Math 60330: Basic Geometry & Topology
Homework 2

1. Let $f\colon M^m \to N^n$ be a smooth map. Assume that M^m is connected and that $D_x f = 0$ for all $x \in M^m$. Prove that f is a constant map.

2. For $n, m \geqslant 1$, identify $\operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ with $\mathbb{R}^{nm}$. Let $\operatorname{Sur}(\mathbb{R}^n, \mathbb{R}^m)$ be the subset of $\operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ consisting of surjections. Prove that $\operatorname{Sur}(\mathbb{R}^n, \mathbb{R}^m)$ is open in $\operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^m)$.

3. Let M^n be a smooth n-dimensional submanifold of $\mathbb{R}^d$. Define
$$TM = \left\{(x, v) \in \mathbb{R}^d \times \mathbb{R}^d \mid x \in M^n \text{ and } v \in T_x M\right\}.$$
   (a) Prove that TM is a smooth $2n$-dimensional submanifold of $\mathbb{R}^{2d}$.

   (b) Define the *normal bundle* to M to be
   $$T^{\perp}M = \left\{(x, v) \in \mathbb{R}^d \times \mathbb{R}^d \mid x \in M^n \text{ and } v \in (T_x M)^{\perp}\right\},$$
   where $\perp$ is taken with respect to the ordinary dot product on $\mathbb{R}^d$. Prove that $T^{\perp}M$ is a smooth d-dimensional submanifold of $\mathbb{R}^{2d}$.

   (c) Let M^n be a smooth n-dimensional submanifold of $\mathbb{R}^d$ and let N^m be a smooth m-dimensional submanifold of $\mathbb{R}^e$. Prove that if $f\colon M^n \to N^m$ is a smooth map, then the map $Df\colon TM \to TN$ defined by the formula
   $$(Df)(x, v) = (f(x), (D_x f)(v)) \in TN \quad \text{for } (x, v) \in TM$$
   is smooth.

4. Fix $n, d \geqslant 1$. A polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ is *homogeneous* of degree d if $f(tx_1, \ldots, tx_n) = t^d f(x_1, \ldots, x_n)$ for all $t, x_1, \ldots, x_n \in \mathbb{R}$.

   (a) Prove *Euler's identity*: for a homogeneous polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ of degree $d \geqslant 1$, we have
   $$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = df.$$

   (b) Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $d \geqslant 1$. For $a \in \mathbb{R}$, let $M(a) = \{x \in \mathbb{R}^n \mid f(x) = a\}$. For $a \neq 0$, prove that $M(a)$ is a smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$. Note that $M(a) = \varnothing$ is allowed.

   (c) For $a, a' > 0$, prove that $M(a)$ is diffeomorphic to $M(a')$. Also, for $b, b' < 0$, prove that $M(b)$ is diffeomorphic to $M(b')$.

5. Fix $n \geqslant 1$.

   (a) Identify the space $\operatorname{Mat}_n(\mathbb{R})$ of $n \times n$ real matrices with $\mathbb{R}^{n^2}$. Prove that $\operatorname{SL}_n(\mathbb{R})$ is a smooth $(n^2 - 1)$-dimensional submanifold of $\operatorname{Mat}_n(\mathbb{R})$.

   (b) Identify the tangent spaces of $\operatorname{Mat}_n(\mathbb{R})$ with $\operatorname{Mat}_n(\mathbb{R})$. Prove that the tangent space to $\operatorname{SL}_n(\mathbb{R})$ at the identity matrix consists of all matrices $A \in \operatorname{Mat}_n(\mathbb{R})$ with trace 0.