Yet another book on topology II: covering spaces and the fundamental group

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Basic topics

CHAPTER 1

Covering spaces: definitions and basic examples

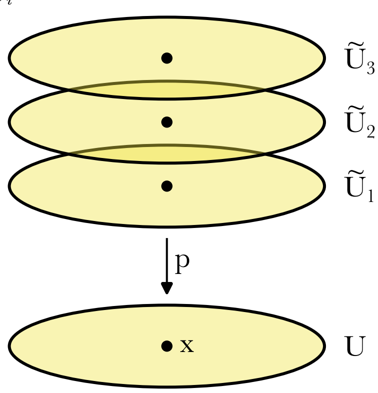
Our first main topic is the theory of covering spaces. This chapter contains some basic definitions and a large number of examples. A first-time reader might be tempted to skip the examples and focus on the theory. This would be a mistake. The richness of the examples is what gives this subject its flavor, and it is impossible to understand the theoretical aspects of covering spaces without having absorbed a large store of these examples.

1.1. Definition and examples

Recall that a *local homeomorphism* is a map $p\colon Z \to X$ such that all $z \in Z$ have open neighborhoods V with $U = p(V)$ open and $p|_V : V \to U$ a homeomorphism. Roughly speaking, in a covering space this condition is strengthened by adding a uniformity condition to these V. The definition is as follows:

Definition 1.1.1. A *covering space* or simply a *cover* of a space X is a space $\widetilde{X}$ equipped with a map $p\colon \widetilde{X} \to X$ such that for all $x \in X$, there is an open neighborhood U of x satisfying:

- the preimage $p^{-1}(U)$ is the disjoint union of open subsets $\{\widetilde{U}_i\}_{i\in\mathcal{I}}$ of $\widetilde{X}$ such that for all $i \in \mathcal{I}$, the restriction $p|_{\widetilde{U}_i} : \widetilde{U}_i \to U$ is a homeomorphism.



We call U a *trivialized neighborhood* of x (or just a *trivialized open set* if we do not want to emphasize x) and each $\widetilde{U}_i$ a *sheet* of $\widetilde{X}$ over U. We will also often call the map $p\colon \widetilde{X} \to X$ a covering space, and refer to X as the *base* of the cover. □

Remark 1.1.2. We allow $p^{-1}(U) = \emptyset$. In particular, for any space X the map $p\colon \emptyset \to X$ is a covering space. This convention is controversial, and some authors require the maps in covering spaces to be surjective. □

Remark 1.1.3. Covering spaces are local homeomorphisms, but the converse does not hold. However, if $\widetilde{X}$ is compact Hausdorff then all local homeomorphisms $p\colon \widetilde{X} \to X$ are covering spaces. We will say more about this in §1.4. □

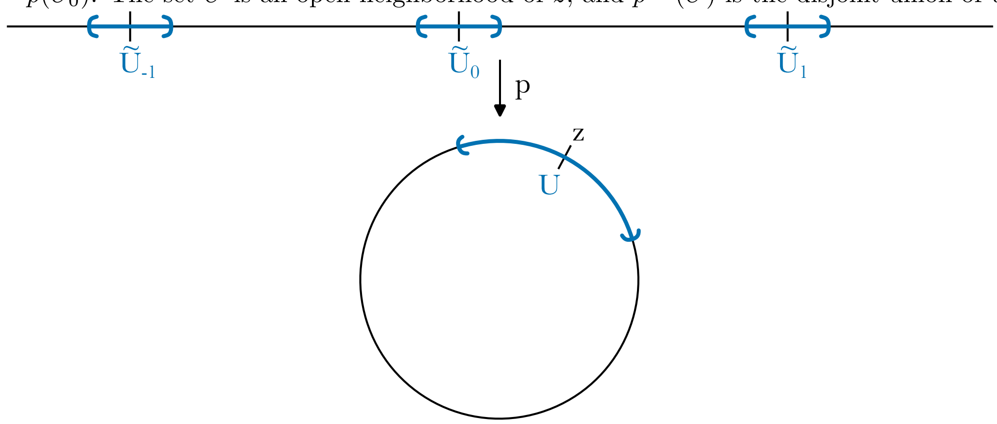
Here are two basic examples:

Example 1.1.4 (Trivial cover). For a space X, the identity map $\mathbb{1}_X : X \to X$ is a covering space. More generally, for any discrete set $\mathcal{I}$ the projection map $p\colon X \times \mathcal{I} \to X$ is a covering space. We will call these the *trivial covers* of X. □

EXAMPLE 1.1.5 (Universal cover of circle). Regard $\mathbb{S}^1$ as the unit circle in the complex plane $\mathbb{C}$. Let $p\colon \mathbb{R} \to \mathbb{S}^1$ be the map $p(\theta) = e^{2\pi i\theta}$. This is a covering space. Indeed, consider $z \in \mathbb{S}^1$. Write $z = e^{2\pi i\theta_0}$. Pick $\epsilon > 0$ with $\epsilon < 1$. For $n \in \mathbb{Z}$, set

$$\widetilde{U}_n = (\theta_0 + n - \epsilon, \theta_0 + n + \epsilon) \subset \mathbb{R}.$$

Let $U = p(\widetilde{U}_0)$. The set U is an open neighborhood of z, and $p^{-1}(U)$ is the disjoint union of the $\widetilde{U}_n$:



Each $\widetilde{U}_n$ projects homeomorphically to U, so U is a trivialized neighborhood of z and the $\widetilde{U}_n$ are the sheets over U. The covering space $p\colon \mathbb{R} \to \mathbb{S}^1$ is called the *universal cover* of $\mathbb{S}^1$. See §1.6 below for why it has this name. □

1.2. Degree of cover

Let $p\colon \widetilde{X} \to X$ be a covering space. The preimages $p^{-1}(x) \subset \widetilde{X}$ of points $x \in X$ are called the *fibers* of $p\colon \widetilde{X} \to X$. For $x \in X$, the fiber $p^{-1}(x)$ is called the *fiber over* x. The first main property of covering spaces is that if X is connected, then the cardinalities of its fibers are all equal. More generally:

LEMMA 1.2.1. *Let $p\colon \widetilde{X} \to X$ be a covering space. Let $f\colon X \to \mathbb{Z} \cup \{\infty\}$ be the function*

$$f(x) = |p^{-1}(x)| \quad \textit{for } x \in X.$$

Then f is locally constant. In particular, if X is connected then f is constant.

PROOF. Consider $x \in X$. Let U be a trivialized neighborhood of x and let $\{\widetilde{U}_i\}_{i \in \mathcal{I}}$ be the sheets of $\widetilde{X}$ over U. For $y \in U$, the preimage $p^{-1}(y)$ consists of one point in each $\widetilde{U}_i$, and thus $f(y) = |\mathcal{I}|$. The lemma follows. □

This suggests the following definition:

DEFINITION 1.2.2. Let $p\colon \widetilde{X} \to X$ be a covering space. We say that $p\colon \widetilde{X} \to X$ has *degree* n if all of its fibers have cardinality n. This degree might be infinity. We will also say that $p\colon \widetilde{X} \to X$ is an n*-sheeted* or an n*-fold* cover. □

Lemma 1.2.1 implies that if X is connected, then every covering space $p\colon \widetilde{X} \to X$ has a degree.[1] For instance, the degree of the universal cover $p\colon \mathbb{R} \to \mathbb{S}^1$ is infinity.

1.3. More examples of covering spaces

Here are some more examples of covering spaces:

EXAMPLE 1.3.1 (Degree n cover of circle). Regard $\mathbb{S}^1$ as the unit circle in $\mathbb{C}$. Fix some $n \geq 1$, and define $p_n\colon \mathbb{S}^1 \to \mathbb{S}^1$ via the formula $p_n(z) = z^n$. This is a degree n covering space. Indeed, consider $z \in \mathbb{S}^1$. The preimage $p_n^{-1}(z)$ consists of n distinct points: writing $z = e^{2\pi i\theta_0}$, we have

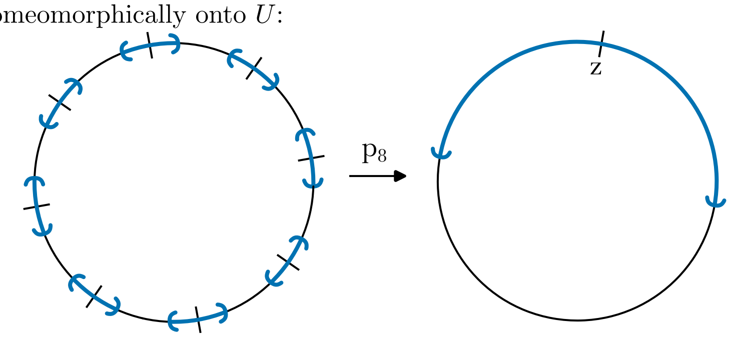
$$p_n^{-1}(z) = \left\{ e^{2\pi i(\theta_0+m)/n} \mid m \text{ is an integer with } 0 \leq m < n \right\}.$$

[1] This is false for non-connected spaces. See Exercise 1.4.

Fix some $\epsilon > 0$ with $\epsilon < 1$, and let

$$U = \left\{ e^{2\pi i\theta} \mid \theta \in (\theta_0 - \epsilon, \theta_0 + \epsilon) \right\}.$$

The set U is an open neighborhood of z, and $p_n^{-1}(U)$ is the disjoint union of n subsets of $\mathbb{S}^1$ each of which projects homeomorphically onto U:



Thus U is a trivialized neighborhood of z and the components of $p_n^{-1}(U)$ are the sheets over U. $\square$

Example 1.3.2 (Cosets of discrete subgroups). Let $\mathbf{G}$ be a topological group, i.e., a group that is a topological space such that the product map $\mathbf{G} \times \mathbf{G} \to \mathbf{G}$ and inversion map $\mathbf{G} \to \mathbf{G}$ are continuous. Let $\mathbf{H}$ be a discrete subgroup of $\mathbf{G}$. Here are two examples to keep in mind:

- $\mathbf{G}$ the additive group $\mathbb{R}^n$, and $\mathbf{H} = \mathbb{Z}^n$; and
- $\mathbf{G} = \mathrm{SL}_n(\mathbb{R})$ and $\mathbf{H} = \mathrm{SL}_n(\mathbb{Z})$.

Endow the set $\mathbf{G}/\mathbf{H} = \{g\mathbf{H} \mid g \in \mathbf{G}\}$ of left cosets with the quotient topology. Then the quotient map $p\colon \mathbf{G} \to \mathbf{G}/\mathbf{H}$ is a cover of degree $|\mathbf{H}|$. Indeed, consider a point $g_0\mathbf{H}$ of $\mathbf{G}/\mathbf{H}$. Since $\mathbf{H}$ is a discrete subgroup of $\mathbf{G}$, we can find an open neighborhood V of $1 \in \mathbf{G}$ whose translates $\{Vh \mid h \in \mathbf{H}\}$ are all disjoint.[2] Set $U = p(g_0 V)$, so

$$p^{-1}(U) = \bigsqcup_{h \in \mathbf{H}} g_0 V h.$$

These are all disjoint sets that project homeomorphically to U, so U is a trivialized neighborhood and the sets $g_0 V h$ with $h \in \mathbf{H}$ are the sheets above U. $\square$

Example 1.3.3. Two of our previous examples are special cases of Example 1.3.2:

- The universal cover $p\colon \mathbb{R} \to \mathbb{S}^1$. Indeed, the additive topological group $\mathbb{R}$ contains the discrete subgroup $\mathbb{Z}$. The satisfies $\mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$, and this homeomorphism fits into a commutative diagram

$$\begin{tikzcd} & \mathbb{R} \arrow[dl] \arrow[dr, "p"] & \\ \mathbb{R}/\mathbb{Z} \arrow[rr, "\cong"] & & \mathbb{S}^1 \end{tikzcd}$$

  Using this, we can identify the covers $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ and $p\colon \mathbb{R} \to \mathbb{S}^1$.
- The covers $p_n\colon \mathbb{S}^1 \to \mathbb{S}^1$ defined by $p_n(z) = z^n$. Indeed, $\mathbb{S}^1 \subset \mathbb{C}$ is a topological group under multiplication, and it contains the discrete group μ_n of n^{th} roots of unity. The quotient $\mathbb{S}^1/\mu_n$ is homeomorphic to $\mathbb{S}^1$, and just like above we can identify the covers $\mathbb{S}^1 \to \mathbb{S}^1/\mu_n$ and $p_n\colon \mathbb{S}^1 \to \mathbb{S}^1$.

As another example, as we noted in Example 1.3.2 the additive group $\mathbb{R}^n$ contains the discrete subgroup $\mathbb{Z}^n$. As the following figure illustrates, the quotient $\mathbb{R}^n/\mathbb{Z}^n$ is homeomorphic to an n-dimensional torus $\mathbb{T}^n = (\mathbb{S}^1)^{\times n}$:

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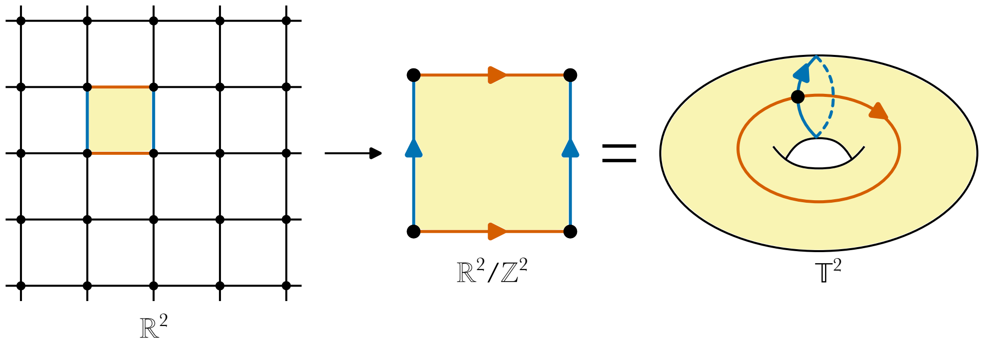
[2] Here are some more details. Since $\mathbf{H}$ is discrete, we can find an open neighborhood W of $1 \in \mathbf{G}$ such that $W \cap \mathbf{H} = \{1\}$. Let $f\colon \mathbf{G} \times \mathbf{G} \to \mathbf{G}$ be the map $f(xy) = xy^{-1}$. Since f is continuous, the set $f^{-1}(W)$ is an open neighborhood of $(1,1)$ and thus we can find open neighborhoods V_1 and V_2 of 1 such that $V_1 \times V_2 \subset f^{-1}(W)$. Letting $V = V_1 \cap V_2$, we then have $f(V \times V) \subset W$. We now claim that the sets $\{Vh \mid h \in \mathbf{H}\}$ are all disjoint. Indeed, if $h_1, h_2 \in \mathbf{H}$ are such that $(Vh_1) \cap (Vh_2) \neq \emptyset$, then we can find $v_1, v_2 \in V$ with $v_1 h_1 = v_2 h_2$, and hence

$$h_2 h_1^{-1} = v_1 v_2^{-1} \in f(V \times V) \cap \mathbf{H} \subset W \cap \mathbf{H} = \{1\}.$$

In other words, $h_1 = h_2$.

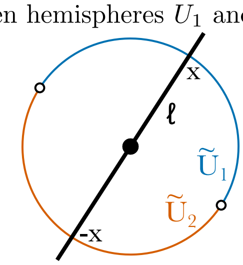


This figure shows the case $n = 2$, and in it the indicated region is a fundamental domain which in the quotient becomes a square with sides identified as indicated. Identifying $\mathbb{R}^n/\mathbb{Z}^n$ with $\mathbb{T}^n$, we get an infinite-degree cover $p\colon \mathbb{R}^n \to \mathbb{T}^n$. □

EXAMPLE 1.3.4 (Real projective space). Let $\mathbb{RP}^n$ be n-dimensional real projective space, that is, the set of lines through the origin in $\mathbb{R}^{n+1}$. Topologize $\mathbb{RP}^n$ as follows:

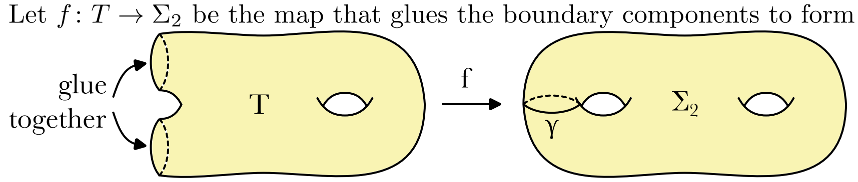
- Let $\pi\colon \mathbb{R}^{n+1} \setminus 0 \to \mathbb{RP}^n$ be the map taking a nonzero point $x \in \mathbb{R}^{n+1}$ to the line determined by 0 and x. Give $\mathbb{RP}^n$ the quotient topology determined by π, so a set $U \subset \mathbb{RP}^n$ is open if and only if $\pi^{-1}(U)$ is open.

We have $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. Let $p\colon \mathbb{S}^n \to \mathbb{RP}^n$ be the restriction of π to $\mathbb{S}^n$. This is a degree 2 covering space. Indeed, consider $\ell \in \mathbb{RP}^n$. The line ℓ intersects $\mathbb{S}^n$ in two antipodal points $x, -x \in \mathbb{S}^n$. Let $U \subset \mathbb{RP}^n$ be the set of lines ℓ' that are *not* orthogonal to ℓ. This is an open set, and the preimage $p^{-1}(U)$ is the disjoint union of two open hemispheres $\widetilde{U}_1$ and $\widetilde{U}_2$ centered at x and $-x$, respectively:

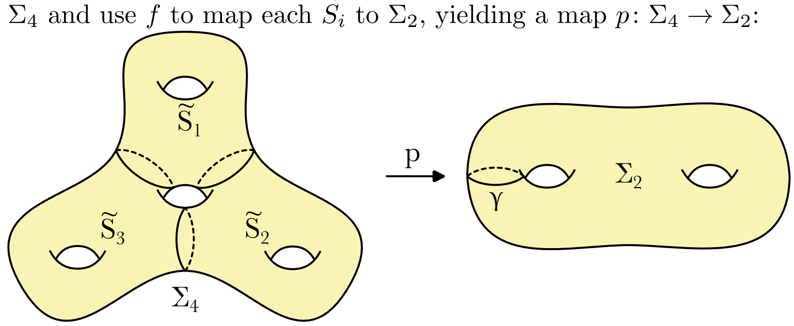


Each $\widetilde{U}_i$ projects homeomorphically to U, so U is a trivialized neighborhood and the $\widetilde{U}_i$ are the sheets over U. □

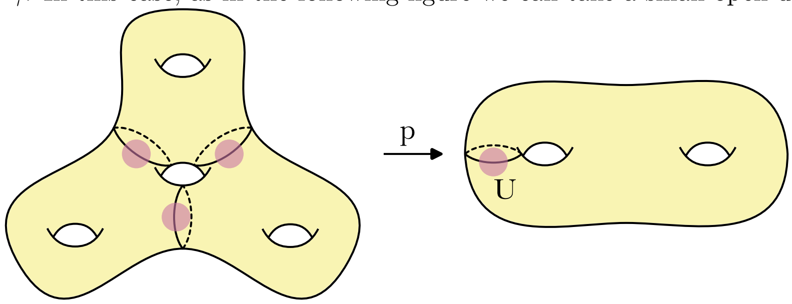
EXAMPLE 1.3.5. Let Σ_2 be a genus 2 surface and let T be a genus 1 surface with two boundary components. Let $f\colon T \to \Sigma_2$ be the map that glues the boundary components to form a loop γ:



For $1 \le i \le 3$, let $\widetilde{S}_i$ be a copy of T. As in the following figure, we can glue the $\widetilde{S}_i$ together to form a genus 4 surface Σ_4 and use f to map each $\widetilde{S}_i$ to Σ_2, yielding a map $p\colon \Sigma_4 \to \Sigma_2$:



Each of the three black loops in Σ_4 maps homeomorphically onto the black loop γ in Σ_2. The map $p\colon \Sigma_4 \to \Sigma_2$ is a degree 3 covering space. Indeed, consider a point $x \in \Sigma_2$. If $x \notin \gamma$, then for our trivialized neighborhood we can take $U = \Sigma_2 \setminus \gamma$. The sheets above U are the $\mathrm{Int}(\widetilde{S}_i)$. If instead $x \notin S$, then $x \in \gamma$. In this case, as in the following figure we can take a small open disk U around x:



The three disks shown in Σ_4 are each mapped homeomorphically to U, so U is a trivialized neighborhood. $\square$

1.4. Covers versus local homeomorphisms

We now give a condition that ensures that a local homeomorphism is a covering space. We start by recalling some definitions from point set topology. See Volume 1 for more details. Let Z be a space. A *compact neighborhood* of a point $z \in Z$ is a compact set $K \subset Z$ such that $z \in \mathrm{Int}(K)$. The space Z is *locally compact* if for all $z \in Z$ and all open neighborhoods U of z, there is a compact neighborhood K of z with $K \subset U$.

Example 1.4.1. All open subsets of $\mathbb{R}^n$ and all closed subsets of $\mathbb{R}^n$ are locally compact. $\square$

Assume that Y and Z are locally compact Hausdorff spaces. A map $f\colon Y \to Z$ is *proper* if for all compact subsets $K \subset Z$, the preimage $f^{-1}(K)$ is compact.[3]

Example 1.4.2. The Heine–Borel theorem says that a subspace of Eulidean space is compact if and only if it is closed and bounded. If Y and Z are both closed subspaces of Euclidean spaces, it follows that a continuous map $f\colon Y \to Z$ is proper if and only if preimages of bounded subspaces are bounded. Equivalently, if $\{y_k\}_{k\geq 1}$ is a sequence of points in Y with $\lim_{k\mapsto\infty} y_k = \infty$, then we must have $\lim_{k\mapsto\infty} f(y_k) = \infty$. $\square$

With these definitions, we have:

Lemma 1.4.3. *Let $p\colon \widetilde{X} \to X$ be a proper local homeomorphism between locally compact Hausdorff spaces. Then $p\colon \widetilde{X} \to X$ is a covering space.*

Proof. Consider $x \in X$. Since p is proper, the set $p^{-1}(x)$ is compact. Since p is a local homeomorphism at each point of $p^{-1}(x)$, the set $p^{-1}(x)$ is also discrete. We deduce that $p^{-1}(x)$ is finite. Enumerate it as $p^{-1}(x) = \{\widetilde{x}_1, \ldots, \widetilde{x}_n\}$. For each $1 \leq i \leq n$, there exists a neighborhood $\widetilde{V}_i$ of $\widetilde{x}_i$ such that $p|_{\widetilde{V}_i}$ is a homeomorphism onto its image $V_i \subset X$. Since $\widetilde{X}$ is Hausdorff, we can shrink the $\widetilde{V}_i$ and assume they are all disjoint. Set $U = V_1 \cap \cdots \cap V_n$ and $\widetilde{U}_i = \widetilde{V}_i \cap p^{-1}(U)$. The set $\widetilde{U}_i$ is an open neighborhood of $\widetilde{x}_i$, and $p|_{\widetilde{U}_i}$ is a homeomorphism onto U.

By construction, $p^{-1}(U)$ contains $\widetilde{U}_1 \sqcup \cdots \sqcup \widetilde{U}_n$. However, we are not done since $p^{-1}(U)$ might contain points that do not lie in some $\widetilde{U}_i$. We want to shrink U to ensure that this does not happen. Since we need U to be open, we need to delete a closed set C of "bad points" from U.

The first step is to shrink U to ensure that $\overline{U}$ is compact. This is possible since X is locally compact: letting K be a compact neighborhood of x with $K \subset U$, we replace U with $\mathrm{Int}(K)$. Since X is Hausdorff the compact set K is closed, so $\overline{U} \subset K$ and thus $\overline{U}$ is compact. Since p is proper $p^{-1}(\overline{U})$ is compact, so since $\widetilde{X}$ is Hausdorff $p^{-1}(\overline{U})$ is closed. Let

$$\widetilde{C} = p^{-1}(\overline{U}) \setminus \bigcup_{i=1}^{n} \widetilde{U}_i.$$

[3]This is not quite the right definition if Y and Z are not locally compact Hausdorff spaces. See Chapter 9 of Volume 1 for more details.

Since $\widetilde{C}$ is a closed subset of the compact set $p^{-1}(\overline{U})$, it follows that $\widetilde{C}$ is compact. This implies that $C = p(\widetilde{C})$ is compact, and hence closed. Replacing U with $U \setminus C$ and each $\widetilde{U}_i$ with $\widetilde{U}_i \setminus p^{-1}(C)$, we now have $p^{-1}(U) = \widetilde{U}_1 \sqcup \cdots \sqcup \widetilde{U}_n$, as desired. □

Here is an example of how Lemma 1.4.3 can be used.

Example 1.4.4 (Roots of square-free polynomials). For some $n \geq 1$, let Poly_n be the space of degree-n monic polynomials over $\mathbb{C}$. Such an $f \in \mathrm{Poly}_n$ can be written as

$$f(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n \quad \text{with } a_1, \ldots, a_n \in \mathbb{C}.$$

The topology comes from the coefficients, so $\mathrm{Poly}_n \cong \mathbb{C}^n$. By the fundamental theorem of algebra, such a polynomial has n roots (counted with multiplicity). Define

$$\mathrm{RPoly}_n = \{(f, x) \in \mathrm{Poly}_n \times \mathbb{C} \mid f(x) = 0\}.$$

In other words, RPoly_n is the space of polynomials equipped with a root. Let $p\colon \mathrm{RPoly}_n \to \mathrm{Poly}_n$ be the map $p(f, x) = f$. For $n \geq 2$ this is not a covering space since the fibers of p have different cardinalities. For example,

$$|p^{-1}(z^n)| = |\{(z^n, 0)\}| = 1 \quad \text{but} \quad |p^{-1}(z^n - 1)| = |\left\{(z^n - 1, \mu) \mid \mu \text{ an } n^{\text{th}} \text{ root of unity}\right\}| = n.$$

As suggested by this, the issue arises because of polynomials with repeated roots. Define

$$\mathrm{Poly}_n^{\mathrm{sf}} = \{f \in \mathrm{Poly}_n \mid f \text{ has } n \text{ distinct roots}\}$$

and

$$\mathrm{RPoly}_n^{\mathrm{sf}} = \left\{(f, x) \in \mathrm{Poly}_n^{\mathrm{sf}} \times \mathbb{C} \mid f(x) = 0\right\}.$$

The "sf" stands for "square-free". The spaces $\mathrm{Poly}_n^{\mathrm{sf}}$ and $\mathrm{RPoly}_n^{\mathrm{sf}}$ are open subsets of Poly_n and RPoly_n, respectively.[4] The projection $p\colon \mathrm{RPoly}_n^{\mathrm{sf}} \to \mathrm{Poly}_n^{\mathrm{sf}}$ is a degree-n covering space. Indeed, since p is a proper map[5] whose fibers all have cardinality n, by Lemma 1.4.3 it is enough to prove that $p\colon \mathrm{RPoly}_n^{\mathrm{sf}} \to \mathrm{Poly}_n^{\mathrm{sf}}$ is a local homeomorphism. But this is easy: for $(f, x) \in \mathrm{RPoly}_n^{\mathrm{sf}}$, since $f(z)$ has no repeated roots we have $f'(x) \neq 0$, so by the implicit function theorem there is a neighborhood $U \subset \mathrm{Poly}_n$ of f such that around (f, x) the subspace

$$\mathrm{RPoly}_n^{\mathrm{sf}} \subset \mathrm{Poly}_n \times \mathbb{C} \subset \mathbb{C}^n \times \mathbb{C}$$

is the graph of a function $U \to \mathbb{C}$. □

1.5. Isomorphisms between covering spaces

We would like to classify the covers of a space X. To do this, we must first define what it means for two covers to be be the same, i.e., we must say what it means to have an isomorphism between two covers of X. The definition is as follows:

Definition 1.5.1. Let X be a space and let $p_1\colon \widetilde{X}_1 \to X$ and $p_2\colon \widetilde{X}_2 \to X$ be two covers of X. A *covering space isomorphism* from $\widetilde{X}_1$ to $\widetilde{X}_2$ is a homeomorphism $f\colon \widetilde{X}_1 \to \widetilde{X}_2$ such that the diagram

$$\begin{tikzcd} \widetilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \widetilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

commutes, i.e., such that $p_2 \circ f = p_1$. If a covering space isomorphism from $\widetilde{X}_1$ to $\widetilde{X}_2$ exists, we say that $\widetilde{X}_1$ and $\widetilde{X}_2$ are *isomorphic* covers of X. This is clearly an equivalence relation. □

[4] This is an elementary exercise. A sophisticated way to see it is to use the fact that having a multiple root is equivalent to the vanishing of the discriminant, which is a polynomial in the coefficients of the polynomial.

[5] To show that $p\colon \mathrm{RPoly}_n^{\mathrm{sf}} \to \mathrm{Poly}_n^{\mathrm{sf}}$ is proper, it is enough to prove that $p\colon \mathrm{RPoly}_n \to \mathrm{Poly}_n$ is proper. This is a consequence of the elementary fact that for all $C > 0$, there exists some $D > 0$ such that if $x \in \mathbb{C}$ is a root of $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ and $|a_k| \leq C$ for all $1 \leq k \leq n$, then $|x| \leq D$.

REMARK 1.5.2. This can be rephrased using categorical language as follows. Recall that $\mathtt{Top}$ is the category of topological spaces and continuous maps. For a space X, let $\mathtt{Top}_{/X}$ be the category whose objects are spaces Y equipped with maps $\phi\colon Y \to X$ and whose morphisms from $\phi_1\colon Y_1 \to X$ to $\phi_2\colon Y_2 \to X$ are maps $f\colon Y_1 \to Y_2$ such that the diagram

$$\begin{tikzcd} Y_1 \arrow[rr, "f"] \arrow[dr, "\phi_1"'] & & Y_2 \arrow[dl, "\phi_2"] \\ & X & \end{tikzcd}$$

commutes. A covering space $p\colon \widetilde{X} \to X$ is an object of $\mathtt{Top}_{/X}$, and a covering space isomorphism is an isomorphism in $\mathtt{Top}_{/X}$ between two covering spaces. □

Here are two basic examples.

EXAMPLE 1.5.3. For $\lambda \neq 0$, define $p^\lambda\colon \mathbb{R} \to \mathbb{S}^1$ via the formula $p_\lambda(\theta) = e^{2\pi i \lambda\theta}$. The universal cover of $\mathbb{S}^1$ is thus $p^1\colon \mathbb{R} \to \mathbb{S}^1$. Each $p^\lambda\colon \mathbb{R} \to \mathbb{S}^1$ is also a covering space, but is isomorphic to the universal cover. Indeed, letting $f\colon \mathbb{R} \to \mathbb{R}$ be the homeomorphism $f(\theta) = \lambda\theta$, the diagram

$$\begin{tikzcd} \mathbb{R} \arrow[rr, "f"] \arrow[dr, "p^\lambda"'] & & \mathbb{R} \arrow[dl, "p^1"] \\ & \mathbb{S}^1 & \end{tikzcd}$$

commutes, so f is a covering space isomorphism from $p^\lambda\colon \mathbb{R} \to \mathbb{S}^1$ to $p^1\colon \mathbb{R} \to \mathbb{S}^1$. □

EXAMPLE 1.5.4. For $n \geq 1$, let $p_n\colon \mathbb{S}^1 \to \mathbb{S}^1$ be the covering space defined by the formula $p_n(z) = z^n$ and let $q_n\colon \mathbb{S}^1 \to \mathbb{S}^1$ be the covering space defined by the formula $q_n(z) = z^{-n}$. The covers $p_n\colon \mathbb{S}^1 \to \mathbb{S}^1$ and $q_n\colon \mathbb{S}^1 \to \mathbb{S}^1$ are isomorphic. Indeed, letting $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ be the homeomorphism $f(z) = z^{-1}$, the diagram

$$\begin{tikzcd} \mathbb{S}^1 \arrow[rr, "f"] \arrow[dr, "p_n"'] & & \mathbb{S}^1 \arrow[dl, "q_n"] \\ & \mathbb{S}^1 & \end{tikzcd}$$

commutes, so f is a covering space homomorphism from $p_n\colon \mathbb{S}^1 \to \mathbb{S}^1$ to $q_n\colon \mathbb{S}^1 \to \mathbb{S}^1$. □

1.6. Goal

Our of our main goals is to classify all the covers of a space up to isomorphism. Remarkably, for a reasonable space X there is a simple *algebraic* classification of covers of X. We will describe this classification later after we define the fundamental group of X (see Chapter 10). It resembles the classical Galois correspondence.

To a reasonable path connected space X, we will associate a group G called its *fundamental group*. Isomorphism classes of covers $p\colon \widetilde{X} \to X$ with $\widetilde{X}$ path connected will correspond[6] to subgroups $K < G$. The cover corresponding to the trivial subgroup $1 < G$ will be called the *universal cover*, and it will cover all the other covers. Here is an example:

EXAMPLE 1.6.1. The fundamental group of $\mathbb{S}^1$ will turn out to be $\mathbb{Z}$. There are two kinds of subgroups of $\mathbb{Z}$:

- For $n \geq 1$, the subgroup $n\mathbb{Z}$. Ths will correspond to the cover $p_n\colon \mathbb{S}^1 \to \mathbb{S}^1$ defined by $p_n(z) = z^n$ for $z \in \mathbb{S}^1$.
- The trivial subgroup $0 < \mathbb{Z}$. This will correspond to the universal cover $p\colon \mathbb{R} \to \mathbb{S}^1$ defined by $p(\theta) = e^{2\pi i/\theta}$ for $\theta \in \mathbb{R}$.

The universal cover $p\colon \mathbb{R} \to \mathbb{S}^1$ covers the cover $p_n\colon \mathbb{S}^1 \to \mathbb{S}^1$ in the following sense. Define $q_n\colon \mathbb{R} \to \mathbb{S}^1$ via $q_n(\theta) = e^{2\pi i\theta/n}$ for $\theta \in \mathbb{R}$. This is a covering space, and we have a factorization

$$\begin{tikzcd} \mathbb{R} \arrow[r, "q_n"] \arrow[rr, bend right, "p"'] & \mathbb{S}^1 \arrow[r, "p_n"] & \mathbb{S}^1. \end{tikzcd}$$ □

[6]There is a small issue with basepoints we are ignoring here to simplify our story.

1.7. Exercises

EXERCISE 1.1. Carefully prove that the following are covering spaces. Let $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

(a) The map $p\colon \mathbb{C} \to \mathbb{C}^\times$ defined by $p(z) = e^z$.
(b) For $n \in \mathbb{Z} \setminus \{0\}$, the map $p\colon \mathbb{C}^\times \to \mathbb{C}^\times$ defined by $p(z) = z^n$. □

EXERCISE 1.2. Prove that the map $p\colon \mathbb{C} \to \mathbb{C}$ defined by $p(z) = z^2$ is not a covering space. □

EXERCISE 1.3. Do the following:

(a) Give an example of a surjective local homeomorphism that is not a covering space.
(b) Give an example of a local homeomorphism $f\colon X \to Y$ and a subset $A \subset X$ such that $f|_A\colon A \to f(A)$ is not a local homeomorphism. □

EXERCISE 1.4. Let $p_1\colon \widetilde{X}_1 \to X_1$ and $p_2\colon \widetilde{X}_2 \to X_2$ be covering spaces. Define $q\colon \widetilde{X}_1 \sqcup \widetilde{X}_2 \to X_1 \sqcup X_2$ via the formula

$$q(z) = \begin{cases} p_1(z) & \text{if } z \in \widetilde{X}_1, \\ p_2(z) & \text{if } z \in \widetilde{X}_2. \end{cases}$$

Prove that $q\colon \widetilde{X}_1 \sqcup \widetilde{X}_2 \to X_1 \sqcup X_2$ is a covering space. Use this construction to find a covering space over a non-connected base that does not have a degree. □

EXERCISE 1.5. Let $p_1\colon \widetilde{X}_1 \to X_1$ and $p_2\colon \widetilde{X}_2 \to X_2$ be covering spaces. Define $q\colon \widetilde{X}_1 \times \widetilde{X}_2 \to X_1 \times X_2$ via the formula $q(z_1, z_2) = (p_1(z_1), p_2(z_2))$. Prove that $q\colon \widetilde{X}_1 \times \widetilde{X}_2 \to X_1 \times X_2$ is a covering space. □

EXERCISE 1.6. In this exercise, you will prove that Exercise 1.5 is false for infinite products. For $n \geq 1$, let $X_n = \mathbb{S}^1$ and $\widetilde{X}_n = \mathbb{R}^1$ and let $p_n\colon \widetilde{X}_n \to X_n$ be the universal cover. Give $\prod_{n\geq 1} \widetilde{X}_n$ and $\prod_{n\geq 1} X_n$ the product topologies, and define a map $p\colon \prod_{n\geq 1} \widetilde{X}_n \to \prod_{n\geq 1} X_n$ via the formula

$$p(z_1, z_2, \ldots) = (p_1(z_1), p_2(z_2), \ldots) \quad \text{for all } (z_1, z_2, \ldots) \in \prod_{n\geq 1} \widetilde{X}_n.$$

Prove that $p\colon \prod_{n\geq 1} \widetilde{X}_n \to \prod_{n\geq 1} X_n$ is not a covering space. □

EXERCISE 1.7. Prove the following:

(a) Let $p\colon \widetilde{X} \to X$ be a cover and let $X' \subset X$ be a subspace. Define $\widetilde{X}' = f^{-1}(X')$ and $p' = p|_{\widetilde{X}'}$. Prove that $p'\colon \widetilde{X}' \to X'$ is a covering space. We will call this the *restriction* of p to X'.
(b) Let X be a locally connected space with connected components $\{X_j\}_{j\in J}$. For each $j \in J$, let $q_j\colon Y_j \to X_j$ be a covering space. Define

$$Y = \bigsqcup_{j\in J} Y_j,$$

and let $q\colon Y \to X$ be the map that for $j \in J$ and $y \in Y_j$ satisfies $q(y) = q_j(y) \in Y_j \subset Y$. Prove that $q\colon Y \to X$ is a covering space.
(c) Construct a counterexample to part (b) in the case where X is not locally connected. □

EXERCISE 1.8. Let $p\colon \widetilde{X} \to X$ be a cover. Prove the following:

(a) Let $f\colon Y \to X$ be a map. Set

$$f^*(\widetilde{X}) = \left\{ (y, \widetilde{x}) \in Y \times \widetilde{X} \mid f(y) = p(\widetilde{x}) \right\},$$

and let $f^*(p)\colon f^*(\widetilde{X}) \to Y$ be the projection onto the first coordinate. Prove that $f^*(p)\colon f^*(\widetilde{X}) \to Y$ is a covering space. We call $f^*(p)\colon f^*(\widetilde{X}) \to Y$ the *pullback* of $p\colon \widetilde{X} \to X$ along $f\colon Y \to X$.
(b) Let $X' \subset X$ be a subspace and let $\iota\colon X' \to X$ be the inclusion. Prove that the covering space $\iota^*(p)\colon \iota^*(\widetilde{X}) \to X'$ is isomorphic to the restriction of $p\colon \widetilde{X} \to X$ to X' discussed in Exercise 1.7. □

EXERCISE 1.9. Let $p\colon \widetilde{X} \to X$ be a covering space such that $p^{-1}(x)$ is finite and nonempty for all $x \in X$. Prove that X is compact Hausdorff if and only if $\widetilde{X}$ is compact Hausdorff. $\square$

EXERCISE 1.10. Let $p\colon \widetilde{X} \to X$ be a cover. This exercise shows that many point-set topological properties of X are reflected in $\widetilde{X}$.

(a) If X is Hausdorff, then prove that $\widetilde{X}$ is Hausdorff.
(b) If X is regular, then prove that $\widetilde{X}$ is regular.
(c) If X is paracompact, then prove that $\widetilde{X}$ is paracompact. Hint: first prove that there is a locally finite collection of closed sets $\{C_i\}_{i \in I}$ of $\widetilde{X}$ such that each C_i is paracompact, and then prove that this implies that $\widetilde{X}$ is paracompact.
(d) If X is metrizable, then prove that $\widetilde{X}$ is metrizable. Hint: Use the Smirnov metrization theorem, which says that a space is metrizable if and only if it is paracompact and locally metrizable. $\square$

CHAPTER 2

Covering spaces: deck transformations and regular covers

We now discuss symmetries of a cover, and single out the covers that have as many symmetries as possible.

2.1. Deck transformations

We defined isomorphisms of covers in §1.5. The isomorphisms from a cover to itself are called deck transformations:

DEFINITION 2.1.1. Let $p\colon \widetilde{X} \to X$ be a covering space. A *deck transformation* of $p\colon \widetilde{X} \to X$. is a covering space isomorphism $f\colon \widetilde{X} \to \widetilde{X}$. These form a group under composition called the *deck group* of $p\colon \widetilde{X} \to X$, denoted $\text{Deck}(p\colon \widetilde{X} \to X)$ or simply $\text{Deck}(\widetilde{X})$. □

Here is an example:

EXAMPLE 2.1.2. Let $p\colon \mathbb{R} \to \mathbb{S}^1$ be the universal cover, so $p(\theta) = e^{2\pi i\theta}$ for all $\theta \in R$. For each $n \in \mathbb{Z}$, we can define a deck transformation $f_n\colon \mathbb{R} \to \mathbb{R}$ via the formula $f_n(\theta) = \theta + n$. □

2.2. Determining the deck group

The key to understanding the deck group is the following lemma, which says that in favorable situations deck transformations are completely determined by what they do to a single point.

LEMMA 2.2.1. *Let $p\colon \widetilde{X} \to X$ be a covering space with $\widetilde{X}$ connected. Let $f, g\colon \widetilde{X} \to \widetilde{X}$ be two deck transformations such that there exists some $z_0 \in \widetilde{X}$ with $f(z_0) = g(z_0)$. Then $f = g$.*

PROOF. Let $E = \left\{z \in \widetilde{X} \mid f(z) = g(z)\right\}$. Our goal is to prove that $E = \widetilde{X}$. By assumption $z_0 \in E$, so since $\widetilde{X}$ is connected it is enough to prove that E is both open and closed.[1] Consider $z \in \widetilde{X}$. We must prove that if $z \in E$ (resp. $z \notin E$) then there is an open neighborhood of z contained in E (resp. disjoint from E). Let U be a trivialized neighborhood of $p(z)$.

Assume first that $z \in E$. Let $\widetilde{U}$ be the sheet above U containing $f(z) = g(z)$. Set $V = f^{-1}(\widetilde{U}) \cap g^{-1}(\widetilde{U})$, so V is an open neighborhood of z with $f(V), g(V) \subset \widetilde{U}$. For $z' \in V$, both $f(z')$ and $g(z')$ are the unique point of $\widetilde{U}$ projecting to $p(z') \in U$, so in particular $f(z') = g(z')$. This implies that $V \subset E$, as desired.

Assume now that $z \notin E$, so $f(z) \neq g(z)$. Let $\widetilde{U}_1$ and $\widetilde{U}_2$ be the sheets above U with $f(z) \in \widetilde{U}_1$ and $g(z) \in \widetilde{U}_2$. Since $f(z) \neq g(z)$, the sheets $\widetilde{U}_1$ and $\widetilde{U}_2$ are distinct and hence disjoint. Set $W = f^{-1}(\widetilde{U}_1) \cap g^{-1}(\widetilde{U}_2)$, so W is an open neighborhood of z with $f(W) \subset \widetilde{U}_1$ and $g(W) \subset \widetilde{U}_2$. Since $\widetilde{U}_1 \cap \widetilde{U}_2 = \emptyset$, this implies that $f(z') \neq g(z')$ for all $z' \in W$, so W is disjoint from E, as desired. □

The following is a typical example of how to use Lemma 2.2.1 to determine the deck group of a covering space:

EXAMPLE 2.2.2 (Universal cover of circle). Let $p\colon \mathbb{R} \to \mathbb{S}^1$ be the universal cover of $\mathbb{S}^1$. For $n \in \mathbb{Z}$, let $f_n\colon \mathbb{R} \to \mathbb{R}$ be the deck transformation defined by the formula $f_n(\theta) = \theta + n$. We claim that

$$\text{Deck}(p\colon \mathbb{R} \to \mathbb{S}^1) = \{f_n \mid n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

To see this, consider an arbitrary deck transformation $f\colon \mathbb{R} \to \mathbb{R}$. Since $p(f(0)) = p(0)$, we must have $f(0) = n$ for some $n \in \mathbb{Z}$. Since $f(0) = f_n(0)$, Lemma 2.2.1 implies that $f = f_n$. □

[1] Note that if X is Hausdorff (like most spaces in this book) it is automatic that E is closed; indeed, if X is Hausdorff then for any continuous maps $f, g\colon Y \to X$ the set of $y \in Y$ with $f(y) = g(y)$ is closed.

2.3. Regular covers

Roughly speaking, a regular cover[2] is a cover with a deck group that is as large as possible. Let $p\colon \widetilde{X} \to X$ be a covering space. The group $\mathrm{Deck}(\widetilde{X})$ acts on $\widetilde{X}$. For $x \in X$, the action of $\mathrm{Deck}(\widetilde{X})$ on $\widetilde{X}$ preserves the fiber $f^{-1}(x)$, so $\mathrm{Deck}(\widetilde{X})$ acts on $f^{-1}(x)$. For $z_1, z_2 \in f^{-1}(x)$, Lemma 2.2.1 implies that if $\widetilde{X}$ is connected then there exists at most one $f \in \mathrm{Deck}(\widetilde{X})$ with $f(z_1) = z_2$. A regular cover is a cover where such an f always exists:

Definition 2.3.1. A *regular cover* is a cover $p\colon \widetilde{X} \to X$ such that for all $x \in X$, the group $\mathrm{Deck}(\widetilde{X})$ acts transitively on $p^{-1}(x)$, i.e., for all $\widetilde{x}_1, \widetilde{x}_2 \in p^{-1}(x)$ there exists some $f \in \mathrm{Deck}(\widetilde{X})$ with $f(\widetilde{x}_1) = \widetilde{x}_2$. A cover that is not regular is *irregular*. □

Example 2.3.2 (Universal cover of circle). The calculation in Example 2.2.2 shows that the universal cover $p\colon \mathbb{R} \to \mathbb{S}^1$ is regular. □

Example 2.3.3 (Trivial cover). Let X be a space and $\mathcal{I}$ be a discrete set. Consider the trivial cover $p\colon X \times \mathcal{I} \to X$. Set $G = \mathrm{Deck}(p\colon X \times \mathcal{I} \to X)$. For each bijection $\sigma\colon \mathcal{I} \to \mathcal{I}$, we can define an element $f_\sigma \in G$ via the formula $f_\sigma(x, i) = (x, \sigma(i))$. These elements act transitively on the fibers, so $p\colon X \times \mathcal{I} \to X$ is regular. One can check that all elements of G are of the form f_σ if X is connected (see Exercise 2.8). □

Remark 2.3.4. It is actually harder to show that a cover is irregular. We will give an example below in Example 2.7.3. □

2.4. More examples of regular covers

Most of the covers we have seen so far are regular:

Example 2.4.1 (Degree n cover of circle). Let $p_n\colon \mathbb{S}^1 \to \mathbb{S}^1$ be the cover defined by the formula $p_n(z) = z^n$. We claim that $p_n\colon \mathbb{S}^1 \to \mathbb{S}^1$ is a regular cover with deck group isomorphic to the cyclic group C_n of order n. Indeed, let $G = \mathrm{Deck}(p_n\colon \mathbb{S}^1 \to \mathbb{S}^1)$. Let $f \in G$ be the map $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ defined by the formula $f(z) = e^{2\pi i/n} z$. The element f has order n and its powers act transitively on the fiber $p_n^{-1}(1)$, which equals the n^{th} roots of unity. This implies that the cover is regular.

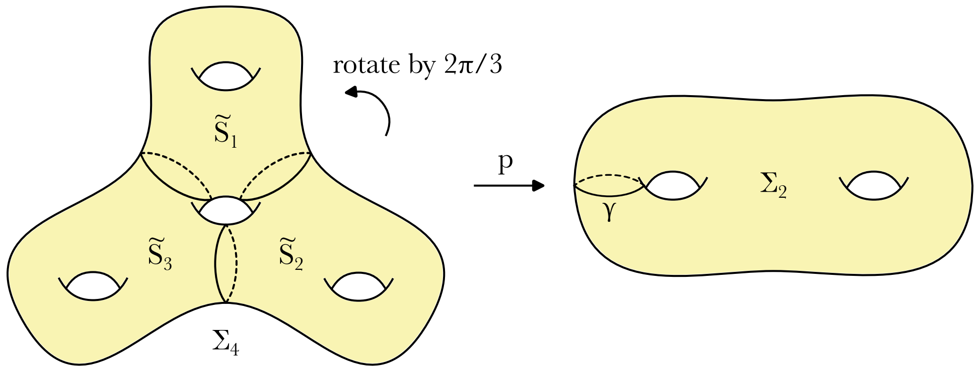
The same argument using Lemma 2.2.1 that we used in Example 2.2.2 shows that G is the cyclic group of order n generated by f. We repeat that argument one more time: for $g \in G$, we have $g(1) = e^{2\pi k i/n}$ for some $k \in \mathbb{Z}$ (well-defined modulo n). Since g and f^k both take 1 to $e^{2\pi k i/n}$, it follows from Lemma 2.2.1 that $g = f^k$. □

Example 2.4.2 (Cosets of discrete subgroups). Let $\mathbf{G}$ be a topological group and let $\mathbf{H} < \mathbf{G}$ be a discrete subgroup. Then the projection $p\colon \mathbf{G} \to \mathbf{G}/\mathbf{H}$ is a regular cover. Indeed, for $h \in \mathbf{H}$ define $f_h\colon \mathbf{G} \to \mathbf{G}$ via the formula $f_h(g) = gh$. Then $f_h \in \mathrm{Deck}(p\colon \mathbf{G} \to \mathbf{G}/\mathbf{H})$, and the f_h act transitively on the fibers of $p\colon \mathbf{G} \to \mathbf{G}/\mathbf{H}$. If $\mathbf{G}$ is connected, then by Lemma 2.2.1 this is the entire deck group, so $\mathrm{Deck}(p\colon \mathbf{G} \to \mathbf{G}/\mathbf{H}) \cong \mathbf{H}$. As a special case, the deck group of the cover $p\colon \mathbb{R}^n \to \mathbb{T}^n$ is the group $\mathbb{Z}^n$, which acts on $\mathbb{R}^n$ by translations. □

Example 2.4.3 (Real projective space). The cover $p\colon \mathbb{S}^n \to \mathbb{RP}^n$ is regular. Indeed, the map $f\colon \mathbb{S}^n \to \mathbb{S}^n$ defined by $f(z) = -z$ is an element of the deck group that swaps the two elements in the fiber over any point of $\mathbb{RP}^n$. By Lemma 2.2.1, the deck group of $p\colon \mathbb{S}^n \to \mathbb{RP}^n$ is the cyclic group C_2 of order 2 generated by f. □

Example 2.4.4 (Cover of surface). Consider the covering space $p\colon \Sigma_4 \to \Sigma_2$ from Example 1.3.5. Set $G = \mathrm{Deck}(p\colon \Sigma_4 \to \Sigma_2)$. There is a deck transformation $f \in G$ that rotates Σ_4 by $2\pi/3$ as follows:

[2] These are also often called *normal covers*.



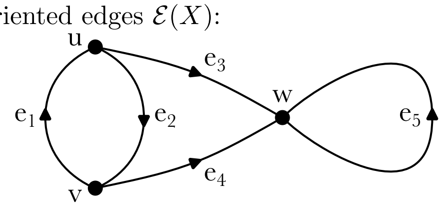
The element f has order 3 and its powers act transitively on all the fibers. This implies that the cover is regular, and also by Lemma 2.2.1 that G is the cyclic group of order 3 generated by f. $\square$

EXAMPLE 2.4.5 (Roots of square-free polynomials). The degree-n covering space $p\colon \mathrm{RPoly}_n^{\mathrm{sf}} \to \mathrm{Poly}_n^{\mathrm{sf}}$ discussed in Example 1.4.4 is regular for $n = 2$ (see Exercise 2.7), but is irregular for $n \geq 3$. We do not have the technology to prove this yet (see Exercise 9.9 of Chapter 9 for the proof), but it should not be surprising. Indeed, if it was regular then the deck group G would act simply transitively on the roots of every degree-n polynomial with distinct roots, and if such a canonical group action existed then we would surely teach about it in elementary abstract algebra classes.[3] $\square$

We will meet examples of irregular covers whose irregularity is easy to verify below when we discuss covers of graphs.

2.5. Graphs

Graphs[4] provide a rich source of examples of covering spaces. Recall that a graph X is a set of vertices $\mathcal{V}(X)$ connected by oriented edges $\mathcal{E}(X)$:



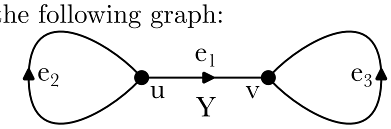
For each vertex $v \in \mathcal{V}(X)$, the *degree* of v, denoted $\deg(v)$, is the number of edges that start or end at v. If an edge is a loop based at v, it contributes 2 to $\deg(v)$. We call X a *finite* graph if the sets $\mathcal{V}(X)$ and $\mathcal{E}(X)$ are both finite. In this case, it is clear how to regard X as a topological space: the vertices are a discrete set of points, and the edges are copies of $I = [0, 1]$ that are glued to the vertices. More generally, we say that X is *locally finite* if for each $v \in \mathcal{V}(X)$ its degree $\deg(v)$ is finite. For locally finite graphs X, it is also clear how to regard X as a topological space.

