Math 20580 Name:_________________________

Midterm 3 Instructor:_________________________

April 16, 2026 Section:_________________________

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an $\times$ through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

____

1. a b ~~c~~ d e

2. ~~a~~ b c d e

3. a b ~~c~~ d e

4. a b c d ~~e~~

5. a b c ~~d~~ e

6. ~~a~~ b c d e

7. a ~~b~~ c d e

8. a b c ~~d~~ e

Multiple Choice.

9.

10.

11.

12.

Total.

Part I: Multiple choice questions (7 points each)

1. Find the orthogonal projection of the vector $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ onto the vector $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

(a) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ (e) $\vec{0}$

$\vec{v} \cdot \vec{u} = 1$

$\vec{u} \cdot \vec{u} = 2$

$proj_{\vec{u}}(\vec{v}) = \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} = \frac{1}{2} \vec{u}$

$= \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}$

2. If A is a 3×5 matrix of rank 2, what is the dimension of the orthogonal complement of the row space of A^T?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

$rank(A^T) = 2$

$\Rightarrow$ $W = Row(A^T)$ has dimension 2

W is a subspace of $\mathbb{R}^3$

$\Rightarrow \dim W^\perp = 3 - 2 = 1$

$$\begin{array}{cc} \vec{v}_1 & \vec{v}_2 \\ \shortparallel & \shortparallel \end{array}$$

3. Find the distance between $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ and the subspace $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(a) 1 (b) $\sqrt{2}$ (c) 2 (d) $2\sqrt{2}$ (e) 3

orthogonal

$$\text{proj}_W(\vec{w}) = \frac{\vec{w}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 + \frac{\vec{w}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2$$

$$= \frac{3}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \text{perp}_W(\vec{w}) = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \Rightarrow \text{dist}(\vec{w}, W) = \left\| \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \right\| = 2$$

4. Which of the following functions is the general solution of the equation $y' - 4y = 0$?

(a) $e^{-4x} + C$ (b) $4C + e^{x}$ (c) $e^{4x} - C$ (d) $e^{-x} - 4$ (e) $C \cdot e^{4x}$

$$\int \frac{dy}{y} = \int 4\,dx$$

$$\ln|y| = 4x + K$$

$$\Rightarrow |y| = e^{4x} \cdot e^{K}$$

$$\Rightarrow y = C \cdot e^{4x} \quad \text{where } C = \pm e^{K}$$

5. Find the solution of the initial value problem

$$\begin{cases} y' + y = 2e^x, \\ y(0) = 0 \end{cases}$$

(a) e^x (b) $2e^x - e^{-x}$ (c) xe^x (d) $e^x - e^{-x}$ (e) $1 - e^{-x}$

$y' + \underbrace{P(x)}_{1} y = \underbrace{f(x)}_{2e^x}$

$\mu(x) = e^{\int P(x)dx} = e^x$

$y(x) = \dfrac{\int e^x \cdot 2e^x \, dx}{e^x}$

$= \dfrac{e^{2x} + C}{e^x} = e^x + C \cdot e^{-x}$

$y(0) = 0 \Rightarrow 1 + C = 0 \Rightarrow C = -1$

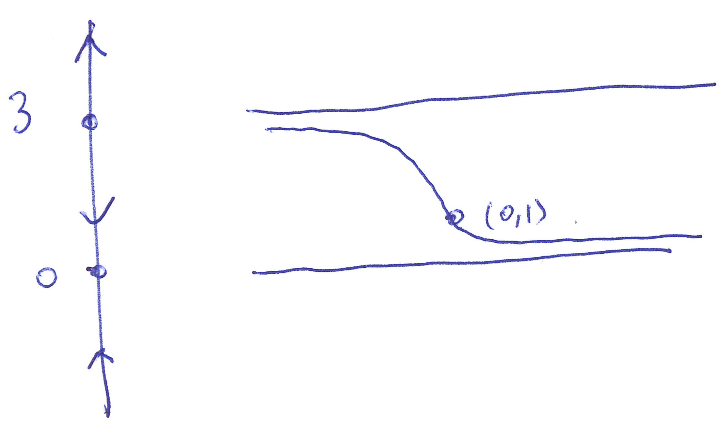
$\Rightarrow y(x) = e^x - e^{-x}$

6. Consider the initial value problem

$$\frac{dy}{dt} = y(y-3), \qquad y(0) = 1.$$

Which of the following describes the nature of the solution?

(a) $\lim_{t\to-\infty} y(t) = 3;$ $\lim_{t\to\infty} y(t) = 0$
(b) $\lim_{t\to-\infty} y(t) = 0;$ $\lim_{t\to\infty} y(t) = 3$
(c) $\lim_{t\to-\infty} y(t) = 3;$ $\lim_{t\to\infty} y(t) = \infty$
(d) $\lim_{t\to-\infty} y(t) = -\infty;$ $\lim_{t\to\infty} y(t) = 0$
(e) $\lim_{t\to-\infty} y(t) = 0;$ $\lim_{t\to\infty} y(t) = -\infty$



7. The differential equation

$$\frac{\partial y}{\partial t} + 2\frac{\partial y}{\partial x} = y + x$$

PDE

is

(a) an equation of order 2 (b) a partial differential equation (c) exact
(d) separable (e) an ordinary differential equation

8. Consider the orthogonal basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathbb{R}^3$, where

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Find the coordinate vector $[\vec{w}]_{\mathcal{B}}$ if $\vec{w} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$.

