ so $\lambda = 0, 2, 4$

3. The vector $\vec{v} = \begin{bmatrix} -1-3i \\ 2 \end{bmatrix}$ is a complex eigenvector of the matrix $A = \begin{bmatrix} 3 & -5 \\ 2 & 5 \end{bmatrix}$. What is the corresponding complex eigenvalue?

(a) $3+2i$ (b) $3-4i$ (c) 2 (d) $4-3i$ (e) $5+5i$

$$A\vec{v} = \begin{bmatrix} 3 & -5 \\ 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} -1-3i \\ 2 \end{bmatrix} = \begin{bmatrix} -13-9i \\ 8-6i \end{bmatrix}$$

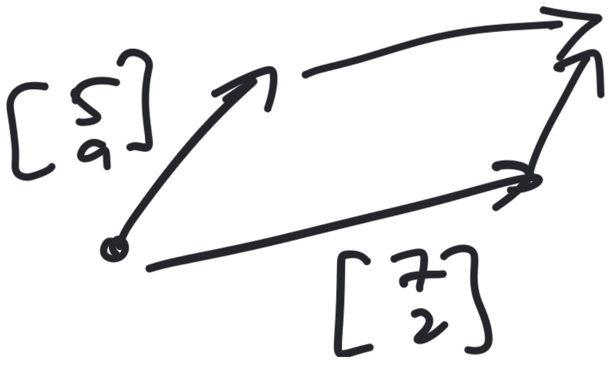
$$= \lambda \begin{bmatrix} -1-3i \\ 2 \end{bmatrix}$$

So $\lambda = \dfrac{8-6i}{2} = 4-3i$

4. Find the area of the parallelogram whose vertices are

$$(0,0), \quad (5,9), \quad (7,2), \quad (12,11).$$

(a) 14 (b) 132 (c) 53 (d) 72 (e) 59



$$\det \begin{bmatrix} 7 & 5 \\ 2 & 9 \end{bmatrix} = 7 \cdot 9 - 2 \cdot 5 = 53$$

5. Consider the vector space V of continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ and the subspace

$$W = \operatorname{Span}\left\{1, 1+e^x, (1+e^x)^2, \cancel{(1-e^x)^2}, \cancel{1+e^{2x}}\right\}.$$

What is the dimension of W?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

$(1+e^x)^2 = 1+2e^x+e^{2x}$ not in $\operatorname{Span}\{1, 1+e^x\}$

$(1-e^x)^2 = (1+e^x)^2 - 4e^x$

$= (1+e^x)^2 - 4(1+e^x) + 4$ in $\operatorname{Span}\{1, 1+e^x, (1+e^x)^2\}$

so redundant

$1+e^{2x} = (1+e^x)^2 - 2e^x = (1+e^x)^2 - 2(1+e^x) + 2$ also

in $\operatorname{Span}\{\underbrace{1, 1+e^x, (1+e^x)^2}_{\text{basis}}\}$

6. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

which has determinant equal to 1. Find the $(2,1)$-entry of A^{-1}, that is, the entry in row 2 and column 1 of the inverse of A.

(a) -2 (b) 18 (c) 15 (d) 20 (e) -4

$$\frac{C_{12}}{\det A} = \frac{(-1)^{1+2} \cdot \begin{vmatrix} 0 & 4 \\ 5 & 0 \end{vmatrix}}{1} = \frac{(-1)\cdot(-20)}{1}$$

$$= 20$$

7. Recall that $M_{2,2}$ is the vector space of 2×2 matrices, and consider the linear transformation $T : \mathbb{R}^3 \longrightarrow M_{2,2}$ defined by

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 & x_2 - x_3 \\ 0 & x_3 - x_1 \end{bmatrix}$$

What is the dimension of the kernel of T?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

$\vec{x}$ in $\ker T \iff T(\vec{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \iff x_1 = x_2 = x_3$

So $\ker T = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$ has dimension 1

8. Recall that $M_{2,2}$ is the vector space of 2×2 matrices, and that $\mathcal{P}_2$ is the vector space of polynomials of degree at most 2. Consider the transformation

$$T : M_{2,2} \to \mathcal{P}_2, \qquad T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a + bx) \cdot (c + dx).$$

Which of the following statements are true?

✗ I. T is a linear transformation.

✗ II. T is not a linear transformation because $T(\vec{0}) \neq \vec{0}$.

~~III. T is not a linear transformation because the domain consists of matrices while the codomain consists of polynomials.~~

✓ IV. T is not a linear transformation because there exist matrices A, B in $M_{2,2}$ such that $T(A + B) \neq T(A) + T(B)$.

(a) IV only (b) I only (c) II, III only (d) III only (e) II, IV only

$T\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \cdot 0 = 0$ so II is false

$T\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1 \cdot 0 = 0$

$T\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 0 \cdot 1 = 0$

take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$A+B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $T(A+B) = 1 \cdot 1 = 1 \neq T(A) + T(B)$

Part II: Partial credit questions (11 points each). Show your work.

9. Consider the bases

$$\mathcal{B} = \{\overset{\vec{b}_1}{1+2x+x^2}, \overset{\vec{b}_2}{1}, \overset{\vec{b}_3}{5x+2x^2}\} \quad \text{and} \quad \mathcal{C} = \{\overset{\vec{c}_1}{x^2}, \overset{\vec{c}_2}{1}, \overset{\vec{c}_3}{-x}\}$$

of $\mathcal{P}_2$ (the vector space of polynomials of degree at most 2 in the variable x).