REMARK 2.5.1. In the general case, we regard X as 1-dimensional CW complex. See Essay K in Volume 1 for more details. $\square$

2.6. Maps of graphs

Let X and Y be graphs. To define a continuous map $\phi\colon X \to Y$, we must specify where ϕ sends each vertex and edge. This is particularly easy to do if we require our map to take vertices to vertices and oriented edges to oriented edges, which will suffice for the examples in this section. This is best explained by an example:

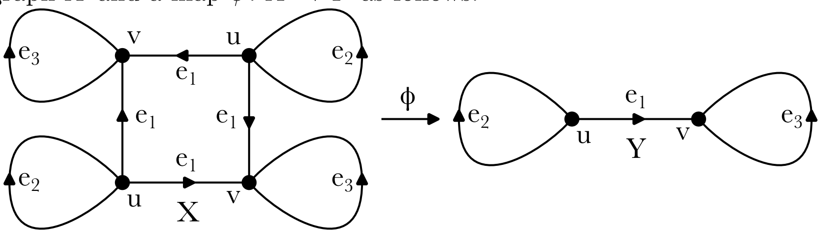
EXAMPLE 2.6.1. Let Y be the following graph:



[3]For $n = 2$, this group action just exchanges the two roots.

[4]Our conventions about graphs are that unless otherwise specified they are oriented and we allow multiple edges and loops.

We specify a graph X and a map $\phi\colon X \to Y$ as follows:

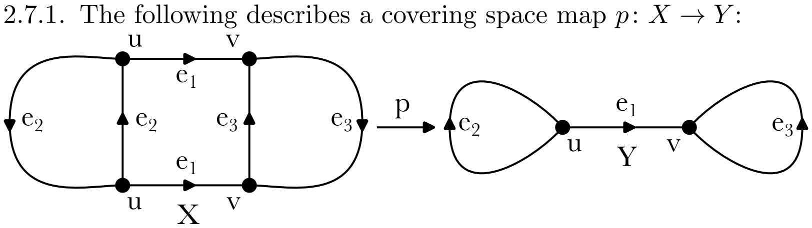


Here we label the vertices and oriented edges of X by the vertices and oriented edges they map to. What this map does is map each edge of the central square of X to the single non-loop edge of Y and map each loop in X to the appropriate loop in Y. The interiors of the edges are identified with copies of the open interval $(0,1)$, and ϕ respects these identifications. □

2.7. Covers of graphs

The above example is not a covering map. The problem is that it is not a local homeomorphism at the vertices. What is needed for a covering map is informally that for each vertex "the same edges enter and exist as in the target". Here is an example of a covering map with the same Y as above but a different X:

EXAMPLE 2.7.1. The following describes a covering space map $p\colon X \to Y$:

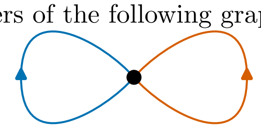


This is a covering space map since:

- for both vertices of X mapping to u, one edge exits mapping to e_1, one edge exits mapping to e_2, and one edge enters mapping to e_2; and
- for both vertices of X mapping to v, one edge enters mapping to e_1, one edge exits mapping to e_3, and one edge enters mapping to e_3; and

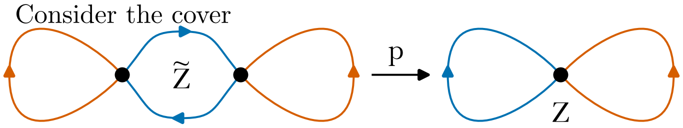
This is a regular cover with deck group isomorphic to C_2. The generator of C_2 acts on X by the involution that swaps the two vertices labeled u, the two vertices labeled v, and for $i = 1, 2, 3$ the two oriented edges labeled e_i. □

We now give several different covers of the following graph Z:



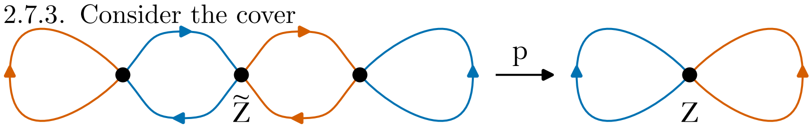
Since Z has only one vertex, there is no need to give it a name since all vertices of a cover map to that one vertex. We also use colors rather than letters to distinguish the two edges of Z, and label the edges in the domain of our covering space maps by coloring them with the appropriate colors.

EXAMPLE 2.7.2. Consider the cover



This is a degree 2 regular cover. The deck group is isomorphic to C_2, and acts on $\widetilde{Z}$ by the involution that swaps the two vertices, the two orange loops, and the two blue edges. □

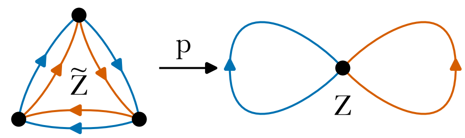
EXAMPLE 2.7.3. Consider the cover



This is an irregular cover with trivial deck group. To see this, note that a deck transformation of

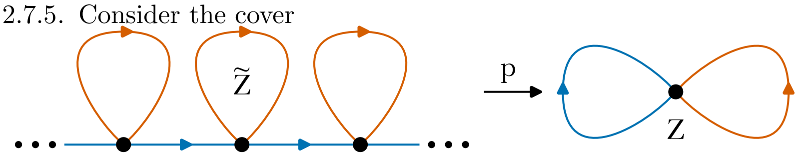
$\widetilde{Z}$ must take vertices to vertices and oriented edges to oriented edges, and also must preserve the coloring on the edges. Since there is only one orange loop in $\widetilde{Z}$, any deck transformation must fix that orange loop and therefore be the identity. □

Example 2.7.4. Consider the cover



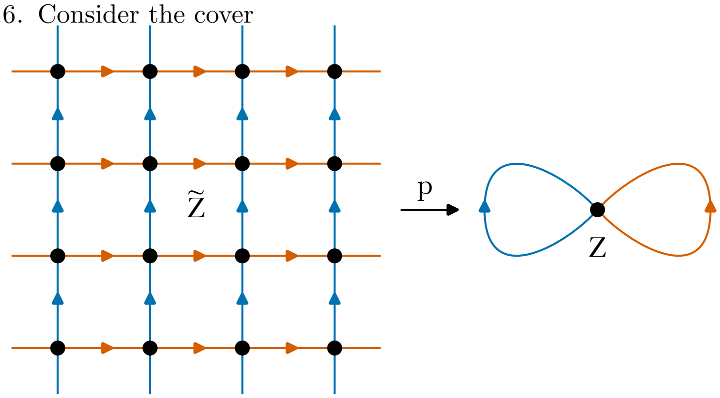
This ia a degree 3 regular cover. The deck group is C_3, which acts on $\widetilde{Z}$ by rotations. □

Example 2.7.5. Consider the cover



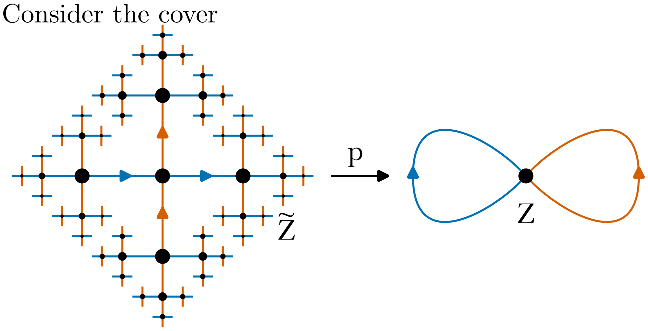
This is an infinite degree regular cover. The deck group is isomorphic to $\mathbb{Z}$, which acts on $\widetilde{Z}$ as translations. □

Example 2.7.6. Consider the cover



Here $\widetilde{Z}$ is the graph embedded in $\mathbb{R}^2$ whose vertices are at $\mathbb{Z}^2$ and whose edges are horizontal and vertical lines. This is an infinite degree regular cover. The deck group is isomorphic to $\mathbb{Z}^2$, which acts on $\widetilde{Z} \subset \mathbb{R}^2$ via the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ by integer translations. □

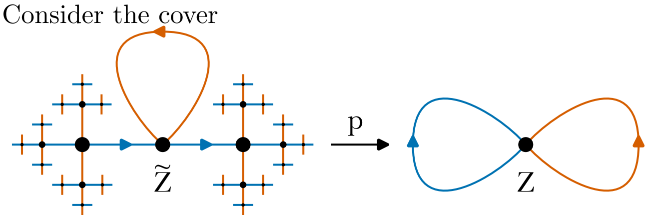
Example 2.7.7. Consider the cover



The indicated pattern in the domain repeats infinitely often, making it an infinite degree-4 tree.[5] The horizontal edges are oriented going right, and the vertical edges are oriented going up. This is an infinite degree regular cover (we leave this as Exercise 2.10). □

[5] A tree is a nonempty graph with no cycles, that is, no embedded circles.

Example 2.7.8. Consider the cover



This is an infinite degree irregular cover with trivial deck group. To see this, note that just like in Example 2.7.3 any deck transformation of $\widetilde{Z}$ must fix the unique orange loop in $\widetilde{Z}$, and thus must be the identity. □

2.8. Covering space actions

Let $p\colon \widetilde{X} \to X$ be a cover with deck group G. The group G acts on $\widetilde{X}$. If $\widetilde{X}$ is connected, then Lemma 2.2.1 says that action is *free*, i.e., that for all $z \in \widetilde{X}$ the stabilizer subgroup G_z is trivial. In fact, even more is true, and this section is devoted to studying this action.

Remark 2.8.1. In this book, a group action is always assumed to preserve any structure a set has. In particular, an action of a group G on a topological space Z is assumed to be continuous. In other words, for all $g \in G$ the map $Z \to Z$ that multiplies points by g is assumed to be a homeomorphism. □

The following lemma isolates the key property of the action of the deck group of a connected cover:

Lemma 2.8.2. *Let $p\colon \widetilde{X} \to X$ be a covering space with $\widetilde{X}$ connected, let $G = \mathrm{Deck}(p\colon \widetilde{X} \to X)$, and let $z \in \widetilde{X}$. Then there is an open neighborhood V of z whose translates $\{g{\cdot}V \mid g \in G\}$ are all disjoint.*

Proof. Let U be a trivialized neighborhood of $p(z)$ and let $\widetilde{U}$ be the sheet lying above U with $z \in \widetilde{U}$. We claim that $V = \widetilde{U}$ has the indicated property. Indeed, let $g_1, g_2 \in G$ satisfy $(g_1{\cdot}\widetilde{U}) \cap (g_2{\cdot}\widetilde{U}) \neq \emptyset$. We must prove that $g_1 = g_2$. Pick $z_1, z_2 \in \widetilde{U}$ with $g_1{\cdot}z_1 = g_2{\cdot}z_2$. Since the action of G preserves the fibers of $p\colon \widetilde{X} \to X$, the points $z_1, z_2 \in \widetilde{U}$ must lie in the same fiber. Since the restriction of $p\colon \widetilde{X} \to X$ to $\widetilde{U}$ is injective, this implies that $z_1 = z_2$. Letting $w = z_1 = z_2$ be this common value, we have $g_1{\cdot}w = g_2{\cdot}w$. Lemma 2.2.1 now implies that $g_1 = g_2$, as desired. □

Actions satisfying the conclusions of this lemma are important, so we give them a special name:

Definition 2.8.3. A *covering space action* is an action of a group G on a space Z such that for all $z \in Z$, there exists an open neighborhood V of z such that the translates $\{g{\cdot}V \mid g \in G\}$ are all disjoint. □

Remark 2.8.4. All covering space actions are free. If G is finite and Z is Hausdorff, then the converse is true: all free action of G on Z are covering space actions (see Exercise 2.4). □

2.9. Quotients by covering space actions

Let G be a group and let Z be a space equipped with a left action of G. Endow the quotient[6] X/G with the quotient topology. In other words, if $q\colon Z \to Z/G$ is the projection then a set $U \subset Z/G$ is open if and only if $q^{-1}(U)$ is open. If the action of G on Z is a covering space action, then the quotient map $q\colon Z \to Z/G$ is a regular covering space:

Lemma 2.9.1. *Let G be a group acting a space Z by a covering space action. Then quotient map $q\colon Z \to Z/G$ is a regular covering space. Moreover, if Z is connected then $G = \mathrm{Deck}(q\colon Z \to Z/G)$.*

[6]This is potentially confusing notation since G is acting on the left. A purist would insist that Z/G is the quotient of Z by an action of G on the right, and denote the quotient of Z by an action of G on the left by $G \backslash Z$. However, our notation is common and traditional, and we will follow it. There will be a few situations where we will have both left and right actions, and we will work hard to be clear about what our notation means in those cases.

PROOF. Consider $x \in Z/G$. Write $x = q(z)$ with $z \in Z$. Let V be an open neighborhood of z such that the sets in the G-orbit of V are disjoint. Set $U = q(V)$. We have

$$q^{-1}(U) = \bigcup_{g \in G} g \cdot V.$$

Since each $g \cdot V$ is open, it follows that $q^{-1}(U)$ is open and thus by definition U is open. The $g \cdot V$ are disjoint open subsets of Z and each projects homeomorphically onto U. We conclude that U is a trivialized neighborhood of x and the $g \cdot V$ are the sheets lying above U. This implies that $q \colon Z \to Z/G$ is a covering space. By construction, the action of G on Z is by deck transformations of $q \colon Z \to Z/G$, so $G < \mathrm{Deck}(q \colon Z \to Z/G)$. This action is transitive on fibers, so if Z is connected then Lemma 2.2.1 implies that $G = \mathrm{Deck}(q \colon Z \to Z/G)$. □

All of our examples of regular covering spaces could have been constructed using Lemma 2.8.2. For instance, C_2 acts on $\mathbb{S}^n$ via the antipodal map $z \mapsto -z$. This is a free action, so since C_2 is finite it is a covering space action (c.f. Remark 2.8.4). We could have defined $\mathbb{RP}^n = \mathbb{S}^n / C_2$ and identified the covering space $p \colon \mathbb{S}^n \to \mathbb{RP}^n$ with the quotient projection. This would be a little artificial, but here is an example where this point of view is essential:

EXAMPLE 2.9.2 (Configuration space). Let X be any space. The *ordered configuration space* of n points on X is the space[7]

$$\mathrm{PConf}_n(X) = \left\{ (x_1, \ldots, x_n) \in X^{\times n} \mid x_i \neq x_j \text{ for all distinct } 1 \leq i, j \leq n \right\}.$$

Topologize this as a subspace of $X^{\times n}$. The symmetric group $\mathfrak{S}_n$ on n letters acts on $\mathrm{Conf}_n(X)$ via the formula

$$\sigma \cdot (x_1, \ldots, x_n) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}) \quad \text{for } \sigma \in \mathfrak{S}_n \text{ and } (x_1, \ldots, x_n) \in \mathrm{PConf}_n(X).$$

The inverses are there to make this a left action.[8] This is a free action since the x_i are all distinct, and thus since $\mathfrak{S}_n$ is finite it is a covering space action. The *configuration space* of n points on X is the quotient $\mathrm{Conf}_n(X) = \mathrm{PConf}_n(X)/\mathfrak{S}_n$. Points of $\mathrm{Conf}_n(X)$ can be viewed as unordered sets $\{x_1, \ldots, x_n\}$ of n distinct points in X. The projection $p \colon \mathrm{PConf}_n(X) \to \mathrm{Conf}_n(X)$ is a regular covering space. □

2.10. Exercises

EXERCISE 2.1. Prove the following:

(a) Let G be a group acting on a space X by a covering space action. Let Y be a subspace of X that is preserved by the action of G, so G acts on Y. Prove that the action of G on Y is a covering space action.

(b) For $i = 1, 2$, let X_i be a space and let G_i be a group acting on X_i by a covering space action. Prove that the action of $G_1 \times G_2$ on $X_1 \times X_2$ is a covering space action. □

EXERCISE 2.2. Construct a covering space action of $C_2 \times C_2$ on a compact oriented surface Σ. □

EXERCISE 2.3. Let $\alpha \in \mathbb{R}$ be an irrational number. Let $G \cong \mathbb{Z}$ be an infinite cyclic group generated by $s \in G$. Let G act on $\mathbb{S}^1$ via the formula

$$t \cdot z = e^{2\pi i \alpha} z \quad \text{for } z \in \mathbb{S}^1.$$

Let $p \colon \mathbb{S}^1 \to \mathbb{S}^1/G$ be the quotient map.

(a) Prove directly that p is not a covering space.

(b) Prove that $\mathbb{S}^1/G$ is not Hausdorff. □

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[7]This is sometimes also called the *pure configuration space*, which is why it is written $\mathrm{PConf}_n(X)$.

[8]This is the same reason that inverses appear in the action of $\mathrm{GL}(V)$ on the dual of a vector space V.

EXERCISE 2.4. Let X be a Hausdorff space and let G be a group acting freely on X. For all $x \in X$, assume that there is an open neighborhood U of x such that the set[9]

$$\{g \in G \mid g{\cdot}U \cap U \neq \emptyset\}$$

is finite. Prove that the action of G on X is a covering space action. The above condition is automatically satisfied if G is finite, so this shows that a free action of a finite group on a Hausdorff space is a covering space action. $\square$

EXERCISE 2.5. Let Z be a Hausdorff topological space and let G be a group acting on Z by a covering space action. Assume that Z/G is Hausdorff (cf. Exercise 2.11), and let $x, y \in Z$ be two points that project to different points of Z/G. Prove that there exist open neighborhoods U of x and V of y such that the translates $\{g{\cdot}U \mid g \in G\}$ and $\{g{\cdot}V \mid g \in G\}$ are all disjoint from each other. $\square$

EXERCISE 2.6. Let Z be a first countable topological space. Let G be a group acting on Z by a covering space action. Assume that Z/G is Hausdorff (cf. Exercise 2.11), and let $K \subset Z$ be compact. Prove that the set

$$\{g \in G \mid g{\cdot}K \cap K \neq \emptyset\}$$

is finite. Hint: the previous exercise will be useful. $\square$

EXERCISE 2.7. Let $p\colon \widetilde{X} \to X$ be a degree 2 cover. Prove that $\widetilde{X}$ is a regular cover. $\square$

EXERCISE 2.8. Let X be a space and $\mathcal{I}$ be a discrete set, and let $p\colon X \times \mathcal{I} \to X$ be the trivial cover.

(a) If X is connected, prove that all elements of the deck group of $p\colon X \times \mathcal{I} \to X$ are of the form $f_\sigma(x, i) = (x, \sigma(i))$ for some bijection $\sigma\colon \mathcal{I} \to \mathcal{I}$.
(b) If X is not connected, construct elements of the deck group that are not of this form. $\square$

EXERCISE 2.9. Let Z be a graph with one vertex and two edges. Construct degree 4 covers $p\colon \widetilde{Z} \to Z$ and $q\colon \widetilde{Z}' \to Z$ with $\widetilde{Z}$ and $\widetilde{Z}'$ path connected such that $p\colon \widetilde{Z} \to Z$ is regular and $q\colon \widetilde{Z}' \to Z$ is irregular. $\square$

EXERCISE 2.10. Verify that the cover in Example 2.7.7 is regular. $\square$

EXERCISE 2.11. In this exercise you will see that non-Hausdorff spaces can have Hausdorff covers. Set $X = \mathbb{R}^2 \setminus 0$. Let $G = \mathbb{Z}$ with generator $t = 1$. Define an action of G on X by letting

$$t^n{\cdot}(x, y) = (2^n x, 2^{-n} y) \quad \text{for all } (x, y) \in X \text{ and } n \in \mathbb{Z}.$$

Prove the following:

(a) The action of G on X is a covering space action.
(b) The quotient X/G is not Hausdorff. $\square$

EXERCISE 2.12. In this exercise, you will see that the composition of two covering maps is not necessarily a covering map (though later we will see that this does hold for reasonable spaces). For $n \geq 1$, let $C_n \subset \mathbb{R}^2$ be the circle of radius $1/n$ with center $(0, 1/n)$. Let $X = \cup_{n=1}^{\infty} C_n$, topologized as a subspace of $\mathbb{R}^2$. This is sometimes called the "earring space" or the "shrinking wedge of circles". Let $p\colon Y \to X$ be the regular cover of X with deck group $\mathbb{Z}$ shown here:

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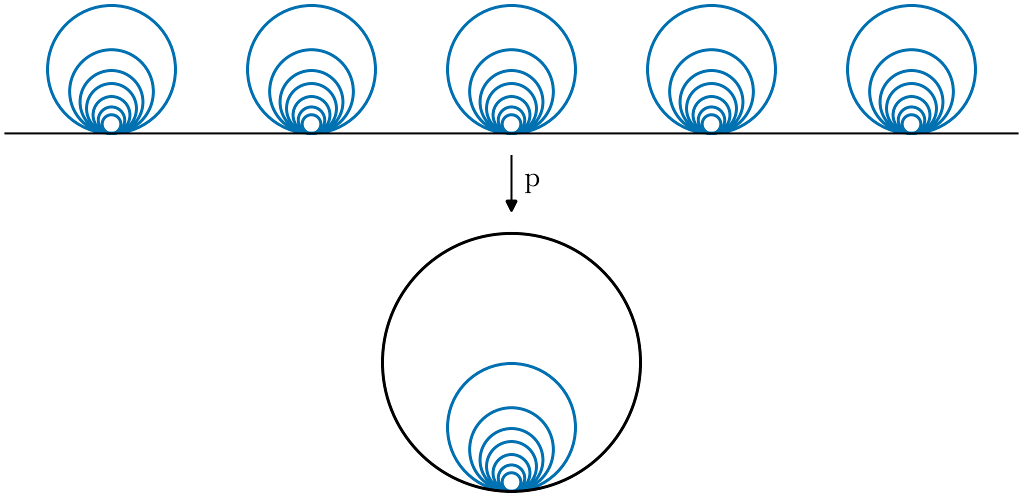
[9]Some authors call actions satisfying this property "properly discontinuous", but the literature contains multiple non-equivalent definitions of what it means for an action to be properly discontinuous, so we prefer to not use this term.



Construct a degree 2 cover $q\colon Z \to Y$ such that the composition $p \circ q\colon Z \to X$ is *not* a covering space. $\square$

CHAPTER 3

Covering spaces: lifting paths and homotopies

This chapter studies lifting problems, which play a key role in both the classification of covering spaces and their applications. As an application, we develop the theory of winding numbers and degrees of maps from $\mathbb{S}^1$ to iteself.

3.1. Lifting problems in general

Let $p\colon \widetilde{X} \to X$ be a covering space and let $f\colon Y \to X$ be a map. A *lift* of f through p is a map $\widetilde{f}\colon Y \to \widetilde{X}$ such that the diagram

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\widetilde{f}"] \arrow[r, "f"] & X \end{tikzcd}$$

commutes, i.e., such that $f = p \circ \widetilde{f}$.

Example 3.1.1. A deck transformation $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ is a lift of the covering space map $p\colon \widetilde{X} \to X$ itself:

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ \widetilde{X} \arrow[ur, "\widetilde{f}"] \arrow[r, "p"] & X \end{tikzcd}$$

Of course, it is possible that a lift of $p\colon \widetilde{X} \to X$ to a map $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ exists such that $\widetilde{f}$ is not a homeomorphism, so not all such lifts are deck transformations. $\square$

A lift might or might not exist. However, just like a deck transformation if a lift exists then under favorable hypotheses it is determined by what it does to a single point:

Lemma 3.1.2. *Let $p\colon \widetilde{X} \to X$ be a covering space and let $f\colon Y \to X$ be a map. Assume that Y is connected. Let $\widetilde{f}_1, \widetilde{f}_2\colon Y \to \widetilde{X}$ be two lifts of f through p such that there is some $y_0 \in Y$ with $\widetilde{f}_1(y_0) = \widetilde{f}_2(y_0)$. Then $\widetilde{f}_1 = \widetilde{f}_2$.*

Proof. The proof is identical to that of Lemma 2.2.1, which is the analgous result for deck transformations. $\square$

3.2. Sections

We now discuss a special kind of lifting problem. Let $p\colon \widetilde{X} \to X$ be a covering space. A *section* of p is a lift $\sigma\colon X \to \widetilde{X}$ of the identity map $\mathbb{1}_X\colon X \to X$ through p. In other words, $\sigma\colon X \to \widetilde{X}$ is a map such that $p(\sigma(x)) = x$ for all $x \in X$.

Example 3.2.1. Let X be a space and $\mathcal{I}$ be a discrete set. Consider the trivial cover $p\colon X \times \mathcal{I} \to X$. Let $i_0 \in \mathcal{I}$, and define $\sigma\colon X \to X \times \mathcal{I}$ via the formula $\sigma(x) = (x, i_0)$. Then σ is a section. $\square$

The above example might be unsatisfying; however, covers typically have no sections:

Lemma 3.2.2. *Let $p\colon \widetilde{X} \to X$ be a covering space with $\widetilde{X}$ connected. Assume that there exists a section $\sigma\colon X \to \widetilde{X}$. Then $p\colon \widetilde{X} \to X$ is a homeomorphism, and in particular has degree 1.*

Proof. It is enough to prove that $p\colon \widetilde{X} \to X$ has degree 1; indeed, this will imply that p is a bijection, and since covering space maps are open maps we will be able to conclude that p is a homeomorphism.[1] To prove that p has degree 1, it is enough to prove that σ is surjective. Since $\widetilde{X}$ is connected, this will follow if we prove that the image of σ is both open and closed.

Consider $x \in X$. It is enough to prove that $\sigma(x)$ lies in the interior of $\sigma(X)$ and that all points of $p^{-1}(x)$ other than $\sigma(x)$ lie in the interior of $\widetilde{X} \setminus \sigma(X)$. Let U be a trivialized neighborhood of x and let $\{\widetilde{U}_i\}_{i \in \mathcal{I}}$ be the sheets lying above U. Let $i_0 \in \mathcal{I}$ be such that $\sigma(x) \in \widetilde{U}_{i_0}$. Naively, one might expect that $\sigma(U) = \widetilde{U}_{i_0}$; however, without further assumptions (like U being connected, which could only be ensured if X is locally connected) this need not hold.

However, let $V = \sigma^{-1}(\widetilde{U}_{i_0})$. The set V is also a trivialized neighborhood of x. Let $\{\widetilde{V}_i\}_{i \in \mathcal{I}}$ be the sheets lying above V, enumerated such that $\widetilde{V}_i \subset \widetilde{U}_i$ for all $i \in \mathcal{I}$. Then $\widetilde{V}_{i_0} = \sigma(V)$ is an open neighborhood of $\sigma(x)$ lying in $\sigma(X)$. Also, the union of the $\widetilde{V}_i$ with $i \neq i_0$ is an open neighborhood of $p^{-1}(x) \setminus \sigma(x)$ lying in $\widetilde{X} \setminus \sigma(X)$. The lemma follows. $\square$

3.3. Formulas for roots of polynomials

We explain an interesting application of Lemma 3.2.2. As discussed in Example 1.4.4, let $\mathrm{Poly}_n^{\mathrm{sf}}$ be the space of monic degree-n polynomials without repeated roots, let $\mathrm{RPoly}_n^{\mathrm{sf}}$ be the space of pairs (f, x) with $f \in \mathrm{Poly}_n^{\mathrm{sf}}$ and $f(x) = 0$, and let $p\colon \mathrm{RPoly}_n^{\mathrm{sf}} \to \mathrm{Poly}_n^{\mathrm{sf}}$ be the map $p(f, x) = f$, so p is a degree n covering space. We start by proving that $\mathrm{RPoly}_n^{\mathrm{sf}}$ is path-connected:

Lemma 3.3.1. *For $n \geq 1$, the space $\mathrm{RPoly}_n^{\mathrm{sf}}$ is path-connected.*

Proof. Let (f_1, x_1) and (f_2, x_2) be two points of $\mathrm{RPoly}_n^{\mathrm{sf}}$. We want to find a path from (f_1, x_1) to (f_2, x_2). Since the polynomial $f_i(z)$ has no repeated roots, we can factor it as

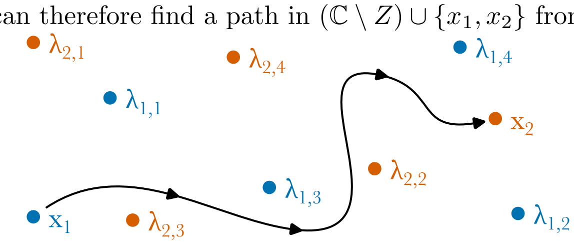
$$f_i(z) = (z - x_i)(z - \lambda_{i,1}) \cdots (z - \lambda_{i,n-1}).$$

Here the $\lambda_{i,j}$ are distinct complex numbers that are different from x_i. We remark that the ordering on $\{\lambda_{i,1}, \ldots, \lambda_{i,n-1}\}$ is not canonical. We can move (f_i, x_i) in $\mathrm{RPoly}_n^{\mathrm{sf}}$ by moving x_i and the $\lambda_{i,j}$ while keeping them distinct. Moving x_1 and the $\lambda_{1,j}$ slightly, we can ensure that the numbers

$$Z = \{x_1, \lambda_{1,1}, \ldots, \lambda_{1,n-1}, x_2, \lambda_{2,1}, \ldots, \lambda_{2,n-1}\}$$

are all distinct. We will now move the points $x_1, \lambda_{1,1}, \ldots, \lambda_{1,n-1}$ to $x_2, \lambda_{2,1}, \ldots, \lambda_{2,n-1}$ one at a time, starting with x_1.

Since removing finitely many points from $\mathbb{C}$ does not disconnect it, the space $(\mathbb{C} \setminus Z) \cup \{x_1, x_2\}$ is path-connected. We can therefore find a path in $(\mathbb{C} \setminus Z) \cup \{x_1, x_2\}$ from x_1 to x_2:



By moving x_1 along this path, we move (f_1, x_1) and reduce ourselves to the case where $x_1 = x_2$. Next, the space $(\mathbb{C} \setminus Z) \cup \{\lambda_{1,1}, \lambda_{2,1}\}$ is path-connected, so we can find a path in it from $\lambda_{1,1}$ to $\lambda_{2,1}$. By moving $\lambda_{1,1}$ along this path, we move (f_1, x_1) and reduce ourselves to the case where $x_1 = x_2$ and $\lambda_{1,1} = \lambda_{2,1}$. Repeating this process, we move (f_1, x_1) to (f_2, x_2). $\square$

Combining this with Lemma 3.2.2, we deduce the following:

Corollary 3.3.2. *For $n \geq 2$, the covering space $p\colon \mathrm{RPoly}_n^{\mathrm{sf}} \to \mathrm{Poly}_n^{\mathrm{sf}}$ has no section.*

[1] We remark that there is a simpler proof that p has degree 1 if $\widetilde{X}$ is path connected. Consider $x \in X$. Set $z_1 = \sigma(x)$, so $z_1 \in p^{-1}(x)$. Consider $z_2 \in p^{-1}(x)$. We must prove that $z_1 = z_2$. Let $\widetilde{\gamma}\colon [0,1] \to \widetilde{X}$ be a path with $\widetilde{\gamma}(0) = z_1$ and $\widetilde{\gamma}(1) = z_2$. Set $\gamma = p \circ \gamma$, so $\gamma\colon [0,1] \to X$ is a path in X from $x = p(z_1)$ to $x = p(z_2)$. Define $\widetilde{\gamma}' = \sigma \circ \gamma$. Both $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ are lifts of $\gamma\colon [0,1] \to X$ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{\gamma}'(0) = z_1$, so by Lemma 3.1.2 we have $\widetilde{\gamma} = \widetilde{\gamma}'$. We conclude that $z_1 = \widetilde{\gamma}(1) = \widetilde{\gamma}'(1) = z_2$, as desired.

Why is this interesting? Recall that $\mathrm{Poly}_n^{\mathrm{sf}} \subset \mathbb{C}^n$, where

$$f(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n \in \mathrm{Poly}_n^{\mathrm{sf}}$$

is identified with $(a_1, \ldots, a_n) \in \mathbb{C}^n$. A section of $p\colon \mathrm{RPoly}_n^{\mathrm{sf}} \to \mathrm{Poly}_n^{\mathrm{sf}}$ is thus a function that takes as input the coefficients of a polynomial $f(z)$ with no repeated roots and returns (f, x), where $x \in \mathbb{C}$ is a root of $f(z)$. In other words, it is a continuous "formula" for the roots of a degree-n polynomial.

The fact that these do not exist for $n \geq 2$ might seem to contradict that we do in fact have such formulas in low degrees. For instance, we have the quadratic formula: for a quadratic polynomial $f(z) = z^2 + bz^1 + c$, its roots are

$$\frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

The point here is that this is not really a well-defined function because of the $\pm$, and indeed there is no way to choose a canonical square root of a complex number in a continuous way. In Essay A, we will see that this forms the germ of a beautiful proof of Arnold of the classical fact (usually proved with Galois theory) that there is no elementary formula for the roots of a degree-n polynomial for $n \geq 5$, even if you allow multivalued k^{th} roots like in the quadratic formula.

3.4. Lifting paths

Once the basic theory of the fundamental group is in place, we will be able to give a satisfying necessary and sufficient condition for a lift to exist, at least for reasonable spaces (see Chapter 9). Before we can do this, we need to solve some important special cases. As notation, let $I = [0, 1]$. A *path* in a space X is a map $\gamma\colon I \to X$. The *initial point* of γ is $\gamma(0)$ and the *terminal point* is $\gamma(1)$, and we say that γ goes from $\gamma(0)$ to $\gamma(1)$. Paths can always be lifted:

LEMMA 3.4.1. *Let $p\colon \widetilde{X} \to X$ be a covering space and let $\gamma\colon I \to X$ be a path. For all $\widetilde{x}_0 \in \widetilde{X}$ with $p(\widetilde{x}_0) = \gamma(0)$, there exists a unique lift $\widetilde{\gamma}\colon I \to \widetilde{X}$ of γ through p with $\widetilde{\gamma}(0) = \widetilde{x}_0$.*

PROOF. Uniqueness follows from Lemma 3.1.2, so we must only prove existence. Using the Lebesgue number lemma,[2] we can partition I into subintervals

$$0 = \epsilon_1 < \epsilon_2 < \cdots < \epsilon_n = 1$$

such that for all $1 \leq k < n$ the image $\gamma([\epsilon_k, \epsilon_{k+1}])$ is contained in a trivialized open set in X. We construct our lift $\widetilde{\gamma}$ inductively as follows.

First, define $\widetilde{\gamma}(0) = \widetilde{x}_0$. Next, assume that for some $1 \leq k < n$ we have constructed a lift $\widetilde{\gamma}\colon [0, \epsilon_k] \to \widetilde{X}$ of $\gamma|_{[0,\epsilon_k]}\colon [0, \epsilon_k] \to X$. We extend $\widetilde{\gamma}$ to $[0, \epsilon_{k+1}]$ as follows. Let U be a trivialized open set in X such that $\gamma([\epsilon_k, \epsilon_{k+1}]) \subset U$. Let $\widetilde{U}$ be the sheet lying above U with $\widetilde{\gamma}(\epsilon_k) \in \widetilde{U}$. The restriction $p|_{\widetilde{U}}\colon \widetilde{U} \to U$ is a homeomorphism, and on the interval $[\epsilon_k, \epsilon_{k+1}]$ we define $\widetilde{\gamma}$ to be the composition

$$[\epsilon_k, \epsilon_{k+1}] \xrightarrow{\ \gamma\ } U \xrightarrow{(p|_{\widetilde{U}})^{-1}} \widetilde{U} \hookrightarrow \widetilde{X}.$$

By construction, this agrees with our already-constructed partial lift $\widetilde{\gamma}\colon [0, \epsilon_k] \to \widetilde{X}$ at ϵ_k. □

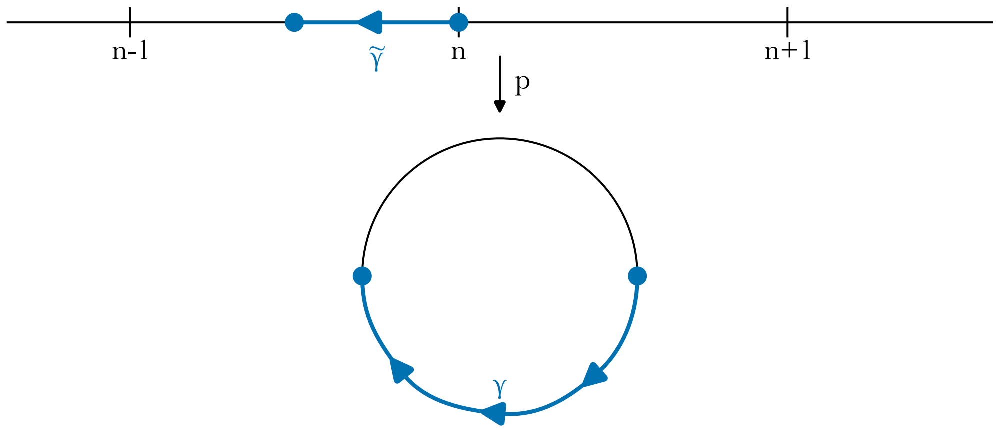
To help the reader understand the content of this lemma, we give several examples.

EXAMPLE 3.4.2 (Circle). Let $p\colon \mathbb{R} \to \mathbb{S}^1$ be the universal cover of $\mathbb{S}^1$, so $p(\theta) = e^{2\pi i\theta}$. Let $\gamma\colon [0, 1] \to \mathbb{S}^1$ be the path that starts at $1 \in \mathbb{S}^1 \subset \mathbb{C}$ and travels clockwise half-way around the circle:

$$\gamma(t) = e^{-\pi i t} \quad \text{for } 0 \leq t \leq 1.$$

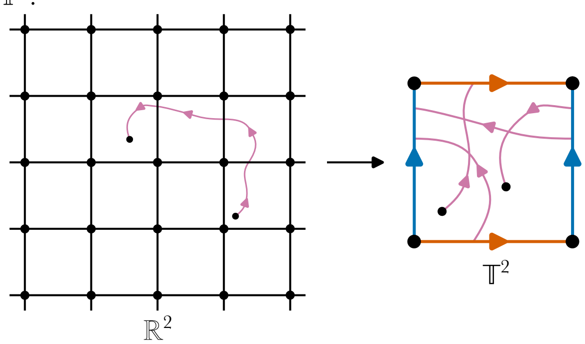
The points of $\mathbb{R}$ that project to $\gamma(0) = 1$ are precisely the integers. For $n \in \mathbb{Z}$, the lift $\widetilde{\gamma}\colon [0, 1] \to \mathbb{R}$ of γ with $\widetilde{\gamma}(0) = n$ is the map that looks like this:

[2] Recall that the Lebesgue number lemma says that if Z is a compact metric space and $\{W_j\}_{j \in J}$ is an open cover of Z, then we can find some $\epsilon > 0$ such that for all $z \in Z$ the ϵ-ball $B_\epsilon(z)$ is contained in some W_j. To find the indicated partition of I, apply this to the cover of I by preimages of trivialized open subsets of X and choose the partition such that each segment $[\epsilon_k, \epsilon_{k+1}]$ has diameter at most the $\epsilon > 0$ given by the lemma.



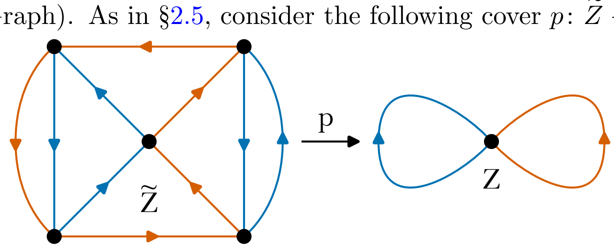
In coordinates, $\widetilde{\gamma}(t) = n - t/2$ for $0 \leq t \leq 1$. □

Example 3.4.3 (Torus). As in Example 1.3.3, identify $\mathbb{R}^2/\mathbb{Z}^2$ with the 2-torus $\mathbb{T}^2$ and let $p\colon \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$ be the associated cover. Here is an example of a path $\gamma\colon [0,1] \to \mathbb{T}^2$ and one choice of lift $\widetilde{\gamma}\colon [0,1] \to \mathbb{T}^2$:

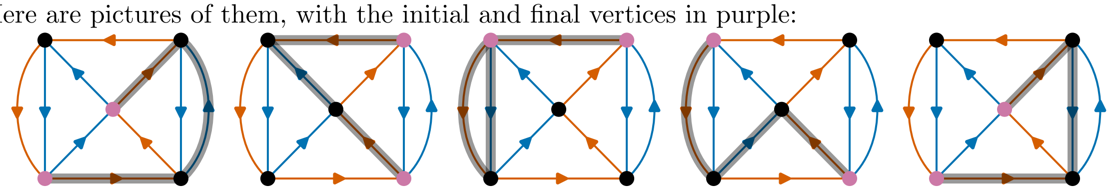


The torus on the right is obtained by gluing the sides of the square together as indicated. Because of this gluing, a path can e.g. pass through the top edge of the square and come out of the bottom edge. The other possible lifts are obtained by varying the initial point, which results in translating the entire lift by some element of $\mathbb{Z}^2$. □

Example 3.4.4 (Graph). As in §2.5, consider the following cover $p\colon \widetilde{Z} \to Z$:



Let $\gamma\colon [0,1] \to Z$ be the path that starts at the vertex, goes around the orange circle in the positive direction, then goes around the blue circle in the positive direction, and finally goes around the orange circle in the negative direction. There are five possible lifts, one starting at each vertex of $\widetilde{Z}$. Here are pictures of them, with the initial and final vertices in purple:



Constructing these illustrates the necessity that each vertex of $\widetilde{Z}$ has one incoming edge of each color and one outgoing edge of each color. □

EXAMPLE 3.4.5 (Polar coordinates). Points of $\mathbb{R}^2 \setminus 0$ can be expressed using polar coordinates (r, θ) with $r > 0$ and $\theta \in \mathbb{R}$:

$$(x, y) = (r\cos(\theta), r\sin(\theta)).$$

While $r = \sqrt{x^2 + y^2}$ is unambiguous, the θ-coordinate is ambiguous since $(r, \theta) = (r, \theta + 2\pi n)$ for all $n \in \mathbb{Z}$. Letting $p \colon \mathbb{R} \to \mathbb{S}^1$ be the universal cover, a choice of polar coordinates for $(x, y) \in \mathbb{R}^2 \setminus 0$ is the same as a choice of lift $\theta \in \mathbb{R}$ for the point

$$(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}) \in \mathbb{S}^1.$$

Lemma 3.4.1 explains why maps $f \colon I \to \mathbb{R}^2 \setminus 0$ can always be continuously expressed using polar coordinates. This is depends on the topology of I, and I cannot be replaced by an arbitrary space.

For instance, the inclusion $\iota \colon \mathbb{S}^1 \hookrightarrow \mathbb{R}^2 \setminus 0$ cannot be continuously described using polar coordinates as $\iota(x) = (r(x), \theta(x))$ for some $r \colon \mathbb{S}^1 \to \mathbb{R}_{>0}$ and $\theta \colon \mathbb{S}^1 \to \mathbb{R}$. Indeed, in such an expression the function r would be identically 1, but the function θ would be a section of the cover $p \colon \mathbb{R} \to \mathbb{S}^1$, and Lemma 3.2.2 implies that such a section does not exist. □

3.5. Homotopies

In algebraic topology, spaces are modeled by algebra. Spaces can vary continuously, while algebraic objects are typically discrete. In this section, we introduce a formalism called homotopy for studying deformations of maps. The algebraic invariants we later study will be insensitive to these deformations.

Consider two maps $f, g \colon X \to Y$. We say that f and g are *homotopic* if there exists a continuous map $H \colon X \times I \to Y$ such that $f(x) = H(x, 0)$ and $g(x) = H(x, 1)$ for all $x \in X$. For $t \in I$, let $h_t \colon X \to Y$ be the map $h_t(x) = H(x, t)$. We thus have $f = h_0$ and $g = h_1$, and we view the h_t as a continuous family of maps witnessing f being deformed to g. Typically we will demonstrate that f and g are homotopic by describing the h_t rather than H, and will call h_t a *homotopy from f to g*. This is an equivalence relations on the set of maps from X to Y (see Exercise 3.3).

EXAMPLE 3.5.1. Let X be a space. Any two maps $f, g \colon X \to \mathbb{R}^n$ are homotopic via the straight-line homotopy $h_t \colon X \to \mathbb{R}^n$ defined via the formula $h_t(x) = (1-t)f(x) + tg(x)$ for all $x \in X$ and $t \in I$. In this, we have $h_0 = f$ and $h_1 = g$. □

We say that a map $f \colon X \to Y$ is *null-homotopic* if f is homotopic to a constant map. As Example 3.5.1 shows, any map $f \colon X \to \mathbb{R}^n$ is null-homotopic. Here is another example:

EXAMPLE 3.5.2. Let X be a space. Then any map $f \colon \mathbb{R}^n \to X$ is null-homotopic via the homotopy $h_t \colon \mathbb{R}^n \to X$ defined via the formula $h_t(x) = f((1-t)x)$ for all $x \in X$ and $t \in I$. In this, we have $h_0 = f$ and h_1 is the constant map taking all points of $\mathbb{R}^n$ to $f(0)$. □

To show that two maps are homotopic, one typically exhibits an explicit homotopy. It is harder to show that two maps are *not* homotopic. This requires invariants of maps. For instance, the identity map $\mathbb{S}^1 \to \mathbb{S}^1$ is not null-homotopic, but this is not so easy to prove directly. In §3.7 below we will prove this by developing the theory of degrees and winding numbers. In fact, using this we will completely describe *all* homotopy classes of maps $\mathbb{S}^1 \to \mathbb{S}^1$. Doing this requires studying the interaction between homotopies and lifting problems.

3.6. Lifting homotopies

Roughly speaking, our goal in this section is to prove that lifting problems are insensitive to homotopies. To make this precise, consider a covering space $p \colon \widetilde{X} \to X$. Let $f, g \colon Y \to X$ be two homotopic maps. One thing we would like to prove is that a lift $\widetilde{f} \colon Y \to \widetilde{X}$ of f exists if and only if a lift $\widetilde{g} \colon Y \to \widetilde{X}$ exists. We would also like to prove that if these lifts exist then we can choose lifts $\widetilde{f} \colon Y \to \widetilde{X}$ and $\widetilde{g} \colon Y \to \widetilde{X}$ such that $\widetilde{f}$ and $\widetilde{g}$ are themselves homotopic. The following result implies both of these claims.

LEMMA 3.6.1. *Let $p \colon \widetilde{X} \to X$ be a covering space. Let $f \colon Y \to X$ be a map and let $\widetilde{f} \colon Y \to \widetilde{X}$ be a lift of f through p. Let $h_t \colon Y \to X$ be a homotopy with $h_0 = f$. There is then a unique lift of h_t through p to a homotopy $\widetilde{h}_t \colon Y \to \widetilde{X}$ such that $\widetilde{h}_0 = \widetilde{f}$.*

PROOF. Let $H\colon Y \times I \to X$ be the map with $H(y,t) = h_t(y)$ for all $y \in Y$ and $t \in I$. Our goal is to prove that there is a unique lift $\widetilde{H}\colon Y \times I \to \widetilde{X}$ of H through p such that $\widetilde{H}(y,0) = \widetilde{f}(y)$ for all $y \in Y$. Uniqueness follows from Lemma 3.1.2, so we must only prove existence. In fact, even more is true. For $y \in Y$, let $\gamma_y\colon I \to X$ be the path $\gamma_y(t) = H(y,t)$. By path-lifting (Lemma 3.4.1), we can lift γ_y to a path $\widetilde{\gamma}_y\colon I \to \widetilde{X}$ such that $\widetilde{\gamma}_y(0) = \widetilde{f}(0)$. Define $\widetilde{H}\colon Y \times I \to \widetilde{X}$ via the formula

$$\widetilde{H}(y,t) = \widetilde{\gamma}_y(t) \quad \text{for all } y \in Y \text{ and } t \in I.$$

By construction, $\widetilde{H}$ is a lift of H with $\widetilde{H}(y,0) = \widetilde{f}(y)$ for all $y \in Y$.

There is only one problem: it is not obvious that this H is continuous. To see that it is, fix some $y_0 \in Y$. We will prove that $\widetilde{H}$ is continuous at all points of the form (y_0,t) by imitating our proof of the path-lifting lemma (Lemma 3.4.1). Just like in that proof, using the Lebesgue number lemma we can partition I into subintervals

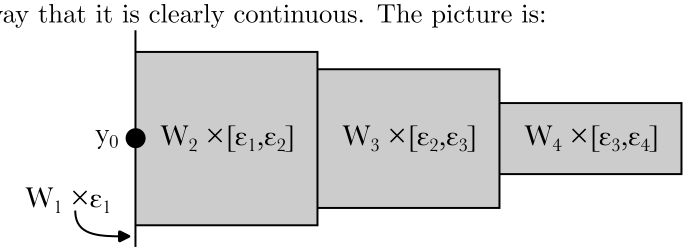
$$0 = \epsilon_1 < \epsilon_2 < \cdots < \epsilon_n = 1$$

such that for all $1 \leq k < n$ the image $H(y_0 \times [\epsilon_k, \epsilon_{k+1}])$ is contained in a trivialized open set in X. In fact, we can even find some open neighborhood V_k of y_0 such that the image $H(V_k \times [\epsilon_k, \epsilon_{k+1}])$ is contained in a trivialized open set in X.

Define $W_1 = V_1 \cap \cdots \cap V_{n-1}$, so W_1 is an open neighborhood of y_0. We will find a nested sequence

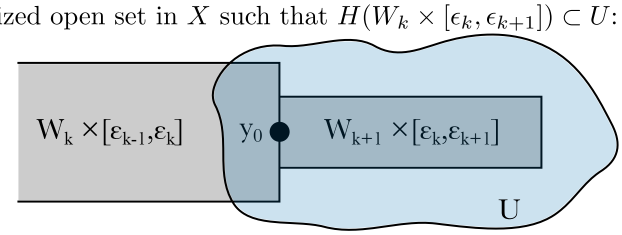
$$W_1 \supset W_2 \supset \cdots \supset W_n$$

of open neighborhoods of y_0 such that $\widetilde{H}$ is continuous on each $W_k \times [0, \epsilon_k]$ by constructing $\widetilde{H}$ on this set in such a way that it is clearly continuous. The picture is:



The construction will be inductive. First, define $\widetilde{H}\colon W_1 \times 0 \to \widetilde{X}$ via the formula $\widetilde{H}(y,0) = \widetilde{f}(y)$. Next, assume that for some $1 \leq k < n$ we have constructed a continuous lift $\widetilde{H}\colon W_k \times [0,\epsilon_k] \to \widetilde{X}$ of $H|_{W_k \times [0,\epsilon_k]}\colon W_k \times [0,\epsilon_k] \to X$. We find an open neighborhood W_{k+1} of y_0 with $W_{k+1} \subset W_k$ and an extension of $\widetilde{H}$ to $W_{k+1} \times [0, \epsilon_{k+1}]$ as follows.

Let U be a trivialized open set in X such that $H(W_k \times [\epsilon_k, \epsilon_{k+1}]) \subset U$:



Let $\widetilde{U}$ be the sheet lying above U with $H(y_0, \epsilon_k) \in \widetilde{U}$. Let W_{k+1} be the preimage of $\widetilde{U}$ under the map[3]

$$W_k = W_k \times \epsilon_k \xrightarrow{\widetilde{H}(-,\epsilon_k)} \widetilde{X}.$$

The set W_{k+1} is an open neighborhood of y_0 and $\widetilde{H}$ takes $W_{k+1} \times \epsilon_k$ to $\widetilde{U}$. The restriction $p|_{\widetilde{U}}\colon \widetilde{U} \to U$ is a homeomorphism, and on $W_{k+1} \times [\epsilon_k, \epsilon_{k+1}]$ we define $\widetilde{H}$ to be the composition

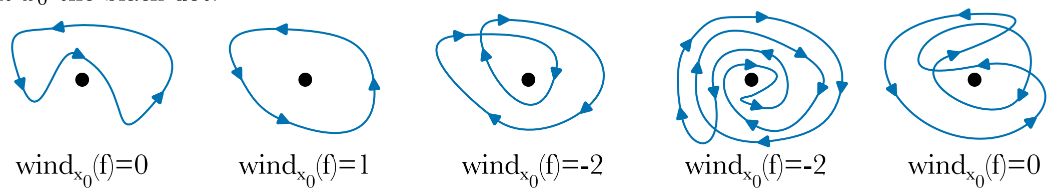
$$W_{k+1} \times [\epsilon_k, \epsilon_{k+1}] \xrightarrow{H} U \xrightarrow{(p|_{\widetilde{U}})^{-1}} \widetilde{U} \hookrightarrow \widetilde{X}.$$

[3]If we could ensure that $\widetilde{H}(W_k \times \epsilon_k) \subset \widetilde{U}$ (which would hold, for instance, if W_k and $\widetilde{U}$ were connected), then there would be no need to pass to the nested sequence $W_1 \supset W_2 \supset \cdots$. Since we are not assuming that our spaces are locally connected, this is unfortunately necessary.

By construction, this agrees with our already-constructed partial lift $\widetilde{H}\colon W_{k+1} \times [0, \epsilon_k] \to \widetilde{X}$ on $W_{k+1} \times \epsilon_k$. $\square$

3.7. Winding numbers

To illustrate the meaning of all this machinery, we study the classical subject of winding numbers. Fix a point $x_0 \in \mathbb{C}$. We start with an intuitive discussion. Consider a map $f\colon \mathbb{S}^1 \to \mathbb{C} \setminus x_0$. Roughly speaking, the winding number $\mathrm{wind}_{x_0}(f)$ of f around x_0 measures the number of times the vector $f(z) - x_0$ rotates as z moves around $\mathbb{S}^1$. The "number" here includes a sign: a counterclockwise rotation counts as $+1$, while a clockwise rotation counts as -1. Here are several examples, with the point x_0 the black dot:



One feature of the winding number is that it is invariant under homotopies of f through maps that avoid x_0. In fact, it is a *complete* invariant of such maps (see Exercise 3.1). It is enlightening to verify that the different f above with the same winding number are homotopic.

