# Math 60330: Basic Geometry & Topology, Homework 3

1. Let $M$ be a smooth manifold and let $f, g\colon M \to \mathbb{S}^n$ be two smooth maps such that $\|f(p) - g(p)\| < 2$ for all $p \in M$. Prove that $f$ and $g$ are smoothly homotopic.

2. Let $f\colon M_1^n \to M_2^n$ and $g\colon M_2^n \to M_3^n$ be smooth maps between $n$-dimensional manifolds with $M_1$ and $M_2$ compact and connected. Letting $\deg_2$ denote the mod-2 degree, prove that $\deg_2(g \circ f) = \deg_2(g)\deg_2(f)$. Hint: part of this is showing that you can pick regular values that work for all the maps in sight!

3. Let $f\colon M_1^n \to M_2^n$ be a smooth map between connected $n$-manifolds with $M_1$ compact and $M_2$ non-compact. Prove that $\deg_2(f) = 0$.

4. Let $M$ be a smooth manifold and let $\nu$ be a smooth vector field on $M$. Recall from Homework 1 that given a smooth function $f\colon M \to \mathbb{R}$ and a tangent vector $\vec{v} \in T_pM$, we can define $\nabla_{\vec{v}}(f) \in \mathbb{R}$. Given a smooth function $f\colon M \to \mathbb{R}$, we define $\nu(f)\colon M \to \mathbb{R}$ via the formula

$$\nu(f)(p) = \nabla_{\nu(p)}(f) \qquad (p \in M).$$

   In the following problems, you are welcome to assume that $M$ is a smooth submanifold of $\mathbb{R}^d$ if you find it easier to think about vector fields on submanifolds of Euclidean space.

   (a) Assume that $\nu$ has compact support, so we can talk abou the flow $\phi_t\colon M \to M$ generated by $\nu$. Prove that

$$\nu(f)(p) = \frac{\partial\left((f \circ \phi_t)(p)\right)}{\partial t}\Big|_{t=0}$$

   for all $p \in M$.

   (b) Given smooth vector fields $\nu_1$ and $\nu_2$ on $M$, prove that there exists a unique smooth vector field $[\nu_1, \nu_2]$ on $M$ such that

$$[\nu_1, \nu_2](f) = \nu_1(\nu_2(f)) - \nu_2(\nu_1(f))$$

   for all smooth functions $f\colon M \to \mathbb{R}$.

   (c) Prove that if $\nu_1$ and $\nu_2$ and $\nu_3$ are smooth vector fields on $M$, then

$$[\nu_1, [\nu_2, \nu_3]] + [\nu_2, [\nu_3, \nu_1]] + [\nu_3, [\nu_1, \nu_2]] = 0.$$

5. Let $G$ be a Lie group, that is, a smooth manifold that is also a group such that the following hold:

   - The multiplication map $m\colon G \times G \to G$ is smooth.
   - The inversion map $\iota\colon G \to G$ is smooth.

For instance, $G$ might be $\mathrm{GL}_n(\mathbb{R})$. For $g \in G$, let $m_g \colon G \to G$ be the map that multiplies elements on the left by $g$, so $m_g(h) = m(gh)$. A vector field $\nu$ on $G$ is said to be *left invariant* if for all $g \in G$, we have

$$D_x m_g(\nu(x)) = \nu(gx) \quad \text{for all } x \in G.$$

The set of all left-invariant vector fields on $G$ is a vector space called the *Lie algebra* of $G$.

(a) Letting $e$ be the identity element of $G$, construct a vector space isomorphism between the Lie algebra of $G$ and $T_e G$.

(b) Prove that if $\nu_1$ and $\nu_2$ are left-invariant vector fields on $G$, then $[\nu_1, \nu_2]$ is also a left invariant vector field on $G$.

(c) Look up the definition of a Lie algebra (say on Wikipedia) and verify that with the aforementioned bracket operation the Lie algebra of $G$ is indeed a Lie algebra.

(d) Prove that the Lie algebra of $\mathrm{GL}_n(\mathbb{R})$ is precisely the set of $n \times n$ real matrices with the bracket

$$[A, B] = AB - BA.$$