(a) $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ (e) $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

$$[\vec{w}]_\beta = \begin{bmatrix} \frac{\vec{w}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1} \\ \frac{\vec{w}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2} \\ \frac{\vec{w}\cdot\vec{v}_3}{\vec{v}_3\cdot\vec{v}_3} \end{bmatrix} = \begin{bmatrix} 2/2 \\ 4/2 \\ 2/1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Part II: Partial credit questions (11 points each). Show your work.

9. Let $W = \text{Span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$, where

$$\vec{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{w}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

(a) Apply the Gram-Schmidt process to find an orthonormal basis for W.

First orthogonal basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$$\vec{v}_1 = \vec{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \vec{w}_2 - \frac{\vec{w}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\vec{v}_3 = \vec{w}_3 - \frac{\vec{w}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{w}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ -1 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -2 \end{bmatrix}$$

Normalize to get orthonormal basis

$$\{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = \left\{ \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}$$

(b) Find the QR decomposition of the matrix A with columns $\vec{w}_1, \vec{w}_2, \vec{w}_3$.

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 0 & 0 & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} \sqrt{3} & \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & \sqrt{6} \end{bmatrix}$$

10. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

(a) Find the least squares solution to the equation $A\vec{x} = \vec{b}$.

$$A^T A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\Rightarrow \hat{x} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\hat{x} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\boxed{\hat{x} = \begin{bmatrix} 2/3 \\ 5/3 \end{bmatrix}}$$

(b) Find the vector in the column space of A which is closest to $\vec{b}$.

$$\text{proj}_{\text{Col}A}(\vec{b}) = A \cdot \hat{x}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2/3 \\ 5/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 5/3 \\ 7/3 \end{bmatrix}$$

11. Consider the differential equation

$$(\underbrace{3xy^2 + 4}_{M})\, dx + \underbrace{2x^2y}_{N}\, dy = 0.$$

(a) Explain why the equation is not exact.

$M_y = 6xy \neq N_x = 4xy$ so NOT exact

(b) Find an integrating factor μ which only depends on the variable x.

$$\frac{M_y - N_x}{N} = \frac{2xy}{2x^2y} = \frac{1}{x} \text{ only depends on } x$$

$$\Rightarrow \mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

(c) Determine the explicit solution which satisfies the initial condition $y(1) = 1$.

Multiply equation by $\mu(x) = x$ to make it exact

$$\underbrace{(3x^2y^2 + 4x)}_{\text{new } M} dx + \underbrace{2x^3y}_{\text{new } N}\, dy = 0$$

$$f(x,y) = \int M(x,y)\,dx = x^3y^2 + 2x^2 + g(y)$$

$$\frac{\partial f}{\partial y} = N \Rightarrow \cancel{2x^3y} + g'(y) = \cancel{2x^3y} \Rightarrow g'(y) = 0$$ can take $\Rightarrow g(y) = 0$

$$\Rightarrow f(x,y) = x^3y^2 + 2x^2$$

Implicit Solution $x^3y^2 + 2x^2 = C$

$y(1) = 1 \Rightarrow \boxed{C = 3}$

get $x^3y^2 = 3 - 2x^2$

$$y^2 = \frac{3 - 2x^2}{x^3} \Rightarrow \boxed{y(x) = \sqrt{\frac{3 - 2x^2}{x^3}}}$$ explicit solution

positive because $y(1) = 1$

12. (a) Find the solution of the initial value problem

$$\begin{cases} \dfrac{dy}{dx} - 2xy^2 = 0 \\ y(0) = 1 \end{cases}$$

$$\frac{dy}{dx} = 2xy^2 \quad \Leftrightarrow \quad \int \frac{dy}{y^2} = \int 2x\, dx$$

$$\Leftrightarrow \quad -\frac{1}{y} = x^2 + C$$

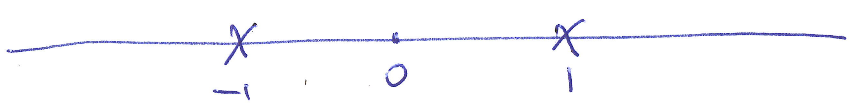
$$y(0) = 1 \quad \Big| \Rightarrow \quad -1 = 0 + C \quad \Rightarrow \boxed{C = -1}$$

$$\Rightarrow \quad -\frac{1}{y} = x^2 - 1 \quad \Rightarrow \quad \boxed{y = \frac{1}{1-x^2}}$$

(b) Find the maximal interval on which the solution in (a) is defined.

interval contains 0

avoids $x^2 = 1$, that is $x = \pm 1$



$I = (-1, 1)$