We now give a precise definition. Let $p\colon \mathbb{R} \to \mathbb{S}^1$ be the universal cover. Consider some $f\colon \mathbb{S}^1 \to \mathbb{C} \setminus x_0$. Define a map $F\colon I \to \mathbb{S}^1$ via the formula

$$F(t) = \frac{f(e^{2\pi i t}) - x_0}{\|f(e^{2\pi i t}) - x_0\|} \quad \text{for } t \in I.$$

This definition makes sense since $f(z) \neq x_0$ for all $z \in \mathbb{S}^1$. Pick some $\theta_0 \in \mathbb{R}$ such that $p(\theta_0) = F(0)$, and use path lifting (Lemma 3.4.1) to lift F through p to $\widetilde{F}\colon I \to \mathbb{R}$ with $\widetilde{F}(0) = \theta_0$. Since $F(0) = F(1)$, the lifts $\widetilde{F}(0) = \theta_0$ and $\widetilde{F}(1)$ differ by an integer. We define

$$\mathrm{wind}_{x_0}(f) = \widetilde{F}(1) - \widetilde{F}(0) \in \mathbb{Z}.$$

The only arbitrary choice we made was the lift θ_0. Any other choice of θ_0 is of the form $\theta_0 + m$ for some $m \in \mathbb{Z}$, and using $\theta_0 + m$ as our initial lift would change $\widetilde{F}$ to $\widetilde{F} + m$. Since

$$(\widetilde{F}(1) + m) - (\widetilde{F}(0) + m) = \widetilde{F}(1) - \widetilde{F}(0),$$

this would not change $\mathrm{wind}_{x_0}(f)$. In other words, $\mathrm{wind}_{x_0}(f) \in \mathbb{Z}$ is well-defined.

EXAMPLE 3.7.1. Fix $k \in \mathbb{Z}$, and define $f\colon \mathbb{S}^1 \to \mathbb{C} \setminus x_0$ via the formula

$$f(z) = x_0 + z^k \quad \text{for } z \in \mathbb{S}^1 \subset \mathbb{C}.$$

In the above recipe, we then have

$$F(t) = \frac{f(e^{2\pi i t}) - x_0}{\|f(e^{2\pi i t}) - x_0\|} = e^{2\pi i k t} \quad \text{for } t \in I.$$

We can take $\theta_0 = 0$, and then

$$\widetilde{F}(\theta) = k\theta \quad \text{for } \theta \in \mathbb{R}.$$

It follows that $\mathrm{wind}_{x_0}(f) = k$. We thus see that all integers can be winding numbers. $\square$

One of the main properties of the winding number is that it is unchanged under homotopies:

LEMMA 3.7.2. *Let $x_0 \in \mathbb{C}$, and let $f, g\colon \mathbb{S}^1 \to \mathbb{C} \setminus x_0$ be homotopic maps. Then $\mathrm{wind}_{x_0}(f) = \mathrm{wind}_{x_0}(g)$.*

PROOF. Let $p\colon \mathbb{R} \to \mathbb{S}^1$ be the universal cover. As in the definition of the winding number, define $F\colon I \to \mathbb{S}^1$ and $G\colon I \to \mathbb{S}^1$ via the formulas[4]

$$F(s) = \frac{f(e^{2\pi i s}) - x_0}{\|f(e^{2\pi i s}) - x_0\|} \quad \text{and} \quad G(s) = \frac{g(e^{2\pi i s}) - x_0}{\|g(e^{2\pi i s}) - x_0\|} \quad \text{for } s \in I.$$

[4] We use s instead of t since we will later use t when we talk about homotopies.

Pick some $\theta_0 \in \mathbb{R}$ such that $p(\theta_0) = F(0)$, and use path lifting (Lemma 3.4.1) to lift F through $p\colon \mathbb{R} \to \mathbb{S}^1$ to $\widetilde{F}\colon I \to \mathbb{R}$ with $\widetilde{F}(0) = \theta_0$. We then have $\text{wind}_{x_0}(f) = \widetilde{F}(1) - \widetilde{F}(0)$. We will hold off on constructing the lift of G that would determine $\text{wind}_{x_0}(g)$.

Now let $h_t\colon \mathbb{S}^1 \to \mathbb{C} \setminus x_0$ be a homotopy from f to g. Define $H_t\colon I \to \mathbb{S}^1$ via the formula

$$H_t(s) = \frac{h_t(e^{2\pi i s}) - x_0}{\|h_t(e^{2\pi i s}) - x_0\|} \quad \text{for } s \in I,$$

so H_t is a homotopy from $H_0 = F$ to $H_1 = G$. Use the homotopy lifting lemma (Lemma 3.6.1) to lift H_t to a homotopy $\widetilde{H}_t\colon I \to \mathbb{R}$ with $\widetilde{H}_0 = \widetilde{F}$. It follows that $\widetilde{H}_1$ is a lift of G, so $\text{wind}_{x_0}(g) = \widetilde{H}_1(1) - \widetilde{H}_1(0)$. More generally, we have $\text{wind}_{x_0}(h_t) = \widetilde{H}_t(1) - \widetilde{H}_t(0)$ for all $t \in I$. This implies that the map

$$t \mapsto \widetilde{H}_t(1) - \widetilde{H}_t(0) \quad \text{for } t \in I$$

is a continuous integer-valued function. It is thus constant, so

$$\text{wind}_{x_0}(f) = \widetilde{H}_0(1) - \widetilde{H}_0(0) = \widetilde{H}_1(1) - \widetilde{H}_1(0) = \text{wind}_{x_0}(g). \qquad \square$$

3.8. Degree of map of circle

Consider a map $f\colon \mathbb{S}^1 \to \mathbb{S}^1$. We can regard f as a map to $\mathbb{C} \setminus 0$, giving an integer $\text{wind}_0(f)$ that we will call the *degree*[5] of f. Denote this by $\deg(f)$. Lemma 3.7.2 implies that $\deg(f)$ is invariant under homotopy, and just like in Example 3.7.1 we have $\deg(z^n) = n$ for all $n \in \mathbb{Z}$. In particular, the degree of the identity is 1 and the degree of a constant map is 0. Since these are different, we deduce the following, which was promised at the end of §3.5:

LEMMA 3.8.1. *The identity map $\mathbb{1}\colon \mathbb{S}^1 \to \mathbb{S}^1$ is not nullhomotopic.*

The following basic result says that the degree is a *complete* invariant of homotopy classes of maps from $\mathbb{S}^1$ to itself:

LEMMA 3.8.2. *Let $f, g\colon \mathbb{S}^1 \to \mathbb{S}^1$ be maps with $\deg(f) = \deg(g)$. Then f is homotopic to g.*

PROOF. By postcomposing f and g with paths of rotations of $\mathbb{S}^1$, we can homotope them to maps with $f(1) = g(1) = 1$. Define $F\colon I \to \mathbb{S}^1$ and $G\colon I \to \mathbb{S}^1$ via the formulas

$$F(s) = f(e^{2\pi i s}) \quad \text{and} \quad G(s) = g(e^{2\pi i s}) \quad \text{for } s \in I.$$

We thus have $F(0) = G(0) = 1$. Letting $p\colon \mathbb{R} \to \mathbb{S}^1$ be the universal cover, by the path lifting lemma (Lemma 3.4.1) we can lift F and G through p to maps $\widetilde{F}, \widetilde{G}\colon I \to \mathbb{R}$ with $\widetilde{F}(0) = \widetilde{G}(0) = 0$. We have $\widetilde{F}(1) = \deg(f)$ and $\widetilde{G}(1) = \deg(g)$, which are equal by assumption. Define $H_t\colon I \to \mathbb{S}^1$ via the formula

$$H_t(s) = p((1-t)\widetilde{F}(s) + t\widetilde{G}(s)) \quad \text{for } s \in I.$$

The maps H_t are a homotopy from F to G, and since $\widetilde{F}(0) = \widetilde{G}(0) = 0$ and $\widetilde{F}(1) = \widetilde{G}(1) \in \mathbb{Z}$ we have $H_t(0) = 1$ and $H_t(1) = 1$ for all $t \in I$. This implies that there exists some $h_t\colon \mathbb{S}^1 \to \mathbb{S}^1$ with

$$H_t(s) = h_t(e^{2\pi i s}) \quad \text{for } s \in I.$$

This h_t is a homotopy from f to g. $\square$

REMARK 3.8.3. In later volumes when we develop some basic results about homology, we will generalize the notion of degree to an integer-valued invariant of maps $f\colon M^n \to N^n$ with M^n and N^n compact oriented n-manifolds. Lemma 3.8.2 generalizes to a deep theorem of Hopf saying that this degree is a complete invariant for maps $f\colon M^n \to \mathbb{S}^n$. $\square$

[5] If f is a covering space, this is different from the degree of f as a covering space. For instance, it can be negative.

3.9. Exercises

EXERCISE 3.1. Let $x_0 \in \mathbb{C}$ and let $f, g\colon \mathbb{S}^1 \to \mathbb{C} \setminus x_0$ be maps with $\mathrm{wind}_{x_0}(f) = \mathrm{wind}_{x_0}(g)$. Prove that f is homotopic to g. Hint: first homotope f and g such that their images lie in $\mathbb{S}^1$, then appeal to Lemma 3.8.2. □

EXERCISE 3.2. Let $p\colon \widetilde{X} \to X$ be a covering space and let $f\colon \mathbb{D}^n \to X$ be a map. Set $x_0 = f(0)$. For each $\widetilde{x}_0 \in p^{-1}(x_0)$, prove that there is a unique lift $\widetilde{f}\colon \mathbb{D}^n \to \widetilde{X}$ with $\widetilde{f}(0) = \widetilde{x}_0$. □

EXERCISE 3.3. Let X and Y be spaces. Prove that the relation of being homotopic is an equivalence relation on maps from X to Y. □

EXERCISE 3.4. Let X and Y and Z be spaces. For $i = 0, 1$, let $f_i\colon X \to Y$ and $g_i\colon Y \to Z$ be maps. Assume that f_0 is homotopic to f_1 and that g_0 is homotopic to g_1. Prove that $g_0 \circ f_0\colon X \to Z$ is homotopic to $g_1 \circ f_1\colon X \to Z$. □

EXERCISE 3.5. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be maps. Prove that $g \circ f\colon X \to Z$ is nullhomotopic if either f or g is nullhomotopic. □

EXERCISE 3.6. Let X be a space and let $f\colon \mathbb{S}^n \to X$ be a map. Prove that f is null-homotopic if and only if f extends to a map $F\colon \mathbb{D}^{n+1} \to X$. □

EXERCISE 3.7. A space Y is *contractible* if the identity map $\mathbb{1}_Y\colon Y \to Y$ is null-homotopic (we will say more about this concept in Chapter 6). Let $p\colon \widetilde{X} \to X$ be a covering space with $\widetilde{X} \neq \emptyset$ and let $f\colon Y \to X$ be a continuous map with Y contractible. Prove that f can be lifted to $\widetilde{f}\colon Y \to \widetilde{X}$. Hint: First prove that f is null-homotopic. □

EXERCISE 3.8. Let $p\colon \mathbb{R} \to \mathbb{S}^1$ be the universal cover. For a space X, prove that a map $f\colon X \to \mathbb{S}^1$ can be lifted to a map $\widetilde{f}\colon X \to \mathbb{R}$ if and only if f is null-homotopic. □

EXERCISE 3.9. Let X be a topological space and let $f\colon X \to \mathbb{C}$ be a continuous function such that $f(x) \neq 0$ for all $x \in X$.

(a) Construct a degree 2 cover $p\colon \widetilde{X} \to X$ such that $f \circ p\colon \widetilde{X} \to \mathbb{C}$ has a continuous square root, i.e., there exists a continuous function $g\colon \widetilde{X} \to \mathbb{C}$ such that $f(x) = g(x)^2$ for all $x \in X$.
(b) Prove that $p\colon \widetilde{X} \to X$ is a trivial cover if and only if $f\colon X \to \mathbb{C}$ has a continuous square root. □

EXERCISE 3.10. Define a map $f\colon \mathbb{S}^1 \times I \to \mathbb{S}^1 \times I$ via the formula

$$f(z, s) = (e^{2\pi i s} z, s) \quad \text{for all } z \in \mathbb{S}^1 \subset \mathbb{C} \text{ and } s \in I.$$

Prove the following:

(a) There is a homotopy $f_t\colon \mathbb{S}^1 \times I \to \mathbb{S}^1 \times I$ with $f_0 = f$ and $f_1 = \text{unit}$ such that $f_t(z, 0) = (z, 0)$ for all $z \in \mathbb{S}^1$ and $t \in I$.
(b) There does not exist a homotopy $f_t\colon \mathbb{S}^1 \times I \to \mathbb{S}^1 \times I$ with $f_0 = f$ and $f_1 = \text{unit}$ such that $f_t(z, 0) = (z, 0)$ and $f_t(z, 1) = (z, 1)$ for all $z \in \mathbb{S}^1$ and $t \in I$. □

EXERCISE 3.11. Let $f\colon \mathbb{S}^1 \to \mathbb{C} \setminus \{x_0\}$ be a smooth map. Prove that you can compute the winding number of f around x_0 using the following formula from complex analysis:

$$\mathrm{wind}_{x_0}(f) = \frac{1}{2\pi i} \int_f \frac{1}{z - x_0}\, \mathrm{dz}\,.$$

Here the integral is the usual path integral. □

EXERCISE 3.12. Let $f, g\colon \mathbb{S}^1 \to \mathbb{S}^1$ be maps. Prove that $\deg(f \circ g) = \deg(f)\deg(g)$. □

EXERCISE 3.13. Let $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ be a map with $\deg(f) \neq 1$. Prove that there exists some $x \in \mathbb{S}^1$ such that $f(x) = x$. □

EXERCISE 3.14. Let $A \in \mathrm{GL}_2(\mathbb{R})$. Identify $\mathbb{C}$ with $\mathbb{R}^2$, so for $z \in \mathbb{C}$ we have $Az \in \mathbb{C}$. Define $f\colon \mathbb{S}^1 \to \mathbb{C} \setminus 0$ via the formula $f(z) = Az$. Prove that

$$\mathrm{wind}_0(f) = \begin{cases} 1 & \text{if } \det(A) > 0, \\ -1 & \text{if } \det(A) < 0. \end{cases}$$

In other words, $\mathrm{wind}_0(f)$ is the sign of $\det(A)$. □

EXERCISE 3.15. Let $f\colon \mathbb{S}^1 \to \mathbb{C}$ be a smooth map. We say that f is an *immersion* if for all $x \in \mathbb{S}^1$ the derivative map $D_x f\colon T_x\mathbb{S}^1 \to T_{f(x)}\mathbb{C}$ is an injective map from the 1-dimensional tangent space $T_x\mathbb{S}^1$ to the 2-dimensional tangent space $T_{f(x)}\mathbb{C} = \mathbb{C}$. A homotopy $f_t\colon \mathbb{S}^1 \to \mathbb{C}$ is a *regular homotopy* if each f_t is an immersion. Define $g\colon \mathbb{S}^1 \to \mathbb{C}$ and $h\colon \mathbb{S}^1 \to \mathbb{C}$ via the formulas

$$g(z) = z \quad \text{and} \quad h(z) = z^{-1} \quad \text{for } z \in \mathbb{S}^1.$$

Prove that there does not exist a regular homotopy from g to h. Hint: for $x \in \mathbb{S}^1$, we can identify $T_x\mathbb{S}^1$ with a subspace of $T_x\mathbb{C} = \mathbb{C}$. Using this identification, we have $ix \in T_x\mathbb{S}^1$. For an immersion $f\colon \mathbb{S}^1 \to \mathbb{C}$, consider the winding number around 0 of the map $\tau_f\colon \mathbb{S}^1 \to \mathbb{C} \setminus 0$ defined by $\tau_f(x) = (D_x f)(ix)$. □

EXERCISE 3.16. For $a < b$, a map $\gamma\colon [a,b] \to \mathbb{R}^n$ is *linear* if for some $x, y \in \mathbb{R}^n$ it can be written in the form

$$\gamma(t) = x + ty \quad \text{for } t \in [a,b].$$

A map $\gamma\colon I \to \mathbb{R}^n$ is *piecewise linear* if there exist

$$0 = \epsilon_1 < \epsilon_2 < \cdots < \gamma_n = 1$$

such that $\gamma|_{[\epsilon_k, \epsilon_{k+1}]}$ is linear for all $1 \leq k < n$. For an open set $U \subset \mathbb{R}^n$ and a map $\gamma\colon I \to U$, prove that there is a homotopy $\gamma_t\colon I \to U$ such that:

- $\gamma_0 = \gamma$ and γ_1 is piecewise-linear; and
- $\gamma_t(0) = \gamma(0)$ and $\gamma_t(1) = \gamma(1)$ for all $t \in I$. □

CHAPTER 4

Covering spaces: homotopy classes of paths

Setting the stage for introducing the fundamental group in the next chapter, this chapter discusses homotopy classes of paths and how they lift to covers. As an application, for a broad class of spaces X we give a criterion that implies that all covers $p\colon \widetilde{X} \to X$ are trivial.

4.1. Homotopies of paths

Let X be a space and let $x, y \in X$. Recall from §3.4 that a *path* in X from x to y is a map $\gamma\colon I \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. We wish to study paths up to homotopy. This would be uninteresting if we allowed the endpoints of γ to move during the homotopy since then all paths would be homotopic if X is path-connected (see Exercise 4.1). We therefore make the following definition:

DEFINITION 4.1.1. Let X be a space, let $x, y \in X$, and let $\gamma_0, \gamma_1\colon I \to X$ be two paths from x to y. We say that γ_0 and γ_1 are *homotopic* paths from x to y if there exists a homotopy $\gamma_t\colon I \to X$ from γ_0 to γ_1 that fixes the endpoints in the sense that

$$\gamma_t(0) = x \quad \text{and} \quad \gamma_t(1) = y \quad \text{for all } t \in I.$$

The relation of being homotopic is an equivalence relation[1] on paths, and we will call the equivalence classes of paths under this equivalence relation *homotopy classes*. For a path γ, we will write $[\gamma]$ for its homotopy class. □

Here are two examples:

EXAMPLE 4.1.2. For all $x, y \in \mathbb{R}^n$, there is a unique homotopy class of path from x to y. Indeed, let $\gamma_0\colon I \to \mathbb{R}^n$ be the straight line path

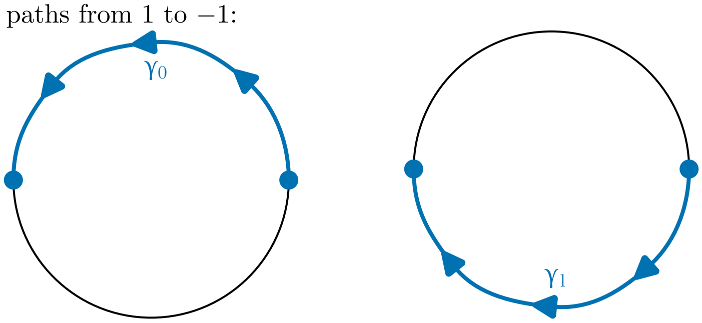
$$\gamma_0(s) = (1 - s)x + sy \quad \text{for } s \in I.$$

If $\gamma\colon I \to \mathbb{R}^n$ is any other path from x to y, then γ_0 is homotopic to γ via the homotopy $\gamma_t\colon I \to \mathbb{R}^n$ defined by $\gamma_t(s) = (1 - t)\gamma_0(s) + t\gamma(s)$ for all $s \in I$ and $t \in I$. □

EXAMPLE 4.1.3. View $\mathbb{S}^1$ as a subspace of $\mathbb{C}$, and let $\gamma_0\colon I \to \mathbb{S}^1$ and $\gamma_1\colon I \to \mathbb{S}^1$ be the paths defined by the formulas

$$\gamma_0(s) = e^{\pi i s} \quad \text{and} \quad \gamma_1(s) = e^{-\pi i s} \quad \text{for } s \in I. \tag{4.1.1}$$

Both γ_0 and γ_1 are paths from 1 to -1:



We claim that γ_0 and γ_1 are not homotopic. To see this, assume that they are homotopic and that

[1] The proof is the same as the one needed for Exercise 3.3.

$\gamma_t\colon I \to \mathbb{S}^1$ is a homotopy. Let $p\colon \mathbb{R} \to \mathbb{S}^1$ be the universal cover, so $p(\theta) = e^{2\pi i\theta}$ for all $\theta \in \mathbb{R}$. The lift of γ_0 to $\mathbb{R}$ starting at 0 is the map $\widetilde{\gamma}_0\colon I \to \mathbb{R}$ defined by

$$\widetilde{\gamma}_0(s) = s/2 \quad \text{for } s \in I. \tag{4.1.2}$$

By Lemma 3.6.1, we can lift the homotopy γ_t to a homotopy $\widetilde{\gamma}_t\colon I \to \mathbb{R}$ with $\widetilde{\gamma}_0$ the map (4.1.2). Since $\gamma_t(0) = 1$ and $\gamma_t(1) = -1$ for all $t \in I$, we have that

$$\widetilde{\gamma}_t(0) \in p^{-1}(1) = 2\pi\mathbb{Z} \quad \text{and} \quad \widetilde{\gamma}_t(1) \in p^{-1}(-1) = 2\pi\mathbb{Z} + \pi \quad \text{for } t \in I.$$

Since $2\pi\mathbb{Z}$ is discrete, it follows that both $\widetilde{\gamma}_t(0)$ and $\widetilde{\gamma}_t(1)$ are constant functions of t, i.e., that $\widetilde{\gamma}_t(0) = 0$ and $\widetilde{\gamma}_t(1) = 1/2$ for all $t \in I$. This implies in particular that $\widetilde{\gamma}_1$ is the lift of γ_1 to $\mathbb{R}$ starting at 0. From (4.1.1), we see that

$$\widetilde{\gamma}_1(s) = -s/2 \quad \text{for } s \in I.$$

In particular, $\widetilde{\gamma}_1(1) = -1/2$, contradicting the fact that $\widetilde{\gamma}_1(1) = 1/2$. $\square$

4.2. Lifting homotopies of paths

Let $p\colon \widetilde{X} \to X$ be a cover and let $\gamma\colon I \to X$ be a path in X from $x \in X$ to $y \in X$. For $\widetilde{x} \in p^{-1}(x)$, Lemma 3.4.1 implies that there exists a unique lift $\widetilde{\gamma}\colon I \to \widetilde{X}$ of γ with $\widetilde{\gamma}(0) = \widetilde{x}$. Generalizing the reasoning from Example 4.1.3, we prove that this lift only depends on the homotopy class $[\gamma]$:

Lemma 4.2.1. *Let $p\colon \widetilde{X} \to X$ be a cover and let $\gamma_0, \gamma_1\colon I \to X$ be two homotopic paths in X from $x \in X$ to $y \in X$. Pick $\widetilde{x} \in p^{-1}(x)$, and for $i = 0, 1$ let $\widetilde{\gamma}_i\colon I \to \widetilde{X}$ be the lift of γ_i to $\widetilde{X}$ with $\widetilde{\gamma}_i(0) = \widetilde{x}$. Then $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ are homotopic, i.e., $[\widetilde{\gamma}_0] = [\widetilde{\gamma}_1]$ In particular, $\widetilde{\gamma}_0(1) = \widetilde{\gamma}_1(1)$.*

Proof. Let $\gamma_t\colon I \to X$ be a homotopy of paths from γ_0 to γ_1. We thus have

$$\gamma_t(0) = x \quad \text{and} \quad \gamma_t(1) = y \quad \text{for all } t \in I.$$

By Lemma 3.6.1, we can lift the homotopy γ_t to a homotopy[2] $\widetilde{\gamma}_t'\colon I \to X$ with $\widetilde{\gamma}_0' = \widetilde{\gamma}_0$. We claim that γ_t' is a homotopy of paths from $\widetilde{\gamma}_0$ to $\widetilde{\gamma}_1$. Indeed, since $\gamma_t(0) = x$ for all $t \in I$ it follows that the map $t \mapsto \widetilde{\gamma}_t'(0)$ is a path in the discrete space $p^{-1}(x)$. This path must be constant, so

$$\widetilde{\gamma}_t'(0) = \widetilde{\gamma}_0'(0) = \widetilde{\gamma}_0(0) = \widetilde{x} \quad \text{for all } t \in I.$$

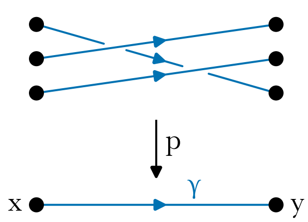
Similarly, $\widetilde{\gamma}_t'(1) = \widetilde{\gamma}_0(1)$ for all $t \in I$, as claimed. It follows that $\widetilde{\gamma}_0' = \widetilde{\gamma}_0$ is homotopic to $\widetilde{\gamma}_1'$. Moreover, $\widetilde{\gamma}_1'$ is a lift of γ_1 with $\widetilde{\gamma}_1'(0) = \widetilde{x}$, so by the uniqueness of lifts of paths we have $\widetilde{\gamma}_1' = \widetilde{\gamma}_1$. The lemma follows. $\square$

4.3. Connecting fibers

Let $p\colon \widetilde{X} \to X$ be a cover and let $\gamma\colon I \to X$ be a path in X from $x \in X$ to $y \in X$. Define a map $\tau_\gamma\colon p^{-1}(x) \to p^{-1}(y)$ as follows:

- For $\widetilde{x} \in p^{-1}(x)$, let $\widetilde{\gamma}$ be the lift of γ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}$. We then set $\tau_\gamma(\widetilde{x}) = \widetilde{\gamma}(1) \in p^{-1}(y)$.

If γ is an embedding, then the lifts $\widetilde{\gamma}$ used to construct τ_γ are all disjoint (see Exercise 4.2). The picture thus looks like the following:



By Lemma 4.2.1, the map τ_γ only depends on the homotopy class $[\gamma]$ of γ. The following lemma says that it is a bijection:

Lemma 4.3.1. *Let $p\colon \widetilde{X} \to X$ be a cover and let $\gamma\colon I \to X$ be a path in X from $x \in X$ to $y \in X$. Then the map $\tau_\gamma\colon p^{-1}(x) \to p^{-1}(y)$ is a bijection.*

---

[2]We call this $\widetilde{\gamma}_t'$ since it is not obvious that $\widetilde{\gamma}_1' = \widetilde{\gamma}_1$, though we will soon prove this.

PROOF. Let $\overline{\gamma}\colon I \to X$ be the path obtained by reversing γ, i.e.,

$$\overline{\gamma}(s) = \gamma(1-s) \quad \text{for } s \in I.$$

The path $\overline{\gamma}$ goes from y to x. To prove the lemma, it is enough to prove that $\tau_{\overline{\gamma}}\colon p^{-1}(y) \to p^{-1}(x)$ is an inverse to τ_γ. To see this, consider $\widetilde{x} \in p^{-1}(x)$. Let $\widetilde{\gamma}$ be the lift of γ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}$, so $\tau_\gamma(\widetilde{x}) = \widetilde{\gamma}(1)$. The lift $\widetilde{\overline{\gamma}}$ of $\overline{\gamma}$ to $\widetilde{X}$ with $\widetilde{\overline{\gamma}}(0) = \widetilde{\gamma}(1)$ is exactly the path obtained by reversing $\widetilde{\gamma}$, so

$$\tau_{\overline{\gamma}}(\tau_\gamma(\widetilde{x})) = \tau_{\overline{\gamma}}(\widetilde{\gamma}(1)) = \widetilde{\gamma}(0) = \widetilde{x}.$$

Similarly, $\tau_\gamma(\tau_{\overline{\gamma}}(\widetilde{y})) = \widetilde{y}$ for all $\widetilde{y} \in p^{-1}(y)$. The lemma follows. □

REMARK 4.3.2. For a cover $p\colon \widetilde{X} \to X$ with X connected, Lemma 1.2.1 implies that the fibers $p^{-1}(x)$ all have the same cardinality. If X is path connected, then, Lemma 4.3.1 also implies this, giving an alternate proof in this special case. □

4.4. Regularity from action on single fiber

Let $p\colon \widetilde{X} \to X$ be a cover with deck group G. Recall that $p\colon \widetilde{X} \to X$ is regular if for all $x_0 \in X$ the group G acts transitively on the fiber $p^{-1}(x_0)$. One application of Lemma 4.3.1 is that if X is path connected it is enough to check this on a single fiber:

LEMMA 4.4.1. *Let $p\colon \widetilde{X} \to X$ be a cover with deck group G. Assume that X is path connected and that there exists some $x_0 \in X$ such that G acts transitively on $p^{-1}(x_0)$. Then $p\colon \widetilde{X} \to X$ is regular.*

PROOF. Let $x_1 \in X$. We must prove that G acts transitively on $p^{-1}(x_1)$. Let γ be a path in X from x_0 to x_1. Lemma 4.3.1 says that the the map $\tau_\gamma\colon p^{-1}(x_0) \to p^{-1}(x_1)$ is a bijection. Since the action of G takes lifts of γ to lifts of γ, it commutes with τ_γ in the sense that

$$\tau_\gamma(g\widetilde{x}_0) = g\tau_\gamma(\widetilde{x}_0) \quad \text{for all } g \in G \text{ and } \widetilde{x}_0 \in p^{-1}(x_0).$$

Since τ_γ is a bijection and G acts transitively on $p^{-1}(x_0)$, it follows that G acts transitively on $p^{-1}(x_1)$, as desired. □

4.5. 1-connectivity, spheres, and general position

We say that a space X is 0-*connected* if it is nonempty and path connected.[3] For each $x, y \in X$, there thus exists a path from x to y. We say that X is 1-*connected* if X is 0-connected and for all $x, y \in X$ there is a *unique* homotopy class of paths from x to y. It is also common to say that a 1-connected space is *simply connected*.

EXAMPLE 4.5.1. We showed in Example 4.1.2 that $\mathbb{R}^n$ is 1-connected. We also showed in Example 4.1.3 that $\mathbb{S}^1$ is not 1-connected. □

Spheres of dimension at least 2 are important examples of 1-connected spaces:

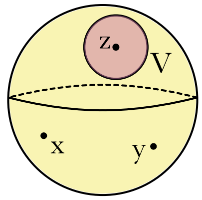
LEMMA 4.5.2. *Let $n \geq 2$. Then $\mathbb{S}^n$ is 1-connected.*

PROOF. Consider $x, y \in \mathbb{S}^n$. We must prove that there is a unique homotopy class of paths from x to y. Let $z \in \mathbb{S}^n$ be a point with $z \neq x, y$. Since $\mathbb{S}^n \setminus z \cong \mathbb{R}^n$, it follows from Example 4.1.2 that there exists a unique homotopy class of path from x to y in $\mathbb{S}^n \setminus r$. Letting γ be a path from x to y in $\mathbb{S}^n$, to prove the claim it is enough to prove that γ can be homotoped into $\mathbb{S}^n \setminus z$. This is nontrivial since there do exist space-filling curves in $\mathbb{S}^n$.

One way to do this is to use smooth manifold techniques. Indeed, it follows from standard results that γ can be homotoped to a smooth map that is transverse to z. The point z is 0-dimensional, and thus is a codimension n submanifold of $\mathbb{S}^n$. It follows that $\gamma^{-1}(z)$ is a codimension $n \geq 2$ submanifold of I, and thus that $\gamma^{-1}(z) = \emptyset$.

Here is another approach that avoids using any technology. As in the following figure, let $V \cong \mathbb{R}^n$ be a small open neighborhood of z with $x, y \notin V$:

[3]See Exercise 4.7 for the origin of this terminology.



The subspace $V \cong \mathbb{R}^n$ is 1-connected (Example 4.1.2), and $V \setminus z$ is path-connected. Intuitively, we should be able to make a small homotopy to the portions of γ that pass through V to make them miss z. Indeed, this is what the smooth manifold approach in the previous paragraph did. Lemma 4.5.3 below shows that this is in fact possible even in more general settings where smooth manifold techniques are unavailable. $\square$

The above proof used the following result:

LEMMA 4.5.3 (General position). *Let X be a space, let $x, y \in X$, and let γ be a path in X from x to y. For some $z \in X$ with $z \neq x, y$, assume there is an open neighborhood V of z such that:*

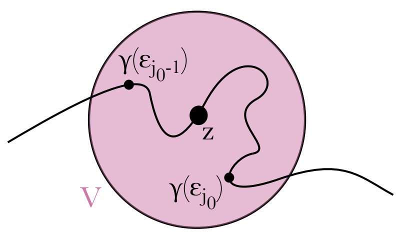
- *V is 1-connected; and*
- *$V \setminus z$ is path-connected.*

Then γ can be homotoped such that its image does not contain z.

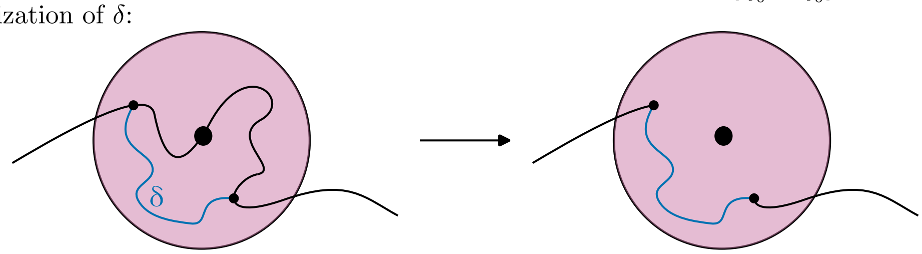
PROOF. Set $U = X \setminus z$. The set $\{U, V\}$ is an open cover of X, so by the Lebesgue number lemma (cf. the proof of Lemma 3.4.1) we can find

$$0 = \epsilon_0 < \epsilon_1 < \cdots < \epsilon_k = 1$$

such that $\gamma([\epsilon_{j-1}, \epsilon_j])$ is contained in either U or V. After possibly deleting some ϵ_j whose adjacent intervals are mapped to the same open set we can also assume that for all $1 \leq j \leq k-1$ we have $\gamma(\epsilon_j) \in U \cap V$. In particular, $\gamma(\epsilon_j) \neq z$ for all $0 \leq j \leq k$. Consider some j_0 such that $\gamma([\epsilon_{j_0-1}, \epsilon_{j_0}]) \subset V$:



Since $V \setminus z$ is path-connected, there is some path δ in $V \setminus z$ from $\gamma(\epsilon_{j_0-1})$ to $\gamma(\epsilon_{j_0})$. Since V is 1-connected, the path obtained by re-parameterizing $\gamma|_{[\epsilon_{j_0-1}, \epsilon_{j_0}]}$ to make its domain $I = [0, 1]$ is homotopic to δ. It follows that we can homotope γ to change $\gamma|_{[\epsilon_{j_0-1}, \epsilon_{j_0}]}$ to a suitable re-parametrization of δ:



This ensures that the image of $\gamma|_{[\epsilon_{i_0-1}, \epsilon_{i_0}]}$ does not contain z. Doing this repeatedly homotopes γ to a path that avoids z, as desired. $\square$

4.6. 1-connectivity and covers

Recall that a trivial cover of a space X is a cover that is isomorphic to a cover of the form $X \times \mathcal{I} \to X$ for some discrete set $\mathcal{I}$. We will prove below that if X is 1-connected and has reasonable local properties, then all covers of X are trivial. For instance, this is why we have not seen any nontrivial covers of $\mathbb{S}^n$ for $n \geq 2$.

A space X is *locally path connected* if for all $x \in X$ and all open neighborhoods U of x, there is a path connected open neighborhood V of x with $V \subset U$. Most geometrically natural spaces are locally path connected; for instance, all manifolds have this property. We have:

THEOREM 4.6.1. *Let X be a 1-connected space that is locally path connected and let $p\colon \widetilde{X} \to X$ be a cover. Then $p\colon \widetilde{X} \to X$ is trivial.*

REMARK 4.6.2. This would be false without the assumption that X is locally path connected. See Exercise 4.11. □

PROOF OF THEOREM 4.6.1. Fix a point $x_0 \in X$. Consider some $x \in X$. Let γ be a path in X from x_0 to x and let

$$\tau_\gamma\colon p^{-1}(x_0) \longrightarrow p^{-1}(x)$$

be the map constructed in §4.3 by lifting γ to $\widetilde{X}$ starting at different points of $p^{-1}(x_0)$. As we observed in §4.3, the map τ_γ only depends on the homotopy class of γ. Since X is 1-connected, all choices of γ are homotopic, so τ_γ only depends on x. We therefore write $\tau_x = \tau_\gamma$.

We proved in Lemma 4.3.1 that τ_x is a bijection from $p^{-1}(x_0)$ to $p^{-1}(x)$. Every point of $\widetilde{X}$ lies in $p^{-1}(x)$ for some unique $x \in X$. Letting $\mathcal{I} = p^{-1}(x_0)$, we can therefore define a bijective set map $\phi\colon X \times \mathcal{I} \to \widetilde{X}$ as follows:

$$\phi(x, \widetilde{x}_0) = \tau_x(\widetilde{x}_0) \quad \text{for } x \in X \text{ and } \widetilde{x}_0 \in \mathcal{I} = p^{-1}(x_0).$$

Let $q\colon X \times \mathcal{I} \to X$ be the projection onto the first factor. By construction, ϕ fits into a commutative diagram

$$\begin{tikzcd} X \times \mathcal{I} \arrow[rr, "\phi"] \arrow[dr, "q"'] & & \widetilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

To prove that ϕ is an isomorphism from the trivial cover $q\colon X \times \mathcal{I} \to X$ to $p\colon \widetilde{X} \to X$, we must prove that ϕ is a homeomorphism. At this point, we remark that we do not even know that ϕ is continuous.

Since ϕ is a bijection, to prove that it is continuous and a homeomorphism it is enough to prove that it is a local homeomorphism. Consider some $x \in X$ and $\widetilde{x}_0 \in \mathcal{I} = p^{-1}(x_0)$. Set $\widetilde{x} = \phi(x, \widetilde{x}_0) = \tau_x(\widetilde{x}_0)$, so $\widetilde{x} \in p^{-1}(x)$. Let $U \subset X$ be a trivialized neighborhood of x and let $\widetilde{U} \subset \widetilde{X}$ be the sheet lying above U with $\widetilde{x} \in \widetilde{U}$. Since X is locally path connected, we can shrink U and assume that it is path connected. We claim that the restriction of ϕ to $U \times \widetilde{x}_0$ is the composition

$$U \times \widetilde{x}_0 \xrightarrow{q} U \xrightarrow{(p|_{\widetilde{U}})^{-1}} \widetilde{U}.$$

Since this is a homeomorphism between the open sets $U \times \widetilde{x}_0$ and $\widetilde{U}$, this will give the theorem.

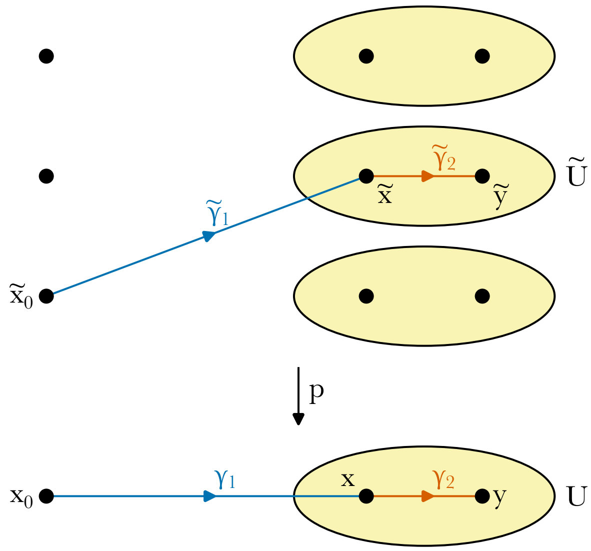
To see this claim, consider some $y \in U$. Let γ be a path in X from x_0 to y, and let $\widetilde{\gamma}$ be the lift of γ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$. Set $\widetilde{y} = \widetilde{\gamma}(1)$, so $\widetilde{y} \in p^{-1}(y)$. What we must prove is that $\widetilde{y} \in \widetilde{U}$. In fact, since X is 1-connected any two paths from x_0 to y are homotopic, so we can choose γ to be any such path we like.

Let γ_1 be a path in X from x_0 to x and let γ_2 be a path in the path-connected subspace U from x to y. Let $\gamma\colon I \to X$ be the path[4]

$$\gamma(s) = \begin{cases} \gamma_1(2s) & \text{if } 0 \le s \le 1/2, \\ \gamma_2(2s-1) & \text{if } 1/2 \le s \le 1. \end{cases}$$

The path γ thus first goes along γ_1 at $2\times$ speed and then goes along γ_2 at $2x$ speed. Let $\widetilde{\gamma}_1$ be the lift of γ_1 to $\widetilde{X}$ with $\widetilde{\gamma}_1(0) = \widetilde{x}_0$. By assumption, $\widetilde{\gamma}_1(1) = \phi(x, \widetilde{x}_0) = \widetilde{x} \in \widetilde{U}$. Let $\widetilde{\gamma}_2$ be the lift of γ_2 to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}$:

[4]In the next chapter, we will systemize this kind of operation. In the notation of that chapter, we have $\gamma = \gamma_1 \cdot \gamma_2$.



As this figure shows, the lift $\widetilde{\gamma}$ of γ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$ is

$$\widetilde{\gamma}(s) = \begin{cases} \widetilde{\gamma}_1(2s) & \text{if } 0 \leq s \leq 1/2, \\ \widetilde{\gamma}_2(2s-1) & \text{if } 1/2 \leq s \leq 1. \end{cases}$$

Since the image of γ_2 lies in U and $\widetilde{\gamma}_2(0) \in \widetilde{U}$, it follows that $\widetilde{\gamma}_2$ is the composition

$$I \xrightarrow{\gamma_2} U \xrightarrow{(p|_{\widetilde{U}})^{-1}} \widetilde{U}.$$

In particular, $\widetilde{y} = \widetilde{\gamma}_2(1) \in \widetilde{U}$, as desired. □

4.7. Exercises

Exercise 4.1. Let X be a path-connected space and let $\gamma_0, \gamma_1 \colon I \to X$ be two paths in X. Prove that γ_0 is homotopic to γ_1 if we do not require the homotopy to fix the endpoints of the path. □

Exercise 4.2. Let $p \colon \widetilde{X} \to X$ be a cover and let $\gamma \colon I \to X$ be a path in X. Assume that γ is an embedding. Let $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ be two lifts of γ to $\widetilde{X}$ such that $\widetilde{\gamma}_1(0) \neq \widetilde{\gamma}_2(0)$. Prove that the images of $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ are disjoint. □

Exercise 4.3. Let $n_1, \ldots, n_m \geq 2$. Fix basepoints $x_i \in \mathbb{S}^{n_i}$ for all $1 \leq i \leq n_m$. Define $\mathbb{S}^{n_1} \vee \cdots \vee \mathbb{S}^{n_m}$ to be the space obtained from the disjoint union $\mathbb{S}^{n_1} \sqcup \cdots \sqcup \mathbb{S}^{n_m}$ by identifying all the x_i to a single point p_0 (this is called the *wedge sum* of the $\mathbb{S}^{n_i}$; see §8.6). Prove that $\mathbb{S}^{n_1} \vee \cdots \vee \mathbb{S}^{n_m}$ is 1-connected. □

Exercise 4.4. Prove that a nonempty space X is 1-connected if and only if any two maps $f, g \colon \mathbb{S}^1 \to X$ are homotopic. □

Exercise 4.5. Calculate a complete set of homotopy classes of paths from $1 \in \mathbb{S}^1 \subset \mathbb{C}$ to $-1 \in \mathbb{S}^1 \in \mathbb{C}$. Hint: the ideas from Example 4.1.3 will be helpful. □

Exercise 4.6. As discussed in Example 1.4.4, let $\mathrm{Poly}_n^{\mathrm{sf}}$ be the space of monic degree-n polynomials without repeated roots, let $\mathrm{RPoly}_n^{\mathrm{sf}}$ be the space of pairs (f, x) with $f \in \mathrm{Poly}_n^{\mathrm{sf}}$ and $f(x) = 0$, and let $p \colon \mathrm{RPoly}_n^{\mathrm{sf}} \to \mathrm{Poly}_n^{\mathrm{sf}}$ be the map $p(f, x) = f$, so p is a degree n covering space. Fix points $f, g \in \mathrm{Poly}_n^{\mathrm{sf}}$. Letting $x_1, \ldots, x_n \in \mathbb{C}$ be the roots of f and $y_1, \ldots, y_n \in \mathbb{C}$ be the roots of g, we have

$$p^{-1}(f) = \{(f, x_1), \ldots, (f, x_n)\},$$
$$p^{-1}(g) = \{(g, y_1), \ldots, (g, y_n)\}.$$

Let $\mathfrak{S}_n$ be the symmetric group on n elements and let $\sigma \in \mathfrak{S}_n$. Prove that there is a path γ in $\mathrm{Poly}_n^{\mathrm{sf}}$ from f to g such that the map $\phi_\gamma\colon p^{-1}(f) \to p^{-1}(g)$ is given by

$$\phi_\gamma(f, x_i) = (g, y_{\sigma(i)}) \quad \text{for all } 1 \leq i \leq n.$$

Hint: the ideas from the proof of Lemma 3.3.1 might be helpful. □

EXERCISE 4.7. This exercise explains why we defined 0- and 1-connectivity the way we did. The sphere $\mathbb{S}^d$ and the disk $\mathbb{D}^d$ are usually defined for $d \geq 0$. As a convention, for any $d < 0$ define $\mathbb{S}^d = \emptyset$ and $\mathbb{D}^d = \emptyset$. For all $d \in \mathbb{Z}$, the space $\mathbb{D}^{d+1}$ contains $\mathbb{S}^d$ as a subspace. Say that a space X is *n-connected* if the following holds for all $d \leq n$:

- All continuous maps $\mathbb{S}^d \to X$ extend to continuous maps $\mathbb{D}^{d+1} \to X$.

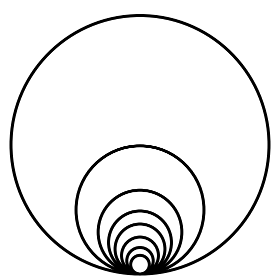
Prove the following using the above definition for 0- and 1-connectivity rather than the one we gave in the text:

(a) All spaces are n-connected for $n \leq -2$.
(b) A space X is -1-connected if and only if it is nonempty.
(c) A space X is 0-connected if and only if it is nonempty and path-connected.
(d) A space X is 1-connected if and only if it is nonempty, path-connected, for all $x, y \in X$ there is a unique homotopy class of paths from x to y. □

EXERCISE 4.8. Let X be a graph and let $p\colon \widetilde{X} \to X$ be a cover. Prove that $\widetilde{X}$ is a graph. Hint: Start by adding vertices to the interiors of edges as necessary to ensure that X has no loops (make sure this does not change the truth of the exercise!). Next, use Theorem 4.6.1 to prove that the restriction of $p\colon \widetilde{X} \to X$ to each edge of X is trivial. □

EXERCISE 4.9. Let $p\colon \widetilde{X} \to X$ be a covering space, let $\widetilde{Y}$ be a path component of $\widetilde{X}$, and let $q = p|_{\widetilde{Y}}$. Assume that X is locally path connected. Prove that $q\colon \widetilde{Y} \to X$ is a covering space. □

EXERCISE 4.10. For $n \geq 1$, let $C_n \subset \mathbb{R}^2$ be the circle of radius $1/n$ with center $(0, 1/n)$. Let $X = \cup_{n=1}^{\infty} C_n$, topologized as a subspace of $\mathbb{R}^2$. This is sometimes called the "earring space" or the "shrinking wedge of circles":

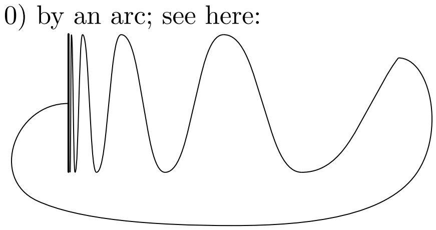


Construct a collection of covers $\{p_i\colon \widetilde{X}_i \to X\}_{i \in I}$ with the following property. Let $\widetilde{X}$ be the disjoint union of the $\widetilde{X}_i$ and let $p\colon \widetilde{X} \to X$ be the map that is p_i on X_i for all $i \in I$. Then $p\colon \widetilde{X} \to X$ is not a cover. This shows that the converse to Exercise 4.9 is false. □

EXERCISE 4.11. The *quasi-circle* is the space Y obtained from the topologist's sine curve

$$X = \{(x, \sin(1/x)) \in \mathbb{R}^2 \mid 0 < x \leq 1\} \cup \{(0, y) \mid -1 \leq y \leq 1\}$$

by connecting $(1, \sin(1))$ to $(0, 0)$ by an arc; see here:



Prove the following:

(a) The space Y is 1-connected.
(b) There exists a nontrivial cover $p\colon \widetilde{Y} \to Y$. □

EXERCISE 4.12. Let $p\colon \mathbb{R} \to \mathbb{S}^1$ be the universal cover. Set $\mathcal{I} = p^{-1}(1)$. For $z \in \mathbb{S}^1$, let $\gamma_z\colon I \to \mathbb{S}^1$ be the path that goes from 1 to z clockwise at constant speed and let

$$\tau_{\gamma_z}\colon \mathcal{I} = p^{-1}(1) \longrightarrow p^{-1}(z)$$

be the map constructed in §4.3 by lifting γ_z to $\mathbb{R}$ starting at different points of $\mathcal{I}$. Define $\phi\colon \mathbb{S}^1 \times \mathcal{I} \to \mathbb{R}$ as follows:

$$\phi(z, \widetilde{z}_0) = \tau_{\gamma_z}(\widetilde{z}_0) \quad \text{for } z \in \mathbb{S}^1 \text{ and } \widetilde{z}_0 \in \mathcal{I}.$$

Give an explicit formula for ϕ, and use this formula to show that ϕ is a non-continuous bijection of sets. $\square$

CHAPTER 5

Fundamental group: definition and basic properties

As the last chapter showed, there is a close connection between covers of a space X and the collection of homotopy classes of paths in X. In this section, we organize the collection of homotopy classes of paths in algebraic objects called the fundamental group and groupoid.

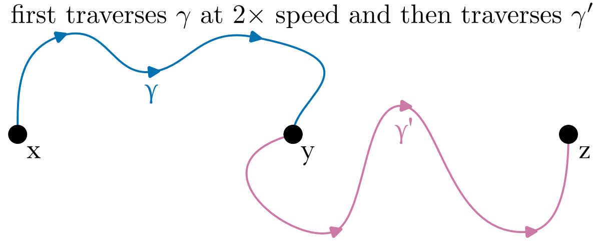
5.1. Multiplying homotopy classes of paths

Let X be a space. Our goal is to endow the set of homotopy classes of paths between points of X with an algebraic structure. In this structure, only some paths can be "multiplied". The definition is as follows:

Definition 5.1.1. Let $\gamma\colon I \to X$ and $\gamma'\colon I \to X$ be paths between points of X. We say that γ and γ' are *composable* if the terminal point of γ equals the initial point of γ'. If γ and γ' are composable, then $\gamma\cdot\gamma'\colon I \to X$ is the path defined by the formula

$$(\gamma\cdot\gamma')(s) = \begin{cases} \gamma(2s) & \text{if } 0 \le s \le 1/2, \\ \gamma'(2s-1) & \text{if } 1/2 \le s \le 1. \end{cases} \quad \text{for } s \in I. \qquad \square$$

In other words, $\gamma\cdot\gamma'$ first traverses γ at $2\times$ speed and then traverses γ' at $2\times$ speed:



If γ goes from x to y and γ' goes from y to z, then $\gamma\cdot\gamma'$ goes from x to z. This only makes sense if γ and γ' are composable, and we do not define $\gamma\cdot\gamma'$ if they are not.

Remark 5.1.2. Being composable is *not* symmetric: if $\gamma\cdot\gamma'$ is defined, then it need not be the case that $\gamma'\cdot\gamma$ is defined. $\square$

For a path $\gamma\colon I \to X$, recall that $[\gamma]$ denotes its homotopy class. The following lemma says that our "multiplication" descends to a multiplication on homotopy classes:

Lemma 5.1.3. *Let X be a space. Let γ_0 and γ_0' be composable paths in X. Let γ_1 be a path that is homotopic to γ_0 and let γ_1' be a path that is homotopic to γ_0', so $[\gamma_0] = [\gamma_1]$ and $[\gamma_0'] = [\gamma_1']$. Then $[\gamma_0\cdot\gamma_0'] = [\gamma_0'\cdot\gamma_1']$.*

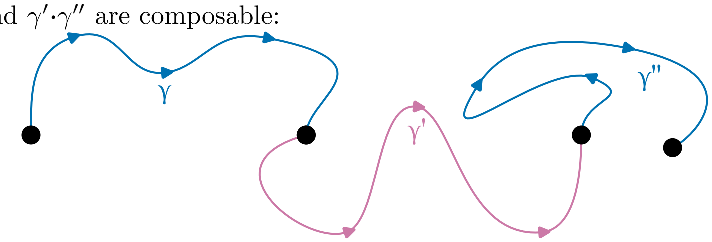
Proof. Assume that γ_0 goes from x to y and that γ_0' goes from y to z. Let γ_t be a homotopy from γ_0 to γ_1 and let γ_t' be a homotopy from γ_0' to γ_1'. For each $t \in I$, we have

$$\gamma_t(0) = x \quad \text{and} \quad \gamma_t(1) = \gamma_t'(0) = y \quad \text{and} \quad \gamma_t'(1) = z,$$

so γ_t and γ_t' are composable and $\gamma_t\cdot\gamma_t'$ is a well-defined path from x to z. As t varies over I, the paths $\gamma_t\cdot\gamma_t'$ form a homotopy from $\gamma_0\cdot\gamma_0'$ to $\gamma_1\cdot\gamma_1'$. $\square$

5.2. Properties of multiplication

Let X be a space. This section explores properties of our multiplication that resemble the properties of a group. We start with associativity. Let γ and γ' and γ'' be paths in X such that γ and γ' are composable and γ' and γ'' are composable. It follows that $\gamma\cdot\gamma'$ and γ'' are composable, and also that γ and $\gamma' \cdot \gamma''$ are composable:



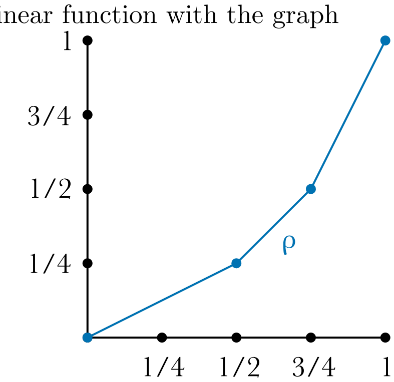
Both $(\gamma \cdot \gamma') \cdot \gamma''$ and $\gamma \cdot (\gamma' \cdot \gamma'')$ thus make sense; however except in degenerate cases we have $(\gamma \cdot \gamma') \cdot \gamma'' \neq \gamma \cdot (\gamma' \cdot \gamma'')$. The following lemma shows that passing to homotopy fixes this:

Lemma 5.2.1 (Associativity). *Let X be a space. Let γ and γ' and γ'' be paths in X such that γ and γ' are composable and γ' and γ'' are composable. Then $[(\gamma \cdot \gamma') \cdot \gamma''] = [\gamma \cdot (\gamma' \cdot \gamma'')]$.*

Proof. The paths $f_1 = (\gamma \cdot \gamma') \cdot \gamma''$ and $f_2 = \gamma \cdot (\gamma' \cdot \gamma'')$ are almost the same. They both traverse γ and then γ' and then γ''; however, they do this at different speeds. As functions on $I = [0, 1]$, we have the following:

- The path f_1 traverses γ at $4\times$ speed on the interval $[0, 1/4]$, then γ' at $4\times$ speed on the interval $[1/4, 1/2]$, and then γ'' at $2\times$ speed on the interval $[1/2, 1]$.
- The path f_2 traverses γ at $2\times$ speed on the interval $[0, 1/2]$, then γ' at $4\times$ speed on the interval $[1/2, 3/4]$, and then γ'' at $4\times$ speed on the interval $[3/4, 1]$.

