Math 20580
Midterm 2
March 5, 2026

Name:____________________
Instructor:____________________
Section:____________________

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an $\times$ through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

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1. ~~a~~ b c d e
2. a b c ~~d~~ e
3. a b c d ~~e~~
4. a b ~~c~~ d e
5. a ~~b~~ c d e
6. a b c d ~~e~~
7. a b c ~~d~~ e
8. ~~a~~ b c d e

____

Multiple Choice.

9.

10.

11.

12.

Part I: Multiple choice questions (7 points each)

1. Suppose that A is a 2×2 matrix such that $\det(A) = -3$, and that $B = \begin{bmatrix} 7 & -15 \\ 23 & -121 \end{bmatrix}$.
   What is $\det(2BA^TB^{-1})$?

   (a) -12 $\quad$ (b) -121 $\quad$ (c) 251 $\quad$ (d) -502 $\quad$ (e) 23

$$2^2 \cdot \det(B) \cdot \det(A^T) \cdot \det(B^{-1})$$

$$= 4 \cdot \cancel{\det B} \cdot \det A \cdot \frac{1}{\cancel{\det B}} = -12$$

2. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 2 & 4 \\ -3 & 0 & 8 \end{bmatrix}$$

   (a) 0,0,2 $\quad$ (b) 2,2,8 $\quad$ (c) -3,2,3 $\quad$ (d) 2,5,5 $\quad$ (e) -1,2,3

$$\begin{vmatrix} 2-\lambda & 0 & 3 \\ -1 & 2-\lambda & 4 \\ -3 & 0 & 8-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & 3 \\ -3 & 8-\lambda \end{vmatrix}$$

$$= (2-\lambda)\left((2-\lambda)(8-\lambda)+9\right)$$

$$= (2-\lambda)(\lambda^2 - 10\lambda + 25)$$

$$= (2-\lambda)(\lambda-5)^2 \Rightarrow 2,5,5$$

3. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Find the $(1,3)$-entry of A^{-1} (the entry in row 1 and column 3 of the inverse of A).

(a) -2 (b) -1 (c) 0 (d) 1 (e) 2

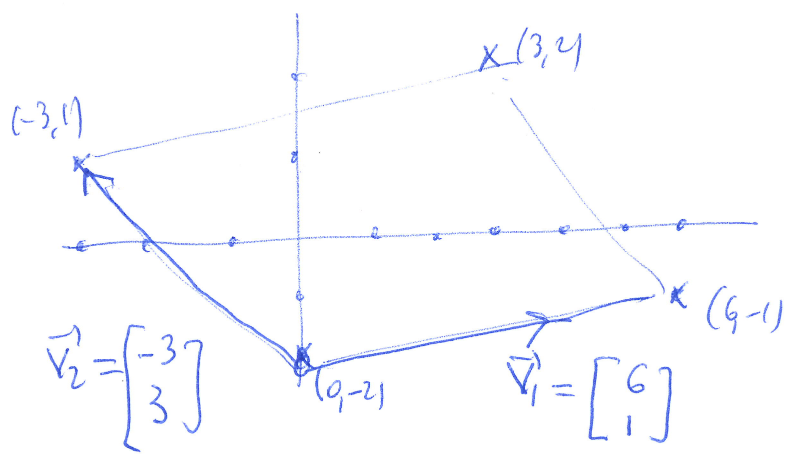
$\det A = 1$ (triangular)

$$A^{-1}_{(1,3)} = \frac{C_{31}}{\det A} = \frac{\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix}}{1} = 2$$

4. Find the area of the parallelogram whose vertices are

$$(0,-2), \quad (6,-1), \quad (-3,1), \quad (3,2).$$

(a) 6 (b) 15 (c) 21 (d) 3 (e) 12



$$\text{area} = \left| \det \begin{bmatrix} 6 & -3 \\ 1 & 3 \end{bmatrix} \right| = 21$$

5. Consider the subspace of $\mathcal{P}_3$ defined by

$$W = \text{Span}\left\{1 - x, x - x^2, x^3 - x^2, 1 - 2x + x^3, 1 - x + 2x^3\right\}.$$

What is the dimension of W?

(a) 5 (b) 4 (c) 3 (d) 2 (e) 1

Use coordinate mapping $\mathcal{P}_3 \to \mathbb{R}^4$ relative to $\mathcal{E} = \{1, x, x^2, x^3\}$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & -2 & -1 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{R_4 \to R_4 + R_3} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \Rightarrow \dim W = 4$$

REF, 4 pivots

6. Suppose that the system of equations

$$\begin{cases} (s-2)x_1 + 3x_2 = 5 \\ x_1 + (s-4)x_2 = 6 \end{cases}$$

has a unique solution. Find the value of x_2 in terms of the parameter s.

(a) $\dfrac{-6s+17}{s^2-6s+8}$ (b) $\dfrac{5s-38}{s^2-6s+8}$ (c) $\dfrac{5s-38}{s^2-6s+5}$ (d) $\dfrac{6s-17}{s^2-6s+8}$ (e) $\dfrac{6s-17}{s^2-6s+5}$

$A\vec{x} = \vec{b}$ $A = \begin{bmatrix} s-2 & 3 \\ 1 & s-4 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

$\det A = (s-2)(s-4) - 3 = s^2 - 6s + 5$

$$x_2 = \frac{\det A_2(\vec{b})}{\det A} = \frac{\det \begin{bmatrix} s-2 & 5 \\ 1 & 6 \end{bmatrix}}{s^2 - 6s + 5} = \frac{6(s-2) - 5}{s^2 - 6s + 5} = \frac{6s - 17}{s^2 - 6s + 5}$$

