Math 60330: Basic Geometry & Topology, Homework 1

1. (a) Let $X \subset \mathbb{R}^n$ and $Y \subset X$ be arbitrary sets. Let $f\colon X \to \mathbb{R}^m$ be a smooth map. Prove that $f|_Y\colon Y \to \mathbb{R}^m$ is smooth.

   (b) Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ and $Z \subset \mathbb{R}^\ell$ be arbitrary sets. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be smooth maps. Prove that $g \circ f\colon X \to Z$ is smooth.

2. Let $V \subset \mathbb{R}^n$ be a linear subspace and let $f\colon V \to \mathbb{R}^m$ be a linear map. Prove that f is smooth.

3. Prove the following statement that was asserted in class without proof: for $p \in \mathbb{S}^n$, the tangent space $T_p\mathbb{S}^n$ is the set of all $v \in T_p\mathbb{R}^{n+1}$ that are orthogonal to p.

4. Fix some real numbers $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{n+1}$. Regarding S^n as a subspace of $\mathbb{R}^{n+1}$, define a map $f : S^n \to \mathbb{R}$ via the formula

   $$f(x_1, \ldots, x_{n+1}) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_{n+1} x_{n+1}^2 \quad \text{for } (x_1, \ldots, x_{n+1}) \in S^n \subset \mathbb{R}^{n+1}.$$

   Recall that a point $p \in S^n$ is a *regular point* of f if the derivative map $D_p f\colon T_p S^n \to T_{f(p)}\mathbb{R}$ is surjective. The *regular values* of f are the set of all $x \in \mathbb{R}$ such that all points of $f^{-1}(x)$ are regular points of f. **Problem**: Prove that the regular values of f are exactly the set $\mathbb{R} \backslash \{\lambda_1, \ldots, \lambda_{n+1}\}$.

5. Let M^n be a smooth manifold and let $p \in M^n$. For this problem, you can assume that M^n is a smooth submanifold of $\mathbb{R}^d$ if you are not comfortable thinking about the tangent space of an arbitrary smooth manifold. Let $C^\infty(M^n)$ be the $\mathbb{R}$-algebra of all smooth functions $M^n \to \mathbb{R}$.

   (a) For $\vec{v} \in T_p M^n$, define $\nabla_{\vec{v}}\colon C^\infty(M^n) \to \mathbb{R}$ by letting $\nabla_{\vec{v}}(f)$ equal the image of $\vec{v}$ under the map

   $$T_p M^n \xrightarrow{D_p f} T_{f(p)}\mathbb{R} = \mathbb{R}.$$

   Prove that this satisfies the following properties.

   i. For $\vec{v} \in T_p M^n$, the map $\nabla_{\vec{v}}$ is $\mathbb{R}$-linear.

   ii. For $\vec{v} \in T_p M^n$ and $f, g \in C^\infty(M^n)$, we have the Leibniz rule

   $$\nabla_{\vec{v}}(fg) = g(p)\nabla_{\vec{v}}(f) + f(p)\nabla_{\vec{v}}(g).$$

   (b) Now assume that $\Psi\colon C^\infty(M^n) \to \mathbb{R}$ is an $\mathbb{R}$-linear map such that

   $$\Psi(fg) = g(p)\Psi(f) + f(p)\Psi(g)$$

   for all $f, g \in C^\infty(M^n)$. Prove the following:

   i. Let $U \subset M^n$ be an open neighborhood of p. For $f \in C^\infty(M^n)$, prove that $\Psi(f)$ only depends on $f|_U$.

ii. Prove that there exists a unique $\vec{v} \in T_p M^n$ such that $\Psi(f) = \nabla_{\vec{v}}(f)$ for all $f \in C^\infty(M^n)$. Hint: Let $\phi\colon V \to U$ be a smooth chart around p, so $p \in U$ and $V \subset \mathbb{R}^n$ is open. For a smooth function $f\colon M^n \to \mathbb{R}$, think about what $\Psi(f)$ must be based on the first few terms of the Taylor series of the smooth function $f \circ \phi\colon V \to \mathbb{R}$.

Remark. Maps $\Psi\colon C^\infty(M^n) \to \mathbb{R}$ satisfying the above two properties are called *derivations at* p. The above exercise shows that you can identify the tangent space of M^n at p as the set of derivations at p.

6. Let M^n be a smooth compact n-manifold. Prove that for some $d \geqslant 1$ the manifold M^n is diffeomorphic to a smooth submanifold of $\mathbb{R}^d$. Hint: Since M^n is compact, we can write it as $M^n = U_1 \cup \cdots \cup U_m$, where each U_k is an open set such that there exists an open set $V_k \subset \mathbb{R}^n$ and a diffeomorphism $\phi_k\colon U_k \to V_k$. Since M^n is paracompact, we can find open sets $\{U_1', \ldots, U_m'\}$ such that $\{U_1', \ldots, U_m'\}$ is an open cover of M^n and

$$\overline{U_k'} \subset U_k \quad \text{for all } 1 \leqslant k \leqslant m.$$

For $1 \leqslant k \leqslant m$, let $f_k\colon M^n \to [0,1]$ be a smooth bump function such that $(f_k)|_{\overline{U_k'}} = 1$ and $\operatorname{supp}(f_k) \subset U_k$. Define $g_k\colon M^n \to \mathbb{R}^n$ via the formula

$$g_k(x) = \begin{cases} f_k(x)\phi_k(x) & \text{for } x \in U_k, \\ 0 & \text{otherwise.} \end{cases}$$

Define a map $\Phi\colon M^n \to \mathbb{R}^{mn+m}$ via the formula

$$\Phi(x) = (g_1(x), \ldots, g_m(x), f_1(x), \ldots, f_m(x)) \quad \text{for all } x \in M^n.$$

Prove that the image of Φ is a smooth submanifold of $\mathbb{R}^{mn+m}$ and that Φ is a diffeomorphism from M^n to $\Phi(M^n)$.