(a) Find the change-of-basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ from $\mathcal{B}$ to $\mathcal{C}$.

$\vec{b}_1 = \vec{c}_2 - 2\vec{c}_3 + \vec{c}_1$ so $[\vec{b}_1]_\mathcal{C} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

$\vec{b}_2 = \vec{c}_2$ so $[\vec{b}_2]_\mathcal{C} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\vec{b}_3 = -5\vec{c}_3 + 2\vec{c}_1$ so $[\vec{b}_3]_\mathcal{C} = \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}$

Thus $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ -2 & 0 & -5 \end{bmatrix}$

(b) Suppose that $p(x)$ is a vector in $\mathcal{P}_2$ with $[p(x)]_\mathcal{B} = \begin{bmatrix} 11 \\ -10 \\ -4 \end{bmatrix}$. Find $p(x)$ and $[p(x)]_\mathcal{C}$.

$p(x) = 11\vec{b}_1 - 10\vec{b}_2 - 4\vec{b}_3 = 11 + 22x + 11x^2 - 10 - 20x - 8x^2$

$= 1 + 2x + 3x^2$

$[p(x)]_\mathcal{C} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \cdot [p(x)]_\mathcal{B} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ -2 & 0 & -5 \end{bmatrix} \begin{bmatrix} 11 \\ -10 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$

(or $p(x) = \vec{c}_2 - 2\vec{c}_3 + 3\vec{c}_1$)

10. Consider the vector space $\mathcal{P}_2$ of polynomials of degree at most two, and the transformation $T : \mathcal{P}_2 \longrightarrow \mathbb{R}^3$ given by

$$T(p(x)) = \begin{bmatrix} p(1) \\ p'(1) \\ p''(1) \end{bmatrix}.$$

(a) Show that $\mathcal{B} = \{1 + x^2, 2 - x, (1 + x)^2\}$ is a basis of $\mathcal{P}_2$.

$1+2x+x^2$

Relative to $\mathcal{E} = \{1, x, x^2\}$

set $\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_2}$

$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -4 \end{bmatrix}$ 3 pivots so basis

(b) Find the matrix of T relative to the basis $\mathcal{B}$ of $\mathcal{P}_2$ from part (a) and the standard basis of $\mathbb{R}^3$ (you may use that T is a linear transformation without explaining why).

$p(x) = 1 + x^2 \Rightarrow p'(x) = 2x \Rightarrow p''(x) = 2$

so $T(1+x^2) = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

$p(x) = 2 - x \Rightarrow p'(x) = -1 \Rightarrow p''(x) = 0$

so $T(2-x) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$p(x) = (1+x)^2 \Rightarrow p'(x) = 2(1+x) \Rightarrow p''(x) = 2$

so $T((1+x)^2) = \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix}$

$$[T]_{\mathcal{E} \leftarrow \mathcal{B}} = \begin{bmatrix} 2 & 1 & 4 \\ 2 & -1 & 4 \\ 2 & 0 & 2 \end{bmatrix}$$

11. Consider the matrix

$$A = \begin{bmatrix} t & 11 & 0 & 2 \\ 0 & -3 & 0 & 0 \\ 4 & -9 & 6 & 12 \\ 2 & -20 & 0 & t \end{bmatrix}$$

where t is some real number.

(a) Calculate the determinant of A (your answer may depend on t).

$$\det A = (-3) \begin{vmatrix} t & 0 & 2 \\ 4 & 6 & 12 \\ 2 & 0 & t \end{vmatrix} = (-3) \cdot 6 \cdot \begin{vmatrix} t & 2 \\ 2 & t \end{vmatrix}$$

$$= -18 \, (t^2 - 4)$$

$$= -18 \, (t-2)(t+2)$$

(b) Find all values of t such that A is invertible.

A invertible $\iff \det A \neq 0$

$\iff t \neq \pm 2$

12. Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 0 & -4 & 0 \\ -5 & -1 & -8 \end{bmatrix}.$$

(a) Find the eigenvalues of A.

$$0 = \det\begin{pmatrix} 1-\lambda & 1 & 4 \\ 0 & -4-\lambda & 0 \\ -5 & -1 & -8-\lambda \end{pmatrix} = (-4-\lambda)\left((1-\lambda)(-8-\lambda)+20\right)$$

$$= (-4-\lambda)\,(\lambda+4)(\lambda+3)$$

$\Rightarrow$ eigenvalues $-4, -4, -3$

(b) Diagonalize A, that is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

$E_{-4} = \mathrm{Nul}\begin{bmatrix} 5 & 1 & 4 \\ 0 & 0 & 0 \\ -5 & -1 & -4 \end{bmatrix}$ has basis $\left\{\begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -5 \end{bmatrix}\right\}$

$E_{-3} = \mathrm{Nul}\begin{bmatrix} 4 & 1 & 4 \\ 0 & -1 & 0 \\ -5 & -1 & -5 \end{bmatrix}$ has basis $\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right\}$

Get $P = \begin{bmatrix} 1 & 4 & 1 \\ -5 & 0 & 0 \\ 0 & -5 & -1 \end{bmatrix}$

$D = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -3 \end{bmatrix}$