Math 20580
Midterm 1
February 12, 2026

Name:____________________________
Instructor:____________________________
Section:____________________________

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 75 minutes to do the test. You may leave earlier if you are finished.

There are 8 multiple choice questions worth 7 points each and 4 partial credit questions each worth 11 points. Record your answers by placing an × through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

____

1. a b ~~c~~ d e
2. a b c ~~d~~ e
3. ~~a~~ b c d e
4. a b c ~~d~~ e
5. a b c d ~~e~~
6. a b c ~~d~~ e
7. a ~~b~~ c d e
8. a b c ~~d~~ e

Multiple Choice.

9.

10.

11.

12.

Total.

Part I: Multiple choice questions (7 points each)

1. Consider the linear system

$$\begin{cases} 3x + 2y = a \\ 5x + 4y = b \end{cases}$$

If (x, y) is a solution, which of the following describes y in terms of a, b?

(a) $y = 2a - b$ (b) $y = 6a + 8b$ (c) $y = (-5a + 3b)/2$ (d) $y = 2a + 5b$

(e) y is not determined by a, b

$$\left[\begin{array}{cc|c} 3 & 2 & a \\ 5 & 4 & b \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c} 3 & 2 & a \\ -1 & 0 & b-2a \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + 3R_2} \left[\begin{array}{cc|c} 0 & 2 & 3b-5a \\ -1 & 0 & b-2a \end{array}\right] \Rightarrow 2y = 3b - 5a$$

so

$$y = \frac{-5a + 3b}{2}$$

2. Consider the vectors $\vec{v}_1 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. Which of the following sets of vectors span $\mathbb{R}^2$?

(I) $\{\vec{v}_1, \vec{v}_2\}$ (II) $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ (III) $\{\vec{v}_2\}$ (IV) $\{\vec{v}_2, \vec{v}_3\}$

(a) III only (b) I and III only (c) I and II only (d) II and IV only

(e) III and IV only

$\{\vec{v}_2, \vec{v}_3\}$ linearly independent in $\mathbb{R}^2$ $\Rightarrow$ $\{\vec{v}_2, \vec{v}_3\}$ basis of $\mathbb{R}^2$

$\Rightarrow \text{Span}(\vec{v}_2, \vec{v}_3) = \mathbb{R}^2$ <u>IV okay</u>

$\Rightarrow \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \mathbb{R}^2$ <u>II okay</u>

Need at least 2 vectors to span $\mathbb{R}^2$ so $\cancel{\text{III}}$

$2\vec{v}_1 = 3\vec{v}_2 \Rightarrow \text{Span}(\vec{v}_1, \vec{v}_2) = \text{Span}(\vec{v}_1)$ so $\cancel{\text{I}}$

3. Let $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear map given by counterclockwise rotation of the plane about the origin by an angle of $30°$ (or $\frac{\pi}{6}$ in radians). Let A be the standard matrix of the transformation T. Which of the following matrices is equal to $A^3 = A \cdot A \cdot A$?

(a) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

A^3 is the standard matrix of rotation by $3 \times 30° = 90°$

so $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

4. Let $\mathcal{P}_3$ denote the vector space of polynomials of degree at most 3,

$$\mathcal{P}_3 = \{p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3\}.$$

Which among the following subsets of $\mathcal{P}_3$ is a subspace?

~~I.~~ The polynomials $p(x)$ in $\mathcal{P}_3$ satisfying $p(0) = 1$. ← no zero vector

II. The polynomials $p(x)$ in $\mathcal{P}_3$ satisfying $p(1) = 0$.

III. The polynomials $p(x)$ in $\mathcal{P}_3$ satisfying $a_1 \geq 1$ and $a_2 \leq 2$. ← no zero vector

IV. The polynomials $p(x)$ in $\mathcal{P}_3$ that have degree at most two.

(a) III and IV only (b) IV only (c) I, III, and IV only

(d) II and IV only (e) III only

II works: if $p(1)=0$, $q(1)=0$ then $(p+q)(1) = 0+0 = 0$ ✓

if $p(1)=0$, c is a scalar then $(c \cdot p)(1) = c \cdot 0 = 0$ ✓

IV works: it is the subspace P_2, or $\mathrm{Span}(1, x, x^2)$

5. Given the matrices

$$A = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

determine $A^{-1}B - AB^T$.

(a) $\begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} -3 & 1 \\ 5 & -2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 4 & 2 \\ -6 & 0 \end{bmatrix}$

$$A^{-1} = \frac{1}{1\cdot(-3)-2(-2)} \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}$$

$$A^{-1}B = \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$$

$$AB^T = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 5 & -2 \end{bmatrix}$$

$$\Rightarrow A^{-1}B - AB^T = \begin{bmatrix} 4 & 2 \\ -6 & 0 \end{bmatrix}$$

6. Consider the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 1-t \\ 1+t \\ 1 \end{bmatrix}$$

For which value of t does the vector $\vec{v}_3$ belong to Span$(\vec{v}_1, \vec{v}_2)$?