Let $\rho \colon I \to I$ be the piecewise linear function with the graph



We then have $f_2 = f_1 \circ \rho$. The lemma now follows from Lemma 5.2.2 below. □

Lemma 5.2.2 (Reparameterization lemma). *Let X be a space and $\gamma \colon I \to X$ be a path. Let $\rho \colon I \to I$ be a function such that $f(0) = 0$ and $f(1) = 1$. Then $[\gamma \circ \rho] = [\gamma]$.*

Proof. The desired homotopy from $\gamma \circ \rho$ to γ is given by

$$\gamma_t(s) = \gamma((1-t)\rho(s) + ts) \quad \text{for } t, s \in I.$$

Here we use the fact that $f(0) = 0$ and $f(1) = 1$ to ensure that the endpoints of γ_t do not move:

$$\gamma_t(0) = \gamma((1-t)\rho(0) + 0) = \gamma(0) \quad \text{and} \quad \gamma_t(1) = \gamma((1-t)\rho(1) + t) = \gamma(1 - t + t) = \gamma(1).$$ □

We now turn to multiplicative identities. For a point $x \in X$, let $\mathfrak{c}_x \colon I \to X$ be the constant path

$$\mathfrak{c}_x(s) = x \quad \text{for } s \in I.$$

This serves as an identity for our multiplication. However, since we can only multiply composable paths an appropriate $\mathfrak{c}_x$ must be chosen for the left- and right-identities of any given path:

Lemma 5.2.3 (Multiplicative identities). *Let X be a space and let γ be a path in X from x to y. Then $[\gamma \cdot \mathfrak{c}_x] = [\gamma]$ and $[\mathfrak{c}_y \cdot \gamma] = [\gamma]$.*

Proof. The path $\gamma \cdot \mathfrak{c}_x$ stays at x on the interval $[0, 1/2]$ and then traverses γ at $2\times$ speed:

$$(\gamma \cdot \mathfrak{c}_x)(s) = \begin{cases} x & \text{if } s \in [0, 1/2], \\ \gamma(2s - 1) & \text{if } s \in [1/2, 1]. \end{cases} \quad \text{for } s \in I.$$

Letting $\rho\colon I \to I$ be the map

$$\rho(s) = \begin{cases} 0 & \text{if } s \in [0, 1/2], \\ 2s - 1 & \text{if } s \in [1/2, 1] \end{cases} \quad \text{for } s \in I,$$

we thus have $\gamma{\cdot}\mathfrak{c}_x = \gamma \circ \rho$. Applying Lemma 5.2.2, we see that $[\gamma{\cdot}\mathfrak{c}_x] = [\gamma \circ \rho] = [\gamma]$, as desired. The proof that $[\mathfrak{c}_y{\cdot}\gamma] = [\gamma]$ is similar. $\square$

Having found identities, our final goal is to find inverses. Let γ be a path in X from x to y. Define $\overline{\gamma}\colon I \to X$ to be the path that traverses γ in the reverse order:

$$\overline{\gamma}(s) = \gamma(1 - s) \quad \text{for } s \in I.$$

The path $\overline{\gamma}$ goes from y to x, and serves as a sort of "inverse" to our multiplication:

Lemma 5.2.4 (Inverses). *Let X be a space and let γ be a path in X from x to y. Then $[\gamma{\cdot}\overline{\gamma}] = [\mathfrak{c}_x]$ and $[\overline{\gamma}{\cdot}\gamma] = [\mathfrak{c}_y]$.*

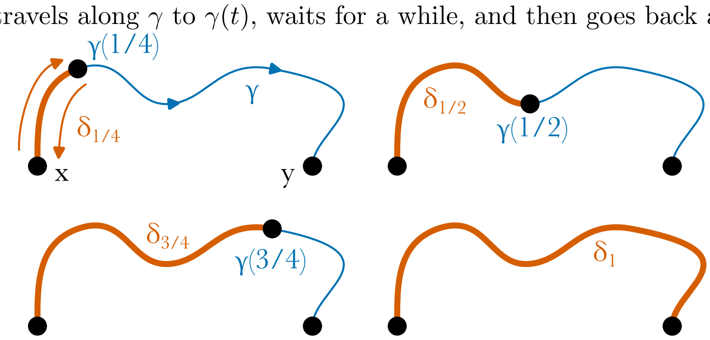
Proof. The path $\gamma{\cdot}\overline{\gamma}$ goes from x to x. For $t \in I$, define $\delta_t\colon I \to X$ to be the path

$$\delta_t(s) = \begin{cases} \gamma(2s) & \text{if } s \in [0, t/2], \\ \gamma(t) & \text{if } s \in [t/2, 1 - t/2], \\ \gamma(2(1-s)) & \text{if } s \in [1 - t/2, 1]. \end{cases} \quad \text{for } s \in I.$$

This makes sense since

$$\gamma(2(t/2)) = \gamma(t) = \gamma(2(1 - (1 - t/2))).$$

Geometrically, δ_t travels along γ to $\gamma(t)$, waits for a while, and then goes back along $\overline{\gamma}$:



Since δ_t is a homotopy from $\mathfrak{c}_x$ to $\gamma{\cdot}\overline{\gamma}$, we deduce that $[\mathfrak{c}_x] = [\gamma{\cdot}\overline{\gamma}]$, as desired. The proof that $[\overline{\gamma}{\cdot}\gamma] = [\mathfrak{c}_y]$ is similar. $\square$

5.3. Categorical language

Let X be a space. In the previous sections, we showed that the set of homotopy classes of paths between points of X has a partially-defined "multiplication" that is associative, has identities, and has inverses. What kind of algebraic structure could this be?

To answer this question, we need the language of category theory. Recall that a category $\mathbf{C}$ consists of the following data:

- A collection of objects. We will write $A \in \mathbf{C}$ to indicate that A is an object of $\mathbf{C}$.
- For all objects $A, B \in \mathbf{C}$, a set $\mathbf{C}(A, B)$ of morphisms. We will often write $f\colon A \to B$ to indicate that f is a morphism from A to B.
- For all objects $A \in \mathbf{C}$, an identity morphism $\mathbb{1}_A\colon A \to A$.

These morphisms can be composed: if $f\colon A \to B$ and $g\colon B \to C$ are morphisms, then we have a morphism $g \circ f\colon A \to C$. This composition should be associative in the sense that if $f\colon A \to B$ and $g\colon B \to C$ and $h\colon C \to D$ are morphisms, then

$$(f \circ g) \circ h = f \circ (g \circ h).$$

Because of this, there is no need to insert parentheses when composing morphisms. Under this composition, the identity morphisms should be units: if $f\colon A \to B$ is a morphism, then $f \circ \mathbb{1}_A = f$ and $\mathbb{1}_B \circ f = f$.

EXAMPLE 5.3.1. The collection of all sets and set maps forms a category $\mathtt{Set}$. □

EXAMPLE 5.3.2. The collection of all topological spaces and continuous maps forms a category $\mathtt{Top}$. □

EXAMPLE 5.3.3. The collection of all groups and homomorphisms forms a category $\mathtt{Group}$. □

EXAMPLE 5.3.4. For a group G, there is a category (also written G) with one object x and with $G(x,x) = G$. □

REMARK 5.3.5. The language of category theorem might seem overly abstract, but it turns out to be very useful and clarifying. Fundamentally, it is just a way of organizing information. Typically you cannot prove interesting new theorems by just defining a category, but the language of category theory often suggests useful constructions. □

5.4. Fundamental groupoid

Our goal now is to encode the homotopy classes of paths in a space X into a category. The objects of this category will be the points of X. For points $x, y \in X$, the morphisms from x to y will be the homotopy classes of paths from x to y. There is one annoying technical point: in a category, composition goes from right to left like functions. However, we multiply paths from left to right: if γ is a path from x to y and γ' is a path from y to z, then $\gamma \cdot \gamma'$ is a path from x to z. To fix this, we introduce the following notation:

NOTATION 5.4.1. Let X be a space. For points $x, y, z \in X$, let γ be a path in X from x to y and let δ be a path in X from y to z. We then define $\gamma' * \gamma = \gamma \cdot \gamma'$. This descends to homotopy classes of paths, and we also write $[\gamma'] * [\gamma] = [\gamma' * \gamma]$. □

We now define the following:

DEFINITION 5.4.2. Let X be a space. The *fundamental groupoid* of X, denoted $\Pi(X)$, is the following category:

- The objects of $\Pi(X)$ are the points of X.
- For points x and y, the $\Pi(X)$-morphisms from x to y are the set of all homotopy classes of paths from x to y. For a path γ from x to y, we will write $[\gamma]\colon x \to y$ for the corresponding morphism from x to y.
- If γ is a path from x to y and γ' is a path from y to z, then the composition of the morphisms $[\gamma]\colon x \to y$ and $[\gamma']\colon y \to z$ is the morphism $[\gamma'] * [\gamma]\colon x \to z$.
- For a point $x \in X$, the identity morphism of x is the constant path $[\mathfrak{c}_x]\colon x \to x$. □

Lemma 5.2.4 says that all the morphisms in the fundamental groupoid $\Pi(X)$ are invertible. This is the defining property of a groupoid:

DEFINITION 5.4.3. A *groupoid* is a category $\mathbf{G}$ in which all morphisms are invertible, i.e., such that for all morphisms $\phi\colon A \to B$, there is a morphism $\overline{\phi}\colon B \to A$ with $\overline{\phi} \circ \phi = \mathbb{1}_A$ and $\phi \circ \overline{\phi} = \mathbb{1}_B$. □

REMARK 5.4.4. Let $\mathbf{G}$ be a groupoid and $\phi\colon A \to B$ be a morphism in $\mathbf{G}$. In Exercise 5.7, you will prove that the inverse to ϕ is unique in the following sense. Consider $\overline{\phi}, \overline{\phi}'\colon B \to A$. Then $\phi = \overline{\phi}'$ if any of the following conditions are satisfied:

- $\overline{\phi} \circ \phi = \overline{\phi}' \circ \phi = \mathbb{1}_A$; or
- $\phi \circ \overline{\phi} = \phi \circ \overline{\phi}' = \mathbb{1}_B$; or
- $\overline{\phi} \circ \phi = \mathbb{1}_A$ and $\phi \circ \overline{\phi}' = \mathbb{1}_B$.

Because of this, we can safely talk about *the* inverse to ϕ. □

As we discussed in Example 5.3.4, a group can be viewed as a category with one object. Under this identification, a group is a groupoid. Conversely, consider a groupoid $\mathbf{G}$. For $A \in \mathbf{G}$, write

$$\mathrm{Aut}_{\mathbf{G}}(A) = \{f \mid f\colon A \to A \text{ is a morphism in } \mathbf{G}\}.$$

Since all morphisms in $\mathbf{G}$ are invertible, this is a group. What is more, for a morphism $\psi\colon A \to B$ in $\mathbf{G}$ there is an isomorphism $\psi_*\colon \mathrm{Aut}_{\mathbf{G}}(A) \to \mathrm{Aut}_{\mathbf{G}}(B)$ defined by

$$\psi_*(\phi) = \psi \circ \phi \circ \overline{\psi} \quad \text{for all } \phi\colon A \to A \text{ in } \mathrm{Aut}_{\mathbf{G}}(A).$$

In this way, a groupoid packages together a collection of groups along with certain isomorphisms between them.

5.5. Fundamental group

Let X be a space. For $x_0 \in X$ the *fundamental group* of X with basepoint x_0, denoted $\pi_1(X, x_0)$, is

$$\pi_1(X, x_0) = \mathrm{Aut}_{\Pi(X)}(x_0).$$

In other words, $\pi_1(X, x_0)$ is the group whose objects are homotopy classes of *loops based at* x_0, i.e., paths γ from x_0 to itself. In the fundamental group, we will use the concatanation product $\cdot$ rather than $*$. This does not change the group (see Exercise 5.8).

If α is a path from x_0 to x_0', then we get an isomorphism

$$\alpha_*\colon \pi_1(X, x_0') \to \pi_1(X, x_0)$$

defined by

$$\alpha_*([\gamma]) = [\alpha\cdot\gamma\cdot\overline{\alpha}] \quad \text{for all } [\gamma] \in \pi_1(X, x_0').$$

From these isomorphisms, we see the following:

LEMMA 5.5.1. *Let X be a path-connected space. Then for all $x_0, x_0' \in X$ we have $\pi_1(X, x_0) \cong \pi_1(X, x_0')$.*

PROOF. Just use the above isomorphism associated to a path from x_0 to x_0'. □

We will give many computations of $\pi_1(X, x_0)$ over the next few chapters. For X path-connected, Lemma 5.5.1 says that the isomorphism type of $\pi_1(X, x_0)$ is independent of the basepoint x_0. The isomorphism type of $\pi_1(X, x_0)$ is thus a useful invariant of path-connected spaces, i.e., if two path-connected spaces have different fundamental groups, then they are not homeomorphic. The fundamental groupoid is not so useful as an invariant since it knows far too much about the space; for instance, its objects are literally the points of the space.

You might wonder why we bothered to introduce the fundamental groupoid at all. There are two reasons:

- While for a path-connected space the isomorphism type of the fundamental group does not depend on the basepoint, the isomorphisms between the fundamental groups at different basepoints are not canonical. The fundamental groupoid packages them all together, and is present at least implictly in all serious treatements of the fundamental group. It seems perverse to refuse to give a name to a structure you use.
- There are many constructions in topology that are most naturally phrased in terms of the fundamental groupoid. For instance, the most general form of the classification of covering spaces uses the fundamental groupoid (see Chapter 11). Later volumes of this book will contain other examples.

We remark that serious applications of $\pi_1(X, x_0)$ often require a careful treatment of the basepoint x_0. Simply identifying the fundamental group at different basepoints will quickly lead you astray. This is analogous to the fact that while all finite-dimensional vector spaces over a field $\mathbf{k}$ are isomorphic to $\mathbf{k}^n$ for some $n \geq 0$, one cannot simply identify vector spaces with $\mathbf{k}^n$. Such an identification requires a choice of basis, and often there is no natural choice. Much of linear algebra focuses on carefully choosing bases adapted to different situations and studying how all these different bases are related.

5.6. Functoriality: fundamental group

Before discussing how to calculate the fundamental group, we must study the way in which the fundamental group and groupoid of X depend on X. We start with the fundamental group. Consider a map $f\colon X \to Y$ and $x_0 \in X$. If γ is a loop in X based at x_0, then $f \circ \gamma$ is a loop in Y based at $f(x_0)$. The loop $[f \circ \gamma]$ only depends on the homotopy class of γ, and the map $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ defined by

$$f_*([\gamma]) = [f \circ \gamma] \quad \text{for all } [\gamma] \in \pi_1(X, x_0)$$

is a homomorphism called the homomorphism *induced* by f.

Since the fundamental group is the group of homotopy classes of loops, one expects the homomorphism induced by a map to only depend on the homotopy class of the map. However, this is not quite right since we have to be careful about the basepoint. To state things properly, we introduce the following terminology:

Definition 5.6.1. A *pointed space* is a pair (X, x_0) with X a space and $x_0 \in X$. A map between pointed space (X, x_0) and (Y, y_0) is a map $f\colon X \to Y$ such that $f(x_0) = y_0$. We will denote such a map by $f\colon (X, x_0) \to (Y, y_0)$. A homotopy of maps from (X, x_0) to (Y, y_0) is a homotopy $f_t\colon X \to Y$ such that $f_t(x_0) = y_0$ for all $t \in I$. Just like for maps, we will denote this by $f_t\colon (X, x_0) \to (Y, y_0)$, and if such an f_t exists we will say that $f_0\colon (X, x_0) \to (Y, y_0)$ and $f_1\colon (X, x_0) \to (Y, y_0)$ are homotopic. $\square$

With this setup, a map $f\colon (X, x_0) \to (Y, y_0)$ between pointed spaces induces a homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$, and if $f\colon (X, x_0) \to (Y, y_0)$ and $g\colon (X, x_0) \to (Y, Y_0)$ are homotopic maps between pointed spaces then $f_* = g_*$. The homomorphisms induced by maps of pointed spaces have the following two simple properties:

- for maps of pointed space $f\colon (X, x_0) \to (Y, y_0)$ and $g\colon (Y, y_0) \to (Z, z_0)$, we have $(g \circ f)_* = g_* \circ f_*$; and
- the identity map $\mathbb{1}\colon (X, x_0) \to (X, x_0)$ induces the identity homomorphism, i.e., $\mathbb{1}_* = \mathbb{1}$.

All of this can be summarized in categorical language as follows. Recall that if $\mathbf{C}$ and $\mathbf{D}$ are categories, then a *functor* $F\colon \mathbf{C} \to \mathbf{D}$ consists of the following data:

- For all objects $C \in \mathbf{C}$, an object $F(D) \in \mathbf{D}$.
- For all morphisms $f\colon C_1 \to C_2$ between objects of $\mathbf{C}$, a morphism $F(f)\colon F(C_1) \to F(C_2)$.

These are required to satisfy:

- for all morphisms $f\colon C_1 \to C_2$ and $g\colon C_2 \to C_3$ between objects of $\mathbf{C}$, we have $F(g \circ f) = F(g) \circ F(f)$; and
- for all identity morphisms $\mathbb{1}_C\colon C \to C$ in $\mathbf{C}$, we have $F(\mathbb{1}_C) = \mathbb{1}_{F(C)}$.

To fit the fundamental group into this, let $\mathtt{Top}_*$ be the category of pointed spaces, so the objects of $\mathtt{Top}_*$ are pointed spaces (X, x_0) and the morphisms in $\mathtt{Top}_*$ are the maps $f\colon (X, x_0) \to (Y, y_0)$ between pointed spaces. We can then summarize our discussion by:

Lemma 5.6.2. *The fundamental group is a functor* $\pi_1\colon \mathtt{Top}_* \to \mathtt{Group}$.

5.7. Homotopies that move the basepoint

Let (X, x_0) be a pointed space and let $f_0, f_1\colon X \to Y$ be two homotopic maps. Set $y_0 = f_0(x_0)$ and $y_1 = f_1(x_0)$. We therefore have maps $f_0\colon (X, x_0) \to (Y, y_0)$ and $f_1\colon (X, x_0) \to (Y, y_1)$. Since their targets are different, it does not make sense to say that the induced maps

$$\begin{aligned} (f_0)_*\colon \pi_1(X, x_0) &\to \pi_1(Y, y_0), \\ (f_1)_*\colon \pi_1(X, x_0) &\to \pi_1(Y, y_1) \end{aligned}$$

are equal. Instead, they are related as follows:

Lemma 5.7.1. *Let X and Y be spaces and let $f_t\colon X \to Y$ be a homotopy of maps from X to Y. Let $x_0 \in X$ be a basepoint, and set $y_0 = f_0(x_0)$ and $y_1 = f_1(x_0)$. Let $\delta\colon I \to Y$ be the following path from y_0 to y_1:*

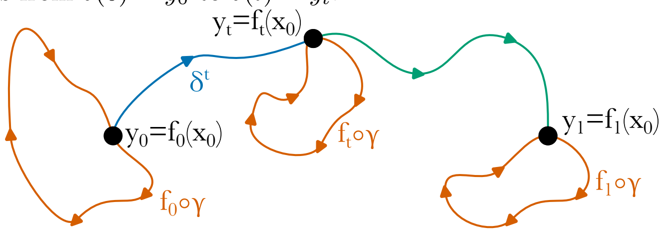
$$\delta(s) = f_s(x_0) \quad \textit{for } s \in I.$$

Then for all $[\gamma] \in \pi_1(X, x_0)$ we have $(f_0)_*([\gamma]) = [\delta](f_1)_*([\gamma])[\overline{\delta}] \in \pi_1(Y, y_0)$.

PROOF. For $t \in I$, let $y_t = \delta(t)$ and let $\delta^t \colon I \to Y$ be the path

$$\delta^t(s) = f_{ts}(x_0) = \delta(ts) \quad \text{for } s \in I.$$

The path δ^t thus goes from $\delta(0) = y_0$ to $\delta(t) = y_t$:



Since $f_t \circ \gamma$ is a loop based at $f_t(x_0) = \delta(t) = y_t$, it follows that the map $t \mapsto \delta^t \cdot (f_t \circ \gamma) \cdot \overline{\delta}^t$ is a homotopy of paths. Since δ^0 is the constant path based at y_0, we conclude that

$$(f_0)_*([\gamma]) = [f_0 \circ \gamma] = [\delta^0 \cdot (f_0 \circ \gamma) \cdot \overline{\delta}^0] = [\delta^1 \cdot (f_1 \circ \gamma) \cdot \overline{\delta}^1] = [\delta](f_1)_*([\gamma])[\overline{\delta}]. \qquad \square$$

5.8. Functoriality: fundamental groupoid

It will play less of a role in this book, but for completeness we now explain how to think about the fundamental groupoid as a functor. Recall that a groupoid is a category in which all morphisms are invertible. For groupoids $\mathbf{G}_1$ and $\mathbf{G}_2$, a groupoid homomorphism from $\mathbf{G}_1$ to $\mathbf{G}_2$ is a functor $F \colon \mathbf{G}_1 \to \mathbf{G}_2$. Unpacking this, F consists of the following data:

- For each object $x \in \mathbf{G}_1$, an object $F(x) \in \mathbf{G}_2$.
- For each morphism $\phi \colon x \to y$ in $\mathbf{G}_1$, a morphism $F(\phi) \colon F(x) \to F(y)$ in $\mathbf{G}_2$.

The morphisms $F(\phi)$ must respect composition in the obvious sense. For each $x \in \mathbf{G}_1$, we have the group $\mathrm{Aut}_{\mathbf{G}_1}(x)$, and $F \colon \mathbf{G}_1 \to \mathbf{G}_2$ induces a group homomorphism $F_* \colon \mathrm{Aut}_{\mathbf{G}_1}(x) \to \mathrm{Aut}_{\mathbf{G}_2}(f(x))$. If we think of a groupoid as a collection of groups connected by isomorphisms, the homomorphism $F \colon \mathbf{G}_1 \to \mathbf{G}_2$ can be regarded as a collection of group homomorphisms that respect the given isomorphisms.

Let $\mathtt{Groupoid}$ be the category whose objects are groupoids and whose objects are groupoid homomorphisms. The fundamental groupoid can then be regarded as a functor $\Pi \colon \mathtt{Top} \to \mathtt{Groupoid}$:

- For a space X, we have the groupoid $\Pi(X)$.
- For a map of space $f \colon X \to Y$, we have the groupoid homomorphism $f_* \colon \Pi(X) \to \Pi(Y)$ defined as follows:
  - An object of $\Pi(X)$ is a point $x \in X$, and $f_*(x) = f(x) \in Y$.
  - A morphism in $\Pi(X)$ from $x \in X$ to $y \in X$ is the homotopy class of a path γ from x to y, and $f_*([\gamma]) = [f \circ \gamma]$.

We remark that unlike for the fundamental group, the groupoid homomorphisms $f_* \colon \Pi(X) \to \Pi(Y)$ are not homotopy invariant, at least not in a naive sense. See Exercise 5.9 for one way to think about this.

5.9. Exercises

EXERCISE 5.1. Let (X, x_0) and (Y, y_0) be pointed spaces. Prove that

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0). \qquad \square$$

EXERCISE 5.2. Let X be a space and let $x_0, x_0' \in X$. For each path α from x_0 to x_0', we defined a change of basepoint isomorphism $\alpha_* \colon \pi_1(X, x_0') \to \pi_1(X, x_0)$ in §5.5:

$$\alpha_*([\gamma]) = [\alpha \cdot \gamma \cdot \overline{\alpha}] \quad \text{for all } [\gamma] \in \pi_1(X, x_0').$$

Prove that α_* is independent of the path α if and only if $\pi_1(X, x_0)$ is abelian. This exercise shows that when the fundamental group is abelian there is a canonical isomorphism between the fundamental groups at different basepoints. $\square$

EXERCISE 5.3. Prove that $\pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z}$. Hint: though this does not follow directly from the fact that the degree is a complete invariant of homotopy classes of maps $\mathbb{S}^1 \to \mathbb{S}^1$ (Lemma 3.8.2), it can be proved by carefully examining the construction of the degree and the proof of Lemma 3.8.2. $\square$

Exercise 5.4. Let (X, x_0) be a pointed space. Do the following:

(a) Prove that there is a bijection between elements of $\pi_1(X, x_0)$ and homotopy classes of maps of pointed spaces $f\colon (\mathbb{S}^1, 1) \to (X, x_0)$.

(b) Let $f\colon (\mathbb{S}^1, 1) \to (X, x_0)$ be a map of pointed spaces. Prove that f represents the trivial element of $\pi_1(X, x_0)$ if and only if f extends to a map $F\colon (\mathbb{D}^2, 1) \to (X, x_0)$.

(c) Let Z be a graph with a single vertex $*$ and two loops ℓ_1 and ℓ_2 based at $*$. Regarding the ℓ_i as circles, for maps $f_1, f_2\colon (\mathbb{S}^1, 1) \to (X, x_0)$ let $f_1 \vee f_2\colon (Z, *) \to (X, x_0)$ be the map that equals f_1 on ℓ_1 and f_2 on ℓ_2. **Problem**: Construct a map $m\colon (\mathbb{S}^1, 1) \to (Z, *)$ with the following property:
   - Consider elements $[\gamma_1], [\gamma_2] \in \pi_1(X, x_0)$. Represent $[\gamma_i]$ by a map $f_i\colon (\mathbb{S}^1, 1) \to (X, x_0)$. Then $[\gamma_1 \cdot \gamma_2]$ is represented by the map $(f_1 \vee f_2) \circ m\colon (\mathbb{S}^1, 1) \to (X, x_0)$. □

Exercise 5.5. Let X be a space and let $f_t\colon X \to X$ be a homotopy from $f_0 = \mathbb{1}_X$ to $f_1 = \mathbb{1}_X$. For a basepoint $x_0 \in X$, let $\gamma\colon I \to X$ be the path $\gamma(s) = f_s(x_0)$. Prove that $[\gamma] \in \pi_1(X, x_0)$ is a central element, that is, that $[\gamma]$ commutes with all elements of $\pi_1(X, x_0)$. □

Exercise 5.6. Let $\mathbf{G}$ be a topological group, that is, a space $\mathbf{G}$ that is also a group such that the multiplication map $\mathbf{G} \times \mathbf{G} \to \mathbf{G}$ and the inversion map $\mathbf{G} \to \mathbf{G}$ are continuous. Letting $1 \in \mathbf{G}$ be the identity, prove the following:

(a) Define an alternate multiplication on $\pi_1(\mathbf{G}, 1)$ using the multiplication on $\mathbf{G}$ as follows: for $[\gamma_1], [\gamma_2] \in \pi_1(\mathbf{G}, 1)$, define $[\gamma_1] * [\gamma_2] = [\gamma_3]$ where $\gamma_3\colon I \to \mathbf{G}$ is the loop
$$\gamma_3(s) = \gamma_1(s)\gamma_2(s) \quad \text{for } s \in I.$$
Prove that $*$ is the same as the usual multiplication on $\pi_1(\mathbf{G}, 1)$.

(b) Prove that $\pi_1(G, 1)$ is abelian. □

Exercise 5.7. Let $\mathbf{G}$ be a groupoid and $\phi\colon A \to B$ be a morphism in $\mathbf{G}$. Consider $\overline{\phi}, \overline{\phi}'\colon B \to A$. Then $\phi = \overline{\phi}'$ if any of the following conditions are satisfied:

- $\overline{\phi} \circ \phi = \overline{\phi}' \circ \phi = \mathbb{1}_A$; or
- $\phi \circ \overline{\phi} = \phi \circ \overline{\phi}' = \mathbb{1}_B$; or
- $\overline{\phi} \circ \phi = \mathbb{1}_A$ and $\phi \circ \overline{\phi}' = \mathbb{1}_B$. □

Exercise 5.8. Let G be a group with multiplication $*$. The *opposite group*, denoted G^{op}, has the same elements as G. However, the multiplication $\cdot$ in G^{op} is the "opposite" multiplication to G:
$$x \cdot y = y * x \quad \text{for all } x, y \in G^{\mathrm{op}}.$$
Prove the following:

(a) The opposite group G^{op} is a group.

(b) The group G^{op} is isomorphic to G. □

Exercise 5.9. Let $f_t\colon X \to Y$ be a homotopy of maps between spaces. Prove that f_t induces a natural isomorphism between the functors $f_0\colon \Pi_1(X) \to \Pi_1(Y)$ and $f_1\colon \Pi_1(X) \to \Pi_1(Y)$ giving the induced maps between fundamental groupoids. Here recall that if $F, G\colon \mathbf{C} \to \mathbf{D}$ are functors between categories $\mathbf{C}$ and $\mathbf{D}$, then a *natural isomorphism* $\Psi\colon F \to G$ consists of the following data:

- For all objects A of $\mathbf{C}$, a $\mathbf{D}$-isomorphism $\Psi(A)\colon F(A) \to G(A)$.

These must satisfy the following:

- For all morphisms $\lambda\colon A \to B$ between objects of $\mathbf{C}$, the diagram
$$\begin{array}{ccc} F(A) & \xrightarrow{F(\lambda)} & F(B) \\ \downarrow{\scriptstyle \Psi(A)} & & \downarrow{\scriptstyle \Psi B} \\ G(A) & \xrightarrow{G(\lambda)} & G(B) \end{array}$$
must commute. □

CHAPTER 6

Fundamental group: triviality

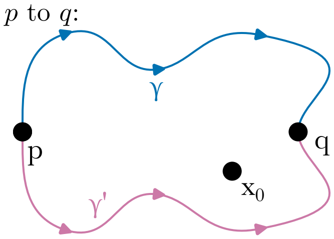
In this chapter and the next, we calculate the fundamental groups of many spaces. This chapter focuses on spaces with trivial fundamental groups. The results we prove will be needed in the next chapter to handle spaces with nontrivial fundamental groups.

6.1. 1-connectivity and the fundamental group

Recall that a space X is 1-connected if it is nonempty and for all $x, y \in X$ there is a unique homotopy class of paths from x to y. If X is 1-connected and locally path connected, then Theorem 4.6.1 says that all covers of X are trivial. We can relate this to the fundamental group as follows:

LEMMA 6.1.1. *Let (X, x_0) be a path-connected pointed space.*[1] *Then X is 1-connected if and only if $\pi_1(X, x_0) = 1$.*

PROOF. If X is 1-connected, then in particular there is only one homotopy class of paths from x_0 to itself, so $\pi_1(X, x_0) = 1$. Conversely, assume that $\pi_1(X, x_0) = 1$. Let p and q be two points of X, and let γ and γ' be paths from p to q:



Since X is path-connected, Lemma 5.5.1 implies that $\pi_1(X, p) = 0$. The path $\gamma \cdot \overline{\gamma}'$ is a path from p to p, so $[\gamma \cdot \overline{\gamma}'] \in \pi_1(X, p)$ must be trivial. We therefore have

$$[\gamma'] = 1[\gamma'] = [\gamma \cdot \overline{\gamma}'][\gamma'] = [\gamma][\overline{\gamma}'][\gamma'] = [\gamma],$$

as desired. $\square$

Lemma 4.5.2 says that $\mathbb{S}^n$ is 1-connected for $n \geq 2$, so we deduce the following:

LEMMA 6.1.2. *For $n \geq 2$, we have $\pi_1(\mathbb{S}^n, x_0) = 1$ for all $x_0 \in \mathbb{S}^n$.*

6.2. Retracts and deformation retracts

Let X be a space and let $A \subset X$ be a subspace. A *retract* of X to A is a map $r \colon X \to A$ such that $r(a) = a$ for all $a \in A$, i.e., such that $r|_A = \mathbb{1}$. A *deformation retraction* of X to A is a homotopy $r_t \colon X \to X$ from the identity $\mathbb{1} \colon X \to X$ to a map $r_1 \colon X \to X$ such that:

- the map r_1 is a retraction of X to A; and
- for all $t \in I$ and $a \in A$, we have $r_t(a) = a$.

If there exists a deformation retraction of X to A, then we say that A is a *deformation retract* of X and that X *deformation retracts* to A. Here are several examples:

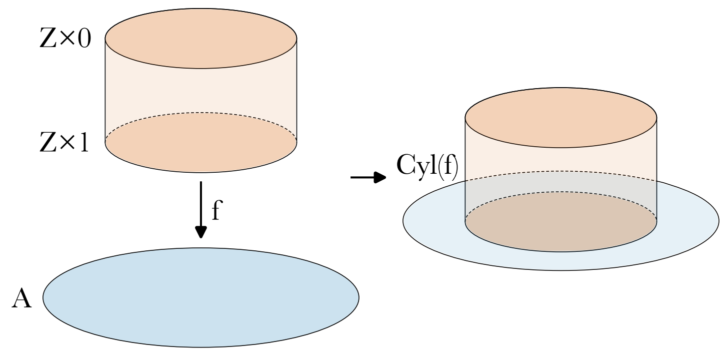
EXAMPLE 6.2.1. Let $U \subset \mathbb{R}^n$ be a set that is *star-shaped*, i.e., such that there exists a point $p_0 \in U$ such that for all $x \in U$ the line segment from p_0 to x is contained in U. For instance, U might be convex. We claim that U deformation retracts to p_0. Indeed, the maps $r_t \colon U \to U$ defined by

$$r_t(x) = (1-t)x + tp_0 \quad \text{for } x \in U \text{ and } t \in I$$

[1] Since $x_0 \in X$, the space X is nonempty and thus 0-connected.

form a deformation retraction. □

Example 6.2.2. Let $f\colon Z \to A$ be a map between spaces. The *mapping cylinder* of f, denoted $\mathrm{Cyl}(f)$, is the quotient of the disjont union $(Z \times I) \sqcup A$ that identifies $(z, 1) \in Z \times I$ with $f(z) \in A$ for all $z \in Z$:

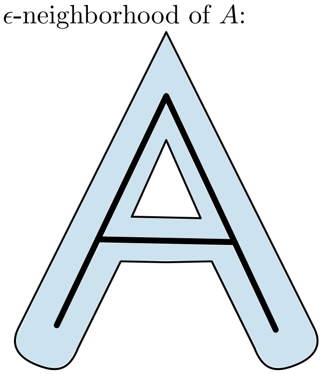


For $z \in Z$ and $s \in I$, let $\overline{(z,s)}$ be the image of $(z, s) \in Z \times I$ in $\mathrm{Cyl}(f)$. Both Z and A are subspaces of $\mathrm{Cyl}(f)$: the space Z can be identified with $\left\{\overline{(z,0)} \mid z \in Z\right\}$, and the copy of A in $(Z \times I) \sqcup A$ maps homeomorphically to a copy of A in $\mathrm{Cyl}(f)$. The space $\mathrm{Cyl}(f)$ deformation retracts to A via the deformation retract $r_t\colon \mathrm{Cyl}(f) \to A$ defined by

$$\begin{cases} r_t\overline{(z,s)} = \overline{(z,(1-t)s)} & \text{for } (z,s) \in Z \times [0,1], \\ r_t(a) = a & \text{for } a \in A. \end{cases}$$

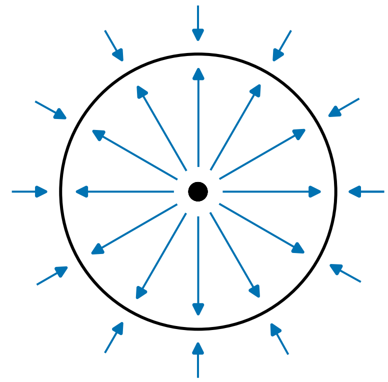
The reader can easily check that this makes sense and is continuous. □

Example 6.2.3. As in the following figure, let A be the letter A embedded in the plane and for some small $\epsilon > 0$ let X be a closed ϵ-neighborhood of A:



Then X deformation retracts to A via a deformation retraction during which points travel along straight line segments to A. In fact, this is a special case of the previous example: the boundary of X consists of two circles $\mathbb{S}^1 \sqcup \mathbb{S}^1$, and X is homeomorphic to the mapping cylinder of a map $f\colon \mathbb{S}^1 \sqcup \mathbb{S}^1 \to A$. □

Example 6.2.4. We claim that $\mathbb{S}^{n-1}$ is a deformation retract of $\mathbb{R}^n \setminus 0$. Geometrically, the picture is as follows, where the blue arrows show the paths points of $\mathbb{R}^n \setminus 0$ travel during the deformation retraction:



In formulas, this deformation retraction is given by the maps $r_t\colon \mathbb{R}^n \setminus 0 \to \mathbb{R}^n \setminus 0$ defined by

$$r_t(x) = \left((1-t) + \frac{t}{\|x\|}\right) x \quad \text{for } x \in \mathbb{R}^n \setminus 0 \text{ and } t \in I. \qquad \square$$

Recall from §5.6 that the fundamental group is functorial under maps of pointed spaces. Using this, we have:

LEMMA 6.2.5. *Let X be a space, let $A \subset X$ be a subspace, and let $a_0 \in A$. Let $\iota\colon (A, a_0) \to (X, a_0)$ be the inclusion. Then:*

(i) If A is a retract of X, then the map $\iota_*\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$ is injective.
(ii) If A is a deformation retraction of X, then the map $\iota_*\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$ is an isomorphism.

PROOF. We start with (i). Let $r\colon X \to A$ be a retraction. Since $r \circ \iota = \mathbb{1}_A$, it follows that $r_* \circ \iota_* = \mathbb{1}_{\pi_1(A,a_0)}$, i.e., that the following composition is the identity:

$$\pi_1(A, a_0) \xrightarrow{\iota_*} \pi_1(X, a_0) \xrightarrow{r_*} \pi_1(A, a_0).$$

This implies that $\ker(\iota_*)$ is trivial, so ι_* is injective.

We now prove (ii). Let $r_t\colon X \to X$ be a deformation retraction. In light of (i), it is enough to prove that the map $\iota_*\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$ is surjective. Consider a loop $\delta\colon I \to X$ based at p. We must prove that δ can be homotoped to a loop lying in A. Since $r_t(a_0) = a_0$ for all $t \in I$, we have a homotopy of paths $r_t \circ \delta$. Since $r_1(X) \subset A$, the image of the endpoint $r_1 \circ \delta$ of this homotopy lies in A, as desired. $\square$

Here is one consequence:

LEMMA 6.2.6. *Let $n \geq 3$. For $x_0 \in \mathbb{R}^n \setminus 0$, we have $\pi_1(\mathbb{R}^n \setminus 0, x_0) = 0$.*

PROOF. Since $\mathbb{R}^n \setminus 0$ is path-connected, we can change x_0 without changing the fundamental group. Choose x_0 such that $x_0 \in \mathbb{S}^{n-1} \subset \mathbb{R}^n \setminus 0$. Since $\mathbb{R}^n \setminus 0$ deformation retracts to $\mathbb{S}^{n-1}$, Lemma 6.2.5 implies that

$$\pi_1(\mathbb{R}^n \setminus 0, x_0) \cong \pi_1(\mathbb{S}^{n-1}, x_0).$$

Since $n \geq 3$, this vanishes by Lemma 4.5.2. $\square$

6.3. Contractibility

A nonempty space X is said to be *contractible* if the identity map $\mathbb{1}\colon X \to X$ is homotopic to a constant map. This holds, for instance, if X deformation retracts to any one-point subspace x_0. Star-shaped or convex subspaces of $\mathbb{R}^n$ are therefore contractible. However, being contractible is more general than this since none of the points of X need to be fixed during the contraction. See Exercise 6.12 for an example where these are genuinely different notions.

If a space X deformation retracts to a point $x_0 \in X$, then it follows from Lemma 6.2.5 that

$$\pi_1(X, x_0) \cong \pi_1(x_0, x_0) = 1.$$

The following shows that this vanishing holds more generally if X is merely contractible.

LEMMA 6.3.1. *Let X be a contractible space and let $x_0 \in X$. Then $\pi_1(X, x_0) = 1$.*

PROOF. Let $f_t\colon X \to X$ be a homotopy from the identity $\mathbb{1}\colon X \to X$ to a constant map. Since X is path connected, its fundamental groups at different basepoints are all isomorphic. We can therefore assume without loss of generality that x_0 is the constant value of f_1. Let $\delta\colon I \to X$ be the following path from x_0 to x_0:

$$\delta(s) = f_s(x_0) \quad \text{for } s \in I.$$

Consider $[\gamma] \in \pi_1(X, x_0)$. Since $f_0 = \mathbb{1}$ we have $(f_0)_*([\gamma]) = [\gamma]$, and since f_1 is the constant map x_0 we have $(f_1)_*([\gamma]) = 1$. By Lemma 5.7.1, we therefore have

$$[\gamma] = (f_0)_*([\gamma]) = [\delta](f_1)_*([\gamma])[\overline{\delta}] = [\delta][\overline{\delta}] = 1. \qquad \square$$

6.4. Trees

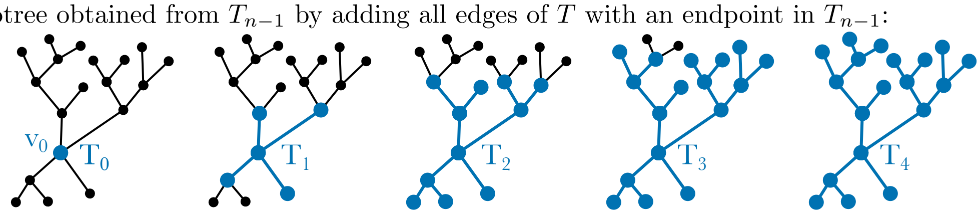
Here is an important example. Recall that we discussed graphs in §2.5. A tree is a nonempty connected graph with no cycles. We will prove:

LEMMA 6.4.1. *Let T be a tree and let v_0 be a vertex of T. Then T deformation retracts to v_0, and in particular T is contractible.*

PROOF. We will omit some of the point-set details, and invite the reader in Exercise 6.9 to verify that all the maps we construct are continuous. Inductively define subtrees

$$T_0 \subset T_1 \subset T_2 \subset \cdots$$

of T in the following way. Start by letting $T_0 = v_0$. Next, if T_{n-1} has been constructed, let T_n be the subtree obtained from T_{n-1} by adding all edges of T with an endpoint in T_{n-1}:



Since T is a tree, each new edge e added to T_{n-1} to form T_n has the property that exactly one endpoint of e lies in T_{n-1}; otherwise, e would form part of a cycle in T_n. This implies that T_n deformation retracts to T_{n-1} via a deformation retract where the points of these new edges e move along e to the vertex lying in T_{n-1}. Let $r_t^n\colon T_n \to T_n$ be this deformation retract. Since T is connected, we have

$$T = \bigcup_{n=0}^{\infty} T_n.$$

For each $n \geq 1$ and $m \geq 0$, consider the retractions

$$R_m^n = r_1^n \circ \cdots \circ r_1^{n+m}\colon T_{n+m} \to T_{n-1}.$$

For $m_1 \geq m_2 \geq 0$, the retractions $R_{m_1}^n$ and $R_{m_2}^n$ agree where they both are defined, namely on T_{n+m_2}. It follows that for a fixed $n \geq 1$ the different R_m^n glue together to give a retraction $R^n\colon T \to T_{n-1}$.

Assume first that $T = T_{n_1}$ for some $n_1 \gg 0$ (which holds, for instance, if T is a finite tree). In this case, we can deformation retract $T = T_{n_1}$ to $T_0 = v_0$ by first using $r_t^{n_1}$ to deformation retract T_{n_1} to T_{n_1-1}, then using $r_t^{n_1-1}$ to deformation retract T_{n_1-1} to T_{n_1-2}, etc. For the general case, we have to be a bit more careful. Write

$$I = \{0\} \cup \bigcup_{n=1}^{\infty} I_n \quad \text{with } I_n = [1/2^n, 1/2^{n-1}],$$

so I_n has length $1/2^n$. Define $r_t\colon T \to T$ in the following way:

- For $t \in I_n$ and $x \in T$, let $r_t(x) = r^n_{2^n(t-1/2^n)}(R^{n+1}(x))$.
- For $t = 0$ and $x \in T$, define $r_0(x) = x$.

The reader will check in Exercise 6.9 that this definition makes sense and is continuous. By definition we have $r_0 = \mathbb{1}$, and since $1 \in I_1$ we have

$$r_1(x) = r_1^1(R^2(x)) = v_0 \quad \text{for } x \in T,$$

where we recall that T_0 is the vertex v_0. It follows that r_t is a deformation retraction of T to v_0, as desired. □

6.5. Projective spaces

Recall that real projective space $\mathbb{RP}^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$. This has the degree 2 cover $p\colon \mathbb{S}^n \to \mathbb{RP}^n$ taking $x \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$ to the line through x. This reflects the fact that $\mathbb{RP}^n$ is not 1-connected, as we will see rigorously in Chapter 7.

An important relative of $\mathbb{RP}^n$ is *complex projective space*, that is, the space $\mathbb{CP}^n$ of lines through the origin in $\mathbb{C}^{n+1}$. This can be topologized just like $\mathbb{RP}^n$: letting $\mathbb{C}^{n+1} \setminus 0 \to \mathbb{CP}^n$ be the map taking $z \in \mathbb{C}^{n+1}$ to the line through z, we give $\mathbb{CP}^n$ the quotient topology:

- A set $U \subset \mathbb{CP}^n$ is open if and only if its preimage in $\mathbb{C}^{n+1} \setminus 0$ is open.

The space $\mathbb{CP}^n$ will play an important role in subsequent volumes when we discuss homology and cohomology. However, it is less important in this volume since it turns out to be 1-connected. We outline a proof of this in Exercise 6.3.

6.6. Exercises

EXERCISE 6.1. Let X be a contractible space and let $A \subset X$ be a subspace such that there is a retract $r\colon X \to A$. Prove that A is contractible. □

EXERCISE 6.2. Let X be a space. Letting p be a one-point space, the *cone* on X, denoted $\mathrm{Cone}(X)$, is the mapping cylinder of the constant map $X \to p$. The space X is a subspace of $\mathrm{Cone}(X)$. Also, the image of p in $\mathrm{Cone}(X)$ is called the *cone point*. The *suspension* of X, denoted ΣX, is the quotient of the disjoint union $\mathrm{Cone}(X) \sqcup \mathrm{Cone}(X)$ that identifies the copies of X in the two cones. Do the following:

(a) Prove that $\mathrm{Cone}(X)$ deformation retracts to the cone point, and in particular is contractible.
(b) Prove that $\Sigma \mathbb{S}^n \cong \mathbb{S}^{n+1}$.
(c) If X is 0-connected, then prove that ΣX is 1-connected. Note that by (b) this generalizes the fact that $\mathbb{S}^n$ is 1-connected for $n \geq 2$. Hint: apply Lemma 4.5.3 (general position). □

EXERCISE 6.3. This exercise outlines a proof that $\mathbb{CP}^n$ is 1-connected. As notation, for $(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \setminus 0$ let $[z_1, \ldots, z_{n+1}]$ be the corresponding point in $\mathbb{CP}^n$, i.e., the line in $\mathbb{C}^{n+1}$ though the origin and $(z_1, \ldots, z_{n+1})$. The proof will be by induction.

(a) For the base case, prove that $\mathbb{CP}^1 \cong \mathbb{S}^2$, so $\mathbb{CP}^1$ is 1-connected.
(b) Now assume that $n \geq 2$ and that $\mathbb{CP}^{n-1}$ is 1-connected. Fix a basepoint $x_0 \in \mathbb{CP}^{n-1}$. Embed $\mathbb{CP}^{n-1}$ into $\mathbb{CP}^n$ via the map taking $[z_1, \ldots, z_n] \in \mathbb{CP}^{n-1}$ to $[z_1, \ldots, z_n, 0] \in \mathbb{CP}^n$. Using this, we identify x_0 with a point in $\mathbb{CP}^n$. Finally, set $r = [0, \ldots, 0, 1] \in \mathbb{CP}^n$. **Problem**: for $[\gamma] \in \pi_1(\mathbb{CP}^n, x_0)$, prove that γ can be homotoped such that its image does not contain r. Hint: use Lemma 4.5.3 (general position).
(c) Prove that $\mathbb{CP}^n \setminus r$ deformation retracts to $\mathbb{CP}^{n-1}$. Since our inductive hypothesis implies that $\pi_1(\mathbb{CP}^{n-1}, x_0) = 1$, this will imply that $\pi_1(\mathbb{CP}^n \setminus r, x_0) = 0$ and thus by (b) that $\pi_1(\mathbb{CP}^n, x_0) = 1$. We conclude that $\mathbb{CP}^n$ is 1-connected. □

EXERCISE 6.4. Let X be a space. Prove the following:

(a) The space X is contractible if and only if for all spaces Y, every map $f\colon X \to Y$ is nullhomotopic.
(b) The space X is contractible if and only if for all spaces Z, every map $g\colon Z \to X$ is nullhomotopic. □

EXERCISE 6.5. Let X be a space. Let $\mathrm{Cone}(X)$ be the cone on X from Exercise 6.2. Prove that X is contractible if and only if X is a retract of $\mathrm{Cone}(X)$. □

EXERCISE 6.6. Let $f\colon X \to Y$ be a map. Prove that there exists a map $g\colon Y \to X$ such that $g \circ f\colon X \to X$ is homotopic to the identity if and only if X is a retract of the mapping cylinder $\mathrm{Cyl}(f)$. □

EXERCISE 6.7. Do the following:

(a) Let X be a space. Assume that $X = U \cup V$ where U and V are 1-connected open sets such that $U \cap V$ is 0-connected. For $x_0 \in U \cap V$, prove that $\pi_1(X, x_0) = 1$. Hint: Start with some $[\gamma] \in \pi_1(X, x_0)$ and try to write $[\gamma] = [\gamma_1] \cdots [\gamma_n]$ where each $[\gamma_i] \in \pi_1(X, x_0)$ is such that the image of γ_i lies in either U or V. The Lebesgue number lemma will be useful.
(b) Use part (a) to give an alternate proof that $\mathbb{S}^n$ is 1-connected for $n \geq 2$.
(c) Generalize part (a) as follows. Let X be a space and let $X = \cup_{i \in I} U_i$ with each U_i open. Let $x_0 \in X$ be a basepoint such that $x_0 \in U_i$ for all $i \in I$. Assume the following:
   - The open set U_i is 0-connected for all $i \in I$.
   - The open set $U_i \cap U_j$ is 0-connected for all $i, j \in I$.
   - The map $\pi_1(U_i, x_0) \to \pi_1(X, x_0)$ is trivial for all $i \in I$.

   Prove that $\pi_1(X, x_0) = 1$.

□

EXERCISE 6.8. Let X and Y be spaces. The *join* of X and Y, denoted $X * Y$, is

$$X * Y = X \sqcup Y \sqcup (X \times Y \times I)/\sim,$$

where $\sim$ makes the following identifications for all $x \in X$ and $y \in Y$:

$$(x, y, 0) \sim x \quad \text{and} \quad (x, y, 1) \sim y.$$

Identify X and Y with their images in $X * Y$. The space $X * Y$ can be viewed as the space of all lines connecting points of X to points of Y. In analogy with the usual way of writing a line segment between points of $\mathbb{R}^n$ using barycentric coordinates, it is useful to denote the image in $X * Y$ of $(x, y, t) \in X \times Y \times I$ by the formal sum $(1-t)x + ty$. Do the following:

(a) For $n, m \geq 0$, we have $\mathbb{S}^n * \mathbb{S}^m \cong \mathbb{S}^{n+m+1}$.
(b) For $n, m \geq 0$, we have $\mathbb{D}^n * \mathbb{D}^m \cong \mathbb{D}^{n+m+1}$.
(c) Prove that $(X * Y) \setminus X$ deformation retracts to Y.
(d) Assume that Y is nonempty. For $x_0 \in X$, prove that the map $\pi_1(X, x_0) \to \pi_1(X * Y, x_0)$ is trivial.
(e) Assume that Y is 0-connected and that Y is nonempty and locally path connected. Prove that $X * Y$ is 1-connected. Hint: Part (c) of Exercise 6.7 might be useful. It also might be easier to first prove this when Y is 0-connected. For the general case, since Y is locally path connected, we can decompose it into clopen path components $Y = \sqcup_{i \in I} Y_i$. □

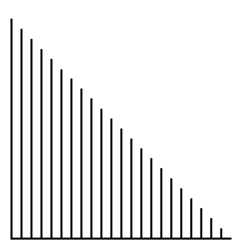
EXERCISE 6.9. Verify that the maps constructed in the proof of Lemma 6.4.1 are well-defined and continuous. □

EXERCISE 6.10. Let $\mathrm{O}(n)$ be the n-dimensional orthogonal group. Prove that $\mathrm{GL}_n(\mathbb{R})$ deformation retracts to $\mathrm{O}(n)$. Hint: analyze the Gram–Schmidt orthogonalization process. □

EXERCISE 6.11. Let $X \subset \mathbb{R}^2$ be the union of the following subspaces:

- the horizontal segment $[0, 1] \times 0$; and
- for each $r \in \mathbb{Q}$, the vertical segment $r \times [0, 1 - r]$.

