Math 20580 Name:______________________________

Midterm 2 Instructor:______________________________

October 26, 2023 Section:______________________________

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an $\times$ through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

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1. a b c d e
2. a b c d e
3. a b c d e
4. a b c d e
5. a b c d e
6. a b c d e
7. a b c d e
8. a b c d e

Multiple Choice.

9.

10.

11.

12.

Total.

Part I: Multiple choice questions (7 points each)

1. Suppose that A, B, C are 2×2 matrices such that $\det(A) = 1/3$, $\det(B) = 2$ and $\det(C) = -2$. What is $\det(3A^T B^{-1} C)$?

   (a) 1 (b) -3 (c) 2 (d) -1 (e) $1/3$

2. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

   (a) 0,0,2 (b) 0,1,2 (c) 2,2,2 (d) 2,4,6 (e) 0,2,4

3. ~~The vector $\vec{v} = \begin{bmatrix} -1 - 3i \\ 2 \end{bmatrix}$ is a complex eigenvector of the matrix $A = \begin{bmatrix} 3 & -5 \\ 2 & 5 \end{bmatrix}$.~~
~~What is the corresponding complex eigenvalue?~~

~~(a) $3 + 2i$~~ ~~(b) $3 - 4i$~~ ~~(c) 2~~ ~~(d) $4 - 3i$~~ ~~(e) $5 + 5i$~~

4. Find the area of the parallelogram whose vertices are

$$(0, 0), \quad (5, 9), \quad (7, 2), \quad (12, 11).$$

(a) 14 (b) 132 (c) 53 (d) 72 (e) 59

5. Consider the vector space V of continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ and the subspace

$$W = \text{Span}\left\{1, 1+e^x, (1+e^x)^2, (1-e^x)^2, 1+e^{2x}\right\}.$$

What is the dimension of W?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

6. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

which has determinant equal to 1. Find the $(2,1)$-entry of A^{-1}, that is, the entry in row 2 and column 1 of the inverse of A.

(a) -2 (b) 18 (c) 15 (d) 20 (e) -4

7. Recall that $M_{2,2}$ is the vector space of 2×2 matrices, and consider the linear transformation $T : \mathbb{R}^3 \longrightarrow M_{2,2}$ defined by

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 & x_2 - x_3 \\ 0 & x_3 - x_1 \end{bmatrix}$$

What is the dimension of the kernel of T?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

8. Recall that $M_{2,2}$ is the vector space of 2×2 matrices, and that $\mathcal{P}_2$ is the vector space of polynomials of degree at most 2. Consider the transformation

$$T : M_{2,2} \to \mathcal{P}_2, \qquad T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a + bx) \cdot (c + dx).$$

Which of the following statements are true?

I. T is a linear transformation.

II. T is not a linear transformation because $T(\vec{0}) \neq \vec{0}$.

III. T is not a linear transformation because the domain consists of matrices while the codomain consists of polynomials.

IV. T is not a linear transformation because there exist matrices A, B in $M_{2,2}$ such that $T(A + B) \neq T(A) + T(B)$.

(a) IV only (b) I only (c) II, III only (d) III only (e) II, IV only

Part II: Partial credit questions (11 points each). Show your work.

9. Consider the bases

$$\mathcal{B} = \left\{1 + 2x + x^2, 1, 5x + 2x^2\right\} \quad \text{and} \quad \mathcal{C} = \{x^2, 1, -x\}$$

of $\mathcal{P}_2$ (the vector space of polynomials of degree at most 2 in the variable x).

(a) Find the change-of-basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ from $\mathcal{B}$ to $\mathcal{C}$.

(b) Suppose that $p(x)$ is a vector in $\mathcal{P}_2$ with $[p(x)]_{\mathcal{B}} = \begin{bmatrix} 11 \\ -10 \\ -4 \end{bmatrix}$. Find $p(x)$ and $[p(x)]_{\mathcal{C}}$.

10. Consider the vector space $\mathcal{P}_2$ of polynomials of degree at most two, and the transformation $T : \mathcal{P}_2 \longrightarrow \mathbb{R}^3$ given by

$$T(p(x)) = \begin{bmatrix} p(1) \\ p'(1) \\ p''(1) \end{bmatrix}.$$

(a) Show that $\mathcal{B} = \{1 + x^2, 2 - x, (1 + x)^2\}$ is a basis of $\mathcal{P}_2$.

(b) Find the matrix of T relative to the basis $\mathcal{B}$ of $\mathcal{P}_2$ from part (a) and the standard basis of $\mathbb{R}^3$ (you may use that T is a linear transformation without explaining why).

11. Consider the matrix

$$A = \begin{bmatrix} t & 11 & 0 & 2 \\ 0 & -3 & 0 & 0 \\ 4 & -9 & 6 & 12 \\ 2 & -20 & 0 & t \end{bmatrix}$$

where t is some real number.

(a) Calculate the determinant of A (your answer may depend on t).

(b) Find all values of t such that A is invertible.

12. Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 0 & -4 & 0 \\ -5 & -1 & -8 \end{bmatrix}.$$

(a) Find the eigenvalues of A.

(b) Diagonalize A, that is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.