(a) all $t \leq 2$ (b) $t = 2$ and $t = -1$ (c) $t = -3$ only (d) $t = 0$ only

(e) no value of t

$$\left[\begin{array}{cc|c} 1 & 1 & 1-t \\ -1 & 1 & 1+t \\ 2 & 1 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + R_1} \left[\begin{array}{cc|c} 1 & 1 & 1-t \\ 0 & 2 & 2 \\ 0 & -1 & -1+2t \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + \frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 1 & 1-t \\ 0 & 2 & 2 \\ 0 & 0 & 2t \end{array}\right]$$

Should not be pivot so $t=0$ only

7. Which of the following sets is linearly independent?

dependent

(~~I~~) $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\6\end{bmatrix}\right\}$ (II) $\left\{\begin{bmatrix}0\\1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$ (~~III~~) $\left\{\begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}\right\}$ (IV) $\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$

$3\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}3\\6\end{bmatrix}$

(a) I, II, III only (b) II and IV only (c) I and II only

(d) III and IV only (e) I, III, IV only

independent because $\neq \vec{0}$

independent, not scalar multiples

8. Let A be a 8×7 matrix of rank 3. Which of the following is equal to the dimension of the null space of the transpose matrix A^T?

(a) 0 (b) 3 (c) 4 (d) 5 (e) 7

A^T has 8 columns

$\text{rank}(A^T) = \text{rank}(A) = 3$ $\Rightarrow$ $\text{nullity}(A^T) = 8-3=5$

Part II: Partial credit questions (11 points each). Show your work.

9. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & -1 \\ 1 & 0 & 1 & 0 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2}$$

(a) Find a basis for Col(A) (the column space of A).

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 1 & 0 & 1 & 0 & 3 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_2]{R_4 \to R_4 - R_1} \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{REF}$$

Pivot columns $1^{st}, 2^{nd}, 3^{rd}$ $\Rightarrow$

basis for Col A is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(b) Find a basis for Row(A) (the row space of A).

non-zero rows in REF:

basis for Row A is $\{ [1\ 0\ 1\ 0\ 3], [0\ 1\ 1\ 2\ 1], [0\ 0\ -1\ -1\ -2] \}$

(c) Find a basis for Nul(A) (the null space of A).

$x_4 = s$
$x_5 = t$ free variables

$x_3 = -x_4 - 2x_5 = -s - 2t$

$x_2 = -x_3 - 2x_4 - x_5 = -s + t$

$x_1 = -x_3 - 3x_5 = +s - t$

$$\vec{x} = \begin{bmatrix} s-t \\ -s+t \\ -s-2t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Basis for Nul A is $\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

10. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & 2 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to -R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 & 2 & -1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ -2 & 2 & -1 \end{bmatrix}$$

11. Consider the linear transformation $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 + x_3 \end{bmatrix},$$

and the matrix transformation $S(\vec{x}) = A\vec{x}$, where

$$A = \begin{bmatrix} -3 & -1 \\ 3 & 2 \\ -2 & -2 \end{bmatrix}.$$

(a) Find the standard matrix of T.

$$[T] = \begin{bmatrix} T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

(b) Find the standard matrices of the compositions $S \circ T$ and $T \circ S$.

$$[S \circ T] = \begin{bmatrix} -3 & -1 \\ 3 & 2 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -4 & -1 \\ 3 & 5 & 2 \\ -2 & -4 & -2 \end{bmatrix}$$

$$[T \circ S] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & -1 \\ 3 & 2 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(c) Find a vector $\vec{v}$ in $\mathbb{R}^3$ with $T(\vec{v}) = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$.

$\vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\quad \begin{cases} x_1 + x_2 = 3 \\ x_2 + x_3 = -5 \end{cases}$ Can take $x_1 = 3$, $x_2 = 0$, $x_3 = -5$ So $\vec{v} = \begin{bmatrix} 3 \\ 0 \\ -5 \end{bmatrix}$

12. Consider the bases $\mathcal{B}$ and $\mathcal{C}$ of $\mathbb{R}^3$ given by

$$\mathcal{B} = \left\{ \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\5\\2\end{bmatrix} \right\}, \qquad \mathcal{C} = \left\{ \begin{bmatrix}0\\0\\-1\end{bmatrix}, \begin{bmatrix}-1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}.$$

(a) Find the change of coordinate matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ from $\mathcal{B}$ to $\mathcal{C}$ (recall that $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is the matrix such that $[\vec{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \cdot [\vec{x}]_{\mathcal{B}}$ for all vectors $\vec{x}$ in $\mathbb{R}^3$).

$$\left[\begin{array}{ccc|ccc} 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 5 \\ -1 & 0 & 0 & 1 & 0 & 2 \end{array}\right] \xrightarrow[\text{to the top}]{\text{move row 3}}$$

$$\left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & 0 & 2 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 5 \end{array}\right] \xrightarrow[R_2 \to -R_2]{R_1 \to -R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 5 \end{array}\right] \quad \text{so} \quad \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} -1 & 0 & -2 \\ -1 & -1 & 0 \\ 2 & 0 & 5 \end{bmatrix}$$

(b) If $\vec{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, determine the coordinate vectors $[\vec{v}]_{\mathcal{B}}$ and $[\vec{v}]_{\mathcal{C}}$.

to find $[\vec{v}]_{\mathcal{B}}$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & 0 & 5 & 2 \\ 1 & 0 & 2 & 3 \end{array}\right] \xrightarrow[\text{to the top}]{\text{move row 3}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 5 & 2 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & -4 \end{array}\right] \xrightarrow[R_1 \to R_1 - 2R_3]{R_2 \to R_2 + 2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

$$\boxed{\text{So } [\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 11 \\ -10 \\ -4 \end{bmatrix}}$$

$$[\vec{v}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} [\vec{v}]_{\mathcal{B}} = \begin{bmatrix} -1 & 0 & -2 \\ -1 & -1 & 0 \\ 2 & 0 & 5 \end{bmatrix} \begin{bmatrix} 11 \\ -10 \\ -4 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix}$$