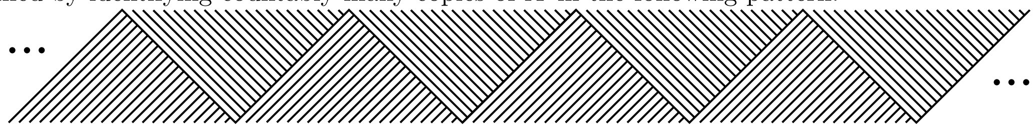
See here:



Prove the following:

(a) The space X deformation retracts to any point on the horizontal segment $[0,1] \times 0$.
(b) The space X does not deformation retract to any point that does not lie on $[0,1] \times 0$. $\square$

EXERCISE 6.12. Let $X \subset \mathbb{R}^2$ be the subspace from Exercise 6.11. Let $Y \subset \mathbb{R}^2$ be the space obtained by identifying countably many copies of X in the following pattern:



Prove the following:

(a) The space Y is contractible.
(b) The space Y does not deformation retract to any point. $\square$

CHAPTER 7

Fundamental group: basic calculations

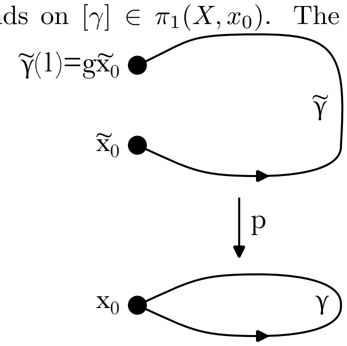
We now calculate a number of nontrivial fundamental groups. One of the main places areas to which the fundamental group can be applied is group theory, where it allows geometric arguments that would be difficult to express purely algebraically. We give a first example of this at the end of this chapter, where we use the fundamental group to construct free groups.

7.1. Calculating the fundamental group using covering spaces

The following is our main tool for calculating fundamental groups:

THEOREM 7.1.1. *Let $p\colon \widetilde{X} \to X$ be a regular cover such that $\widetilde{X}$ is 1-connected. Set $G = \mathrm{Deck}(\widetilde{X})$. Then for $x_0 \in X$ we have $\pi_1(X, x_0) \cong G$.*

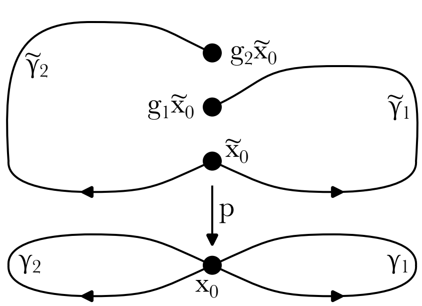
PROOF. Pick $\widetilde{x}_0 \in \widetilde{X}$ with $p(\widetilde{x}_0) = x_0$. Define a set map $f\colon \pi_1(X, x_0) \to G$ as follows. Consider $[\gamma] \in \pi_1(X, x_0)$. By path lifting (Lemma 3.4.1), we can lift γ to a path $\widetilde{\gamma}$ in $\widetilde{X}$ starting at $\widetilde{x}_0$. Lemma 4.2.1 implies that the homotopy class of $\widetilde{\gamma}$ only depends on the homotopy class of γ. In particular, $\widetilde{\gamma}(1) \in \widetilde{X}$ only depends on $[\gamma] \in \pi_1(X, x_0)$. The point $\widetilde{\gamma}(1)$ projects to $\gamma(1) = x_0$:

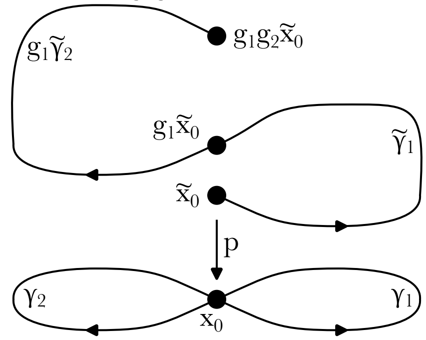


Since $\widetilde{X}$ is a path-connected regular cover of X, there exists a unique $g \in G$ with $g\widetilde{x}_0 = \widetilde{\gamma}(1)$; see Lemma 2.2.1. Define $f([\gamma]) = g$. To prove the theorem, it is enough to prove that f is a group homomorphism that is injective and surjective. We do this in the following three claims:

CLAIM 1. *The set map f is a group homomorphism.*

Consider $[\gamma_1], [\gamma_2] \in \pi_1(X, x_0)$. For $i = 1, 2$, let $\widetilde{\gamma}_i$ be the lift of γ_i to $\widetilde{X}$ with $\widetilde{\gamma}_i(0) = \widetilde{x}_0$. Letting $g_i = f([\gamma_i])$, the path $\widetilde{\gamma}_i$ thus goes from $\widetilde{x}_0$ to $g_i\widetilde{x}_0$. The deck group G acts not only on $\widetilde{X}$, but also on paths in $\widetilde{X}$. Under this group action, the path $g_1\widetilde{\gamma}_2$ goes from $g_1\widetilde{x}_0$ to $g_1 g_2\widetilde{x}_0$. It follows that $\widetilde{\gamma}_1$ and $g_1\widetilde{\gamma}_2$ are composable paths, and $\widetilde{\gamma}_1 \cdot (g_1\widetilde{\gamma}_2)$ goes from $\widetilde{x}_0$ to $g_1 g_2\widetilde{x}_0$:





The path $\widetilde{\gamma}_1 \cdot (g_1\widetilde{\gamma}_2)$ is the lift of $\gamma_1 \cdot \gamma_2$, so by definition this implies that $f([\gamma_1 \cdot \gamma_2]) = g_1 g_2$, as desired.

Claim 2. *The homomorphism f is surjective.*

Consider $g \in G$. Since $\widetilde{X}$ is path-connected, we can find a path $\widetilde{\gamma}$ in $\widetilde{X}$ from $\widetilde{x}_0$ to $g\widetilde{x}_0$. The path $\widetilde{\gamma}$ projects to a path γ in X from x_0 to x_0, so $[\gamma] \in \pi_1(X, x_0)$. By definition, $f([\gamma]) = g$.

Claim 3. *The homomorphism f is injective.*

Consider $[\gamma] \in \pi_1(X, x_0)$ such that $f([\gamma]) = 1$. Let $\widetilde{\gamma}$ be the lift of γ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$. Since $f([\gamma]) = 1$, we must have $\widetilde{\gamma}(1) = \widetilde{x}_0$, so $\widetilde{\gamma}$ is a loop based at $\widetilde{x}_0$. Since $\widetilde{X}$ is 1-connected, the loop $\widetilde{\gamma}$ is homotopic to a constant loop. Composing this homotopy with the map $p\colon \widetilde{X} \to X$, we obtain a homotopy from γ to a constant loop, so $[\gamma] = 1$, as desired. □

7.2. The lifting map

Before giving examples of Theorem 7.1.1, we give a name to the isomorphism underlying its conclusion. Let $p\colon \widetilde{X} \to X$ be a regular cover such that $\widetilde{X}$ is 1-connected. Let $\widetilde{x}_0 \in \widetilde{X}$, and set $x_0 = p(\widetilde{x}_0)$ and $G = \mathrm{Deck}(\widetilde{X})$. Theorem 7.1.1 says that $\pi_1(X, x_0) \cong G$. In the proof of that theorem, this isomorphism is given by the following set map $f\colon \pi_1(X, x_0) \to G$:

- Consider $[\gamma] \in \pi_1(X, x_0)$. Let $\widetilde{\gamma}$ be the lift of γ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$. There exists a unique $g \in G$ with $\widetilde{\gamma}(1) = g\widetilde{x}_0$. We then have $f([\gamma]) = g$.

We will call this isomorphism $f\colon \pi_1(X, x_0) \to G$ the *lifting map*. The proof that it is a well-defined homomorphism does not use the fact that $\widetilde{X}$ is 1-connected. Moreover, the proof that it is surjective only uses the fact that $\widetilde{X}$ is path connected. The 1-connectedness of $\widetilde{X}$ is only used in the proof that f is injective. We record these observations in the following lemma:

Lemma 7.2.1. *Let $p\colon \widetilde{X} \to X$ be a regular cover. Let $\widetilde{x}_0 \in \widetilde{X}$, and set $x_0 = p(\widetilde{x}_0)$ and $G = \mathrm{Deck}(\widetilde{X})$. The following hold:*

(i) The lifting map $f\colon \pi_1(X, x_0) \to G$ is a homomorphism.
(ii) If $\widetilde{X}$ is path connected, then the lifting map $f\colon \pi_1(X, x_0) \to G$ is surjective.

7.3. Circle and torus

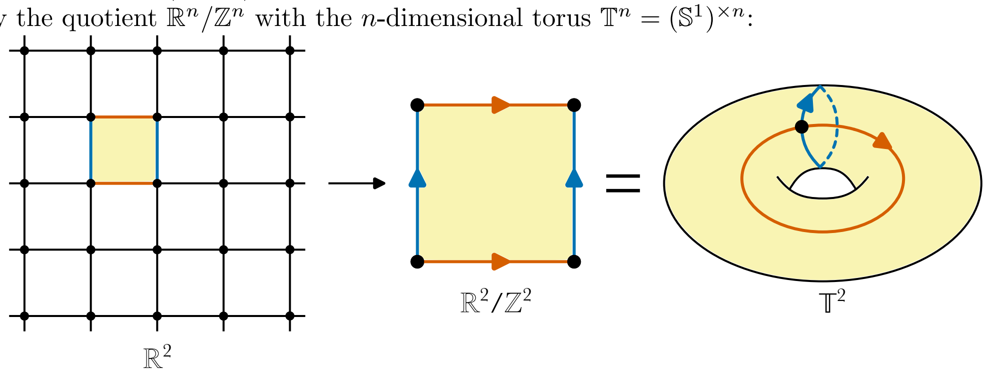
We now give some calculations using Theorem 7.1.1. Our first is important enough that we single it out as a lemma. Recall that we identify $\mathbb{S}^1$ with a subset of $\mathbb{C}$, so $1 \in \mathbb{S}^1$.

Lemma 7.3.1 (Circle). *We have $\pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z}$, where $n \in \mathbb{Z}$ corresponds to the loop $\gamma_n\colon I \to \mathbb{S}^1$ defined by $\gamma_n(s) = e^{2\pi i n s}$ for $s \in I$.*

Proof. Consider the universal cover $p\colon \mathbb{R} \to \mathbb{S}^1$ of $\mathbb{S}^1$, so $p(\theta) = e^{2\pi i \theta}$. As we observed in Example 2.2.2, this is a regular cover with deck group $\mathbb{Z}$, which acts on $\mathbb{R}$ by integer translations. Since $\mathbb{R}$ is contractible, it is 1-connected. We can therefore apply Theorem 7.1.1 to see that $\pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z}$. This isomorphism is given by the lifting map, and under the lifting map $n \in \mathbb{Z}$ corresponds to the loop γ_n. □

Our next example generalizes this:

Example 7.3.2 (Torus). As in Example 1.3.3 let $\mathbb{Z}^n$ act on $\mathbb{R}^n$ by integer translations and identify the quotient $\mathbb{R}^n/\mathbb{Z}^n$ with the n-dimensional torus $\mathbb{T}^n = (\mathbb{S}^1)^{\times n}$:



This figure shows the case $n = 2$. The projection $p\colon \mathbb{R}^n \to \mathbb{Z}^n$ is a regular cover with deck group $\mathbb{Z}^n$. Set $x_0 = p(0)$. Since $\mathbb{R}^n$ is contractible, it is 1-connected. We can thus apply Theorem 7.1.1 to see that $\pi_1(\mathbb{T}^n, x_0) = \pi_1((\mathbb{S}^1)^{\times n}, x_0) \cong \mathbb{Z}^n$. $\square$

REMARK 7.3.3. More generally, by Exercise 5.1 if X and Y are spaces with basepoints $x_0 \in X$ and $y_0 \in Y$, then $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$. $\square$

7.4. Brouwer fixed point theorem

Since $\pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z}$ but $\mathbb{S}^n$ is 1-connected for all $n \geq 2$, it follows that $\mathbb{S}^1$ is not homeomorphic to $\mathbb{S}^n$ for any $n \geq 2$. This is not a particularly profound result, and can be proved in many other ways.[1] In the next chapter we will talk about homotopy equivalences, and we will be able to deduce the more interesting result that $\mathbb{S}^1$ and $\mathbb{S}^n$ are not homotopy equivalent for any $n \geq 2$.

Here, however, we give an interesting application of a different flavor that also illustrates how the functorality of the fundamental group can be used to obstruct the existence of maps. Regard $\mathbb{S}^1$ as the boundary of $\mathbb{D}^2$.

LEMMA 7.4.1. *There does not exist a retraction $r\colon \mathbb{D}^2 \to \mathbb{S}^1$.*

PROOF. Assume for the sake of contradiction that a retraction $r\colon \mathbb{D}^2 \to \mathbb{S}^1$ exists. Let $\iota\colon \mathbb{S}^1 \to \mathbb{D}^2$ be the inclusion, and fix some $x_0 \in \mathbb{S}^1$. Since $r \circ \iota = \mathbb{1}_{\mathbb{S}^1}$, the following composition is the identity:

$$\begin{array}{ccccc} \pi_1(\mathbb{S}^1, x_0) & \xrightarrow{\iota_*} & \pi_1(\mathbb{D}^2, x_0) & \xrightarrow{r_*} & \pi_1(\mathbb{S}^1, x_0). \\ \| & & \| & & \| \\ \mathbb{Z} & & 0 & & \mathbb{Z} \end{array}$$

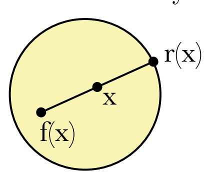
However, since the middle group is 0 this composition is also the 0 map, a contradiction. $\square$

This has the following consequence:

THEOREM 7.4.2 (Brouwer fixed point theorem). *Let $f\colon \mathbb{D}^2 \to \mathbb{D}^2$ be a continuous map. Then f has a fixed point, i.e., there exists some $x \in \mathbb{D}^2$ with $f(x) = x$.*

PROOF. Assume for the sake of contradiction that $f(x) \neq x$ for all $x \in \mathbb{D}^2$. Define $r\colon \mathbb{D}^2 \to \mathbb{S}^1$ to be following map:

- For $x \in \mathbb{D}^2$, let $r(x)$ be the point where the ray from $f(x)$ to x intersects $\mathbb{S}^1$:



By construction, f is a continuous retraction, contradicting Lemma 7.4.1. $\square$

REMARK 7.4.3. The Brouwer fixed point theorem (Theorem 7.4.2) actually holds for maps $f\colon \mathbb{D}^n \to \mathbb{D}^n$ with $n \geq 1$ arbitrary. See Exercise 7.5 for the (easy) proof when $n = 1$. Similarly, Lemma 7.4.1 holds in all dimension. For general $n \geq 1$, we gave an elementary proof of both of these results in Volume 1 using a combinatorial result called Sperner's Lemma. Once we develop the theory of homology, we will be able to give a proof of the general case that is very similar to the proof we gave above for $n = 2$. $\square$

7.5. Real projective space

We now turn to real projective space:

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[1] For instance, by observing that $\mathbb{S}^1 \setminus \{p, q\}$ has two path components for any distinct $p, q \in \mathbb{S}^1$, while removing two points from $\mathbb{S}^n$ cannot disconnect $\mathbb{S}^n$ for any $n \geq 2$.

EXAMPLE 7.5.1 (Real projective space). Let $n \geq 2$. Recall that $\mathbb{RP}^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$. As we described in Example 1.3.4, there is a 2-fold cover $p\colon \mathbb{S}^n \to \mathbb{RP}^n$ taking $x \in \mathbb{S}^n$ to the line through x. This is a regular cover with deck group the cyclic group C_2 of order 2, which acts on $\mathbb{S}^n$ by the antipodal map $x \mapsto -x$. Fix some $x_0 \in \mathbb{S}^n$, and let $\ell_0 = p(x_0) \in \mathbb{RP}^n$. Since $n \geq 2$, we have $\pi_1(\mathbb{S}^n, x_0) = 1$ (Lemma 4.5.2). We can therefore apply Theorem 7.1.1 and see that

$$\pi_1(\mathbb{RP}^n, \ell_0) \cong C_2.$$

This isomorphism is given by the lifting map, and under the lifting map the generator of C_2 corresponds to the loop in $\pi_1(\mathbb{RP}^n, \ell_0)$ that rotates the line ℓ_0 around an axis by an angle of π, coming back to itself but with the reversed orientation. $\square$

REMARK 7.5.2. We have $\mathbb{RP}^1 \cong \mathbb{S}^1$ (see Exercise 7.7), so $\pi_1(\mathbb{RP}^1, \ell_0) \cong \mathbb{Z}$. $\square$

7.6. Free groups: definition

Before we can give our next example, we need some background about free groups. Let S be a set. Roughly speaking, a free group on S is a group that is easy to map out of. One need only say where the elements of S must go. Here is the formal definition:

DEFINITION 7.6.1. Let S be a set. A *free group* on S is a group $F(S)$ equipped with a map of sets $\iota\colon S \to F(S)$ such that the following holds:

(†) Let G be a group and $h\colon S \to G$ be a map of sets. Then there is a unique homomorphism $H\colon F(S) \to G$ such that $h = H \circ \iota$.

The set S is called a *free basis* for $F(S)$. $\square$

REMARK 7.6.2. The condition (†) is an example of a *universal mapping property*. $\square$

The map ι in the definition of a free group is necessarily injective:

LEMMA 7.6.3. *Let S be a set and let $F(S)$ be a free group on S. Then the associated map $\iota\colon S \to F(S)$ is injective.*

PROOF. Let G be a group of cardinality at least $|S|$ and let $h\colon S \to G$ be an injection. By the universal property (†), there is a homomorphism $H\colon F(S) \to G$ such that $h = H \circ \iota$. Since h is injective, it follows that ι is injective. $\square$

Let $F(S)$ be a free group on a set S. By Lemma 7.6.3, we can identify S with a subset of $F(S)$ via the corresponding map ι. Having done this, we can now rephrase (†) as follows:

(†′) Let G be a group and $h\colon S \to G$ be a map of sets. Then h extends uniquely to a homomorphism $H\colon F(S) \to G$.

Whenever we work with free groups in this book, we will identify S with a subset of $F(S)$ and use (†′) as the defining property of $F(S)$. The subset S generates $F(S)$:

LEMMA 7.6.4. *Let S be a set and let $F(S)$ be a free group on S. Then S generates $F(S)$.*

PROOF. Let $G < F(S)$ be the subgroup generated by S. By (†′), the inclusion $h\colon S \hookrightarrow G$ extends uniquely to a homomorphism $H\colon F(S) \to G$. The composition

$$F(S) \xrightarrow{H} G \hookrightarrow F(S)$$

and the identity map $\mathbb{1}_{F(S)}\colon F(S) \to F(S)$ both extend $\mathbb{1}_S\colon S \to S$, and thus must be equal. We conclude that $G = F(S)$, as desired. $\square$

It is not obvious that a free group on a set S exists. We will soon construct one, but first we prove that they are unique in the following sense:

LEMMA 7.6.5. *Let S be a set and let $F(S)$ and $F'(S)$ be free groups on S. There is then a unique isomorphism $\phi\colon F(S) \to F'(S)$ with $\phi|_S = \mathbb{1}_S$.*

PROOF. Applying (†′) to the identity maps between $S \subset F(S)$ and $S \subset F'(S)$, we get:

- a homomorphism $\phi\colon F(S) \to F'(S)$ such that $\phi|_S = \mathbb{1}_S$; and

- a homomorphism $\phi'\colon F'(S) \to F(S)$ such that $\phi'_S = \mathbb{1}_S$.

The composition $\phi' \circ \phi\colon F(S) \to F(S)$ fixes each element of the generating set S, so $\phi' \circ \phi = \mathbb{1}_{F(S)}$. Similarly, $\phi \circ \phi' = \mathbb{1}_{F'(S)}$. We conclude that ϕ and ϕ' are inverse isomorphisms. □

REMARK 7.6.6. A similar argument shows uniqueness for other mathematical objects defined by universal mapping properties. □

7.7. Free groups: reduced words

Let S be a set. A *word* in S is a formal expression $w = s_1^{\epsilon_1} \cdots s_n^{\epsilon_n}$ with $s_i \in S$ and $\epsilon_i \in \{\pm 1\}$ for all $1 \leq i \leq n$. This word is *reduced* if for all $1 \leq i < n$ we do *not* have

$$s_i^{\epsilon_i} s_{i+1}^{\epsilon_{i+1}} \in \left\{ss^{-1},\, s^{-1}s \mid s \in S\right\}.$$

By cancelling terms of the form ss^{-1} and $s^{-1}s$ with $s \in S$ we can reduce any word to a reduced word. If S is a subset of a group Γ, then we can regard words in S as elements of Γ. Cancelling terms of the form ss^{-1} and ss^{-1} does not change the associated element of Γ, so every element of Γ that can be written as a word in S can be written as a reduced word in S.

The main existence theorem for free groups is:

THEOREM 7.7.1. *Let S be a set. There then exists a group $F(S)$ with $S \subset F(S)$ such that:*

(i) the group $F(S)$ is a free group on S; and

(ii) every element of $F(S)$ can be represented by a unique reduced word in S.

The standard proof of this is algebraic, and proves the two parts separately:

- First, free groups are proved to exist, establishing (i).
- Next, (ii) is proved using the universal property of the free group.

See Exercises 7.13 and 7.14 for an outline of this proof. We will prove Theorem 7.7.1 geometrically in the next section. Our argument will reverse the logic of the algebraic proof:

- First, we will use geometry to construct a group $F(S)$ containing a set S such that every element of $F(S)$ can be represented by a unique reduced word in S. This establishes part (ii) of Theorem 7.7.1.
- Next, we will prove that $F(S)$ has the desired universal property, establishing part (i).

In fact, the second step is quite formal:

LEMMA 7.7.2. *Let S be a set and let Γ be a group containing S such that every element of Γ can be represented by a unique reduced word in S. Then Γ is a free group on S.*

PROOF. Let G be a group and let $h\colon S \to G$ be a set map. We extend h to $H\colon \Gamma \to G$ as follows. Consider $g \in \Gamma$. We can uniquely write g as a reduced word in S:

$$g = s_1^{\epsilon_1} \cdots s_n^{\epsilon_n} \quad \text{with } s_1, \ldots, s_n \in S \text{ and } \epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}.$$

We then define

$$H(g) = h(s_1)^{\epsilon_1} \cdots h(s_n)^{\epsilon_n}.$$

This is a homomorphism. Indeed, consider $g, g' \in \Gamma$. Write them as reduced words in S:

$$\begin{aligned} g &= s_1^{\epsilon_1} \cdots s_n^{\epsilon_n} \quad \text{with } s_1, \ldots, s_n \in S \text{ and } \epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}, \\ g' &= t_1^{e_1} \cdots t_m^{e_m} \quad \text{with } t_1, \ldots, t_m \in S \text{ and } e_1, \ldots, e_m \in \{\pm 1\}. \end{aligned}$$

Then the reduced word representing gg' is obtained from

$$s_1^{\epsilon_1} \cdots s_n^{\epsilon_n} t_1^{e_1} \cdots t_m^{e_m}$$

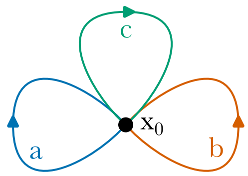
by cancelling terms. The images of those terms under H also cancel, so

$$H(gg') = h(s_1)^{\epsilon_1} \cdots h(s_n)^{\epsilon_n} h(t_1)^{e_1} \cdots h(t_m)^{e_m} = H(g)H(g').$$

It follows that H is a well-defined homomorphism. Since S generates Γ, it is the only possible homomorphism extending h. □

7.8. Constructing free groups using graphs

We now prove Theorem 7.7.1. In fact, we prove something better. Recall our convention from §2.5 that all graphs are oriented. Let S be a set. Denote by X_S the graph with one vertex x_0 and with $|S|$ oriented edges, each labeled with an element of S. For instance, if $S = \{a, b, c\}$ then X_S is



Each $s \in S$ corresponds to a loop in X_S based at x_0, and we will identify s with the corresponding element of $\pi_1(X_S, x_0)$. The elements of S correspond to distinct elements of $\pi_1(X_S, x_0)$. Indeed, for $s \in S$ let $r_s \colon (X_S, x_0) \to (\mathbb{S}^1, 1)$ be the retraction onto the loop labeled s. The induced map

$$(r_s)_* \colon \pi_1(X_S, x_0) \to \pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z}$$

takes s to $1 \in \mathbb{Z}$ and each $s' \in S \setminus \{s\}$ to $0 \in \mathbb{Z}$, so $s \neq s'$ for all $s' \in S \setminus \{s\}$. We can therefore identify S with a subset of $\pi_1(X_S, x_0)$. We then have:

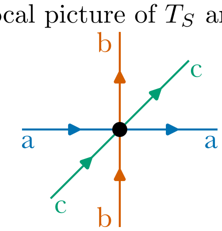
THEOREM 7.8.1. *Let S be a set. The following hold:*

(i) the group $\pi_1(X_S, x_0)$ is a free group on S; and
(ii) every element of $\pi_1(X_S, x_0)$ can be represented by a unique reduced word in S.

PROOF. By Lemma 7.7.2, it is enough to prove (ii). Define T_S to be an infinite tree each of whose vertices has valence $2|S|$. Label the oriented edges of T_S by elements of S such that for each vertex v of T_S there are:

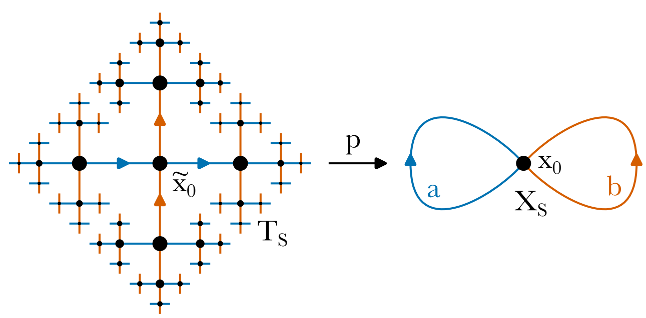
- $|S|$ edges coming out of v labeled by elements of S; and
- $|S|$ edges going into v labeled by elements of S.

For instance, if $S = \{a, b, c\}$ then the local picture of T_S around v looks like



Fix a vertex $\widetilde{x}_0$ of T_S.

There is a covering space $p \colon T_S \to X_S$ taking each vertex of T_S to x_0 and each oriented edge of T_S labeled by $s \in S$ to the corresponding loop in X_S labeled by s. For instance, if $S = \{a, b\}$ this is the cover



Just like in the $S = \{a, b\}$ case, the cover $p \colon T_S \to X_S$ is regular.[2] Since T_S is a tree, it is contractible and hence 1-connected (see Lemma 6.4.1).

Letting G be the deck group of $p \colon T_S \to X$, we can apply Theorem 7.1.1 to see that

$$\pi_1(X_S, x_0) \cong G. \tag{7.8.1}$$

[2]The point here is that T is a tree each of whose vertices has the same local picture, so there are edge-label preserving graph automorphisms of T taking any vertex to any other vertex. These are deck transformations.

The group G acts simply transitively on the vertices of T, so each vertex is of the form $g\widetilde{x}_0$ for some unique $g \in G$. The isomorphism (7.8.1) is given by the lifting map, so the element of $\pi_1(X, x_0)$ corresponding to $g \in G$ is the homotopy class of the loop in X based at x_0 obtained by taking a path in T from $\widetilde{x}_0$ to $g\widetilde{x}_0$ and projecting it to X.

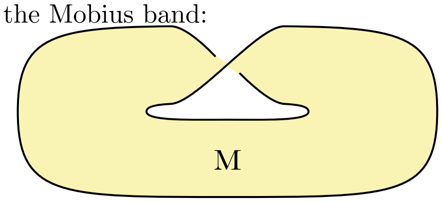
For each vertex $\widetilde{x}_1$ of T, there is a unique sequence of edges connecting $\widetilde{x}_0$ to $\widetilde{x}_1$ that does not backtrack, that is, traverse an edge in one direction and then go backwards along the same edge. This non-backtracking condition is exactly what is needed to ensure that this edge-path corresponds to an element of $\pi_1(X_S, x_0)$ represented by a reduced word in S. In this way, we see that every element of $\pi_1(X_S, x_0)$ is represented by a unique reduced word in S, as desired. $\square$

7.9. Exercises

Exercise 7.1. For $d \geq 2$, construct a pointed space (X, x_0) with $\pi_1(X, x_0) \cong C_d$. Hint: generalize the construction of $\mathbb{RP}^n$ as $\mathbb{S}^n/C_2$. It might be helpful to view $\mathbb{S}^{2n-1}$ as a subspace of $\mathbb{C}^n \cong \mathbb{R}^{2n}$. $\square$

Exercise 7.2. Let A be a finitely generated abelian group. Construct a pointed space (X, x_0) with $\pi_1(X, x_0) \cong A$. Hint: the previous exercise might be helpful here. $\square$

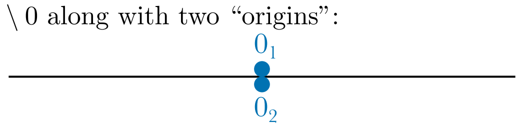
Exercise 7.3. Let M be the Mobius band:



Fixing a basepoint $x_0 \in M$, prove that $\pi_1(M, x_0) \cong \mathbb{Z}$ in two ways:

(a) By identifying M as $Z/\mathbb{Z}$ for an explicit 1-connected space Z equipped with a covering space action of $\mathbb{Z}$.
(b) By showing that M deformation retracts to a subspace $A \subset M$ with $A \cong \mathbb{S}^1$. $\square$

Exercise 7.4. Let X be the "line with two origins", i.e., the quotient of $\mathbb{R} \sqcup \mathbb{R}$ that for $x \in \mathbb{R}$ nonzero identifies the points x in the two copies of $\mathbb{R}$ to a single point. This is a non-Hausdorff space composed of an open set $\mathbb{R} \setminus 0$ along with two "origins":



Letting $x_0 \in X$, prove that $\pi_1(X, x_0) \cong \mathbb{Z}$. Hint: Exercise 6.7 might be useful for proving that the cover you produce is 1-connected. $\square$

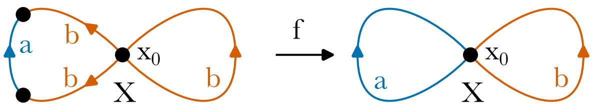
Exercise 7.5. Prove the 1-dimensional Brouwer fixed point theorem: every continuous map $f : \mathbb{D}^1 \to \mathbb{D}^1$ has a fixed point. $\square$

Exercise 7.6. Say that a space X has the *fixed point property* if every continuous map $f : X \to X$ has a fixed point.

(a) Prove that if X has the fixed point property and $Y \subset X$ is a retract of X, then Y has the fixed point property.
(b) Prove that every finite tree T has the fixed point property. Hint: embed T into $\mathbb{D}^2$ such that $\mathbb{D}^2$ retracts to T, and apply the Brouwer fixed point theorem.
(c) A challenging problem: give a direct proof of part (b) that does not use the Brouwer fixed point theorem. $\square$

Exercise 7.7. Prove that $\mathbb{RP}^1 \cong \mathbb{S}^1$, and thus that for all $x_0 \in \mathbb{RP}^1$ we have $\pi_1(\mathbb{RP}^1, x_0) \cong \mathbb{Z}$. $\square$

Exercise 7.8. Let X be a graph with one vertex x_0 and two oriented edges labeled a and b, so $\pi_1(X, x_0)$ is the free group $F(a, b)$ on a and b. Define a map $f : (X, x_0) \to (X, x_0)$ as in the following figure:



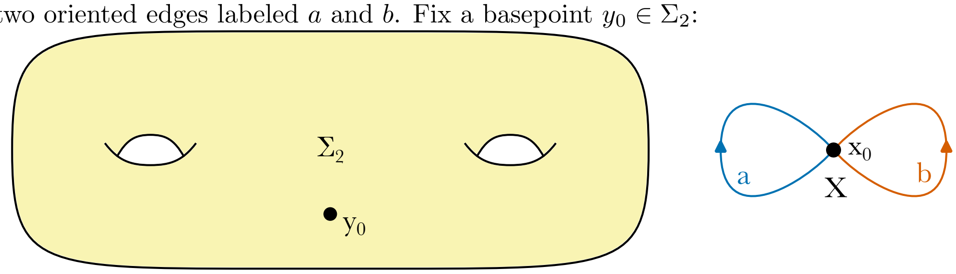
Here the three vertices on the left hand side all map to the single vertex x_0 on the right hand side, and the edges connecting those three vertices map to the edges on the right hand side as indicated. Do the following:

(a) Calculate the induced map $f_*\colon \pi_1(X, x_0) \to \pi_1(X, x_0)$ by determining the images of the two generators $a, b \in F(a, b) = \pi_1(X, x_0)$ in $\pi_1(X, x_0) = F(a, b)$.

(b) Prove that the map f_* you computed in the previous part is not the identity, and conclude that $f\colon (X, x_0) \to (X, x_0)$ is not homotopic to the identity as a map of pointed spaces.

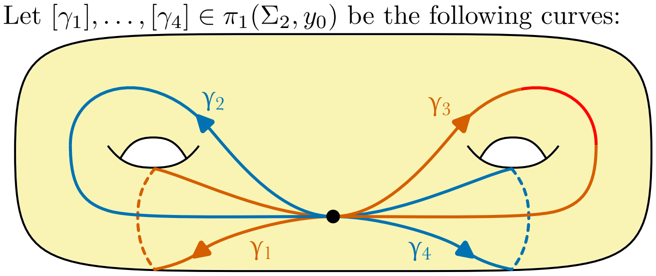
(c) Prove $f\colon X \to X$ is homotopic to the identity as a map of unpointed spaces. □

Exercise 7.9. Let Σ_2 be a closed oriented genus 2 surface and let X be a graph with one vertex x_0 and two oriented edges labeled a and b. Fix a basepoint $y_0 \in \Sigma_2$:



We do not yet know how to calculate $\pi_1(\Sigma_2, y_0)$, but in this exercise and the next we will prove some things about it. Your goal in this exercise is to construct a map $f\colon (\Sigma_2, y_0) \to (X, x_0)$ such that the induced map $f_*\colon \pi_1(\Sigma_2, y_0) \to \pi_1(X, x_0)$ is surjective. Hint: To prove it is surjective, draw explicit elements of $\pi_1(\Sigma_2, y_0)$ mapping to a and to b. □

Exercise 7.10. Just like in the previous exercise, let Σ_2 be a closed oriented genus 2 surface with a basepoint y_0. Let $[\gamma_1], \ldots, [\gamma_4] \in \pi_1(\Sigma_2, y_0)$ be the following curves:



Let $\mathbb{T}^4 = (\mathbb{S}^1)^{\times 4}$ be the 4-dimensional torus. Construct a map $f\colon \Sigma_2 \to \mathbb{T}^4$ such that the map $f_*\colon \pi_1(\Sigma_2, y_0) \to \pi_1(\mathbb{T}^4, f(y_0)) \cong \mathbb{Z}^4$ takes $[\gamma_i]$ to the i^{th} basis vector of $\mathbb{Z}^4$. Hint: It might be easier to construct maps $f_1, f_2\colon \Sigma_2 \to \mathbb{T}^2$ such that:

- $(f_1)_*$ takes $[\gamma_1]$ and $[\gamma_2]$ to the basis elements of $\pi_1(\mathbb{T}^2, f_1(y_0)) \cong \mathbb{Z}^2$ and has $[\gamma_3], [\gamma_4] \in \ker((f_1)_*)$; and
- $(f_2)_*$ takes $[\gamma_3]$ and $[\gamma_4]$ to the basis elements of $\pi_1(\mathbb{T}^2, f_1(y_0)) \cong \mathbb{Z}^2$ and has $[\gamma_1], [\gamma_2] \in \ker((f_2)_*)$.

You can then take $f = f_1 \times f_2$. To construct the f_i, think about collapsing subsurfaces of Σ_2 to points. □

Exercise 7.11. Let G be a group. The goal of this exercise is to construct a pointed space (X, x_0) with $\pi_1(X, x_0) \cong G$. This exercise uses the notion of the join of spaces from Exercise 6.8. Do the following:

(a) Let A and B be spaces equipped with left G-actions. Prove that there is an induced action of G on the join $A * B$.

(b) Let $X = G$ and $Y = G$, both viewed as discrete spaces. Let J be the join $J = X * Y$. Prove that J is 0-connected.

(c) Let $Z = G$, again viewed as a discrete space. Prove that $K = Z * J$ is 1-connected. Hint: Exercise 6.8 will be useful.

(d) The group G acts on itself on the left. By part (a), we get induced actions of G on J and K. Prove that the action of G on K is a covering space action. Letting $X = G/K$ and fixing a basepoint $x_0 \in X$, deduce that $\pi_1(X, x_0) \cong G$. ☐

EXERCISE 7.12. Let $F(a_1, \ldots, a_n)$ be the free group on elements $a_1, \ldots, a_n$. Recall that the *abelianization* of a group G is the abelian group $G/[G,G]$, where $[G,G]$ is the normal subgroup generated by elements of the form $[g,h] = ghg^{-1}h^{-1}$ with $g, h \in G$. Prove that the abelianization of $F(a_1, \ldots, a_n)$ is isomorphic to $\mathbb{Z}^n$. Hint: start by constructing a map $F(a_1, \ldots, a_n) \to \mathbb{Z}^n$ using the universal property of the free group. ☐

EXERCISE 7.13. In this exercise, we outline one of the classical constructions of a free group on a set S.

(a) Let M be the set of all words $w = s_1^{\epsilon_1} \cdots s_n^{\epsilon_n}$, including the empty word. Prove that M is an associative monoid under concatanation. Here an associative monoid is a set equipped with an associative multiplication with an identity, but unlike in a group there need not be inverses.

(b) Let $\sim$ be the equivalence relation on M where $w \sim w'$ if w' can be obtained from w by a sequence of the following two moves:
   - For some $s \in S$, insert either ss^{-1} or $s^{-1}s$ somewhere in w.
   - For some $s \in S$, delete a subword of the form ss^{-1} or $s^{-1}s$ from w.

   Set $F = M/\sim$. Prove that the multiplication on M descends to an associative multiplication on M, and that F is a group.

(c) Prove that F along with the evident map $\iota\colon S \to F$ satisfies the universal property of a free group. Hint: first prove that M satisfies an appropriate universal property to be a "free associative monoid" on S. ☐

EXERCISE 7.14. Let S be a set and let $F(S)$ be a free group on S. This exercise outlines the classical algebraic proof that every element of $F(S)$ is represented by a unique reduced word in S.

(a) Let $\mathcal{W}$ be the set of reduced words in S. Use the universal property of the free group to construct a left action of $F(S)$ on $\mathcal{W}$ that for $s \in S$ multiplies a word $w \in \mathcal{W}$ by s to get sw and then reduces appropriately to get a reduced word. Hint: an action is the same as a homomorphism $F(S) \to \mathrm{Sym}(\mathcal{W})$ where $\mathrm{Sym}(\mathcal{W})$ is the symmetric group consisting of all bijections of $\mathcal{W}$. You can construct such homomorphisms using the universal property of $F(S)$.

(b) Prove that every element of $F(S)$ is represented by a unique reduced word. Hint: consider $w \in F(S)$, and think about where w takes the trivial word $1 \in \mathcal{W}$. ☐

EXERCISE 7.15. For a group G generated by a set S, we have the following two classic decision problems:

- The *word problem*: decide if two words in S represent the same element of G.
- The *conjugacy problem*: decide if two words in S represent conjugate elements of G, that is, elements $g_1, g_2 \in G$ such that there is some $h \in G$ with $g_2 = hg_1h^{-1}$.

For a free group $F(S)$ on a set S, the word problem is easy since two words in S represent the same element of G if and only if they become the same reduced word after repeatedly cancelling terms of the form ss^{-1} and $s^{-1}s$ with $s \in S$. In this exercise, you will solve the conjugacy problem in $F(S)$.

(a) Consider a word $w = s_1^{\epsilon_1} \cdots s_n^{\epsilon_n}$ in S. For some $k \geq 1$, let
$$w' = s_{1+k}^{\epsilon_{1+k}} \cdots s_{n+k}^{\epsilon_{n+k}},$$
where the subscripts are interpreted modulo n. Prove that w and w' are conjugate elements of $F(S)$. We say that they are *cyclically conjugate*.

(b) Say that a word $w = s_1^{\epsilon_1} \cdots s_n^{\epsilon_n}$ is *cyclically reduced* if w is reduced and $s_1^{\epsilon_1} \neq s_n^{-\epsilon_n}$. Prove that every element of $F(S)$ is conjugate to a cyclically reduced word.

(c) Prove that if w and w' are cyclically reduced words, then w and w are conjugate elements of $F(S)$ if and only if they are cyclically conjugate. Since we can check this by just enumerating all the cyclic conjugates of w, this gives an algorithm for solving the conjugacy problem. ☐

CHAPTER 8

Fundamental group: homotopy equivalences

In this chapter, we discuss a weakening of a homeomorphism called a homotopy equivalences that plays an important role in algebraic topology.

8.1. Pointed homotopy equivalences

A map $f\colon (X, x_0) \to (Y, y_0)$ between pointed spaces is a *homotopy equivalence* if there exists a map $g\colon (Y, y_0) \to (X, x_0)$ such that $g \circ f\colon (X, x_0) \to (X, x_0)$ and $f \circ g\colon (Y, y_0) \to (Y, y_0)$ are both homotopic to the identity. We call g a *homotopy inverse* to f, and if there is a homotopy equivalence between (X, x_0) and (Y, y_0) then we will say that (X, x_0) is *homotopy equivalent* to (Y, y_0) and write $(X, x_0) \simeq (Y, y_0)$. Here is an example:

Example 8.1.1. Let X be a space, let $A \subset X$ be a subspace, and let $r_t\colon X \to X$ be a deformation retract to A. We can therefore regard r_1 as a retraction $r_1\colon X \to A$. Pick a basepoint $a_0 \in A$, and let $\iota\colon (A, a_0) \to (X, a_0)$ be the inclusion. Then ι is a homotopy equivalence with homotopy inverse $r_1\colon (X, a_0) \to (A, a_0)$. Indeed, $r_1 \circ \iota\colon (A, a_0) \to (A, a_0)$ is literally the identity, and r_t is a homotopy from the identity $r_0 = \mathbb{1}_X\colon (X, x_0) \to (X, x_0)$ to $r_1 = \iota \circ r_1$. □

It will become more and more clear as we delve deeper into algebraic topology that homotopy equivalent pointed spaces are in many ways the "same" from the perspective of the tools of the subject. Here is one easy way in which this is true, which generalizes the corresponding fact for deformation retracts (Lemma 6.2.5):

Lemma 8.1.2. *Let $f\colon (X, x_0) \to (Y, y_0)$ be a homotopy equivalence between pointed spaces. Then $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.*

Proof. Let $g\colon (Y, y_0) \to (X, x_0)$ be a homotopy inverse to f. Since $g \circ f\colon (X, x_0) \to (X, x_0)$ and $f \circ g\colon (Y, y_0) \to (Y, y_0)$ are homotopic to the identity, the induced maps $(g \circ f)_*\colon \pi_1(X, x_0) \to \pi_1(X, x_0)$ and $(f \circ g)_*\colon \pi_1(Y, y_0) \to \pi_1(Y, y_0)$ are the identity. Functorality implies that

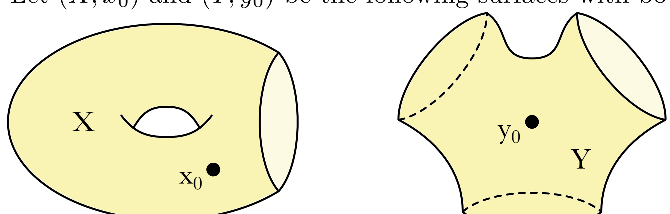
$$\begin{aligned} \mathbb{1}_{\pi_1(X,x_0)} &= (g \circ f)_* = g_* \circ f_* \quad \text{and} \\ \mathbb{1}_{\pi_1(Y,y_0)} &= (f \circ g)_* = f_* \circ g_*, \end{aligned}$$

so f_* and g_* are inverses to each other. This implies that that f_* and g_* are isomorphisms. □

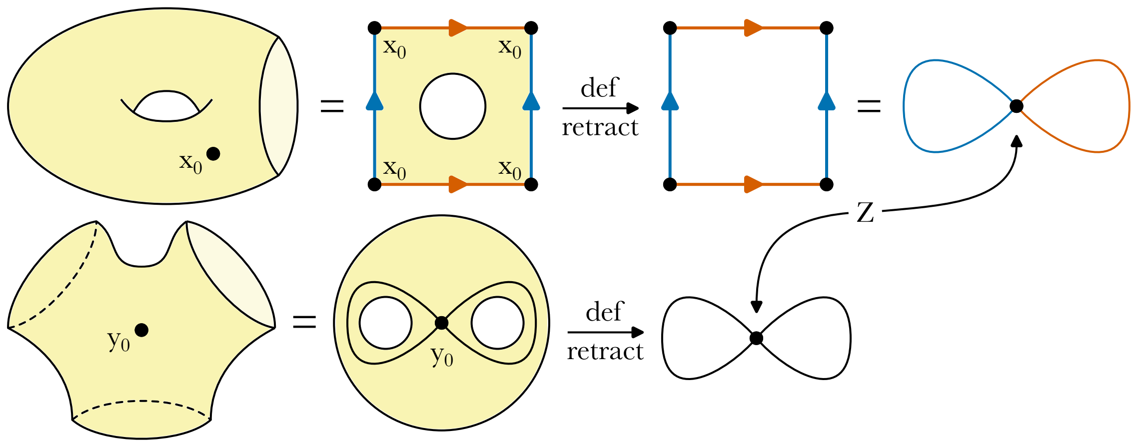
8.2. Composing deformation retractions

As we already noted, if X deformation retracts to a subspace Y and $y_0 \in Y$, then $(Y, y_0) \simeq (X, y_0)$. Being homotopy equivalent is an equivalence relation (see Exercise 8.1), so by applying this multiple times we can get interesting homotopy equivalences. For instance, if Y is a subspace of both X and Z and both X and Z deformation retract to Y, then $(X, y_0) \simeq (Z, y_0)$ even though neither X or Z is contained in the other. Here is an example:

Example 8.2.1. Let (X, x_0) and (Y, y_0) be the following surfaces with boundary:



Both X and Y deformation retract to the same space Z, with x_0 and y_0 corresponding to the same point of Z:



It follows that $(X, x_0) \simeq (Y, y_0)$. Moreover, since the fundamental group of Z is a free group on two generators (Theorem 7.8.1), it follows that $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$ are free groups on two generators. □

8.3. Unpointed homotopy equivalences

We now explain how this works without basepoints. A map $f\colon X \to Y$ between spaces is a *homotopy equivalence* if there exists a map $g\colon Y \to X$ such that $g \circ f\colon X \to X$ and $f \circ g\colon Y \to Y$ are homotopic to the identity. The difference between this and the pointed case is that these homotopies need not fix a basepoint. We call g a *homotopy inverse* to f, and if a homotopy equivalence from X to Y exists we say that X and Y are *homotopy equivalent* and write $X \simeq Y$.

In Lemma 6.3.1, we proved that even though contractions need not fix a basepoint, it is still true that contractible spaces have trivial fundamental groups. By being similarly careful with the basepoint, we prove the following:

LEMMA 8.3.1. *Let $f\colon X \to Y$ be a homotopy equivalence and let $x_0 \in X$. Set $y_0 = f(x_0)$. Then $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.*

REMARK 8.3.2. Since $\pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z}$ but $\mathbb{S}^n$ is 1-connected for $n \geq 2$, it follows from this lemma that $\mathbb{S}^1$ and $\mathbb{S}^n$ are not homotopy equivalent for $n \geq 2$. Similarly, since $\pi_1(\mathbb{RP}^n, 1) \cong C_2$ for $n \geq 2$ it follows that $\mathbb{RP}^n$ and $\mathbb{S}^n$ are not homotopy equivalent for $n \geq 2$. We remark that $\mathbb{RP}^1$ and $\mathbb{S}^1$ are homeomorphic (see Exercise 7.7). □

PROOF OF LEMMA 8.3.1. Let $g\colon Y \to X$ be a homotopy inverse to f. Set $x_1 = g(y_0)$ and $y_1 = f(x_1)$. The naive thing to do would be to prove that the maps $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $g_*\colon \pi_1(Y, y_0) \to \pi_1(X, x_1)$ were inverses to each other. However, this does not make sense since the domain $\pi_1(X, x_0)$ of f_* is not the same as the codomain $\pi_1(X, x_1)$ of g_*.

Instead, what we will prove is that

$$(g \circ f)_*\colon \pi_1(X, x_0) \to \pi_1(X, x_1) \quad \text{and} \quad (f \circ g)_*\colon \pi_1(Y, y_0) \to \pi_1(Y, y_1) \tag{8.3.1}$$

are both isomorphisms. This will imply that $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $g_*\colon \pi_1(Y, y_0) \to \pi_1(X, x_1)$ are both injections. We claim that this implies that f_* is a surjection and hence an isomorphism. Indeed, consider $\zeta \in \pi_1(Y, y_0)$. We want to find some $\eta \in \pi_1(X, x_0)$ with $f_*(\eta) = \zeta$. Since $g_* \circ f_*$ is an isomorphism by (8.3.1), there exists some $\eta \in \pi_1(X, x_0)$ such that $g_*(f_*(\eta)) = g_*(\zeta)$. Since g_* is an injection, we must have $f_*(\eta) = \zeta$, as desired.

It remains to prove that the two maps in (8.3.1) are isomorphisms. The proofs are the same, so we will give the details for $(g \circ f)_*\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$. Since g is a homotopy inverse to f, the map $g \circ f\colon X \to X$ is homotopic to the identity. Let $h_t\colon X \to X$ be a homotopy from $g \circ f$ to $\mathbb{1}_X$. Let $\delta\colon I \to X$ be the following path from $x_1 = (g \circ f)(x_0) = h_0(x_0)$ to $x_0 = h_1(x_0)$:

$$\delta(s) = h_s(x_0) \quad \text{for } s \in I.$$

For $[\gamma] \in \pi_1(X, x_0)$, Lemma 5.7.1 implies that

$$(g \circ f)_*([\gamma]) = (h_0)_*([\gamma]) = [\delta](h_1)_*([\gamma])[\overline{\delta}] = [\delta][\gamma][\overline{\delta}].$$

In other words, $(g \circ f)_*$ equals the change of basepoint isomorphism from $\pi_1(X, x_0)$ to $\pi_1(X, x_1)$ given by the path δ. In particular, $(g \circ f)_*$ is an isomorphism. $\square$

8.4. Mapping cylinder neighborhoods

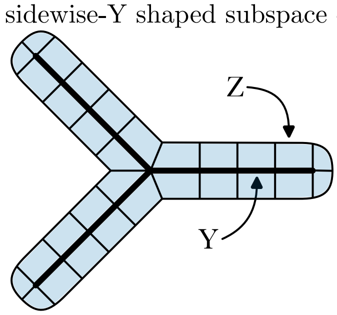
Let X be a space and Y be a contractible subspace of X. Consider the quotient map $q\colon X \to X/Y$. It turns out that in many cases q is a homotopy equivalence. Roughly speaking, this holds as long as Y is embedded into X with reasonable local properties. There are a number of conditions that ensure this. Our next goal is to give one that is fairly easy to state and prove. This requires some preliminary definitions.