7. Recall that $\mathcal{P}_3$ is the vector space of polynomials of degree at most 3, and consider the linear transformation $T : \mathcal{P}_3 \longrightarrow \mathbb{R}$ defined by

$$T(p(x)) = p(1).$$

What is the dimension of the kernel of T?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

T is onto, so $\text{rank}(T) = \dim \mathbb{R} = 1$

$\Rightarrow \text{nullity}(T) = \dim(\mathcal{P}_3) - \text{rank}(T) = 4 - 1 = 3$

Alternatively, $\ker T = \{ p(x) = a + bx + cx^2 + dx^3 \text{ with } a+b+c+d=0 \}$

$\Updownarrow d = -a-b-c$

$= \{ a + bx + cx^2 - (a+b+c)x^3 \} = \{ a(1-x^3) + b(x-x^3) + c(x^2-x^3) \}$

$= \text{Span}\{ 1-x^3, x-x^3, x^2-x^3 \}$

linearly independent $\Rightarrow$ basis for $\ker T \Rightarrow \dim(\ker T) = 3$

8. Which of the following sets is linearly independent in $\mathcal{P}_3$?

(I) $\{1 + 2x,\ 3 + 6x\}$

(II) $\{1,\ x,\ x^2\}$

(III) $\{x^3 + x,\ 2x^3 + 2x,\ x^2\}$

(IV) $\{x^2 + 1\}$

(a) II and IV only (b) I, II, III only (c) I and II only (d) III and IV only
(e) I, III, IV only

~~I~~ $3(1+2x) = 3+6x$ dependent

II ✓ (standard basis for $\mathcal{P}_2$)

~~III~~ use coordinate mapping $\mathcal{P}_3 \to \mathbb{R}^4$ relative to $\mathcal{E} = \{1, x, x^2, x^3\}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

no pivot so dependent

(or observe $2 \cdot (x^3+x) = 2x^3 + \ldots$ so dependent)

IV ✓ one non-zero vector is independent

Part II: Partial credit questions (11 points each). Show your work.

9. Consider the bases

$$\mathcal{B} = \{x, 1 + x^2, x + x^2\} \quad \text{and} \quad \mathcal{C} = \{1, 1 + x, x^2\}$$

of $\mathcal{P}_2$ (the vector space of polynomials of degree at most 2 in the variable x).

(a) Find the change-of-basis matrix $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ from $\mathcal{C}$ to $\mathcal{B}$.

Use $\mathcal{E} = \{1, x, x^2\}$ standard basis for $\mathcal{P}_2$ + Gauss-Jordan

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

(b) Given the vector $p(x) = 4 - 2x + x^2$ in $\mathcal{P}_2$, find $[p(x)]_{\mathcal{B}}$ and $[p(x)]_{\mathcal{C}}$.

Find $[p(x)]_{\mathcal{C}}$ first:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\rightarrow \boxed{[p(x)]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix}}$$

$$[p(x)]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} \cdot [p(x)]_{\mathcal{C}} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix}$$

$$\boxed{[p(x)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}}$$

10. Consider the vector space $\mathcal{P}_2$ of polynomials of degree at most two, the vector space $M_{2,2}$ of 2×2 matrices, and the transformation $T : \mathcal{P}_2 \longrightarrow M_{2,2}$ given by

$$T(p(x)) = \begin{bmatrix} p(0) & p(1) \\ p'(1) & p'(0) \end{bmatrix},$$

where $p'(x)$ denotes the derivative of $p(x)$.

(a) Compute the matrix $T(\underbrace{a + bx + cx^2}_{p(x)})$ explicitly in terms of a, b, c.

$p(0) = a \qquad p(1) = a+b+c$

$p'(x) = b+2cx \Rightarrow p'(0) = b\,,\ p'(1) = b+2c$

Get $\quad T(a+bx+cx^2) = \begin{bmatrix} a & a+b+c \\ b+2c & b \end{bmatrix}$

(b) Find the matrix of T relative to the standard basis $\mathcal{B} = \{1, x, x^2\}$ of $\mathcal{P}_2$ and the standard basis $\mathcal{C} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ of $M_{2,2}$.

$T(1) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow [T(1)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$T(x) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow [T(x)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$T(x^2) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \Rightarrow [T(x^2)]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$

$[T]_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix}$

11. Consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & t \\ 1 & 1 & t & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

where t is some real number.

cofactor

(a) Calculate the determinant of A (your answer may depend on t).

$$\det A = -\begin{vmatrix} 0 & 1 & t \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & t \\ 1 & 1 \end{vmatrix}$$

$$= 1 - t$$

(b) Find all values of t such that A is invertible.

A invertible $\Leftrightarrow \det A \neq 0$

$\Leftrightarrow t \neq 1$

12. Let A be the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues of A.

$$0 = \det \begin{bmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 2 & 1-\lambda \end{bmatrix} = (3-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (3-\lambda)\left((1-\lambda)^2 - 4\right)$$
$$= (3-\lambda)(3-\lambda)(-1-\lambda)$$

solutions $3, 3, -1$

(b) Diagonalize A, that is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

$$E_3 = \text{Nul}\begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{bmatrix} \overset{R_3 \to R_3 + R_2}{=} \text{Nul}\begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overset{R_1 \leftrightarrow R_2}{=} \text{Nul}\begin{bmatrix} 0 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = t$

$x_2 = s \Rightarrow \vec{x} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$x_3 = s$

Basis for E_3 is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

$$E_{-1} = \text{Nul}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix} \overset{R_3 \to R_3 - R_2}{=} \text{Nul}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 0$

$x_2 = -t \Rightarrow \vec{x} = t\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$x_3 = t$

Basis of E_{-1} is $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$