Recall from Example 6.2.2 that for a map $f\colon Z \to Y$ between spaces, the mapping cylinder of f is the space $\mathrm{Cyl}(f)$ obtained by quotienting the disjoint union $(Z \times I) \sqcup Y$ to identify $(z, 1) \in Z \times I$ with $f(z) \in Y$ for all $z \in Z$. For $z \in Z$ and $s \in I$, let $\overline{(z, s)}$ be the image of $(z, s) \in Z \times I$ in $\mathrm{Cyl}(f)$. We now define:

Definition 8.4.1. Let X be a space and let $Y \subset X$ be a subspace. A *mapping cylinder neighborhood* of Y is a closed subset N of X containing Y along with a closed subset $Z \subset N$ such that:

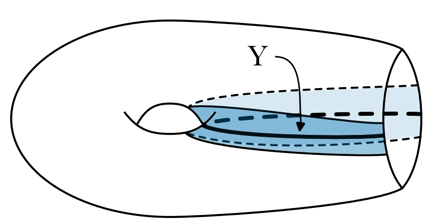
- $N \setminus Z$ is an open neighborhood of Y in X; and
- there exists a map $f\colon Z \to Y$ and a homeomorphism $\phi\colon\ \mathrm{Cyl}(f) \to N$ such that $f(\overline{(z, 0)}) = z$ and $f(y) = y$ for all $z \in Z$ and $y \in Y$. $\square$

Example 8.4.2. Let Y be the sidewise-Y shaped subspace of $\mathbb{R}^2$ shown here:

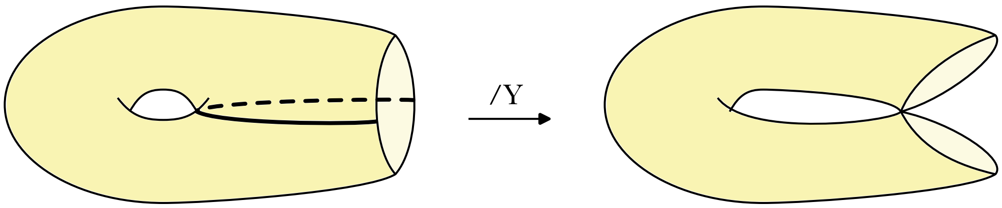


The subspace Y of $\mathbb{R}^2$ has a mapping cylinder neighborhood N indicated in blue. This blue subspace is the mapping cylinder of a map $f\colon Z \to Y$ with $Z \cong \mathbb{S}^1$ the indicated subspace. The lines connect points $z \in Z$ with their images $f(z) \in Y$. Since Y is contractible and has a mapping cylinder neighborhood, it will follow from Theorem 8.5.1 below that the quotient map $\mathbb{R}^2 \to \mathbb{R}^2/Y$ is a homotopy equivalence. In fact, $\mathbb{R}^2/Y \cong \mathbb{R}^2$ (see Exercise 8.4). $\square$

Example 8.4.3. Let X be the following surface with boundary and let $Y \cong I$ be the indicated arc in X:



A mapping cylinder neighborhood N of Y is drawn in blue. Here $N \cong Y \times I$, and N is homeomorphic to the mapping cylinder of the projection $f\colon Y \sqcup Y \to Y$. Since Y is contractible and has a mapping cylinder neighborhood, it will follow from Theorem 8.5.1 below that the quotient map $q\colon X \to X/Y$ is a homotopy equivalence:



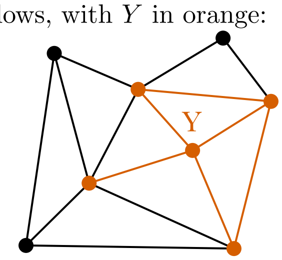
In this case, X and X/Y are not homeomorphic; indeed, X/Y is not a manifold around the point that is the image of Y. We showed in Example 8.2.1 that the fundamental group of X is the free group on two generators, so the same is true for X/Y. □

EXAMPLE 8.4.4. For readers who are familiar with smooth manifolds, here is an important example. Let M^n be a smooth manifold with boundary and let N^d be a properly embedded submanifold of M^n. A closed tubular neighborhood of N^d is then a mapping cylinder neighborhood of N^d. Example 8.4.3 is a special case of this. □

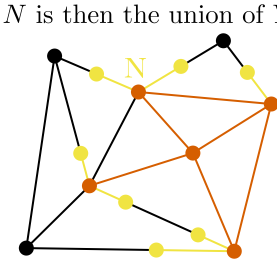
Our final example will be important later, so we separate it out as a lemma:

LEMMA 8.4.5. *Let X be a graph and let $Y \subset X$ be a subgraph. Then Y has a mapping cylinder neighborhood.*

PROOF. Once an example is understood this lemma will be clear, so we give one in lieu of a formal proof. Let X and Y be as follows, with Y in orange:



The mapping cylinder neighborhood N is then the union of Y with the yellow region here:



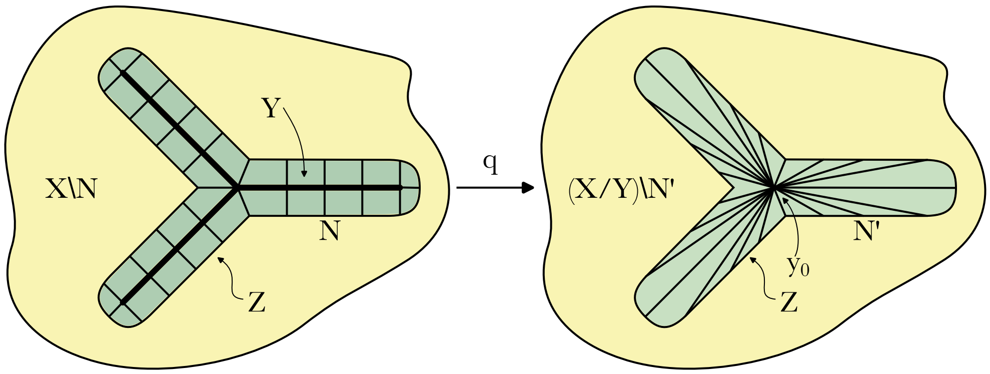
Here Z consists of eight points, and N is the mapping cylinder of a map $f\colon Z \to Y$ taking each of those eight points to a vertex of Y. □

8.5. Collapsing subspaces with mapping cylinder neighborhoods

We now prove:

THEOREM 8.5.1. *Let X be a space and let $Y \subset X$ be a contractible subspace with a mapping cylinder neighborhood. Then the quotient map $q\colon X \to X/Y$ is a homotopy equivalence.*

PROOF. We must use the hypotheses to construct a homotopy inverse $g\colon X/Y \to X$ to q. Let N be a mapping cylinder neighborhood of Y. Identify N with $\mathrm{Cyl}(f)$ for some $Z \subset N$ and some map $f\colon Z \to Y$. Let y_0 be the point of X/Y corresponding to Y, let $f'\colon Z \to y_0$ be the projection, and let $N' = \mathrm{Cyl}(f')$. We can identify N' with N/Y, and after making this identification N' is a mapping cylinder neighborhood of y_0 in X/Y with $X \setminus N = (X/Y) \setminus N'$. See here:



To construct a continuous map $X/Y \to X$, it is enough to construct a continuous map $N' \to N$ that is the identity on Z and then extend $N' \to N$ to X/Y by the identity.[1] In a similar way, we can construct continuous maps $X \to X$ (resp. $X/Y \to X/Y$) by constructing continuous maps $N \to N$ (resp. $N' \to N$) and extending by the identity.

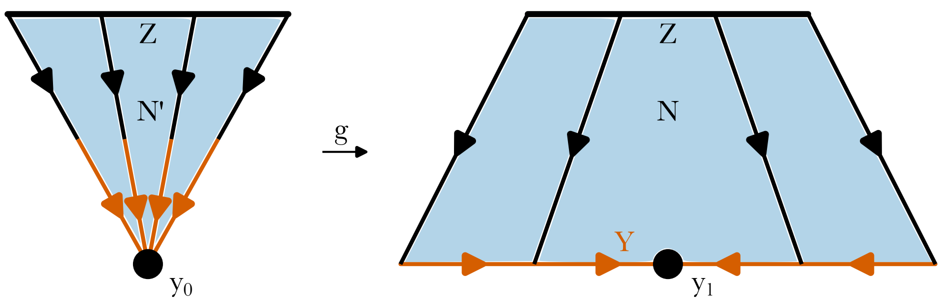
For $z \in Z$ and $s \in I$, let $\overline{(z,s)} \in N$ and $\overline{(z,s)}' \in N'$ be the corresponding points. We now divide the proof into three steps:

STEP 1. *We construct the purported homotopy inverse $g\colon X/Y \to X$.*

Since Y is contractible, there is a homotopy $h_t\colon Y \to Y$ from $\mathbb{1}_Y$ to a constant map. Let $y_1 \in Y$ be the constant value of h_1. Define $g\colon X/Y \to X$ via the formulas

$$\begin{cases} g(\overline{(z,s)}') = \overline{(z,2s)} & \text{for } z \in Z \text{ and } s \in [0,1/2], \\ g(\overline{(z,s)}') = h_{2s-1}(f(z)) & \text{for } z \in Z \text{ and } s \in [1/2,1], \\ g(y_0) = y_1 & \\ g(x) = x & \text{for } x \in (X/Y) \setminus N' = X \setminus N. \end{cases}$$

See here, where the indicated map takes each line segment from $z \in Z$ to y_0 to the path that first goes to $f(z)$ (in black) and then in Y to y_1 (in orange):



By what we said above, this map $g\colon X/Y \to X$ is continuous.

STEP 2. *We prove that the composition $g \circ q\colon X \to X$ is homotopic to the identity.*

The map $g \circ q\colon X \to X$ is given by the formulas

$$\begin{cases} g \circ q(\overline{(z,s)}) = \overline{(z,2s)} & \text{for } z \in Z \text{ and } s \in [0,1/2], \\ g \circ q(\overline{(z,s)}) = h_{2s-1}(f(z)) & \text{for } z \in Z \text{ and } s \in [1/2,1], \\ g \circ q(y) = y_1 & \text{for } y \in Y, \\ g \circ q(x) = x & \text{for } x \in X \setminus N. \end{cases}$$

[1]To see that this is continuous, note that the map $X/Y \to X$ is continuous on the closed sets N' and $(X/Y)\setminus(N'\setminus Z)$; indeed, on the latter set it is the identity. These cover X/Y. Now apply the fact that if $\psi\colon A \to B$ is a map of sets between spaces and $\{C_1, \ldots, C_n\}$ is a cover of A by closed sets such that each $\psi|_{C_i}$ is continuous, then ψ is continuous. Note that this would be false if our cover had infinitely many closed sets in it.

This is homotopic to the identity via the homotopy $\phi_t\colon X \to X$ given by the formulas

$$\begin{cases} \phi_t(\overline{(z,s)}) = \overline{(z,(2-t)s)} & \text{for } z \in Z \text{ and } s \in [0, 1/(2-t)], \\ \phi_t(\overline{(z,s)}) = h_{(2-t)s-1}(f(z)) & \text{for } z \in Z \text{ and } s \in [1/(2-t), 1], \\ \phi_t(y) = h_{1-t}(y) & \text{for } y \in Y, \\ \phi_t(x) = x & \text{for } x \in X \setminus N. \end{cases}$$

By the discussion at the beginning of the proof this is continuous.

STEP 3. *We prove that the composition $q \circ g\colon X/Y \to X/Y$ is homotopic to the identity.*

The map $q \circ g\colon X/Y \to X/Y$ is given by the formulas

$$\begin{cases} q \circ g(\overline{(z,s)}') = \overline{(z,2s)}' & \text{for } z \in Z \text{ and } s \in [0, 1/2], \\ q \circ g(\overline{(z,s)}') = y_0 & \text{for } z \in Z \text{ and } s \in [1/2, 1], \\ q \circ g(y_0) = y_0 & \\ g \circ q(x) = x & \text{for } x \in (X/Y) \setminus N'. \end{cases}$$

This is homotopic to the identity via the homotopy $\psi_t\colon X/Y \to X/Y$ given by the formulas

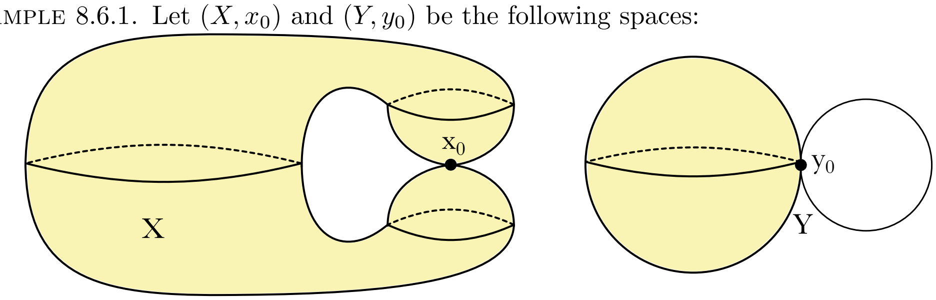
$$\begin{cases} \psi_t(\overline{(z,s)}') = \overline{(z,(2-t)s)}' & \text{for } z \in Z \text{ and } s \in [0, 1/(2-t)], \\ \psi_t(\overline{(z,s)}') = y_0 & \text{for } z \in Z \text{ and } s \in [1/(2-t), 1], \\ \psi_t(y_0) = y_0 & \\ \psi_t(x) = x & \text{for } x \in X \setminus N. \end{cases}$$

By the discussion at the beginning of the proof this is continuous. □

8.6. Example of collapsing

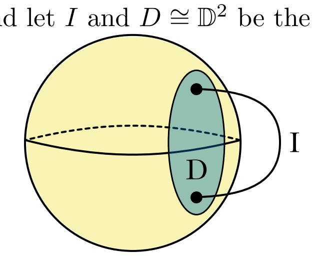
We now give an example of how to this to analyze an interesting example.

EXAMPLE 8.6.1. Let (X, x_0) and (Y, y_0) be the following spaces:



The space X is obtained by quotienting $\mathbb{S}^2$ to identify two points together,[2] and the space Y is obtained by gluing $\mathbb{S}^2$ and $\mathbb{S}^1$ together at a single point. We will prove that $X \simeq Y$ and that $\pi_1(X, x_0) \cong \pi_1(Y, y_0) \cong \mathbb{Z}$.

Let Z be the following space and let I and $D \cong \mathbb{D}^2$ be the indicated subspaces of Z:



We have $Z/I \cong X$ and $Z/D \cong Y$. Since I is contractible and has a mapping cylinder neighborhood in

[2] It does not matter which two points are identified. Indeed, any two points "look the same" in the sense that they differ by a homeomorphism of $\mathbb{S}^2$. More generally, if M^n is a connected n-manifold with $n \geq 2$ and $\{p_1, \ldots, p_k\}$ and $\{q_1, \ldots, q_k\}$ are two sets of k distinct points on M^n, then there exists a homeomorphism $f\colon M^n \to M^n$ with $f(p_i) = q_i$ for all $1 \leq i \leq k$.

Z, it follows that $Z \simeq X$. Similarly, since D is contractible and has a mapping cylinder neighborhood in Z, it follows that $Z \simeq Y$. We conclude that $X \simeq Y$.

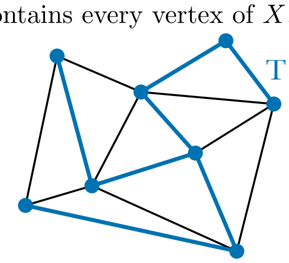
In particular, $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$. It remains to prove that $\pi_1(Y, y_0) \cong \mathbb{Z}$. The space Y is the union of $\mathbb{S}^2$ and $\mathbb{S}^1$. Identifying $\mathbb{S}^2$ and $\mathbb{S}^1$ with their images in Y, we have $\mathbb{S}^2 \cap \mathbb{S}^1 = y_0$. There is a retraction $Y \to \mathbb{S}^1$ that collapses $\mathbb{S}^2$ to y_0. It follows that the inclusion $(\mathbb{S}^1, y_0) \to (Y, y_0)$ induces an injection from $\pi_1(\mathbb{S}^1, y_0) \cong \mathbb{Z}$ to $\pi_1(Y, y_0)$.

We must prove that this injection is surjective. Consider some $[\gamma] \in \pi_1(Y, y_0)$. Let $r \in \mathbb{S}^2$ be a point other than y_0. Just like in our proof that $\mathbb{S}^n$ has a trivial fundamental group (Lemma 4.5.2), we can use Lemma 4.5.3 (general position) to homotope γ such that its image does not contain r. Since $\mathbb{S}^2 \setminus r \cong \mathbb{R}^2$ deformation retracts to y_0, it follows that γ can be homotoped to a loop in $\mathbb{S}^1$, as desired. □

We close this section by introducing some terminology. For pointed spaces $\{(Z_i, z_i)\}_{i \in I}$, the *wedge product* of the Z_i is the space $W = \vee_{i \in I}(Z_i, z_i)$ obtained from the disjoint union of the Z_i by identifying all the basepoints z_i together to a single point w_0. The point w_0 serves as a basepoint, so (W, w_0) is a pointed space. We will often omit explicit mention of the basepoints z_i and just write $W = \vee_{i \in I} Z_i$, and if I is finite we will use the $\vee$ like a sum and e.g. write $Z_1 \vee Z_2$. For instance, the pointed space (Y, y_0) in Example 8.6.1 is $\mathbb{S}^2 \vee \mathbb{S}^1$.

8.7. Maximal trees

Our final goal in this chapter is to use these tools to calculate the fundamental group of an arbitrary connected graph. This requires some preliminaries. Recall that a tree is a nonempty connected graph with no cycles. Each tree is contractible (Lemma 6.4.1). For a graph X, a *maximal tree* in X is a subtree T of X that contains every vertex of X. For instance:



These always exist:

LEMMA 8.7.1. *Let X be a nonempty connected graph. Then X contains a maximal tree.*

PROOF. Inductively define subtrees

$$T_0 \subset T_1 \subset T_2 \subset \cdots$$

of X in the following way. Start by choosing a vertex v_0 of X and letting $T_0 = v_0$. Next, if T_{n-1} has been constructed, let T_n be the subtree obtained from T_{n-1} as follows:

- For each vertex v of X that does not lie in T_{n-1} but is connected by an edge to T_{n-1}, choose an edge connecting T_{n-1} to v and add it to T_n.

Now define

$$T = \bigcup_{n=0}^{\infty} T_n.$$

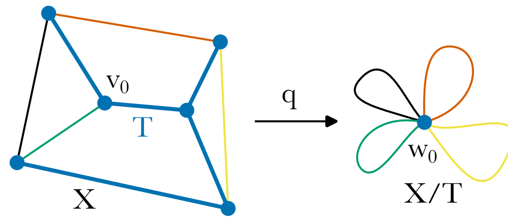
This is a subgraph of G. Since a cycle in T only involves finitely many edges, a cycle of T must be contained in some T_n. Since each T_n is a tree, it follows that T has no cycles, so T is a tree. Since X is connected, each vertex of X must lie in T, so T is a maximal tree. □

8.8. Fundamental groups of graphs

Using maximal trees, we will prove:

THEOREM 8.8.1. *Let X be a connected graph and let v_0 be a vertex of X. Then $\pi_1(X, v_0)$ is a free group.*

Proof. Let T be a maximal tree in X. The quotient graph X/T contains a single vertex w_0 and a loop for each edge of X that does not lie in T:



Note that though our graphs are always oriented, this picture does not indicate the orientations of the edges. We saw in the proof of Theorem 7.7.1 that $\pi_1(X, w_0)$ is a free group. Since T is a contractible subspace of X with a mapping cylinder neighborhood (Lemma 8.4.5), the quotient map $q\colon X \to X/T$ is a homotopy equivalence. The induced map $q_*\colon \pi_1(X, v_0) \to \pi_1(W, w_0)$ is therefore an isomorphism, so $\pi_1(X, v_0)$ is also a free group. □

If X is a finite connected graph with vertices $\mathcal{V}(X)$ and edges $\mathcal{E}(X)$, then we can give a formula for the rank of the free group $\pi_1(X, v_0)$ as follows. The *Euler characteristic* of X is

$$\chi(X) = |\mathcal{V}(X)| - |\mathcal{E}(X)| \in \mathbb{Z}.$$

We then have:

Theorem 8.8.2. *Let X be a finite connected graph and let v_0 be a vertex of X. Then $\pi_1(X, v_0)$ is a free group of rank n, where $\chi(X) = 1 - n$.*

Proof. Let T be a maximal tree of X. The proof of Theorem 8.8.1 shows that $\pi_1(X, v_0)$ is a free group of rank n where n is the number of edges of X/T. Collapsing a single non-loop edge to a point does not change $\chi(X)$ since it causes the number of vertices and edges to both go down by 1. Iterating this, we see that $\chi(X) = \chi(X/T) = 1 - n$, as desired. □

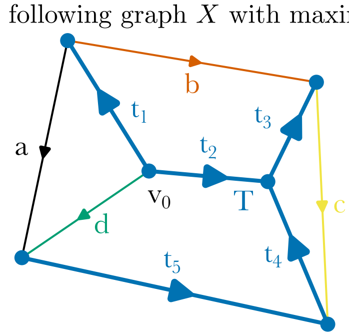
8.9. Free bases for fundamental groups of graphs

Let X be a connected graph and let v_0 be a vertex of X. By analyzing the proof of Theorem 8.8.1, we can construct a free basis for the free group $\pi_1(X, v_0)$. Begin by choosing a maximal tree T of X. Let w_0 be the single vertex of X/T. As in the proof of Theorem 8.8.1, the quotient map $q\colon X \to X/T$ induces an isomorphism $q_*\colon \pi_1(X, v_0) \to \pi_1(X/T, w_0)$.

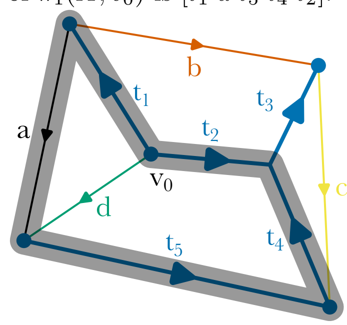
Let $\{e_i \mid i \in I\}$ be the edges of X that do *not* lie in T. The map q maps each e_i to a loop $\overline{e}_i$ in X/T that is based at w_0. Recall our convention that each edge of a graph is oriented (cf. §2.5). Using the orientation on $\overline{e}_i$, we get an element $[\overline{e}_i] \in \pi_1(X/T, w_0)$. By Theorem 7.8.1, the set $\{[\overline{e}_i] \mid i \in I\}$ is a basis for the free group $\pi_1(X/T, w_0)$.

For each $i \in I$, we must lift $\overline{e}_i$ to a loop in X that is based at v_0. To do this, let t_i be a path in T from v_0 to the initial vertex of the edge e_i and let t'_i be a path in T from the terminal vertex of e_i back to v_0. We then have a loop $t_i{\cdot}e_i{\cdot}t'_i$ in X based at v_0, and $q_*([t_i{\cdot}e_i{\cdot}t'_i]) = [\overline{e}_i]$. We deduce that $\{[t_i{\cdot}e_i{\cdot}t'_i] \mid i \in I\}$ is a basis for the free group $\pi_1(X, v_0)$.

Example 8.9.1. Consider the following graph X with maximal tree T:



We have labeled and shown the orientation on each edge of X. Following the above algorithm, we obtain a free basis for $\pi_1(X, v_0)$. There is one element of this basis for each edge $\{a, b, c, d\}$. For the edge a, the corresponding element of $\pi_1(X, v_0)$ is $[t_1{\cdot}a{\cdot}t_5{\cdot}t_4{\cdot}\bar{t}_2]$:



We similarly get basis elements corresponding to b and c and d. In summary, the following is a free basis for $\pi_1(X_S, v_0)$:

$$\{[t_1{\cdot}a{\cdot}t_5{\cdot}t_4{\cdot}\bar{t}_2], [t_1{\cdot}b{\cdot}\bar{t}_3{\cdot}\bar{t}_2], [t_2{\cdot}t_3{\cdot}c{\cdot}\bar{t}_4{\cdot}\bar{t}_2], [d{\cdot}t_5{\cdot}t_4{\cdot}\bar{t}_2]\}. \qquad \square$$

8.10. Exercises

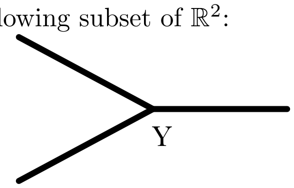
Exercise 8.1. Prove that being homotopy equivalent is an equivalence relation on pointed spaces and on spaces. □

Exercise 8.2. Prove that a space X is contractible if and only if it is homotopy equivalent to a one-point space p_0. □

Exercise 8.3. Let (X, x_0) and (Y, y_0) be pointed spaces. Let $[(X, x_0), (Y, y_0)]_*$ be the set of homotopy classes of maps $f\colon (X, x_0) \to (Y, y_0)$. We will often omit the basepoints and just write $[X, Y]_*$. We remark that set of homotopy classes of maps $f\colon X \to Y$ that do not necessarily preserve the basepoints is written $[X, Y]$. Prove the following:

(a) Precomposition with a pointed homotopy equivalence $h\colon (Z, z_0) \to (X, x_0)$ induces a bijection $h_*\colon [X, Y]_* \to [Z, Y]_*$.
(b) Postcomposition with a pointed homotopy equivalence $h\colon (Y, y_0) \to (Z, z_0)$ induces a bijection $h_*\colon [X, Y]_* \to [X, Z]_*$. □

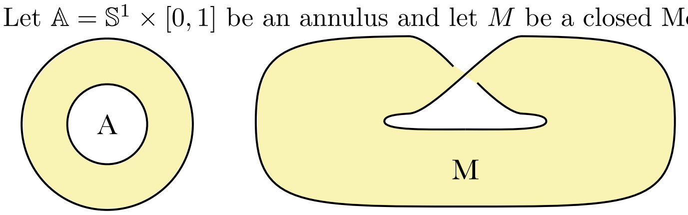
Exercise 8.4. Let Y be the following subset of $\mathbb{R}^2$:



Prove that $\mathbb{R}^2/Y \cong \mathbb{R}^2$. □

Exercise 8.5. For some $n \geq 1$, let $X = \{(x, y) \in \mathbb{S}^n \times \mathbb{S}^n \mid x \neq -y\}$. Define a map $f\colon \mathbb{S}^n \to X$ via the formula $f(x) = (x, x)$. Prove that f is a homotopy equivalence. □

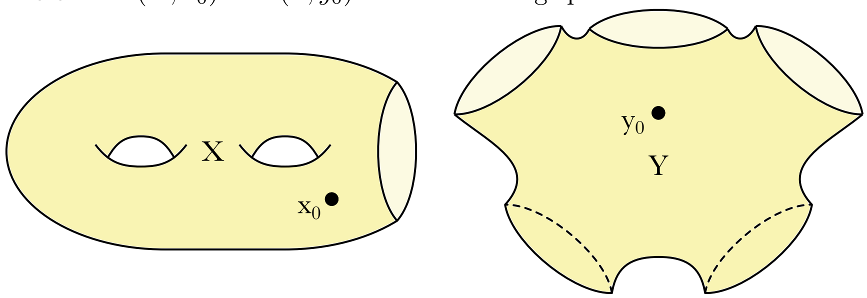
Exercise 8.6. Let $\mathbb{A} = \mathbb{S}^1 \times [0, 1]$ be an annulus and let M be a closed Möbius band:



Prove that $\mathbb{A}$ and M are homotopy equivalent. □

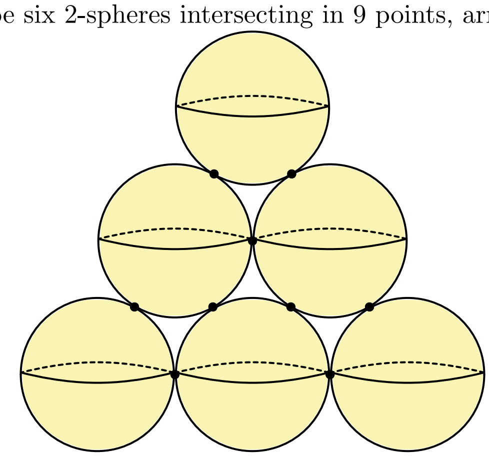
Exercise 8.7. Let $\mathbb{A} = \mathbb{S}^1 \times [0, 1]$ be an annulus and let $\mathrm{Int}(\mathbb{A}) = \mathbb{S}^1 \times (0, 1)$ be an open annulus. Prove that $\mathbb{A}$ and $\mathrm{Int}(\mathbb{A})$ are homotopy equivalent. □

EXERCISE 8.8. Let (X, x_0) and (Y, y_0) be the following spaces:



Prove that $X \simeq Y$, and calculate $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$. □

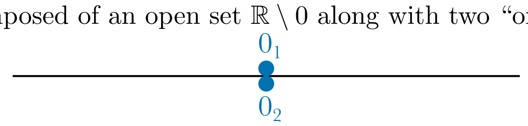
EXERCISE 8.9. Let X be six 2-spheres intersecting in 9 points, arranged as follows:



Do the following:

(a) Prove that X is homotopy equivalent to $(\vee_{i=1}^{6} \mathbb{S}^2) \vee (\vee_{j=1}^{4} \mathbb{S}^1)$.
(b) Letting $x_0 \in X$ be a basepoint, calculate $\pi_1(X, x_0)$. The result will be a free group, and you should also draw loops on X corresponding to generators of this free group. □

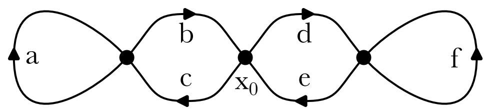
EXERCISE 8.10. As in Exercise 7.4, let X be the "line with two origins", i.e., the quotient of $\mathbb{R} \sqcup \mathbb{R}$ that for $x \in \mathbb{R}$ nonzero identifies the points x in the two copies of $\mathbb{R}$ to a single point. This is a non-Hausdorff space composed of an open set $\mathbb{R} \setminus 0$ along with two "origins":



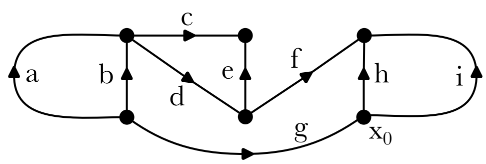
Prove that X is not homotopy equivalent to any Hausdorff space. Hint: you proved in Exercise 7.4 that the fundamental group of X is $\mathbb{Z}$. Prove that any map from X to a Hausdorff space induces the trivial map on the fundamental group. □

EXERCISE 8.11. Calculate free generating sets for $\pi_1(X, x_0)$ where X is one of the following three graphs with the indicated base vertex x_0:

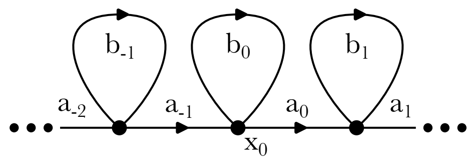
(i)



(ii)



(iii)



The edges are labeled to make it easy to describe loops based at x_0. Hint: first find a maximal tree. □

EXERCISE 8.12. As a silly exercise in pure point-set topology, prove that for all $n \geq 1$ there exist topological spaces X with exactly n points that are contractible. □

CHAPTER 9

Classifying covers: lifting criterion

In this chapter, we use the fundamental group to characterize when a map can be lifted to a cover. Using this, we will prove that in favorable situations covering spaces can be understood using fundamental group information. We will elaborate on this more in the next chapter when we describe the classification of covering spaces.

9.1. Fundamental group of cover

Before giving the lifting criterion, we introduce some terminology and prove a preliminary result. Let $p\colon \widetilde{X} \to X$ be a cover and let $\gamma\colon I \to X$ be a path. For $\widetilde{x} \in \widetilde{X}$ such that $p(\widetilde{x}) = \gamma(0)$, Lemma 3.4.1 (path lifting) says that there is a unique lift $\widetilde{\gamma}\colon I \to \widetilde{X}$ of γ such that $\widetilde{\gamma}(0) = \widetilde{x}$. We will simply call this the *lift of* γ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}$.

We will also need to be careful with basepoints. A *pointed cover* is a pointed map $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that the map $p\colon \widetilde{X} \to X$ is a cover. A pointed cover $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ induces a map $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$. This map is injective; in fact, the following holds:

Theorem 9.1.1. *Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a pointed cover. Then the following hold:*

(i) The map $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective.

(ii) The image of p_* consists of all $[\gamma] \in \pi_1(X, x_0)$ such that the following holds:

(†) Let $\widetilde{\gamma}$ be the lift of γ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$. Then $\widetilde{\gamma}$ is a loop, i.e., $\widetilde{\gamma}(1) = \widetilde{x}_0$.

(iii) If $\widetilde{X}$ is path connected, then the index of $\operatorname{Im}(p_*)$ in $\pi_1(X, x_0)$ equals the degree of the cover.

Proof. We prove each part separately:

Step 1. *The map $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective.*

Consider $[\widetilde{\gamma}] \in \pi_1(\widetilde{X}, \widetilde{x}_0)$ in the kernel of p_*. Our goal is to prove that $[\widetilde{\gamma}] = 1$. Letting $\gamma = p \circ \gamma$, the element $[\gamma] \in \pi_1(X, x_0)$ is trivial. Lemma 4.2.1 says that the homotopy class of the lift $\widetilde{\gamma}$ of γ to $\widetilde{X}$ only depends on the homotopy class of γ. Since $[\gamma] = 1$, it follows that $[\widetilde{\gamma}] = 1$.

Step 2. *The image of p_* consists of all $[\gamma] \in \pi_1(X, x_0)$ such that the following holds:*

(†) Let $\widetilde{\gamma}$ be the lift of γ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$. Then $\widetilde{\gamma}$ is a loop, i.e., $\widetilde{\gamma}(1) = \widetilde{x}_0$.

It is immediate from the definitions that the image of p_* consists of all $[\gamma] \in \pi_1(X, x_0)$ such that γ is homotopic as a path to a loop γ' based at x_0 such that γ' satisfies (†). Just like in Step 1, we can lift a homotopy of paths from γ to γ' to a homotopy of paths in the cover. It follows that γ also satisfies (†). The step follows.

Step 3. *If $\widetilde{X}$ is path connected, then the index of $\operatorname{Im}(p_*)$ in $\pi_1(X, x_0)$ equals the degree of the cover.*

We have $\widetilde{x}_0 \in p^{-1}(x_0)$. Enumerate the entire fiber $p^{-1}(x_0)$ as $\{\widetilde{x}_i \mid i \in I\}$ for some indexing set I with $0 \in I$. The cardinality $|I|$ is thus the degree of the cover. For each $i \in I$, pick a path $\widetilde{\delta}_i$ in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}_i$. Let δ_i be the image of $\widetilde{\delta}_i$ in X, so $[\delta_i] \in \pi_1(X, x_0)$. We claim that $\{[\delta_i] \mid i \in I\}$ is a set of left coset representatives for $\operatorname{Im}(p_*)$.

To see this, consider $[\gamma] \in \pi_1(X, x_0)$. Lift γ to a path $\widetilde{\gamma}$ in $\widetilde{X}$ starting at $\widetilde{x}_0$. Let $\widetilde{x}_{i_0}$ be the endpoint of $\widetilde{\gamma}$. As in the previous steps, $\widetilde{x}_{i_0}$ only depends on $[\gamma]$. We have

$$[\gamma] = [\gamma \cdot \overline{\delta}_{i_0}][\delta_{i_0}]. \tag{9.1.1}$$

Since $\gamma\cdot\overline{\delta}_{i_0}$ lifts to the loop $\widetilde{\gamma}\cdot\overline{\widetilde{\delta}}_{i_0}$ based at $\widetilde{x}_0$, Step 2 implies that $[\gamma\cdot\overline{\delta}_{i_0}] \in \operatorname{Im}(p_*)$. The equation (9.1.1) exhibits $[\gamma]$ as an element of the left $[\delta_{i_0}]$-coset of $\operatorname{Im}(p_*)$. Since this expression is clearly unique, this proves the claim. $\square$

9.2. Lifting criterion

Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a pointed cover. Consider a pointed map $f\colon (Y, y_0) \to (X, x_0)$. Our goal is to understand when f can be lifted to $\widetilde{X}$, i.e., when there exists a pointed map $\widetilde{f}\colon (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ such that $f = p \circ \widetilde{f}$. In other words, we want the following diagram to commute:

$$\begin{tikzcd} & (\widetilde{X}, \widetilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, "\widetilde{f}"] \arrow[r, "f"] & (X, x_0). \end{tikzcd}$$

There is one obvious necessary condition. Assume that $\widetilde{f}$ exists, and pass to the fundamental group. We get a commutative diagram

$$\begin{tikzcd} & \pi_1(\widetilde{X}, \widetilde{x}_0) \arrow[d, "p_*"] \\ \pi_1(Y, y_0) \arrow[ur, "\widetilde{f}_*"] \arrow[r, "f_*"] & \pi_1(X, x_0). \end{tikzcd}$$

Since $p_* \circ \widetilde{f}_* = f_*$, it follows immediately that the image of f_* is contained in the image of the injective map p_*. It turns out that for reasonable spaces Y, this necessary condition is sufficient. Since the fundamental group only depends on the path component containing the basepoint, we clearly need to assume that Y is path connected.

To avoid pathological local behavior, we also need to assume that Y is locally path connected, i.e., that for all $y \in Y$ and all open neighborhoods U of y there exists a path connected open neighborhood V of y with $V \subset U$. This property passes to covers in the sense that if Y is locally path connected and $q\colon \widetilde{Y} \to Y$ is a cover, then $\widetilde{Y}$ is locally path connected (see Exercise 9.1). Our lifting criterion is as follows:

THEOREM 9.2.1 (Lifting criterion). *Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a pointed cover. Let $f\colon (Y, y_0) \to (X, x_0)$ be a pointed map such that the image of $f_*\colon \pi_1(Y, y_0) \to \pi_1(X, x_0)$ is contained in the image of $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$. Assume that Y is path connected and locally path connected. Then f can be uniquely lifted to a map $\widetilde{f}\colon (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$.*

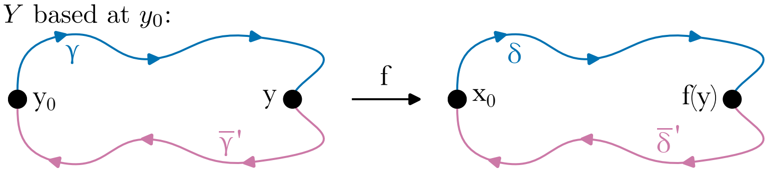
REMARK 9.2.2. See Exercise 9.12 for an example showing that it is necessary to assume that Y is locally path connected. $\square$

PROOF OF THEOREM 9.2.1. We proved in Lemma 3.1.2 that for connected spaces lifts are determined by what they do to a single point. Since we are assuming that our lift $\widetilde{f}$ takes y_0 to $\widetilde{x}_0$, this implies that the lift is unique if it exists. We must prove existence.

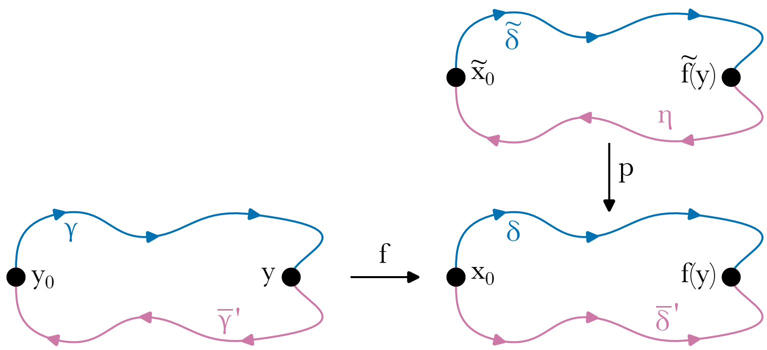
Consider $y \in Y$. We define $\widetilde{f}(y)$ as follows. Since Y is path connected, we can find a path γ from y_0 to y. Its image $\delta = f \circ \gamma$ is a path from $f(y_0) = x_0$ to $f(y)$. Let $\widetilde{\delta}\colon I \to \widetilde{X}$ be the lift of δ to $\widetilde{X}$ with $\widetilde{\delta}(0) = \widetilde{x}_0$. We would like to define $\widetilde{f}(y) = \widetilde{\delta}(1)$. To do this, we must prove that this does not depend on the choice of γ:

CLAIM. *The above definition of $\widetilde{f}(y) \in \widetilde{X}$ does not depend on the choice of γ.*

PROOF OF CLAIM. Let γ' be another path from y_0 to y. Letting $\delta' = f \circ \gamma'$ be its image in X and $\widetilde{\delta}'\colon I \to \widetilde{X}$ be the lift of δ' with $\widetilde{\delta}'(0) = \widetilde{x}_0$, our goal is to prove that $\widetilde{\delta}(1) = \widetilde{\delta}'(1)$. Observe that $\gamma\cdot\overline{\gamma}'$ is a loop in Y based at y_0:



As is shown here, the image of $[\gamma \cdot \overline{\gamma}'] \in \pi_1(Y, y_0)$ under f_* is the loop $[\delta \cdot \overline{\delta}'] \in \pi_1(X, x_0)$. By assumption, this lies in the image of p_*. Theorem 9.1.1 therefore implies that $\delta \cdot \overline{\delta}' : I \to X$ lifts to a loop in $\widetilde{X}$ starting at $\widetilde{x}_0$:



As is shown here, this lifted loop equals $\widetilde{\delta} \cdot \eta$, where:

- as introduced above, the path $\widetilde{\delta}$ is the lift of δ with $\widetilde{\delta}(0) = \widetilde{x}_0$; and
- the path η is the lift of $\overline{\delta}'$ with $\eta(0) = \widetilde{\delta}(1) = \widetilde{f}(y)$.

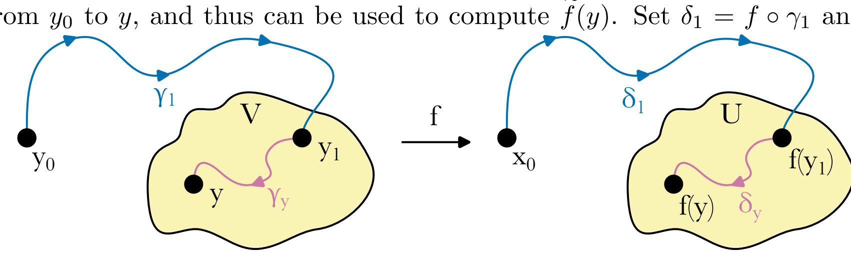
Since $\eta(1) = \widetilde{x}_0$, it follows that $\overline{\eta}$ is the path $\widetilde{\delta}'$ introduced above that we obtained by lifting δ' to a path with $\widetilde{\delta}'(0) = \widetilde{x}_0$. We conclude that $\widetilde{\delta}(1) = \eta(0) = \widetilde{\delta}'(1)$, as desired. $\square$

We have now defined $\widetilde{f} \colon Y \to \widetilde{X}$. This map satisfies $\widetilde{f}(y_0) = \widetilde{x}_0$ since in its definition when calculating $\widetilde{f}(y_0)$ we can take γ to be the constant path at y_0. All that remains to prove is that $\widetilde{f}$ is continuous. Consider some $y_1 \in Y$. Let $U \subset X$ be a trivialized open neighborhood of $f(y_1) \in X$ and let $\widetilde{U} \subset X$ be the sheet above U containing $\widetilde{f}(y_1)$. Since Y is locally path connected, we can find a path connected open neighborhood V of y_1 such that $V \subset f^{-1}(U)$. To prove that $\widetilde{f}$ is continuous at y_1, it is enough to prove that on V the map $\widetilde{f}$ is the composition

$$V \xrightarrow{f} U \xrightarrow{(p|_{\widetilde{u}})^{-1}} \widetilde{U}.$$

To see this, consider some $y \in V$. The image $\widetilde{f}(y) \in \widetilde{X}$ lies in the fiber $p^{-1}(f(y))$, and we must prove that it is the point of this fiber lying in $\widetilde{U}$. In other words, we must prove that $\widetilde{f}(y) \in \widetilde{U}$.

Let γ_1 be a path in Y from y_0 to y_1 and let γ_y be a path in V from y_1 to y. The path $\gamma_1 \cdot \gamma_y$ in Y goes from y_0 to y, and thus can be used to compute $\widetilde{f}(y)$. Set $\delta_1 = f \circ \gamma_1$ and $\delta_y = f \circ \gamma_y$:



By definition, $\widetilde{f}(y)$ is the endpoint of the lift of $\delta_1 \cdot \delta_y$ to $\widetilde{X}$ starting at $\widetilde{x}_0$. We construct this lift in two steps:

- Let $\widetilde{\delta}_1$ be the lift of δ_1 to $\widetilde{X}$ with $\widetilde{\delta}_1(0) = \widetilde{x}_0$. By definition, $\widetilde{f}(y_1) = \widetilde{\delta}_1(1) \in \widetilde{U}$.
- Let $\widetilde{\delta}_y$ be the lift of δ_y to $\widetilde{X}$ with $\widetilde{\delta}_y(0) = \widetilde{\delta}_1(1) \in \widetilde{U}$. By definition, $\widetilde{f}(y) = \widetilde{\delta}_y(1)$.

Since γ_y is a path in V and $f(V) \subset U$, it follows that δ_y is a path in U. Since $\widetilde{\delta}_y(0) \in \widetilde{U}$, we have that $\widetilde{\delta}_y$ is a path in $\widetilde{U}$; in fact, $\widetilde{\delta}_y = (p|_{\widetilde{u}})^{-1} \circ \delta_y$. We conclude that $\widetilde{f}(y) = \widetilde{\delta}_y(1) \in \widetilde{U}$, as desired. $\square$

9.3. Pointed isomorphisms of covers

Our next goal is to apply Theorem 9.2.1 (lifting criterion) to help us classify pointed covers and construct deck transformations using fundamental group information. We defined what it means for two covers to be isomorphic in §1.5. Adding basepoints, we make the following definition:

Definition 9.3.1. Let (X, x_0) be a pointed space and $p_1\colon (\widetilde{X}_1, \widetilde{x}_1) \to (X, x_0)$ and $p_2\colon (\widetilde{X}_2, \widetilde{x}_2) \to (X, x_0)$ be two pointed covers of (X, x_0). A *pointed covering space isomorphism* from $(\widetilde{X}_1, \widetilde{x}_1)$ to $(\widetilde{X}_2, \widetilde{x}_2)$ is a homeomorphism $\widetilde{f}\colon (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ such that the diagram

$$\begin{tikzcd} (\widetilde{X}_1, \widetilde{x}_1) \arrow[rr, "\widetilde{f}"] \arrow[dr, "p_1"'] & & (\widetilde{X}_2, \widetilde{x}_2) \arrow[dl, "p_2"] \\ & (X, x_0) & \end{tikzcd}$$

commutes, i.e., such that $p_2 \circ \widetilde{f} = p_1$. If a pointed covering space isomorphism from $(\widetilde{X}_1, \widetilde{x}_1)$ to $(\widetilde{X}_2, \widetilde{x}_2)$ exists, we say that $(\widetilde{X}_1, \widetilde{x}_1)$ and $(\widetilde{X}_2, \widetilde{x}_2)$ are *isomorphic* pointed covers of X. This is clearly an equivalence relation. □

Using Theorem 9.2.1 (lifting criterion), we will prove that in favorable situations pointed covers are determined up to isomorphism by the images of their fundamental groups:

Theorem 9.3.2. *Let $p_1\colon (\widetilde{X}_1, \widetilde{x}_1) \to (X, x_0)$ and $p_2\colon (\widetilde{X}_2, \widetilde{x}_2) \to (X, x_0)$ be two pointed covers such that:*

- *the images of $(p_1)_*\colon \pi_1(\widetilde{X}_1, \widetilde{x}_1) \to \pi_1(X, x_0)$ and $(p_2)_*\colon \pi_1(\widetilde{X}_2, \widetilde{x}_2) \to \pi_1(X, x_0)$ are the same; and*
- *both $\widetilde{X}_1$ and $\widetilde{X}_2$ are path connected.*

Assume that X is path connected and locally path connected. Then $(\widetilde{X}_1, \widetilde{x}_1)$ and $(\widetilde{X}_2, \widetilde{x}_2)$ are isomorphic pointed covers of (X, x_0).

Proof. Since X is locally path connected, so are $\widetilde{X}_1$ and $\widetilde{X}_2$. Applying Theorem 9.2.1 twice, we get pointed maps $\widetilde{f}\colon (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ and $\widetilde{g}\colon (\widetilde{X}_2, \widetilde{x}_2) \to (\widetilde{X}_1, \widetilde{x}_1)$ such that the following diagrams commute:

$$\begin{tikzcd} & (\widetilde{X}_2, \widetilde{x}_2) \arrow[d, "p_2"] \\ (\widetilde{X}_1, \widetilde{x}_1) \arrow[ur, "\widetilde{f}"] \arrow[r, "p_1"] & (X, x_0) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & (\widetilde{X}_1, \widetilde{x}_1) \arrow[d, "p_1"] \\ (\widetilde{X}_2, \widetilde{x}_2) \arrow[ur, "\widetilde{g}"] \arrow[r, "p_2"] & (X, x_0) \end{tikzcd}$$

commute. To prove that $\widetilde{f}$ is the desired isomorphism of pointed covers, it is enough to prove that it is a homeomorphism. In fact, we will prove that $\widetilde{f}$ and $\widetilde{g}$ are inverse homeomorphisms, i.e., $\widetilde{g} \circ \widetilde{f}\colon (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_1, \widetilde{x}_1)$ and $\widetilde{f} \circ \widetilde{g}\colon (\widetilde{X}_2, \widetilde{x}_2) \to (\widetilde{X}_2, \widetilde{x}_2)$ are both the identity. The two proofs are similar, so we will prove the first. By construction, the diagrams

$$\begin{tikzcd} & (\widetilde{X}_1, \widetilde{x}_1) \arrow[d, "p_1"] \\ (\widetilde{X}_1, \widetilde{x}_1) \arrow[ur, "\widetilde{g} \circ \widetilde{f}"] \arrow[r, "p_1"] & (X, x_0) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & (\widetilde{X}_1, \widetilde{x}_1) \arrow[d, "p_1"] \\ (\widetilde{X}_1, \widetilde{x}_1) \arrow[ur, "\mathbb{1}"] \arrow[r, "p_1"] & (X, x_0) \end{tikzcd}$$

both commute. Since $\widetilde{X}_1$ is path connected and $\widetilde{g} \circ \widetilde{f}$ and $\mathbb{1}$ are both lifts of $p_1\colon (\widetilde{X}_1, \widetilde{x}_1) \to (X, x_0)$ that are equal at the point $\widetilde{x}_1$, it follows from Lemma 3.1.2 that they are equal, as desired. □

9.4. Deck transformations

Let $p\colon \widetilde{X} \to X$ be a cover with deck group G. For $x_0 \in X$, the group G acts on the fiber $p^{-1}(x_0)$. If $\widetilde{X}$ is connected, then Lemma 2.2.1 says that if $g, g' \in G$ are elements such that there is some $\widetilde{x}_0 \in p^{-1}(x_0)$ with $g\widetilde{x}_0 = g'\widetilde{x}_0$, then $g = g'$. In other words, the group G is not only determined by its action on $p^{-1}(x_0)$, but even by its action on any single point of $p^{-1}(x_0)$. In favorable situations, the following theorem therefore completely describes G:

Theorem 9.4.1. *Let $p\colon \widetilde{X} \to X$ be a cover with deck group G. Let $x_0 \in X$ and let $\widetilde{x}_0, \widetilde{x}_0' \in p^{-1}(X)$. Set $\Gamma = \pi_1(X, x_0)$, and let K and K' be the following subgroups of Γ:*

$$K = \mathrm{Im}(p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)),$$
$$K' = \mathrm{Im}(p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0') \to \pi_1(X, x_0)).$$

Assume that $\widetilde{X}$ is path connected and that X is locally path connected. Then there exists $g \in G$ with $g\widetilde{x}_0 = \widetilde{x}_0'$ if and only if $K = K'$.

Proof. If such a g exists, then the corresponding deck transformation $\widetilde{f}_g$ and its inverse $\widetilde{f}_{g^{-1}}$ fit into commutative diagrams

$$\begin{tikzcd} & (\widetilde{X}, \widetilde{x}_0') \arrow[d, "p"] \\ (\widetilde{X}, \widetilde{x}_0) \arrow[ur, "\widetilde{f}_g"] \arrow[r, "p"] & (X, x_0) \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} & (\widetilde{X}, \widetilde{x}_0) \arrow[d, "p"] \\ (\widetilde{X}, \widetilde{x}_0') \arrow[ur, "\widetilde{f}_{g^{-1}}"] \arrow[r, "p"] & (X, x_0). \end{tikzcd}$$

Applying the fundamental group, we get commutative diagrams

$$\begin{tikzcd} & \pi_1(\widetilde{X}, \widetilde{x}_0') \arrow[d, "p"] \\ \pi_1(\widetilde{X}, \widetilde{x}_0) \arrow[ur, "\widetilde{f}_g"] \arrow[r, "p"] & \pi_1(X, x_0) \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} & \pi_1(\widetilde{X}, \widetilde{x}_0) \arrow[d, "p"] \\ \pi_1(\widetilde{X}, \widetilde{x}_0') \arrow[ur, "\widetilde{f}_{g^{-1}}"] \arrow[r, "p"] & \pi_1(X, x_0). \end{tikzcd}$$

From this, we see that $K = K'$. Conversely, if $K = K'$ then Theorem 9.2.1 (lifting criterion) says that there exists $\widetilde{f}_g$ and $\widetilde{f}_{g^{-1}}$ making the above diagrams commute. Just like in the proof of Theorem 9.3.2, the compositions $\widetilde{f}_g \circ \widetilde{f}_{g^{-1}}$ and $\widetilde{f}_{g^{-1}} \circ \widetilde{f}_g$ are both the identity, so they are homeomorphisms and thus the desired deck transformations. $\square$

9.5. Regular covers

Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a pointed cover with deck group G. Recall that the cover is *regular* if for all $x \in X$ the group G acts transitively on the fiber $p^{-1}(x)$. In fact, if X is path connected then Lemma 4.4.1 says that it is enough to check this transitivity on a single fiber, for instance $p^{-1}(x_0)$. Also, recall from §7.2 that the *lifting map* is the set map $f\colon \pi_1(X, x_0) \to G$ defined as follows:

- Consider $[\gamma] \in \Gamma$. Let $\widetilde{\gamma}$ be the lift of γ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$. Letting $f([\gamma]) = g$, we then have $\widetilde{\gamma}(1) = g\widetilde{x}_0$.

The lifting map is a well-defined group homomorphism, and is surjective if $\widetilde{X}$ is path connected (see Lemma 7.2.1). Moreover, when we proved Theorem 7.1.1 we showed that the lifting map is an isomorphism if $\widetilde{X}$ is 1-connected. Our final result in this chapter generalizes this as follows. See the next section for some examples of it.

Theorem 9.5.1. *Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a pointed cover. Let $\Gamma = \pi_1(X, x_0)$, and let K be the following subgroup of Γ:*

$$K = \mathrm{Im}(p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0) = \Gamma).$$

Assume that $\widetilde{X}$ is path connected and that X is locally path connected. The following hold:

(i) The cover $p\colon \widetilde{X} \to X$ is a regular cover if and only if K is a normal subgroup of Γ.
(ii) Assume that $p\colon \widetilde{X} \to X$ is a regular cover, and let G be its deck group. Then the lifting map $f\colon \Gamma \to G$ is a surjective homomorphism with kernel K.

Proof. We prove (i) and (ii) separately:

Step 1. *The cover $p\colon \widetilde{X} \to X$ is a regular cover if and only if K is a normal subgroup of Γ.*

By Lemma 4.4.1, the cover is regular if and only if the deck group G acts transitively on $p^{-1}(x_0)$. For $\widetilde{x}_1 \in p^{-1}(x_0)$, set

$$K_{\widetilde{x}_1} = \operatorname{Im}(p_* \colon \pi_1(\widetilde{X}, \widetilde{x}_1) \to \pi_1(X, x_0)).$$

By Theorem 9.4.1, there exists some $g \in G$ with $g\widetilde{x}_0 = \widetilde{x}_1$ if and only if $K_{\widetilde{x}_1} = K$. We deduce that $p \colon \widetilde{X} \to X$ is regular if and only if $K_{\widetilde{x}_1} = K$ for all $\widetilde{x}_1 \in p^{-1}(x_0)$.

For $\widetilde{x}_1 \in p^{-1}(x_0)$, let $\widetilde{\gamma}_{\widetilde{x}_1}$ be a path in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}_1$. The change of basepoint isomorphism from $\pi_1(\widetilde{X}, \widetilde{x}_1)$ to $\pi_1(\widetilde{X}, \widetilde{x}_0)$ takes $[\eta] \in \pi_1(\widetilde{X}, \widetilde{x}_1)$ to $[\widetilde{\gamma}_{\widetilde{x}_1} \cdot \eta \cdot \overline{\widetilde{\gamma}_{\widetilde{x}_1}}] \in \pi_1(\widetilde{X}, \widetilde{x}_0)$. Let $\gamma_{\widetilde{x}_1}$ be the image of $\widetilde{\gamma}_{\widetilde{x}_1}$ in X, so $\gamma_{\widetilde{x}_1}$ is a path from x_0 to x_0. We thus have $[\widetilde{\gamma}_{\widetilde{x}_1}] \in \pi_1(X, x_0) = \Gamma$. It follows that

$$[\gamma_{\widetilde{x}_1}] K [\gamma_{\widetilde{x}_1}]^{-1} = K_{\widetilde{x}_1}.$$

Moreover, every $g \in \Gamma = \pi_1(X, x_0)$ can be lifted to a path connecting $\widetilde{x}_0$ to $\widetilde{x}_1$ for some choice of $\widetilde{x}_1 \in p^{-1}(x_0)$. We conclude that our cover is regular if and only if $gKg^{-1} = K$ for all $g \in \Gamma$, i.e., if and only if K is a normal subgroup.

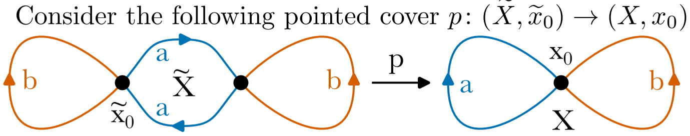
STEP 2. *Assume that $p \colon \widetilde{X} \to X$ is a regular cover, and let G be its deck group. Then the lifting map $f \colon \Gamma \to G$ is a surjective homomorphism with kernel K.*

Lemma 7.2.1 gives all of this except for the fact that the kernel of the lifting map is K, so this is what we must prove. Consider $[\gamma] \in \pi_1(X, x_0)$ such that $f([\gamma]) = 1$. Let $\widetilde{\gamma}$ be the lift of γ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$. Since $f([\gamma]) = 1$, we must have $\widetilde{\gamma}(1) = \widetilde{x}_0$, so $\widetilde{\gamma}$ is a loop based at $\widetilde{x}_0$. It follows that $[\gamma] = p_*([\widetilde{\gamma}]) \in K$. Conversely, reversing the above logic we see that elements of K lie in the kernel of f, as desired. □

9.6. Examples of regular and irregular covers

We now give two examples of Theorem 9.5.1.

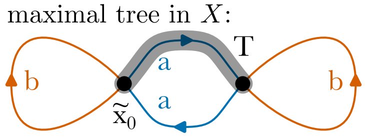
EXAMPLE 9.6.1. Consider the following pointed cover $p \colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$:



This is a regular cover with deck group C_2. Letting t be the generator of C_2, the generator t acts on $\widetilde{X}$ by flipping the two vertices, the two oriented edges labeled a, and the two oriented edges labeled b. We explain how we could see this using Theorem 9.5.1. To do this, we must prove:

- The image of $p_* \colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is a normal subgroup of $\pi_1(X, x_0)$.
- The quotient of $\pi_1(X, x_0)$ by the image of p_* is C_2.

The group $\pi_1(X, x_0)$ is isomorphic to the free group $F(a, b)$ on generators a and b. We have labeled the edges of $\widetilde{X}$ with the loops in X they map to. We calculate the image of $p_* \colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ as follows. Let T be the following maximal tree in $\widetilde{X}$:

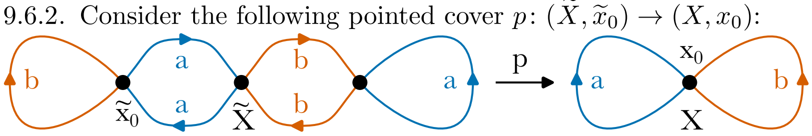


Using the algorithm described in §8.9, we can use T to calculate a free basis S for $\pi_1(\widetilde{X}, \widetilde{x}_0)$. Since p_* is injective, we might as well describe the elements of this free basis by giving their images in $\pi_1(X, x_0) = F(a, b)$:

$$S = \{b, a^2, aba^{-1}\}.$$

The subgroup of $F(a, b)$ generated by S is normal; indeed, it is the kernel of the map $f \colon F(a, b) \to C_2$ taking a to t and b to 1. □

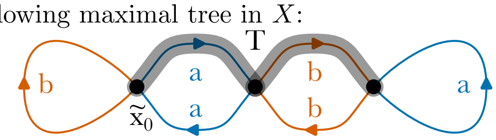
EXAMPLE 9.6.2. Consider the following pointed cover $p \colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$:



As we noted in §2.5, this is an irregular cover. We explain how we could see this using Theorem 9.5.1. To do this, we must prove:

- The image of $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is a non-normal subgroup of $\pi_1(X, x_0)$.

The group $\pi_1(X, x_0)$ is isomorphic to the free group $F(a, b)$ on generators a and b. We have labeled the edges of $\widetilde{X}$ with the loops in X they map to. We calculate the image of $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ as follows. Let T be the following maximal tree in $\widetilde{X}$:



Using the algorithm described in §8.9, we can use T to calculate a free basis S for $\pi_1(\widetilde{X}, \widetilde{x}_0)$. Since p_* is injective, we might as well describe the elements of this free basis by giving their images in $\pi_1(X, x_0) = F(a, b)$:

$$S = \{b, a^2, ab^2a^{-1}, abab^{-1}a^{-1}\}.$$

The subgroup of $F(a, b)$ generated by S is not normal. Indeed, it contains b, but we claim it does not contain aba^{-1}. To see this, let $S' = S \cup \{aba^{-1}\}$. Since

$$abab^{-1}a^{-1} = (aba^{-1})(a^2)(b^{-1})a^{-1},$$

the subgroup generated by S' contains a^{-1} and hence a. But this implies that this subgroup contains both a and b, and hence S' generates all of $F(a, b)$. Since S generates a proper subgroup[1] of $F(a, b)$, this implies that aba^{-1} is not in the subgroup generated by S. $\square$

9.7. Exercises

EXERCISE 9.1. Let X be a locally path connected space and let $p\colon \widetilde{X} \to X$ be a cover. Prove that $\widetilde{X}$ is locally path connected. $\square$

EXERCISE 9.2. Let (X, x_0) be a pointed space that is path connected and locally path connected. Assume that $\pi_1(X, x_0)$ is a finite group. Let $f\colon (X, x_0) \to (\mathbb{S}^1, 1)$ be a map. Prove that f can be lifted to the universal cover $p\colon (\mathbb{R}, 0) \to (\mathbb{S}^1, 1)$. $\square$

EXERCISE 9.3. Let $n \geq 2$ and $m \geq 1$. Let $\mathbb{T}^m = (\mathbb{S}^1)^{\times m}$ be the m-torus. Prove that every continuous map $f\colon \mathbb{S}^n \to \mathbb{T}^m$ is null homotopic. $\square$

EXERCISE 9.4. Let $f\colon X \to Y$ be a continuous map with X and Y both path connected and locally path connected. Let $p\colon \widetilde{X} \to X$ and $q\colon \widetilde{Y} \to Y$ be covers such that $\widetilde{X}$ and $\widetilde{Y}$ are both 1-connected (we will prove in the next couple of chapters that such covers exist if X and Y are reasonable; they are called *universal covers*). Prove that there exists a map $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ such that the diagram

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\widetilde{f}"] \arrow[d, "p"] & \widetilde{Y} \arrow[d, "q"] \\
X \arrow[r, "f"] & X
\end{tikzcd}$$

commutes. $\square$

EXERCISE 9.5. Let X and Y be path connected and locally path connected space. As in the previous exercise, let $p\colon \widetilde{X} \to X$ and $q\colon \widetilde{Y} \to Y$ be covers such that $\widetilde{X}$ and $\widetilde{Y}$ are both 1-connected. Assume that X and Y are homotopy equivalent. Prove that $\widetilde{X}$ and $\widetilde{Y}$ are homotopy equivalent. $\square$

EXERCISE 9.6. Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a pointed cover. Let $\Gamma = \pi_1(X, x_0)$, and let K be the following subgroup of Γ:

$$K = \operatorname{Im}(p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)).$$

Assume that $\widetilde{X}$ is path connected and that X is locally path connected. Let $N(K)$ be the normalizer of K in Γ, i.e., the subgroup of Γ consisting of $g \in \Gamma$ with $gKg^{-1} = K$. Prove that the deck group of $p\colon \widetilde{X} \to X$ is isomorphic to $N(K)/K$. $\square$

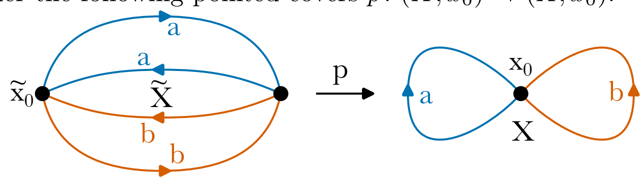
[1]For instance, we know that it generates a rank-4 free group, which has abelianization $\mathbb{Z}^4$ (see Exercise 7.12). On the other hand, $F(a, b)$ has abelianization $\mathbb{Z}^2$.

EXERCISE 9.7. Let $\mathbf{G}$ be a topological group, i.e., a space that is also a group such that the multiplication map $m\colon \mathbf{G} \times \mathbf{G} \to \mathbf{G}$ and the inversion map $i\colon \mathbf{G} \to \mathbf{G}$ are continuous. Assume that $\mathbf{G}$ is path connected and locally path connected. Prove the following:

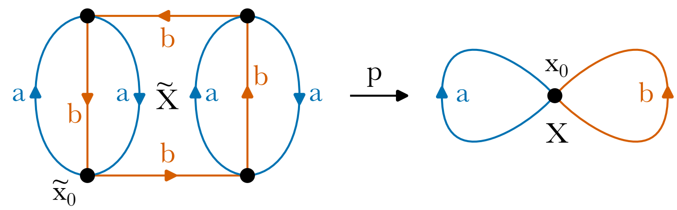
(a) Let $p\colon \widetilde{\mathbf{G}} \to \mathbf{G}$ be a cover. Prove that $\widetilde{\mathbf{G}}$ can be given the structure of a topological group such that p is a group homomorphism.
(b) If $\mathbf{G}$ is abelian, then prove that $\widetilde{\mathbf{G}}$ is abelian.
(c) Prove that $\ker(p)$ is a central subgroup of $\widetilde{\mathbf{G}}$, i.e., that each $g \in \ker(p)$ commutes with all elements of $\widetilde{\mathbf{G}}$. □

EXERCISE 9.8. Consider the following pointed covers $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$:

(a)



(b)



For both, give a free basis for image of $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$. Also, determine if these images are normal. If they are, describe them as the kernels of maps to the deck group of the cover. If they are not, prove it. □

EXERCISE 9.9. As in Example 1.4.4, let Poly_n be the space of degree-n monic polynomials over $\mathbb{C}$. Such an $f \in \mathrm{Poly}_n$ can be written as

$$f(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n \quad \text{with } a_1, \ldots, a_n \in \mathbb{C},$$

so $\mathrm{Poly}_n \cong \mathbb{C}^n$. Let

$$\mathrm{Poly}_n^{\mathrm{sf}} = \{f \in \mathrm{Poly}_n \mid f \text{ has } n \text{ distinct roots}\},$$
$$\mathrm{RPoly}_n^{\mathrm{sf}} = \Big\{(f, x) \in \mathrm{Poly}_n^{\mathrm{sf}} \times \mathbb{C} \mid f(x) = 0\Big\}.$$

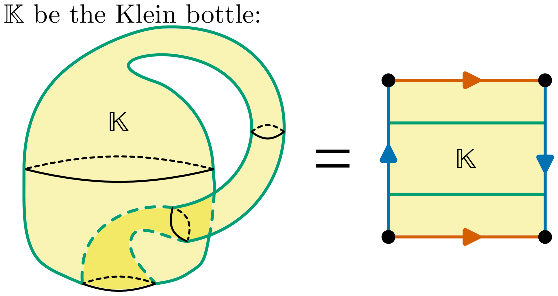
We showed in Example 1.4.4 that the projection $p\colon \mathrm{RPoly}_n^{\mathrm{sf}} \to \mathrm{Poly}_n^{\mathrm{sf}}$ is a degree n cover. In this exercise, you will show that it is an irregular cover for $n \geq 3$. Do the following:

(a) Let
$$C_n = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \mid \lambda_i \neq \lambda_j \text{ for all } 1 \leq i < j \leq n\}.$$
Let $m\colon C_n \to \mathrm{Poly}_n^{\mathrm{sf}}$ be the map
$$m(\lambda_1, \ldots, \lambda_n) = (x - \lambda_1) \cdots (x - \lambda_n) \quad \text{for all } (\lambda_1, \ldots, \lambda_n) \in C_n.$$
Prove that $m\colon C_n \to \mathrm{Poly}_n^{\mathrm{sf}}$ is a regular cover with deck group the symmetric group $\mathfrak{S}_n$ on n generators.
(b) Let $q\colon C_n \to \mathrm{RPoly}_n^{\mathrm{sf}}$ be the map
$$q(\lambda_1, \ldots, \lambda_n) = (m(\lambda_1, \ldots, \lambda_n), \lambda_n) \quad \text{for all } (\lambda_1, \ldots, \lambda_n) \in C_n.$$
Prove that $q\colon C_n \to \mathrm{RPoly}_n^{\mathrm{sf}}$ is a regular cover with deck group the symmetric group $\mathfrak{S}_{n-1}$ on $(n-1)$ generators.
(c) Use the previous two parts to prove that $p\colon \mathrm{RPoly}_n^{\mathrm{sf}} \to \mathrm{Poly}_n^{\mathrm{sf}}$ is an irregular cover for $n \geq 3$. Hint: Fix a basepoint $f_0 \in \mathrm{Poly}_n^{\mathrm{sf}}$. Let $x_1, \ldots, x_n \in \mathbb{C}$ be the roots of f_0. We will then use $r_0 = (x_1, \ldots, x_n)$ for our basepoint of C_n and (f_0, x_n) as our basepoint for $\mathrm{RPoly}_n^{\mathrm{sf}}$. Applying Theorem 9.5.1 to the cover from (a), we get a homomorphism $\phi\colon \pi_1(\mathrm{Poly}_n^{\mathrm{sf}}, f_0) \to \mathfrak{S}_n$

whose kernel is the image of $m_*\colon \pi_1(C_n, r_0) \to \pi_1(\mathrm{Poly}_n^{\mathrm{sf}}, f_0)$. Argue using (b) that the homomorphism ϕ takes the image of $p_*\colon \pi_1(\mathrm{RPoly}_n^{\mathrm{sf}}, (f_0, x_n)) \to \pi_1(\mathrm{Poly}_n^{\mathrm{sf}}, f_0)$ to the subgroup $\mathfrak{S}_{n-1}$ of $\mathfrak{S}_n$. Conclude by using the fact that $\mathfrak{S}_{n-1}$ is a non-normal subgroup of $\mathfrak{S}_n$ for $n \geq 3$. □

EXERCISE 9.10. Let $\mathbb{T}^n = (\mathbb{S}^1)^{\times n}$ be the n-torus. Using the group structure on $\mathbb{S}^1 \subset \mathbb{C}$ coming from the multiplication on $\mathbb{C}$, endow $\mathbb{T}^n$ with the structure of a topological group. Let $\mathrm{Aut}(\mathbb{T}^n)$ be the group of continuous group homomorphisms. Prove that $\mathrm{Aut}(\mathbb{T}^n) \cong \mathrm{GL}_n(\mathbb{Z})$. Hint: Let $0 \in \mathbb{T}^n$ be the identity, and let $f\colon (\mathbb{T}^n, 0) \to (\mathbb{T}^n, 0)$ be a continuous group homomorphism. Let $p\colon (\mathbb{R}^n, 0) \to (\mathbb{T}^n, 0)$ be the pointed covering space obtained by taking the product of the universal cover $\mathbb{R} \to \mathbb{S}^1$. The space $\mathbb{R}^n$ is a topological group under addition, and p is a group homomorphism. Prove that you can lift f to a map $\widetilde{f}\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, then prove that $\widetilde{f}$ is a linear map, and then finally prove that $\widetilde{f} \in \mathrm{GL}_n(\mathbb{Z})$. The uniqueness of lifts will be important here! □

EXERCISE 9.11. Let $\mathbb{K}$ be the Klein bottle:



In this figure, the green loop on the left corresponds to the two green lines on the right, whose ends match up to form a circle. Prove that the fundamental group of $\mathbb{K}$ is the following group Γ:

- Let $\mathbb{Z}$ act on $\mathbb{Z}$ via the homomorphism $\phi\colon \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z})$ defined by

$$\phi(n)(m) = (-1)^n m \quad \text{for all } n \in \mathbb{Z} \text{ and } m \in \mathbb{Z}.$$

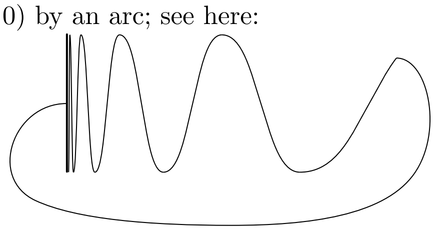
  Then $\Gamma = \mathbb{Z} \ltimes_\phi \mathbb{Z}$, i.e., the semidirect product of $\mathbb{Z}$ and $\mathbb{Z}$ given by the action ϕ.

Hint: Construct a regular degree 2 cover $p\colon \mathbb{T}^2 \to \mathbb{K}$ with deck group C_2 and use Theorem 9.5.1 to analyze the fundamental group of $\mathbb{K}$ using this cover. □

EXERCISE 9.12. The *quasi-circle* is the space Y obtained from the topologist's sine curve

$$X = \left\{(x, \sin(1/x)) \in \mathbb{R}^2 \mid 0 < x \leq 1\right\} \cup \{(0, y) \mid -1 \leq y \leq 1\}$$

by connecting $(1, \sin(1))$ to $(0, 0)$ by an arc; see here:



We saw in Exercise 4.11 that X is 1-connected but has nontrivial covers. Note that this would be impossible if Y were locally path connected (cf. Theorem 4.6.1). Prove the following:

- Let $f\colon Y \to \mathbb{S}^1$ be the map that collapses $\{(0, y) \mid -1 \leq y \leq 1\}$ to a point and identifies the resulting space with $\mathbb{S}^1$. Prove that f cannot be lifted to the universal cover $p\colon \mathbb{R} \to \mathbb{S}^1$. Note that f could be lifted if Y were locally path connected (cf. Theorem 9.2.1). □

CHAPTER 10

Classifying covers: Galois correspondence

We now describe one version of the classification of covering spaces. The proof of the main technical result used in its proof is postponed until later.

10.1. Rough statement of first version of classification

Let (X, x_0) be a pointed space. Assume that X is path connected and locally path connected. Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a pointed cover. We proved in Theorem 9.1.1 that the induced map

$$p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$$

is injective. We call its image the subgroup of $\pi_1(X, x_0)$ corresponding to the cover. Call $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ a connected pointed cover if $\widetilde{X}$ is path connected. We proved in Theorem 9.3.2 that a connected pointed cover of (X, x_0) is determined up to isomorphism by its corresponding subgroup of $\pi_1(X, x_0)$.

Ignoring some technical issues, the first version of the classification of covers says that every subgroup of $\pi_1(X, x_0)$ is the subgroup corresponding to a connected pointed cover. By what we said above this connected pointed cover is unique up to isomorphism, so this establishes a bijection between:

- isomorphism classes of connected pointed covers of (X, x_0); and
- subgroups of $\pi_1(X, x_0)$.

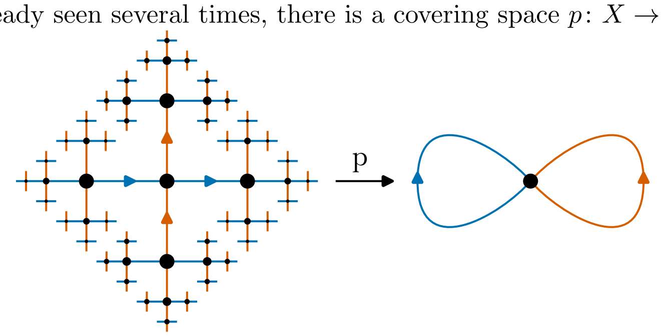
If $p_u\colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ is the connected pointed cover corresponding to the trivial subgroup, then since p_* is injective it follows that $\widetilde{X}_u$ is 1-connected. For reasons we will describe later in this chapter, this is called the *universal cover* of (X, x_0). We will also talk about the unpointed version of this: if $p\colon \widetilde{X}_u \to X$ is a cover with $\widetilde{X}_u$ a 1-connected space, then $\widetilde{X}_u$ will be called the *universal cover* of X.

Example 10.1.1. If X is already 1-connected, then X is its own universal cover with covering space map the identity map $\mathbb{1}\colon X \to X$. □

Example 10.1.2. Since $\mathbb{R}$ is 1-connected, the universal cover $p\colon \mathbb{R} \to \mathbb{S}^1$ we have discussed since the beginning of this book exhibits $\mathbb{R}$ as the universal cover of $\mathbb{S}^1$. □

Example 10.1.3. For $n \geq 2$, the degree 2 cover $p\colon \mathbb{S}^n \to \mathbb{RP}^n$ taking $x \in \mathbb{S}^n$ to the line through x exhibits $\mathbb{S}^n$ as the universal cover of $\mathbb{RP}^n$. □

Example 10.1.4. Let X be a graph with one vertex and two edges. Let $\widetilde{X}$ be a regular 4-valent tree. As we have already seen several times, there is a covering space $p\colon \widetilde{X} \to X$; see here:



Since $\widetilde{X}$ is a tree, it is contractible and hence 1-connected (see Lemma 6.4.1). It follows that $\widetilde{X}$ is the universal cover of X. □

Unfortunately, there are technical issues with the theorem described above, and it does not hold for all (X, x_0) such that X is path connected and locally path connected. In particular, these conditions are not strong enough to ensure that a universal cover exists (see Exercises 10.13–10.14). In the next section, we describe the needed extra hypotheses.

REMARK 10.1.5. We emphasize that in the above we only defined a universal cover of a pointed space (X, x_0) that is path connected and locally path connected. If X is not locally path connected, then a pointed cover $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that $\widetilde{X}$ is 1-connected can have pathological properties. For instance:

- The pointed space $(\widetilde{X}, \widetilde{x}_0)$ might have nontrivial covers, which is impossible if X is locally path connected by Theorem 4.6.1. See Exercise 4.11.
- The pointed cover $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ might not be regular, though Theorem 9.5.1 says that if X is locally path connected this cover is regular with deck group $\pi_1(X, x_0)$. See Exercise 10.15. □

10.2. Semilocal 1-connectedness

Let X be a space that is path connected and locally path connected. As we said in §10.1, there might not exist a universal cover of X. One condition that would suffice for this is the following:

DEFINITION 10.2.1. A space X is *locally* 1*-connected* if for all $x \in X$ and all open neighborhoods U of x, there is a 1-connected open neighborhood V of x with $V \subset U$. □

Most spaces of geometric interest are locally 1-connected; indeed, most of them are not just locally 1-connected, but even locally contractible. For instance, all manifolds are locally contractible. However, we can get away with less. One local property that spaces with universal covers have is as follows:

LEMMA 10.2.2. *Let X be a space. Assume there exists a cover $p\colon \widetilde{X} \to X$ such that $\widetilde{X}$ is* 1*-connected. Then for all $x \in X$, there exists an open neighborhood U of x such that the map $\pi_1(U, x) \to \pi_1(X, x)$ is the trivial map.*

PROOF. Let $x \in X$, let $U \subset X$ be a trivialized open neighborhood of X for $p\colon \widetilde{X} \to X$, and let $\widetilde{U} \subset \widetilde{X}$ be any sheet lying above U. Let $\widetilde{x} \in \widetilde{U}$ be the point lying in the fiber over x. We can factor the inclusion $(U, x) \to (X, x)$ as

$$(U, x) \xrightarrow{(p|_{\widetilde{U}})^{-1}} (\widetilde{U}, \widetilde{x}) \longrightarrow (\widetilde{X}, x) \xrightarrow{p} (X, x),$$

where $(\widetilde{U}, \widetilde{x}) \to (\widetilde{X}, x)$ is the inclusion. Passing to fundamental groups, the map $\pi_1(U, x) \to \pi_1(X, x)$ factors as

$$\begin{array}{c} \pi_1(U, x) \xrightarrow{(p|_{\widetilde{U}})^{-1}_*} \pi_1(\widetilde{U}, \widetilde{x}) \longrightarrow \pi_1(\widetilde{X}, x) \xrightarrow{p_*} \pi_1(X, x). \\ \hphantom{\pi_1(U, x) \xrightarrow{(p|_{\widetilde{U}})^{-1}_*} \pi_1(\widetilde{U}, \widetilde{x}) \longrightarrow} \| \hphantom{\xrightarrow{p_*} \pi_1(X, x).} \\ \hphantom{\pi_1(U, x) \xrightarrow{(p|_{\widetilde{U}})^{-1}_*} \pi_1(\widetilde{U}, \widetilde{x}) \longrightarrow} 1 \hphantom{\xrightarrow{p_*} \pi_1(X, x).} \end{array}$$

It follows that the map $\pi_1(U, x) \to \pi_1(X, x)$ is the trivial map, as desired. □

We call spaces X satisfying the conclusion of Lemma 10.2.2 *semilocally* 1*-connected*. This is an awkward condition, but it holds for instance if X is locally 1-connected. It also passes to covers in the sense that if $p\colon \widetilde{X} \to X$ is a cover and X is semilocally 1-connected, then so is $\widetilde{X}$ (see Exercise 10.1). Most importantly, we have the following converse to Lemma 10.2.2:

THEOREM 10.2.3 (Existence of universal covers). *Let (X, x_0) be a pointed space. Assume that X is path connected, locally path connected, and semilocally* 1*-connected. Then it has a universal cover $p_u\colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$.*

We will prove Theorem 10.2.3 in Chapter 11. The details of the proof are beautiful, but not important for applying it in concrete situations. The rest of this chapter and the next will explore the consequences of Theorem 10.2.3.

Remark 10.2.4. One nice feature of semilocal 1-connectedness is that unlike purely local conditions like being locally 1-connected, it is preserved by homotopy equivalences. See Exercise 10.2. □

10.3. Galois correspondence

The first version of the classification of covering spaces is as follows. For reasons we will describe soon, it should be thought of as analogous to the classical Galois correspondence.

Theorem 10.3.1 (Galois correspondence). *Let (X, x_0) be a pointed space that is path connected, locally path connected, and semilocally 1-connected. There is then a bijection between the following two sets:*

- *The set of isomorphism classes of connected pointed covers $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$.*
- *The set of subgroups of $\pi_1(X, x_0)$.*

This bijection takes a connected pointed cover $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ to its corresponding subgroup, i.e., to the image K of the map $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$. The degree of the cover $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is the index of K in $\pi_1(X, x_0)$.

Remark 10.3.2. Let (X, x_0) be as in Theorem 10.3.1. That theorem classifies connected pointed covers of (X, x_0). Connected covers of X without distinguished basepoints are classified by conjugacy classes of subgroups $K < \pi_1(X, x_0)$. See Exercise 10.3. □

Proof of Theorem 10.3.1. Let $\Gamma = \pi_1(X, x_0)$ and let $K < \Gamma$ be a subgroup. Theorem 9.3.2 says that there is at most one isomorphism class of connected pointed covers corresponding to K, and Theorem 9.1.1 says that its degree is the index of K in Γ. What we must prove is that there exists a connected pointed cover whose corresponding subgroup is K.

Let $p_u\colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ be the universal cover provided by Theorem 10.2.3. Since the trivial subgroup $1 < \Gamma = \pi_1(X, x_0)$ is normal, Theorem 9.5.1 implies that the universal cover is a regular cover with deck group Γ. We can therefore identify X with $\widetilde{X}_u/\Gamma$. Set $\widetilde{X} = \widetilde{X}_u/K$, and let $\widetilde{x}_0$ be the image of $\widetilde{x}_u$ under the projection $q\colon \widetilde{X}_u \to \widetilde{X}_u/K = \widetilde{X}$. Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be the projection

$$\widetilde{X} = \widetilde{X}_u/K \twoheadrightarrow \widetilde{X}_u/\Gamma = X.$$

We claim that $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering space and that the image of $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is K.

To see that $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering space, consider $x \in X$. Let $U \subset X$ be a trivialized neighborhood for $p_u\colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ and let $\widetilde{U}_u \subset \widetilde{X}_u$ be the sheet lying above U with $\widetilde{x}_u \in \widetilde{U}_u$. The deck group Γ acts simply transitively on the sheets of the universal cover lying above U, so

$$p_u^{-1}(U) = \bigsqcup_{g \in \Gamma} g\widetilde{U}_u.$$

Let $\{g_c \mid c \in \Gamma/K\}$ be a set of left coset representatives for K in Γ. Recalling that $q\colon \widetilde{X}_u \to \widetilde{X}_u/K = \widetilde{X}$ is the projection, it follows that

$$p^{-1}(U) = \bigsqcup_{c \in \Gamma/K} q(g_c\widetilde{U})$$

and that that the restriction of p to each $q(g_c\widetilde{U})$ is a homeomorphism. We conclude that U is a trivialized neighborhood for $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ and that the $q(g_c\widetilde{U})$ are the sheets lying above U. In particular, $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering space.

It remains to prove that the image of $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is K. For this, recall that Theorem 9.1.1 says that the image of p_* consists of all $[\gamma] \in \pi_1(X, x_0)$ such that the following holds:

(†) Let $\widetilde{\gamma}$ be the lift of γ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$. Then $\widetilde{\gamma}$ is a loop, i.e., $\widetilde{\gamma}(1) = \widetilde{x}_0$.

Consider some $g = [\gamma] \in \pi_1(X, x_0)$. Let $\widetilde{\gamma}$ be the lift of γ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$ and let $\widetilde{\gamma}_u$ be the lift of γ to the universal cover $\widetilde{X}_u$ with $\widetilde{\gamma}_u(0) = \widetilde{x}_u$. The path $\widetilde{\gamma}_u$ projects to $\widetilde{\gamma}$. By Theorem 9.5.1, we have $\widetilde{\gamma}_u(1) = g\widetilde{x}_u$. It follows that $\widetilde{\gamma}$ is a loop if and only if $g\widetilde{x}_u$ maps to $\widetilde{x}_0 \in \widetilde{X}$, i.e., if and only if $g \in K$, as desired. □

Before giving some examples of Theorem 10.3.1, we extract two things from its proof:

COROLLARY 10.3.3. *Let (X, x_0) be a pointed space that is path connected, locally path connected, and semilocally 1-connected. Set $\Gamma = \pi_1(X, x_0)$, and let $K < \Gamma$ be a subgroup. Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be the connected pointed cover whose corresponding subgroup is K and let $p_u\colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ be the universal cover. The following then hold:*

- *(i) We have $\widetilde{X} = \widetilde{X}_u/K$.*
- *(ii) There is a cover $q\colon (\widetilde{X}_u, \widetilde{x}_u) \to (\widetilde{X}, \widetilde{x}_0)$. In particular, $\widetilde{X}_u$ is the universal cover of $\widetilde{X}$.*

PROOF. Immediate from the proof of Theorem 10.3.1. □

REMARK 10.3.4. Conclusion (ii) of Corollary 10.3.3 explains why it is called the universal cover: it covers all connected covers of the space. □

We now give some examples of Theorem 10.3.1:

EXAMPLE 10.3.5 (Circle). We have $\pi_1(\mathbb{S}^1, 1) = \mathbb{Z}$. The subgroups of $\mathbb{Z}$ are all of the form $n\mathbb{Z}$ for some $n \geq 0$. These correspond to the following covers:

- For $n \geq 1$, the index n subgroup $n\mathbb{Z} < \mathbb{Z}$ corresponds to the degree n cover $p_n\colon (\mathbb{S}^1, 1) \to (\mathbb{S}^1, 1)$ defined by
$$p_n(z) = z^n \quad \text{for } z \in \mathbb{S}^1 \subset \mathbb{C}.$$
In particular, the whole group $1\mathbb{Z} = \mathbb{Z}$ corresponds to the trivial cover $\mathbb{1}_{\mathbb{S}^1}\colon (\mathbb{S}^1, 1) \to (\mathbb{S}^1, 1)$.
- The infinite index trivial subgroup $0 < \mathbb{Z}$ corresponds to the universal cover $p\colon \mathbb{R} \to \mathbb{S}^1$. □

EXAMPLE 10.3.6 (Real projective space). Let $n \geq 2$ and let $x_0 \in \mathbb{RP}^n$ be a basepoint. We have $\pi_1(\mathbb{RP}^n, x_0) = C_2$. There are two subgroups of C_2:

- The whole group C_2 has index 1 and corresponds to the trivial cover $\mathbb{1}_{\mathbb{RP}^n}\colon (\mathbb{RP}^n, x_0) \to (\mathbb{RP}^n, x_0)$.
- The index 2 trivial group $0 < C_2$ corresponds to the degree 2 cover $p\colon (\mathbb{S}^n, \widetilde{x}_0) \to (\mathbb{RP}^n, x_0)$, where $\widetilde{x}_0 \in \mathbb{S}^n$ is a point projecting to x_0. This is the universal cover of $\mathbb{RP}^n$. □

10.4. Comparison with classical Galois correspondence

Theorem 9.3.2 should be viewed as an analogue for spaces of the classical Galois correspondence.[1] Letting L/K be a finite Galois extension of fields, the classical Galois correspondence is a bijection between:

- fields E with $K \subset E \subset L$; and
- subgroups of the Galois group $\mathrm{Gal}(L/K)$, which we recall is the set of automorphisms of L that fix the subfield K.

This bijection tales a subgroup G of $\mathrm{Gal}(L/K)$ to the subfield $L^G = \{\ell \in L \mid g\ell = \ell \text{ for all } g \in G\}$. The degree of the field extension L^G/K is the index of the subgroup G of $\mathrm{Gal}(L/K)$.

We hope the analogy is clear: the field K corresponds to the base of the cover, the Galois group $\mathrm{Gal}(L/K)$ corresponds to the fundamental group, and the field $L = L^1$ corresponds to the universal cover. Another feature of the classical Galois correspondence is that it is order-reversing:

- If $G_1, G_2 < \mathrm{Gal}(L/K)$ are subgroups with $G_1 < G_2$, then the corresponding fields satisfy $L^{G_2} < L^{G_1}$.

Something similar holds for covers, where the relation “is contained in” is replaced with the relation “covers”:

[1]Understanding this analogy is not essential for understanding covering spaces, so a reader should not worry if they are unfamiliar with the classical Galois correspondence.

THEOREM 10.4.1. *Let (X, x_0) be a pointed space that is path connected, locally path connected, and semilocally 1-connected. Let $p_1\colon (\widetilde{X}_1, \widetilde{x}_1) \to (X, x_0)$ and $p_2\colon (\widetilde{X}_2, \widetilde{x}_2) \to (X, x_0)$ be connected pointed covers corresponding to subgroups $K_1, K_2 < \pi_1(X, x_0)$ satisfying $K_1 < K_2$. Then there is a pointed covering map $q\colon (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ such that the following diagram commutes:*

$$\begin{tikzcd} (\widetilde{X}_1, \widetilde{x}_1) \arrow[r, "q"] \arrow[rr, bend right, "p_1"'] & (\widetilde{X}_2, \widetilde{x}_2) \arrow[r, "p_2"] & (X, x_0). \end{tikzcd}$$

PROOF. The proof is very similar to that of Theorem 10.3.1, so we leave it as Exercise 10.5. $\square$

Here is an example:

EXAMPLE 10.4.2 (Circle). For $n \geq 1$, the cover of $(\mathbb{S}^1, 1)$ corresponding to the subgroup $n\mathbb{Z} < \mathbb{Z} = \pi_1(\mathbb{S}^1, 1)$ is the cover $p_n\colon (\mathbb{S}^1, 1) \to (\mathbb{S}^1, 1)$ defined by $p_n(z) = z^n$. For $n, m \geq 1$, we have $n\mathbb{Z} < m\mathbb{Z}$ if and only if m divides n. In this case, we have the covers

$$\begin{tikzcd} (\mathbb{S}^1, 1) \arrow[r, "p_{m/n}"] \arrow[rr, bend right, "p_m"'] & (\mathbb{S}^1, 1) \arrow[r, "p_n"] & (\mathbb{S}^1, 1) \end{tikzcd}$$

as in Theorem 10.4.1. $\square$

10.5. Subgroups of free groups are free

We now explain an application of Theorem 10.3.1 to group theory. The following theorem was originally proved algebraically by Nielsen (who proved it for finitely generated subgroups) and Schreier (who proved it in general).

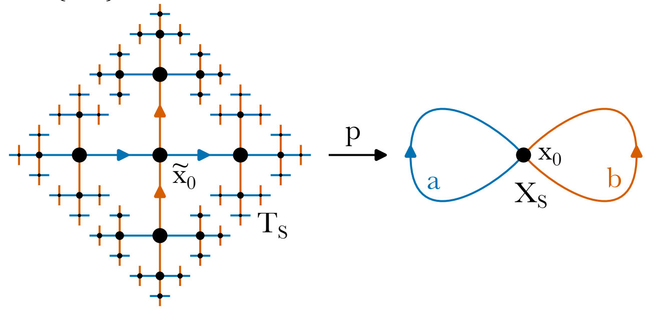
THEOREM 10.5.1. *Let $F(S)$ be the free group on a set S and let G be a subgroup of $F(S)$. Then G is a free group.*

PROOF. Let X_S be the graph with one vertex x_0 and with $|S|$ oriented edges, each labeled with an element of S. Identify each $s \in S$ with the corresponding loop in X_S based at x_0. Theorem 7.8.1 says that $\pi_1(X_S, x_0)$ is a free group on $\{[s] \mid s \in S\}$. By Theorem 10.3.1, there is a pointed cover $q\colon (\widetilde{X}(G), \widetilde{x}_G) \to (X_S, x_0)$ whose corresponding subgroup is G. Since X_S is a graph, one way to proceed would be to appeal to Exercise 4.8, which says that all covers of X_S are also graphs. This would imply that $\widetilde{X}(G)$ is a graph, and thus that $G \cong \pi_1(\widetilde{X}(G), \widetilde{x}_G)$ is a free group (Theorem 8.8.1).

Instead of doing this, we give a more explicit approach that will later allow us to compute free generators for G. In the proof of Theorem 7.8.1, we identified the universal cover of (X_S, x_0), though of course that term had not yet been defined. Namely, let T_S to be an infinite tree each of whose vertices has valence $2|S|$. Label the oriented edges of T_S by elements of S such that for each vertex v of T_S there are:

- $|S|$ edges coming out of v labeled by elements of S; and
- $|S|$ edges going into v labeled by elements of S.

Fix a vertex $\widetilde{x}_0$ of T_S. There is a pointed covering space $p\colon (T_S, \widetilde{x}_0) \to (X_S, x_0)$ taking each vertex of T_S to x_0 and each oriented edge of T_S labeled by $s \in S$ to the corresponding loop in X_S labeled by s. For instance, if $S = \{a, b\}$ this is the cover



The tree T_S is contractible (Lemma 6.4.1), so T_S is the universal cover of X_S. In particular, it is a regular cover with deck group $F(S)$ (Theorem 9.5.1). Let $\widetilde{X}(G) = T_S/G$ and let $\widetilde{x}_G \in \widetilde{X}(G)$ be the image of of $\widetilde{x}_0 \in T_S$. The map p factors through a map $p_G \colon (\widetilde{X}(G), \widetilde{x}_G) \to (X_S, x_0)$. Corollary 10.3.3 says that $p_G \colon (\widetilde{X}(G), \widetilde{x}_G) \to (X_S, x_0)$ is the pointed connected cover corresponding to G. In particular,

$$\pi_1(\widetilde{X}(G), \widetilde{x}_G) \cong G.$$

Since $\widetilde{X}(G)$ is a graph, Theorem 8.8.1 says that $\pi_1(\widetilde{X}(G), \widetilde{x}_G)$ is a free group. The theorem follows. $\square$

10.6. Computing free generators for subgroups of free groups

As in Theorem 10.5.1, let $F(S)$ be a free group on a set S and let G be a subgroup of $F(S)$. The proof of Theorem 10.5.1 actually gives an algorithm to compute free generators for G. We start with the following definition:

Definition 10.6.1. Let H be a group with generating set T. The *Cayley graph* of H with respect to T, denoted $\mathrm{Cay}(H, T)$, is the following graph:

- The vertices of $\mathrm{Cay}(H, T)$ are the elements of H.
- For each $h \in H$ and $t \in T$, there is an oriented edge of $\mathrm{Cay}(H, T)$ connecting the vertices h and ht. $\square$

For a group H with a generating set T, the graph $\mathrm{Cay}(H, T)$ is connected. Indeed, for $h \in H$ we can write $h = t_1^{e_1} \cdots t_n^{e_n}$ with $t_1, \ldots, t_n \in T$ and $e_1, \ldots, e_n \in \{\pm 1\}$. The following is then an edge path from the vertex $1 \in H$ to the vertex $h \in H$:

$$1, t_1^{e_1}, t_1^{e_1} t_2^{e_2}, t_1^{e_1} t_2^{e_2} t_3^{e_3}, \ldots, t_1^{e_1} \cdots t_n^{e_n} = h.$$

Here we are using the fact that for $h' \in H$ and $t \in T$ there is an oriented edge from $h' t^{-1}$ to h'. The group H acts on $\mathrm{Cay}(H, T)$ on the left. This action is transitive on the vertices, and the orbits of the edges are in bijection with T.

Remark 10.6.2. In the above, we allow the generating set T to have repeated elements. It is also allowed to contain the identity element $1 \in H$. The same is true in what we do below. $\square$

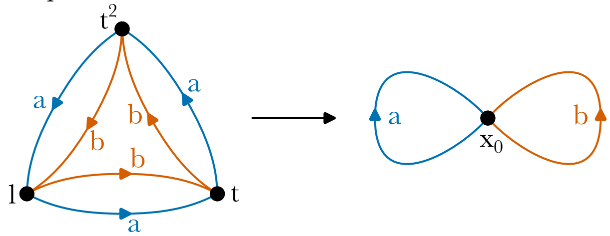
We now return to the setting of free groups. The tree T_S constructed in the proof of Theorem 10.5.1 is exactly $\mathrm{Cay}(F(S), S)$. It follows that the graph $\widetilde{X}(G)$ from that proof with fundamental group $G < F(S)$ is $\mathrm{Cay}(F(S), S)/G$. If G is a normal subgroup, then letting $\overline{S}$ be the image of S in $F(S)/G$ we have $\mathrm{Cay}(F(S), S)/G \cong \mathrm{Cay}(F(S)/G, \overline{S})$. More generally, $\mathrm{Cay}(F(S), S)/G$ is the *Schreier graph* of $F(S)/G$ with respect to S, whose definition is as follows:

Definition 10.6.3. Let H be a group with generating set T and let $K < H$ be a subgroup. The *Schreier graph* of H/K with respect to T, denoted $\mathrm{Sch}(H, K, T)$, is the following graph:

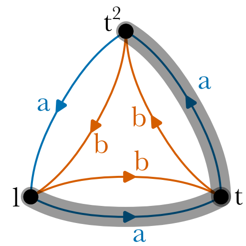
- The vertices of $\mathrm{Sch}(H, K, T)$ are the right cosets Kh with $h \in H$.
- For each right coset Kh and generator $t \in T$, there is an oriented edge of $\mathrm{Sch}(H, K, T)$ connecting the vertices Kh and Kht. $\square$

Here are three examples of how to use all this to compute free generators for subgroup G of free group $F(S)$:

Example 10.6.4. Consider the free group $F(a, b)$ on a and b. Let C_3 be the cyclic group of order 3 generated by t. Let G be the kernel of the homomorphism $F(a, b) \to C_3$ taking a and b to t. Following the above recipe, the group G is the fundamental group of the Cayley graph of $F(a, b)/G \cong C_3$ with respect to the generating set $\{a, b\}$. These map to the same element of C_3, so this generating set has a repeated element in it:

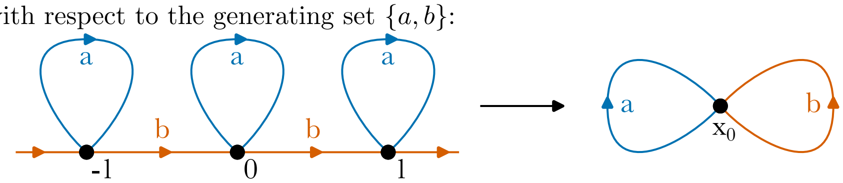


The basepoint is the vertex labeled 1. Let T be the following maximal tree in this Cayley graph:

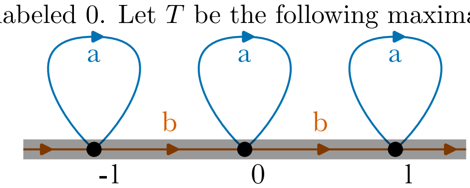


Using the recipe for computing fundamental groups of graphs from §8.9, we see that $G < F(a, b)$ is a free group on the following four generators: $\{ba^{-1}, aba^{-2}, a^3, a^2b\}$. ☐

Example 10.6.5. Consider the free group $F(a, b)$ on a and b. Let G be the normal subgroup of $F(a, b)$ generated by a. We thus have $F(a, b)/G \cong \mathbb{Z}$. Under this isomorphism, a maps to $0 \in \mathbb{Z}$ and b maps to $1 \in \mathbb{Z}$. Following the above recipe, the group G is the fundamental group of the Cayley graph of $\mathbb{Z}$ with respect to the generating set $\{a, b\}$:

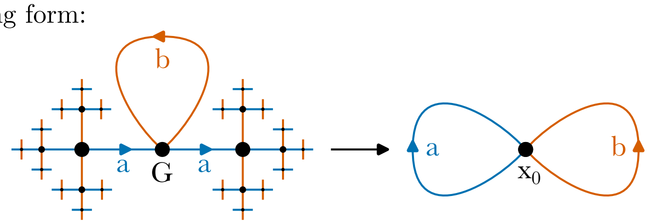


The basepoint is the vertex labeled 0. Let T be the following maximal tree in this Cayley graph:



Using the recipe for computing fundamental groups of graphs from §8.9, we see that $G < F(a, b)$ is the free group on the following set of generators: $\{b^n a b^{-n} \mid n \in \mathbb{Z}\}$. ☐

Example 10.6.6. To illustrate this construction for a non-normal subgroup, let $F(a, b)$ be the free group on a and b and let G be the cyclic subgroup generated by b. We already know that G is free on the single generator b. The cosets of G in $F(a, b)$ are of the form Gw where $w \in F(a, b)$ is a reduced word that does not start with b or b^{-1}. The Schreier graph of G in $F(a, b)$ with respect to S is thus of the following form:



For reasons of space, we only label the vertex corresponding to the trivial coset G, which is the basepoint. Note that this Schreier graph deformation retracts to the single loop labeled by b, so G is indeed the free group on the single generator b. ☐

Remark 10.6.7. It is enlightening to re-interpret the examples in §9.6 from this point of view. ☐

10.7. Exercises

Exercise 10.1. Let X be a space that is semilocally 1-connected and let $p \colon \widetilde{X} \to X$ be a cover. Then $\widetilde{X}$ is semilocally 1-connected. ☐

Exercise 10.2. Let X be a semilocally 1-connected space and let Y be a space that is homotopy equivalent to X. Prove that Y is semilocally 1-connected. ☐

Exercise 10.3. Let X be a space that is path connected, locally path connected, and semilocally 1-connected. Fix a basepoint $x_0 \in X$. Construct a bijection between the following two sets:

- The set of isomorphism classes of connected covers $p \colon \widetilde{X} \to X$.

- The set of conjugacy classes of subgroups of $\pi_1(X, x_0)$. $\square$

Exercise 10.4. Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a connected pointed cover and let $f\colon (Y, y_0) \to (X, x_0)$ be a map. Assume that X and Y are path connected and locally path connected. Define

$$\begin{aligned}\widetilde{Y} &= \left\{(y, \widetilde{x}) \in Y \times \widetilde{X} \mid f(y) = p(\widetilde{x})\right\},\\ \widetilde{y}_0 &= (y_0, \widetilde{x}_0) \in \widetilde{Y},\end{aligned}$$

and let $q\colon (\widetilde{Y}, \widetilde{y}_0) \to (y, y_0)$ be the projection onto the first factor. You proved in Exercise 1.8 that $q\colon (\widetilde{Y}, \widetilde{y}_0) \to (Y, y_0)$ is a covering space. Let $\widetilde{Y}'$ be the path component of $\widetilde{Y}$ containing $\widetilde{y}_0$ and let $q' = q|_{\widetilde{Y}'}$. Prove the following:

(a) The map $q'\colon (\widetilde{Y}', \widetilde{y}_0) \to (Y, y_0)$ is a cover.
(b) Let

$$K = \operatorname{Im}(p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0))$$

be the subgroup corresponding to $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ and let

$$K' = f_*^{-1}(K) \subset \pi_1(Y, y_0).$$

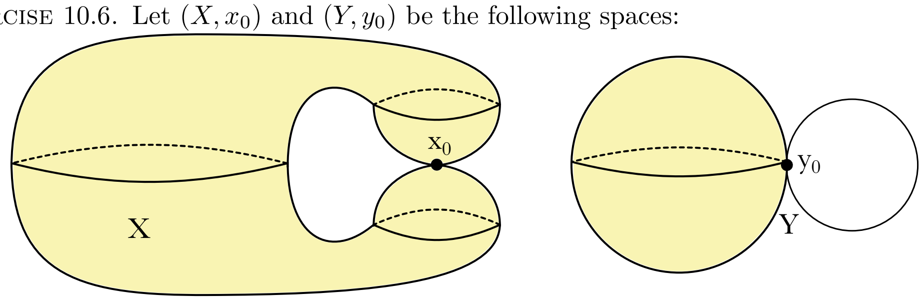
Prove that K' is the subgroup corresponding to $q'\colon (\widetilde{Y}', \widetilde{y}_0) \to (y, y_0)$. $\square$

Exercise 10.5. Let (X, x_0) be a pointed space that is path connected, locally path connected, and semilocally 1-connected. Let $p_1\colon (\widetilde{X}_1, \widetilde{x}_1) \to (X, x_0)$ and $p_2\colon (\widetilde{X}_2, \widetilde{x}_2) \to (X, x_0)$ be connected pointed covers corresponding to subgroups $K_1, K_2 < \pi_1(X, x_0)$ satisfying $K_1 < K_2$. Prove that there is a pointed covering map $q\colon (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ such that the following diagram commutes:

$$\begin{tikzcd}(\widetilde{X}_1, \widetilde{x}_1) \arrow[r, "q"] \arrow[rr, bend right, "p_1"'] & (\widetilde{X}_2, \widetilde{x}_2) \arrow[r, "p_2"] & (X, x_0).\end{tikzcd}$$

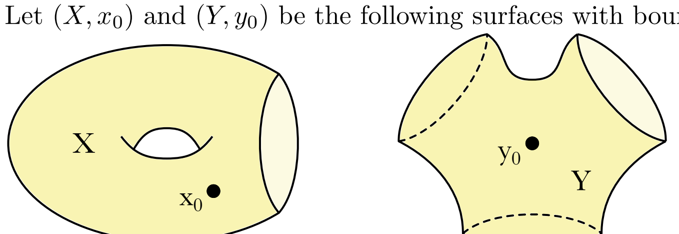
$\square$

Exercise 10.6. Let (X, x_0) and (Y, y_0) be the following spaces:



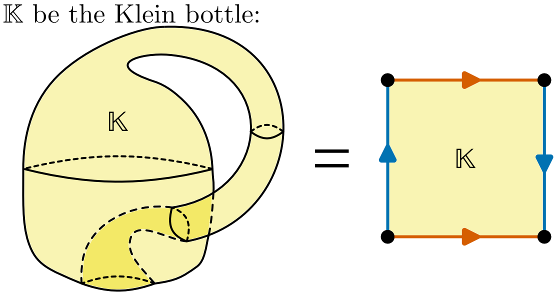
We proved in Example 8.6.1 that these spaces are homotopy equivalent, and also that $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$ are isomorphic to $\mathbb{Z}$. Explictly construct the universal covers of X and Y. $\square$

Exercise 10.7. Let (X, x_0) and (Y, y_0) be the following surfaces with boundary:



We proved in Example 8.2.1 that these spaces are homotopy equivalent, and also that $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$ are free groups on two generators. Explictly construct the universal covers of X and Y. $\square$

Exercise 10.8. Let $\mathbb{K}$ be the Klein bottle:

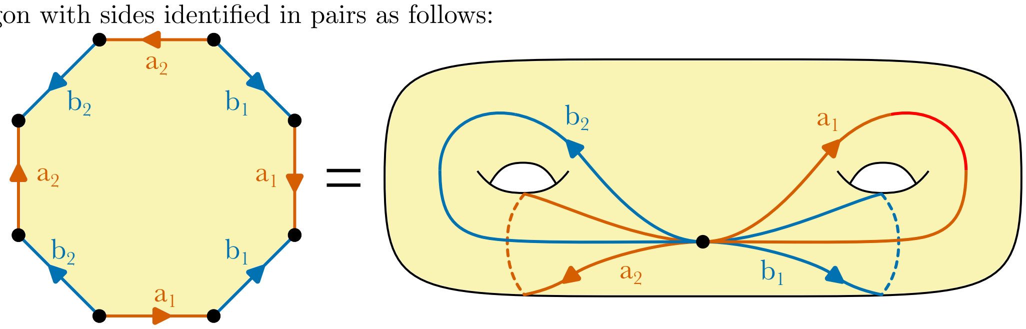


In Exercise 9.11, you proved that the fundamental group of $\mathbb{K}$ is the following group Γ:

- Let $\mathbb{Z}$ act on $\mathbb{Z}$ via the homomorphism $\phi\colon \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z})$ defined by
$$\phi(n)(m) = (-1)^n m \quad \text{for all } n \in \mathbb{Z} \text{ and } m \in \mathbb{Z}.$$
Then $\Gamma = \mathbb{Z} \rtimes_\phi \mathbb{Z}$, i.e., the semidirect product of $\mathbb{Z}$ and $\mathbb{Z}$ given by the action ϕ.

Construct a covering space action of Γ on $\mathbb{R}^2$ such that $\mathbb{R}^2/\Gamma \cong \mathbb{K}$. This shows that the universal cover of $\mathbb{K}$ is $\mathbb{R}^2$, and also gives a new proof that the fundamental group of $\mathbb{K}$ is Γ. □

Exercise 10.9. Let Σ_2 be a closed oriented genus 2 surface, which can be identified with an octagon with sides identified in pairs as follows:



Prove that the universal cover of Σ_2 is homeomorphic to $\mathbb{R}^2$. Hint: Let $P \cong \mathbb{D}^2$ be an octagon, so the above picture shows a surjection $f\colon P \to \Sigma_2$. The map f is an open embedding on the interior of P, but is not injective on the boundary. Construct a space $\widetilde{S} \cong \mathbb{R}^2$ and a covering map $p\colon \widetilde{S} \to \Sigma_2$ by carefully gluing together infinitely many copies of P and letting p equal f on each copy of P. We remark that this same argument will show that for all $g \geq 1$ the universal cover of a closed genus g surface is homeomorphic to $\mathbb{R}^2$. □

Exercise 10.10. Let $\mathbb{T}^2 = (\mathbb{S}^1)^{\times 2}$ be the 2-torus. Fix a basepoint $x_0 \in \mathbb{T}^2$, so $\pi_1(\mathbb{T}^2, x_0) \cong \mathbb{Z}^2$.

(a) Construct the universal cover of $\mathbb{T}^2$.

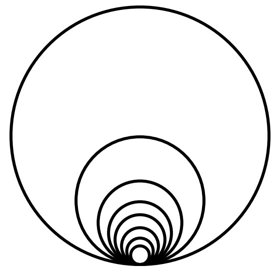
(b) Let $K < \mathbb{Z}^2$ be a finite-index subgroup and let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (\mathbb{T}^2, x_0)$ be the cover whose corresponding subgroup is K. Prove that $\widetilde{X} \cong \mathbb{T}^2$.

(c) Let $K < \mathbb{Z}^2$ be a nontrivial subgroup of infinite index and let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (\mathbb{T}^2, x_0)$ be the cover whose corresponding subgroup is K. Prove that $\widetilde{X} \cong \mathbb{S}^1 \times \mathbb{R}$. □

Exercise 10.11. Let X be a graph with one vertex x_0 and two edges labeled a and b. We can therefore identify $\pi_1(X, x_0)$ with the free group $F(a, b)$. Classify all the connected degree 2 and degree 3 covers of X, and for each cover determine if it is regular and give generators for its corresponding subgroup of $\pi_1(X, x_0) = F(a, b)$. □

Exercise 10.12. Let $F = F(a, b)$ be the free group on a and b. Recall from Exercise 7.12 that the abelianization of F is $\mathbb{Z}^2$. The kernel of the abelianization map $F \to \mathbb{Z}^2$ is the commutator subgroup $[F, F]$. Compute a free basis for the free group $[F, F]$. □

Exercise 10.13. For $n \geq 1$, let $C_n \subset \mathbb{R}^2$ be the circle of radius $1/n$ with center $(0, 1/n)$. Let $X = \cup_{n=1}^{\infty} C_n$, topologized as a subspace of $\mathbb{R}^2$. This is sometimes called the "earring space" or the "shrinking wedge of circles":



Prove the following:

(a) The space X is path connected.
(b) The space X is locally path connected.
(c) The space X has no universal cover. □

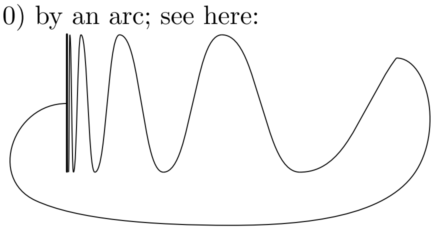
EXERCISE 10.14. Let $X = \prod_{n=1}^{\infty} \mathbb{S}^1$, endowed with the product topology. Prove the following:

(a) The space X is path connected.
(b) The space X is locally path connected.
(c) The space X has no universal cover. □

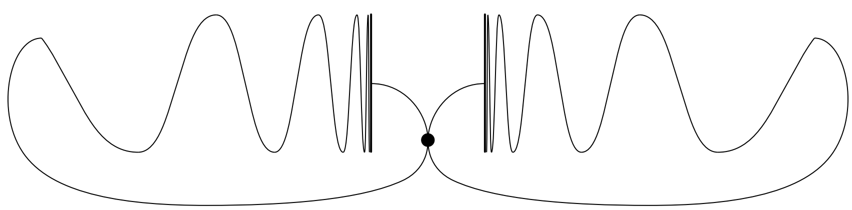
EXERCISE 10.15. The quasi-circle is the space Y obtained from the topologist's sine curve

$$X = \left\{(x, \sin(1/x)) \in \mathbb{R}^2 \mid 0 < x \leq 1\right\} \cup \{(0, y) \mid -1 \leq y \leq 1\}$$

by connecting $(1, \sin(1))$ to $(0, 0)$ by an arc; see here:



Let Z be the following space obtained from two quasi-circles by identifying basepoints on their arcs together:



Prove the following:

(a) The space Z is 1-connected (recall that Exercise 4.11 says that the quasi-circle itself is 1-connected).
(b) There exists a finite irregular cover $p\colon \widetilde{Z} \to Z$ such that $\widetilde{Z}$ is 1-connected. Hint: start with a finite irregular cover of a graph with one vertex and two edges. □

CHAPTER 11

Classifying covers: monodromy and the universal cover

Our main goal in this chapter is to construct universal covers. We do this by proving a much more general version of the classification of covering spaces.

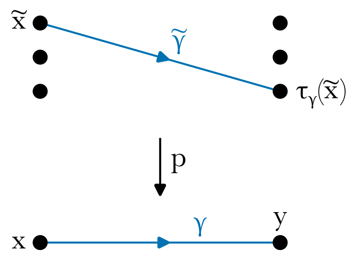
Remark 11.0.1. The arguments in this chapter are more abstract and categorical than those in the other chapters of this book. For a reader who just wants to get to the construction of the universal cover as fast as possible, the traditional proof is in Essay C. In fact, the proof of our very general classification theorem is essentially the same as the traditional construction of universal covers. One of the reasons we wrote this chapter was to put that somewhat mysterious proof in its natural context. □

11.1. Monodromy action of the fundamental group

Let $p\colon \widetilde{X} \to X$ be a cover. For $x \in X$, let $F(x) = p^{-1}(x)$ be the fiber over x. If γ is a path in X from $x \in X$ to $y \in X$, then in §4.3 we defined a map $\tau_\gamma\colon F(x) \to F(y)$ as follows:

- For $\widetilde{x} \in F(x)$, let $\widetilde{\gamma}$ be the lift of γ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}$. We then set $\tau_\gamma(\widetilde{x}) = \widetilde{\gamma}(1) \in F(y)$.

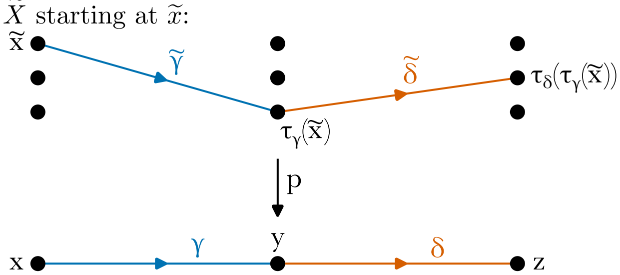
The picture is as follows:



Lemma 4.2.1 says that τ_γ only depends on the homotopy class of γ, and Lemma 4.3.1 says that τ_γ is a bijection. These compose as follows:

Lemma 11.1.1. *Let $p\colon \widetilde{X} \to X$ be a cover. For $x \in X$, let $F(x) = p^{-1}(x)$. Let γ be a path in X from $x \in X$ to $y \in X$ and let δ be a path in X from $y \in X$ to $z \in X$. Then $\tau_{\gamma\cdot\delta} = \tau_\delta \circ \tau_\gamma$.*

Proof. Consider some $\widetilde{x} \in F(x)$. Let $\widetilde{\gamma}$ be the lift of γ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}$. We thus have $\widetilde{\gamma}(1) = \tau_\gamma(\widetilde{x})$. Let $\widetilde{\delta}$ be the lift of δ to $\widetilde{X}$ with $\widetilde{\delta}(0) = \widetilde{\gamma}(1)$. As the following figure shows, the path $\widetilde{\gamma}\cdot\widetilde{\delta}$ is the lift of $\gamma\cdot\delta$ to $\widetilde{X}$ starting at $\widetilde{x}$:



It follows that $\tau_{\gamma\cdot\delta} = \widetilde{\delta}(1) = \tau_\delta(\widetilde{\delta}(0)) = \tau_\delta(\tau_\gamma(\widetilde{x}))$, as desired. □

Fix a basepoint $x_0 \in X$. Define a right action of $\pi_1(X, x_0)$ on $F(x_0)$ as follows:

$$\widetilde{x}[\gamma] = \tau_\gamma(\widetilde{x}) \quad \text{for all } \widetilde{x} \in F(x_0) \text{ and } [\gamma] \in \pi_1(X, x_0).$$

The fact that this is an action uses Lemma 11.1.1: for $\widetilde{x} \in F(x_0)$ and $[\gamma], [\delta] \in \pi_1(X, x_0)$, we have

$$\widetilde{x}[\gamma \cdot \delta] = \tau_{\gamma \cdot \delta}(\widetilde{x}) = \tau_\delta(\tau_\gamma(\widetilde{x})) = \tau_\gamma(\widetilde{x})[\delta] = \widetilde{x}[\gamma][\delta].$$

This is called the *monodromy* action of $\pi_1(X, x_0)$ on $F(x_0)$.

REMARK 11.1.2. In Lemma 11.1.1, the order of γ and δ are reversed in $\tau_{\gamma \cdot \delta} = \tau_\delta \circ \tau_\gamma$. This order reversal is why we get a right action rather than a left action. It happens because we compose paths from left to right, but functions from right to left. □

REMARK 11.1.3. If $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a pointed regular cover with deck group G, then G acts on $F(x_0)$ on the left. Composing this action with the homomorphism $\pi_1(X, x_0) \to G$ given by Lemma 7.2.1, we get a left action of $\pi_1(X, x_0)$ on $F(x_0)$. This is different from the monodromy action, which is a right action and exists for all covers, not just regular ones. This left action also depends on a choice of basepoint $\widetilde{x}_0$ in the cover, while the monodromy action does not. See Exercise 11.1 for how the two actions are related. □

11.2. All monodromy actions come from covers

Let G be a group. A *right G-set* is a set S equipped with a right G-action. Two right G-sets S and T are isomorphic if there exists a bijection $f\colon S \to T$ such that

$$f(sg) = f(s)g \quad \text{for all } s \in S \text{ and } g \in G.$$

If $p\colon \widetilde{X} \to X$ is a cover and $x_0 \in X$, then the fiber $p^{-1}(x_0)$ is a right $\pi_1(X, x_0)$-set via the monodromy action. The following theorem says that for reasonable pointed spaces (X, x_0), covers of X can be identified with right $\pi_1(X, x_0)$-sets:

THEOREM 11.2.1. *Let (X, x_0) be a pointed space. Assume that X is path connected, locally path connected, and semilocally 1-connected. The following then hold:*

(i) Let S be a right $\pi_1(X, x_0)$-set. There then exists a cover $p\colon \widetilde{X} \to X$ such that the fiber $p^{-1}(x_0)$ over x_0 is isomorphic to S as a right G-set.

(ii) Let $p_1\colon \widetilde{X}_1 \to X$ and $p_2\colon \widetilde{X}_2 \to X$ be covers. Assume that the fibers $p_1^{-1}(x_0)$ and $p_2^{-1}(x_0)$ are isomorphic as right $\pi_1(X, x_0)$-sets. Then $p_1\colon \widetilde{X}_1 \to X$ and $p_2\colon \widetilde{X}_2 \to X$ are isomorphic covers.

Just like Theorem 10.3.1 (Galois correspondence), Theorem 11.2.1 is a classification of covers of X. It differs from Theorem 10.3.1 in two ways:

- Theorem 10.3.1 classifies covers with $\widetilde{X}$ path connected, while Theorem 11.2.1 allows $\widetilde{X}$ to not be path connected.
- Theorem 10.3.1 is about pointed covers $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$, while Theorem 11.2.1 does not specify a basepoint in $\widetilde{X}$. The notion of isomorphism of covers in the two theorems is thus slightly different (pointed vs unpointed isomorphisms of covers).

We will prove Theorem 11.2.1 later in this chapter; see §11.6 below. First, we will explain how to use it to construct the covers given by Theorem 10.3.1 (Galois correspondence), including the universal cover.

We start with the following. If G is a group, a right G-set S is *transitive* if for all $s, s' \in S$ there exists some $g \in G$ with $sg = s'$.

LEMMA 11.2.2. *Let $p\colon \widetilde{X} \to X$ be a cover with X path connected and let $x_0 \in X$. The path components of $\widetilde{X}$ are in bijection with the $\pi_1(X, x_0)$-orbits of the right $\pi_1(X, x_0)$-set $p^{-1}(x_0)$. In particular, $\widetilde{X}$ is path connected if and only if the right $\pi_1(X, x_0)$-set $p^{-1}(x_0)$ is transitive.*

PROOF. For $\widetilde{x} \in \widetilde{X}$, a path δ in X from $p(\widetilde{x})$ to x_0 lifts to a path from $\widetilde{x}$ to a point of $p^{-1}(x_0)$. It follows that each path component of $\widetilde{X}$ contains at least one point of $p^{-1}(x_0)$. To prove the lemma, it is therefore enough to prove that two points of $p^{-1}(x_0)$ can be connected by a path if and only if they are in the same $\pi_1(X, x_0)$-orbit.

Consider $\widetilde{x}, \widetilde{x}' \in p^{-1}(x_0)$. A path $\widetilde{\gamma}$ from $\widetilde{x}$ to $\widetilde{x}'$ projects to a loop γ with $[\gamma] \in \pi_1(X, x_0)$ such that $\widetilde{x}[\gamma] = \widetilde{x}'$. Conversely, if $[\gamma] \in \pi_1(X, x_0)$ satisfies $\widetilde{x}[\gamma] = \widetilde{x}'$ then the lift of γ to $\widetilde{X}$ starting at $\widetilde{x}$

is a path from $\widetilde{x}$ to $\widetilde{x}'$. It follows that $\widetilde{x}$ and $\widetilde{x}'$ can be connected by a path if and only if they are in the same $\pi_1(X, x_0)$-orbit, as desired. □

Next, if G is a group and S is a right G-set, for $s \in S$ we will write G_s for the stabilizer subgroup of s, i.e., $G_s = \{g \in G \mid sg = s\}$. We then have:

LEMMA 11.2.3. *Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a pointed cover and let $G = \pi_1(X, x_0)$. Then the subgroup of G corresponding to $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is $G_{\widetilde{x}_0}$.*

PROOF. Recall that the subgroup of $G = \pi_1(X, x_0)$ corresponding to the cover is the image of the induced map $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$. By Theorem 9.1.1, this subgroup consists of all $[\gamma] \in \pi_1(X, x_0)$ such that the following holds:

(†) Let $\widetilde{\gamma}$ be the lift of γ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$. Then $\widetilde{\gamma}$ is a loop, i.e., $\widetilde{\gamma}(1) = \widetilde{x}_0$.

By definition, these are exactly the $[\gamma]$ such that $\widetilde{x}_0[\gamma] = \widetilde{x}_0$. □

For a pointed space (X, x_0) satisfying appropriate hypotheses and a subgroup $H < \pi_1(X, x_0)$, Theorem 10.3.1 gives a connected pointed cover $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ whose corresponding subgroup is H. The following theorem shows how to produce this cover with Theorem 11.2.1. For the statement of the theorem, note that if G is a group and $H < G$ is a subgroup, then the set $H\backslash G$ of right H-cosets is a transitive right G-set. Moreover, if S is a transitive right G-set and $s \in S$, then we have an isomorphism $G_s\backslash G \cong S$ of transitive right G-sets. This isomorphism takes a coset $G_s g$ to sg.

THEOREM 11.2.4. *Let (X, x_0) be a pointed space. Assume that X is path connected, locally path connected, and semilocally 1-connected. Let $G = \pi_1(X, x_0)$ and let $H < G$ be a subgroup. Let $p\colon \widetilde{X} \to X$ be the cover obtained by applying Theorem 11.2.1 to the right G-set $H\backslash G$. Let $\widetilde{x}_0 \in \widetilde{X}$ be the point in the fiber $p^{-1}(x_0)$ corresponding to the trivial coset $H \in H\backslash G$. Then $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a connected pointed cover whose corresponding subgroup is H.*

PROOF. Immediate from Lemmas 11.2.2 and 11.2.3. □

From this, we will deduce the following theorem from Chapter 10 whose proof was postponed:

THEOREM 10.2.3 (Existence of universal covers). *Let (X, x_0) be a pointed space. Assume that X is path connected, locally path connected, and semilocally 1-connected. Then it has a universal cover $p_u\colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$.*

PROOF. Let $p_u\colon \widetilde{X}_u \to X$ be the cover obtained by applying Theorem 11.2.1 to the right $\pi_1(X, x_0)$-set $1\backslash\pi_1(X, x_0)$ and let $\widetilde{x}_u \in \widetilde{X}_u$ be the point in the fiber $p^{-1}(x_0)$ corresponding to the trivial coset 1. It follows from Theorem 11.2.4 that $p_u\colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ is a universal cover. □

11.3. Reminder about fundamental groupoid and functors

To prove Theorem 11.2.1, we will actually prove an even more general result. This more general result is most naturally stated in terms of the fundamental groupoid. Since we have not discussed the fundamental groupoid since introducing it in Chapter 5, we recall some basic facts about it.

Let X be a space. The fundamental groupoid of X is a category that endodes the collection of homotopy classes of paths between poitns of X. As we discussed in §5.4, in a category morphisms are composed right-to-left like functions. However, our conventions for composing paths goes from left to right: if γ is a path from x to y and γ' is a path from y to z, then $\gamma\cdot\gamma'$ is the path from x to z that first traverses γ and then traverses γ'. To fix this mismatch of conventions, we introduced the following notation:

NOTATION 11.3.1. Let X be a space. For points $x, y, z \in X$, let γ be a path in X from x to y and let δ be a path in X from y to z. We then define $\gamma' * \gamma = \gamma\cdot\gamma'$. This descends to homotopy classes of paths, and we also write $[\gamma'] * [\gamma] = [\gamma' * \gamma]$. □

With this notation, the fundamental groupoid of X, denoted $\Pi(X)$, is the following category:

- The objects of $\Pi(X)$ are the points of X.

- For points x and y, the $\Pi(X)$-morphisms from x to y are the set of all homotopy classes of paths from x to y. For a path γ from x to y, we will write $[\gamma]\colon x \to y$ for the corresponding morphism from x to y.
- If γ is a path from x to y and γ' is a path from y to z, then the composition of the morphisms $[\gamma]\colon x \to y$ and $[\gamma']\colon y \to z$ is the morphism $[\gamma'] * [\gamma]\colon x \to z$.
- For a point $x \in X$, the identity morphism of x is the constant path $[\mathfrak{c}_x]\colon x \to x$.

We will encode covers of spaces as functors on the fundamental groupoid. We recall that a functor is defined as follows:

Definition 11.3.2. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. A *functor* F from $\mathbf{C}$ to $\mathbf{D}$ consists of the following data:

- For each object $c \in \mathbf{C}$, an object $F(c) \in \mathbf{D}$.
- For each morphism $f\colon c \to c'$ in $\mathbf{C}$, a morphism $F(f)\colon F(c) \to F(c')$ in $\mathbf{D}$.

This data should respect composition in the sense that if $f\colon c \to c'$ and $g\colon c' \to c''$ are morphisms in $\mathbf{C}$, then $F(g \circ f) = F(g) \circ F(f)$. □

Example 11.3.3. If $\mathtt{Top}_*$ is the category whose objects are pointed spaces (X, x_0) and whose morphisms are pointed maps $f\colon (X, x_0) \to (Y, y_0)$, then the fundamental group π_1 is a functor from $\mathtt{Top}_*$ to the category $\mathtt{Group}$ of groups. □

11.4. Fiber functors

Let X be a space and let $p\colon \widetilde{X} \to X$ be a covering space. The *fiber functor* of $p\colon \widetilde{X} \to X$ is the following functor F from $\Pi(X)$ to the category $\mathtt{Set}$ of sets. For an object $x \in \widetilde{X}$, we define $F(x)$ to be the set of points in the fiber of x, i.e.,

$$F(x) = p^{-1}(x).$$

For a morphism $[\gamma]\colon x \to y$ in $\Pi(X)$, we define $F([\gamma])\colon F(x) \to F(y)$ to be the map τ_γ we discussed in §11.1 above. To see that this is a functor, consider morphisms $[\gamma]\colon X \to y$ and $[\delta]\colon y \to z$ in $\Pi(X)$. We must prove that $F([\delta] * [\gamma]) = F([\delta]) \circ F([\gamma])$, i.e., that

$$\tau_{\gamma \cdot \delta} = \tau_\delta \circ \tau_\gamma.$$

This is exactly Lemma 11.1.1.

Remark 11.4.1. Using $*$ instead of the concatanation product $\cdot$ in the fundamental groupoid accomplised the same thing for the fiber functor that using right actions did for the monodromy action of the fundamental group. □

11.5. Realizing fiber functors

Our main theorem says that every functor from the fundamental groupoid to $\mathtt{Set}$ is the fiber functor of a cover. This result will also have a uniqueness statement. For this, we make the following categorical definition:

Definition 11.5.1. Let F and G be two functors from a category $\mathbf{C}$ to a category $\mathbf{D}$. A *natural isomorphism* Φ from F to G consists of the following data:

- For each object $c \in \mathbf{C}$, an isomorphism $\Phi(c)\colon F(c) \to G(c)$ in $\mathbf{D}$.

These isomorphisms should satisfy the follow consistency condition. Let $f\colon c \to c'$ be a morphism in $\mathbf{C}$. We then require that the diagram

$$\begin{array}{ccc} F(c) & \xrightarrow{F(f)} & F(c') \\ {\scriptstyle \Phi(c)}\big\downarrow & & \big\downarrow{\scriptstyle \Phi(c')} \\ G(c) & \xrightarrow{G(f)} & G(c') \end{array}$$

commutes, i.e., that $\Phi(c') \circ F(f) = G(f) \circ \Phi(c)$. If a natural isomorphism from F to G exists, we say that F and G are *naturally isomorphic*. □

Our theorem is as follows. Note that unlike in Theorem 11.2.1, the space X below is not assumed to be path connected.

THEOREM 11.5.2. *Let X be a space that is locally path connected and semilocally 1-connected. The following then hold:*

(i) For all functors $F\colon \Pi(X) \to \mathtt{Set}$, there exists a cover $p\colon \widetilde{X} \to X$ whose fiber functor is naturally isomorphic to F.
(ii) If $p\colon \widetilde{X} \to X$ and $p'\colon \widetilde{X}' \to X$ are covers whose fiber functors are naturally isomorphic, then $p\colon \widetilde{X} \to X$ and $p'\colon \widetilde{X}' \to X$ are isomorphic.

PROOF. We prove (i), and then deduce (ii) by meditating on our proof of (i).

STEP 1. *Let $F\colon \Pi(X) \to \mathtt{Set}$ be a functor. There then exists a cover $p\colon \widetilde{X} \to X$ whose fiber functor is F.*

Roughly speaking, we start by defining $\widetilde{X}$ to be the disjoint union of all points in all the purported fibers given by F. To ensure that this really is a disjoint union, we actually define this as follows:

$$\widetilde{X} = \{(x, \widetilde{x}) \mid x \in X \text{ and } \widetilde{x} \in F(x)\}.$$

Let $p\colon \widetilde{X} \to X$ be the projection onto the first factor. Our main goal is to construct a topology on $\widetilde{X}$ such that $p\colon \widetilde{X} \to X$ is a covering map.

Say that a set $U \subset X$ is an *sl1c-set* if U is open and path connected, and for all $x \in U$ the map $\pi_1(U, x) \to \pi_1(X, x)$ is the trivial map. These satisfy the following property:

- If U is an sl1c-set and $V \subset U$ is open and path connected, then V is an sl1c-set.

Since X is locally path connected and semilocally 1-connected, every point has a open neighborhood that is an sl1c-set. Combined with the above bullet point, we deduce that the sl1c-sets form a basis for the topology of X.

Let $U \subset X$ be an sl1c-set. For some $x \in U$, let $\widetilde{x} \in F(x)$. Since U is an sl1c-set, for each $y \in U$ there exists a path γ_y in U from x to y. Moreover, any two such paths are homotopic in X. The morphism $[\gamma_y]\colon x \to y$ thus only depends on y. Applying the functor F, we get a bijection $F([\gamma_y])\colon F(x) \to F(y)$. Define

$$\widetilde{U}(\widetilde{x}) = \left\{(y, \widetilde{y}) \in \widetilde{X} \mid y \in U \text{ and } \widetilde{y} = F([\gamma])(\widetilde{x})\right\}.$$

The projection $p\colon \widetilde{X} \to X$ takes $\widetilde{U}(\widetilde{x})$ bijectively to U. By the functorality of F, these sets $\widetilde{U}(\widetilde{x})$ satisfy the following key properties:

(a) Let $U \subset X$ be an sl1c-set, let $x \in U$, and let $\widetilde{x} \in F(x)$. For all $(y, \widetilde{y}) \in U(\widetilde{x})$, we have $U(\widetilde{x}) = U(\widetilde{y})$.
(b) Let $U \subset X$ be an sl1c-set, let $x \in U$, and let $\widetilde{x} \in F(x)$. Let $V \subset U$ be an open path connected subset of U, so V is also an sl1c-set. Assume that $x \in V$. Then $\widetilde{V}(\widetilde{x}) = \widetilde{U}(\widetilde{x})$.

We claim that the collection of all sets of the form $\widetilde{U}(\widetilde{x})$ forms the basis for a topology on $\widetilde{X}$. This requires checking the following:

- Let $U, V \subset X$ be sl1c-sets, let $x \in U$ and $y \in V$, and let $\widetilde{x} \in F(x)$ and $\widetilde{y} \in F(y)$. We must prove that $\widetilde{U}(\widetilde{x}) \cap \widetilde{V}(\widetilde{y})$ is the union of sets of this form. To see this, consider some $(z, \widetilde{z}) \in \widetilde{U}(\widetilde{x}) \cap \widetilde{V}(\widetilde{y})$. Let $W \subset U \cap V$ be a path connected open neighbrohood of z. Using (a) and (b), we have

  $$\widetilde{W}(\widetilde{Z}) \subset \widetilde{U}(\widetilde{z}) \cap \widetilde{V}(\widetilde{z}) = \widetilde{U}(\widetilde{x}) \cap \widetilde{V}(\widetilde{y}).$$

  The claim follows.

We can therefore endow $\widetilde{X}$ with the topology generated by the $\widetilde{U}(\widetilde{x})$.

By construction, this topology makes the projection $p\colon \widetilde{X} \to X$ continuous. In fact, even more is true: if $U \subset X$ is an sl1c-set and $x \in U$ and $\widetilde{x} \in F(x)$, then the restriction of p to $\widetilde{U}(\widetilde{x})$ is a homeomorphism $\widetilde{U}(\widetilde{x}) \to U$. This implies that $p\colon \widetilde{X} \to X$ is a covering map. Indeed, for $x \in X$ letting $U \subset X$ be an sl1c-set containing x we have

$$p^{-1}(U) = \bigsqcup_{\widetilde{x} \in F(x)} \widetilde{U}(\widetilde{x}).$$

These are the sheets lying above U.

It remains to check that the fiber functor of $p\colon \widetilde{X} \to X$ is naturally isomorphic to F. For $x \in X$, we have

$$p^{-1}(x) = \{(x, \widetilde{x}) \mid \widetilde{x} \in F(x)\}.$$

Our natural isomorphism takes this to $F(x)$ via the map $(x, \widetilde{x}) \mapsto \widetilde{x}$. We must check that this purported natural isomorphism respects the morphisms in the fundamental groupoid. Let $[\gamma]\colon x \to y$ be a morphism in $\Pi(X)$ and let $\widetilde{x} \in F(x)$. Unwinding the definitions, what we must check is the following:

(†) Let $\widetilde{\gamma}$ be the lift of γ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = (x, \widetilde{x})$. Then $\widetilde{\gamma}(1) = (y, \widetilde{y})$.

Using the Lebesgue number lemma just like in our proof of path lifting for covers (cf. Lemma 3.4.1), we can divide the domain I of $\gamma\colon I \to X$ into subintervals

$$0 = \epsilon_1 < \epsilon_2 < \cdots < \epsilon_n = 1$$

such that for all $1 \leq k < n$ the image $\gamma([\epsilon_k, \epsilon_{k+1}])$ is contained in an sl1c-set $U_k \subset X$. After re-parameterizing γ (which changes it by a homotopy), we can therefore write $\gamma = \gamma_1 \cdot \cdots \cdot \gamma_n$ such that each γ_k is contained in the sl1c-set U_k. We can lift γ by lifting γ_1, then γ_2, etc. This reduces us to checking (†) for a γ that is contained in an sl1c-set U, and for these paths (†) is immediate from our construction.

Step 2. *Let $p\colon \widetilde{X} \to X$ and $p'\colon \widetilde{X}' \to X$ be covers whose fiber functors are naturally isomorphic. Then $p\colon \widetilde{X} \to X$ and $p'\colon \widetilde{X}' \to X$ are isomorphic.*

Let F be the fiber functor of $p\colon \widetilde{X} \to X$ and F' be the fiber functor of $p'\colon \widetilde{X}' \to X$. Examining our proof in Step 1, it is clear that $p\colon \widetilde{X} \to X$ and $p'\colon \widetilde{X}' \to X$ are isomorphic to the covering spaces obtained by applying the construction in Step 1 to F and F', respectively. We can therefore assume without loss of generality that

$$\widetilde{X} = \{(x, \widetilde{x}) \mid x \in X \text{ and } \widetilde{x} \in F(x)\},$$
$$\widetilde{X}' = \{(x, \widetilde{x}') \mid x \in X \text{ and } \widetilde{x}' \in F'(x)\}$$

with the topologies from Step 1. Let Φ be a natural isomorphism from F to F', so for $x \in X$ we have bijections $\Phi(x)\colon F(x) \to F'(x)$. We can therefore define a map $\phi\colon \widetilde{X} \to \widetilde{X}'$ via the formula

$$\phi(x, \widetilde{x}) = (x, \Phi(x)(\widetilde{x})) \quad \text{for all } x \in X \text{ and } \widetilde{x} \in F(x).$$

It is clear that ϕ is a homeomorphism commuting with the projections p and p', i.e., an isomorphism from $p\colon \widetilde{X} \to X$ to $p'\colon \widetilde{X}' \to X$. □

11.6. Realizing monodromy representations using fiber functors

In this section, we use Theorem 11.5.2 to prove Theorem 11.2.1, whose statement we recall:

Theorem 11.2.1. *Let (X, x_0) be a pointed space. Assume that X is path connected, locally path connected, and semilocally 1-connected. The following then hold:*

(i) Let S be a right $\pi_1(X, x_0)$-set. There then exists a cover $p\colon \widetilde{X} \to X$ such that the fiber $p^{-1}(x_0)$ over x_0 is isomorphic to S as a right G-set.

(ii) Let $p_1\colon \widetilde{X}_1 \to X$ and $p_2\colon \widetilde{X}_2 \to X$ be covers. Assume that the fibers $p_1^{-1}(x_0)$ and $p_2^{-1}(x_0)$ are isomorphic as right $\pi_1(X, x_0)$-sets. Then $p_1\colon \widetilde{X}_1 \to X$ and $p_2\colon \widetilde{X}_2 \to X$ are isomorphic covers.

Proof. We divide the proof into two steps:

Step 1. *Let S be a right $\pi_1(X, x_0)$-set. There then exists a cover $p\colon \widetilde{X} \to X$ such that the fiber $p^{-1}(x_0)$ over x_0 is isomorphic to S as a right G-set.*

We construct a functor $F\colon \Pi(X) \to \mathtt{Set}$ as follows. For $x \in X$, define $F(x) = S$. To define F on morphisms of $\Pi(X)$, for each $x \in X$ fix some arbitrary path δ_x from x_0 to x. The only thing we will assume is that δ_{x_0} is the constant path. We are going to define F such that for all $x \in X$ the map

$$F([\delta_x])\colon F(x_0) \to F(x)$$

is the identity map $S \to S$. Of course, we do not want $F([\gamma])$ to be the identity map for all paths γ since then we would just get the fiber functor of the trivial cover $X \times S \to X$. We must incorporate the right action of $\pi_1(X, x_0)$ on S.

Consider a path γ in X from $x \in X$ to $y \in X$. We therefore have a loop $\delta_x \cdot \gamma \cdot \overline{\delta}_y$ based at x_0. We define

$$F([\gamma])(s) = s[\delta_x \cdot \gamma \cdot \delta_y] \quad \text{for all } s \in F(x) = S.$$

We must check that this is a functor. Let γ be a path from $x \in X$ to $y \in X$ and let γ' be a path from $y \in X$ to $z \in Z$. We must prove that

$$F([\gamma'] * [\gamma]) = F(\gamma \cdot \gamma')$$

equals $F([\gamma']) \circ F([\gamma])$. To see this, note that for $s \in F(x) = S$ we have

$$F(\gamma \cdot \gamma') = s[\delta_x \cdot \gamma \cdot \gamma' \cdot \overline{\delta}_z] = s[\delta_x \cdot \gamma \cdot \overline{\delta}_y][\delta_y \cdot \gamma' \cdot \overline{\delta}_z] = F([\gamma])(s)[\delta_y \cdot \gamma' \cdot \overline{\delta}_z] = F([\gamma']) \circ F([\gamma])(s),$$

as desired.

We can therefore apply Theorem 11.5.2 to construct a cover $p\colon \widetilde{X} \to X$ with fiber functor naturally isomorphic to F. We must check that $p^{-1}(x_0)$ is isomorphic to S as a right $\pi_1(X, x_0)$-set. Since F is naturally isomorphic to the fiber functor of $p\colon \widetilde{X} \to X$, we can identify $p^{-1}(x_0)$ with $F(x_0) = S$. To check that it has the right $\pi_1(X, x_0)$-action, consider some $[\gamma] \in \pi_1(X, x_0)$ and $s \in S$. By our construction, under the monodromy action the image of s under $[\gamma]$ is

$$F([\gamma])(s) = s[\delta_{x_0} \cdot \gamma \cdot \overline{\delta}_{x_0}] = s[\gamma],$$

as desired. Here we are using the fact that δ_{x_0} is the constant path.

STEP 2. *Let $p_1\colon \widetilde{X}_1 \to X$ and $p_2\colon \widetilde{X}_2 \to X$ be covers. Assume that the fibers $p_1^{-1}(x_0)$ and $p_2^{-1}(x_0)$ are isomorphic as right $\pi_1(X, x_0)$-sets. Then $p_1\colon \widetilde{X}_1 \to X$ and $p_2\colon \widetilde{X}_2 \to X$ are isomorphic covers.*

Let $S_1 = p_1^{-1}(x_0)$ and $S_2 = p_2^{-1}(x_0)$. Let $\phi\colon S_1 \to S_2$ be an isomorphism of right $\pi_1(X, x_0)$-sets. Let F_1 be the fiber functor of p_1 and F_2 be the fiber functor of p_2. By the uniqueness part of Theorem 11.5.2, it is enough to prove that F_1 is naturally isomorphic to F_2.

As in Step 1, for each $x \in X$ let δ_x be a path in X from x_0 to x. Choose these paths such that δ_{x_0} is the constant path. We therefore have bijections $F_1([\delta_x])\colon S_1 \to F_1(x)$ and $F_2([\delta_x])\colon S_2 \to F_2(x)$ for all $x \in X$. To simplify our notation,[1] we will rename the points in our sets so that in fact $F_1(x) = S_1$ and $F_2(x) = S_2$ for all $x \in X$, with the bijections $F_1([\delta_x])\colon S_1 \to F_1(x)$ and $F_2([\delta_x])\colon S_2 \to F_2(x)$ the identity maps.

We then define a natural isomorphism Φ from F_1 to F_2 as follows:

- Consider $x \in X$, so $F_1(x) = S_1$ and $F_2(x) = S_2$. Let $\Phi(x)\colon S_1 \to S_2$ equal $\phi\colon S_1 \to S_2$.

We must check that this is compatible with morphisms. Let γ be a path in X from x to y. We must prove that the diagram

$$\begin{array}{ccc} F_1(x) & \xrightarrow{F_1([\gamma])} & F_1(y) \\ {\scriptstyle \Phi(x)}\downarrow & & \downarrow{\scriptstyle \Phi(y)} \\ F_2(x) & \xrightarrow{F_2([\gamma])} & F_2(y) \end{array}$$

commutes. Consider $s_1 \in F_1(x) = S_1$. We must prove that the two ways of applying the maps in this diagram take s_1 to the same element of $F_2(y)$. We trace these through as follows:

[1] This might seem a confusing thing to do, but it makes our notation line up as much as possible with Step 1.

- Recall that we identify s_1 with an element of $F_1(x_0) = S_1$ via the identity map $F_1([\delta_x])\colon S_1 \to F_1(x)$. Similarly, we identify $F_1([\gamma])(s_1) \in F_1(y) = S_1$ with an element of $F_1(x_0) = S_1$ via the identity map $F_1([\delta_y])\colon S_1 \to F_1(y)$. By naturality, under these identifications we have

  $$F_1([\gamma])(s_1) = s_1[\delta_x{\cdot}\gamma{\cdot}\overline{\delta}_y].$$

  Applying $\Phi(y) = \phi$ to this, we get

$$\phi(s_1[\delta_x{\cdot}\gamma{\cdot}\overline{\delta}_y]). \tag{11.6.1}$$

- First apply $\Phi(x) = \phi$ to $s_1 \in F_1(x) = S_1$ to get $\phi(s_1) \in S_2 = F_2(y)$. Just like in the previous bullet point $F_2([\gamma])$ takes this to

$$\phi(s_1)[\delta_x{\cdot}\gamma{\cdot}\overline{\delta}_y]. \tag{11.6.2}$$

The commutativity of our diagram is equivalent to the equality of (11.6.1) and (11.6.2), which follows from the fact that ϕ is a map of right $\pi_1(X, x_0)$-sets. □

11.7. Exercises

Exercise 11.1. WRITE IT!!! □

CHAPTER 12

Classifying covers: regular G-covers

CHAPTER 13

Calculating fundamental groups: preliminaries on group presentations

Our final topic is the Seifert–van Kampen theorem, which describes how the fundamental group of a space is built out of the fundamental groups of subspaces. Before we can describe this theorem, we need some preliminary results about group presentations.

13.1. Some basic groups

There are a vast range of groups that arise throughout mathematics. For instance:

EXAMPLE 13.1.1 (Cyclic groups). The cyclic groups include the infinite cyclic group $C_\infty \cong \mathbb{Z}$ and the finite cyclic group $C_n \cong \mathbb{Z}/n$ of order n. We will typically write the generator of these cyclic group by t, so $C_\infty = \{t^n \mid n \in \mathbb{Z}\}$ and $C_n = \{1, t, \ldots, t^{n-1}\}$. □

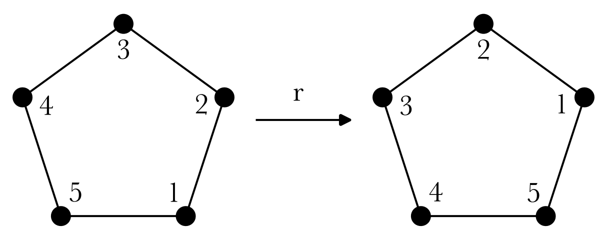
EXAMPLE 13.1.2 (Abelian groups). Let A be an abelian group. One possibility is that A might be cyclic. When we are thinking of the cyclic groups as abelian groups we will often write them as $\mathbb{Z}$ and $\mathbb{Z}/n$. If A is a finitely generated abelian group, then we can write

$$A \cong \mathbb{Z}^n \oplus \bigoplus_{i=1}^{k} \mathbb{Z}/p_i^{d_i}$$

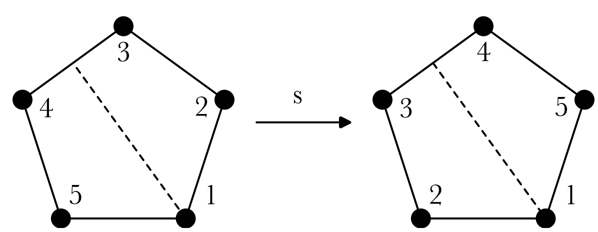
with $n \geq 0$ and $p_1, \ldots, p_k$ prime and $d_1, \ldots, d_k \geq 1$. However, if A abelian but not finitely generated then is no hope for any kind of simple classification. □

EXAMPLE 13.1.3 (Finite dihedral groups). For $n \geq 3$, the dihedral group D_{2n} of order $2n$ is the isometry group of a regular n-gon P_n. Enumerate the vertices of P_n counterclockwise as $[n] = \{1, \ldots, n\}$. The group D_{2n} acts on $[n]$, and an element $\sigma \in D_{2n}$ is determined by $\sigma(1) \in [n]$ and whether or not σ preserves the orientation of P_n. From this, we see that $D_{2n} = \{1, r, \ldots, r^{n-1}, s, rs, \ldots, r^{n-1}s\}$ where r and s are as follows:

- The element r is the counterclockwise rotation by $2\pi/n$:



- The element s is the reflection in the vertex $1 \in [n]$:



These elements satisfy the relations $r^n = 1$ and $s^2 = 1$ and $srs^{-1} = r^{-1}$. There is a homomorphism $\sigma\colon D_{2n} \to C_2$ that records whether or not an element preserves orientation, so $\sigma(r) = t^0 = 1 \in C_2$ and $\sigma(s) = t \in C_2$. We have $\ker(\sigma) = \{1, r, \ldots, r^{n-1}\} \cong C_n$, and σ splits via the map $C_2 \to D_{2n}$ taking $t \in C_2$ to s. We therefore have a semidirect product decomposition[1] $D_{2n} \cong C_n \rtimes C_2$ where the generator of C_2 acts on C_n by the automorphism taking $t^k \in C_n$ to t^{-k}. □

EXAMPLE 13.1.4 (Infinite dihedral group). The infinite dihedral group D_∞ is the group of isometries of $\mathbb{R}$ that preserve the subspace $\mathbb{Z}$. We have $D_\infty = \left\{r^k, sr^k \mid k \in \mathbb{Z}\right\}$, where:

- The element $r\colon \mathbb{R} \to \mathbb{R}$ is the translation map $r(x) = x + 1$.
- The element $s\colon \mathbb{R} \to \mathbb{R}$ is the reflection map $s(x) = -x$.

These elements satisfy the relations $s^2 = 1$ and $srs^{-1} = r^{-1}$. Just like for the finite dihedral groups, we have $D_\infty \cong C_\infty \rtimes C_2$ where C_2 acts on C_∞ by the automorphism taking t^k to t^{-k}. There are also surjections $f_n\colon D_\infty \to D_{2n}$ taking $r \in D_\infty$ to $r \in D_{2n}$ and $s \in D_\infty$ to $s \in D_{2n}$. □

EXAMPLE 13.1.5 (Symmetric groups). Fix $n \geq 2$. Let $[n] = \{1, \ldots, n\}$ and let $\mathfrak{S}_n$ be the symmetric group on $[n]$, i.e., the group of bijections $\sigma\colon [n] \to [n]$. The group $\mathfrak{S}_n$ contains every finite group G of order n as a subgroup. Indeed, enumerate G as

$$G = \{g_1, \ldots, g_n\}.$$

For $g \in G$, let $\sigma_g \in \mathfrak{S}_n$ be the permutation such that $gg_i = g_{\sigma_g(i)}$ for $1 \leq i \leq n$. The map $f\colon G \to \mathfrak{S}_n$ defined by $f(g) = \sigma_g$ for $g \in G$ is then an injective homomorphism whose image is isomorphic to G. More generally, if G is an arbitrary group then an action of G on the set $[n]$ is the same as a homomorphism $G \to \mathfrak{S}_n$. For instance, there is an injective homomorphism $D_{2n} \to \mathfrak{S}_n$ arising from the action of D_{2n} on the n vertices of the regular n-gon P_n. □

EXAMPLE 13.1.6 (Linear groups). For a field $\mathbf{k}$, the group $\mathrm{GL}_n(\mathbf{k}) = \mathrm{Aut}(\mathbf{k}^n)$ plays a basic role in linear algebra. It contains many interesting subgroups; for instance, the orthogonal groups of quadratic forms on $\mathbf{k}^n$. It also contains a copy of $\mathfrak{S}_n$ consisting of the permutation matrices. A subgroup of $\mathrm{GL}_n(\mathbf{k})$ for some n and $\mathbf{k}$ is called a *linear group*. For instance, since $\mathfrak{S}_n$ can be realized as a linear group and all finite groups are subgroups of $\mathfrak{S}_n$ for some n, it follows that all finite groups can be realized as linear groups. □

EXAMPLE 13.1.7 (Free groups). Let S be a set and let $F(S)$ be the free group on S (see §7.6). The group $F(S)$ is generated by S, and each element $w \in F(S)$ can be uniquely expressed as a reduced word $w = s_1^{\epsilon_1} \cdots s_n^{\epsilon_n}$ in S. Recall that this means that:

- $s_i \in S$ and $\epsilon_i \in \{\pm 1\}$ for all $1 \leq i \leq n$; and
- for all $1 \leq i < n$ we do not have $s_i^{\epsilon_i} s_{i+1}^{\epsilon_{i+1}} \in \left\{ss^{-1},\, s^{-1}s \mid s \in S\right\}$. □

13.2. Presentations for groups

The examples in the previous section barely scratch the surface of the world of groups. To write down an arbitrary group, we introduce the notation of a presentation. We start with some notation:

NOTATION 13.2.1. Let G be a group. For $g, h \in G$, write $g^h = h^{-1}gh$ for the conjugate of G by h. If $C \subset G$ is a subset, then $\langle C \rangle$ denotes the subgroup generated by C and $\langle\!\langle C \rangle\!\rangle$ denotes the normal subgroup generated by C. We therefore have $\langle\!\langle C \rangle\!\rangle = \langle c^g \mid c \in C \text{ and } g \in G \rangle$. □

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[1] Let G and H be groups such that G acts on H on the left. For $g \in G$ and $h \in H$, we will write ${}^g h$ for the image of h under the action of g. The corresponding semidirect product $H \rtimes G$ is the following group:

- The elements of $H \rtimes G$ are pairs (h, g) with $h \in H$ and $g \in G$.
- For $(h_1, g_1), (h_2, g_2) \in H \rtimes G$, their product is $(h_1, g_1)(h_2, g_2) = (h_1\, {}^{g_1}h_2, g_1 g_2)$.

It is enlightening to prove that this is a group. If the G-action on H is trivial, this is just the usual direct product $H \times G$. Identify H and G with the subgroups of $H \rtimes G$ consisting of elements of the form $(h, 1)$ and $(1, g)$, respectively. The subgroup H is normal, and every $x \in H \rtimes G$ can be uniquely written as $x = hg$ with $h \in H$ and $g \in G$.

Conversely, let Γ be a group and $H, G < \Gamma$ be subgroups such that H is normal and every $x \in \Gamma$ can be uniquely written as $x = hg$ with $h \in H$ and $g \in G$. Since H is a normal subgroup of Γ, the group Γ acts on H by conjugation. This restricts to an action of G on H, and $\Gamma \cong H \rtimes G$. This isomorphism takes $(h, g) \in H \rtimes G$ to $hg \in \Gamma$. One way this can arise is if Γ is a group and $p\colon \Gamma \to G$ is a surjection that splits via a map $\sigma\colon G \to \Gamma$. Identify G with its image in Γ under the injective map σ, and set $H = \ker(p)$. We are then in the above situation, so $\Gamma \cong H \rtimes G$.

This allows us to make the following key definition:

Definition 13.2.2. Let S be a set and let R be a subset of the free group $F(S)$ on S. Define $G = \langle S \mid R \rangle$ to be the quotient $F(S)/\langle\!\langle R \rangle\!\rangle$. We call S the *generators* and R the *relations* for the presentation, and $\langle S \mid R \rangle$ is called a *group presentation* for G. If both S and R are finite, then $\langle S \mid R \rangle$ is a *finite presentation* for G. A *finitely presented group* is a group with a finite presentation. $\square$

Example 13.2.3. For a set S, we have $F(S) = \langle S \mid \emptyset \rangle$. This is often written $\langle S \mid \, \rangle$. If S is finite, then this is a finite presentation and $F(S)$ is a finitely presented group. $\square$

Example 13.2.4. As a special case of the previous example, $C_\infty \cong \langle t \mid \, \rangle$. Similarly, for $n \geq 2$ we have $C_n \cong \langle t \mid t^n \rangle$. Both C_∞ and C_n are therefore finitely presented. $\square$

Remark 13.2.5. Not all presentations of finitely presented groups are finite. For instance, $C_\infty = \langle t \mid \, \rangle$ can also be written $C_\infty = \langle t, x_1, x_2, \ldots \mid x_1, x_2, \ldots \rangle$. $\square$

Every group has some presentation:

Lemma 13.2.6. *Every group G can be written as $G \cong \langle S \mid R \rangle$.*

Proof. Let S be a generating set for G; e.g., $S = G$. The map $S \to G$ extends to a surjection $\phi\colon F(S) \to G$. Set $R = \ker(\phi)$, so ϕ induces an isomorphism from $F(S)/R \cong \langle S \mid R \rangle$ to G. $\square$

Before we give more examples, we introduce some notation. Let $G = \langle S \mid R \rangle$ be a group equipped with a presentation. For $w \in F(S)$, we write $\overline{w}$ for its image in G. A *relation* in G is an element $r \in \langle\!\langle R \rangle\!\rangle \subset F(S)$. The relations of the presentation are thus relations in G, but except in degenerate cases there are many other relations in G. For instance, here are some relations in $G = \left\langle a, b \mid a^2, b^3 \right\rangle$:

$$a^{-2}, a^{10}, a^2 b^3 a^2, ab^3 a^{-1} a^2 = ab^3 a.$$

If $w, v \in F(S)$ are such that $\overline{w} = \overline{v}$, then wv^{-1} is a relation. We will sometimes write this relation as $w = v$. This convention will also be used when giving presentations. For instance,

$$\left\langle a, b, c \mid ab = c, b^2 = 1, cab = a \right\rangle = \left\langle a, b, c \mid abc^{-1}, b^2, caba^{-1} \right\rangle.$$

For $w, v \in F(S)$, we will also write $w \equiv v$ to mean that $\overline{w} = \overline{v}$, i.e., that $w = v$ is a relation.

13.3. Mapping from groups with presentations

Let $G = \langle S \mid R \rangle$ be a group given by a presentation. For any group H that is well-understood, it is easy to construct homomorphisms $\Phi\colon G \to H$. Indeed, choose a set map $\phi\colon S \to H$. The map ϕ extends to a homomorphism $\phi\colon F(S) \to H$. To check if ϕ descends to a homomorphism on $G = F(S)/\langle\!\langle R \rangle\!\rangle$, we must only verify that $\phi(r) = 1$ for all $r \in R$. In summary, to construct a homomorphism $\Phi\colon G \to H$ we must choose where the generators go and then verify that relations go to relations. Here is an example of this:

Example 13.3.1. Let G be a group. Let S be the set of formal symbols $\{s_g \mid g \in G\}$ and let $R = \{s_g s_h = s_{gh} \mid g, h \in G\}$. Set $\Gamma = \langle S \mid R \rangle$. We claim that $\Gamma \cong \langle S \mid R \rangle$. Indeed, the map $s_g \mapsto g$ gives a map $S \to G$ taking each relation $s_g s_h = s_{gh}$ to a relation in G. We thus get a surjective map $\Phi\colon \Gamma \to G$.

To see that Φ is injective, consider $w \in F(S)$. For $g \in G$ the relation $s_g^{-1} \equiv s_{g^{-1}}$ holds in Γ; indeed, $s_1 \equiv 1$ since $s_1 s_1 \equiv s_1$, so $s_g s_{g^{-1}} \equiv s_1 \equiv 1 \equiv s_g s_g^{-1}$. It follows that w is equivalent to a word in S in which only positive powers of generators appear. Using the relations in Γ, we then see that w is equivalent to a single generator s_g. We conclude that $\Gamma = \{\overline{s}_g \mid g \in G\}$. Since Φ takes $\{\overline{s}_g \mid g \in G\}$ bijectively to G, we conclude that Φ is injective. $\square$

Remark 13.3.2. If G is a finite group, then the presentation from Example 13.3.1 is finite and thus G is finitely presented. $\square$

13.4. Normal forms

We abstract the argument from Example 13.3.1. Let G be a group with a generating set S. For each $g \in G$, pick $w_g \in F(S)$ such that w_g maps to g under the natural map $F(S) \to G$. The collection of elements $\{w_g \in F(S) \mid g \in G\}$ is called a *normal form* for G. Assume now that $R \subset F(S)$ is a collection of relations that hold in G. Letting $\Gamma = \langle S \mid R\rangle$, we want to prove that $\Gamma \cong G$.

Since S generates G and each relation in R holds in G, we get a surjective homomorphism $\Phi\colon \Gamma \to G$. We want to prove that Φ is an isomorphism. Observe that Φ restricts to a bijection from $\{\overline{w}_g \in F(S) \mid g \in G\}$ to G. It is therefore enough to prove that $\{\overline{w}_g \in F(S) \mid g \in G\} = \Gamma$. In other words, we must prove that every $w \in F(S)$ can be reduced to some w_g using the relations in R. Here are some examples:

Example 13.4.1 (Free abelian group). Fix $n \geq 1$. For $1 \leq i \leq n$, let $x_i \in \mathbb{Z}^n$ be the element with a 1 in position i and zeros elsewhere. The elements $S = \{x_1, \ldots, x_n\}$ generate $\mathbb{Z}^n$. They satisfy the relations

$$R = \{x_i x_j = x_j x_i \mid 1 \leq i, j \leq n\}.$$

We claim that $\mathbb{Z}^n \cong \langle S \mid R\rangle$. Indeed, $\mathbb{Z}^n$ has the normal form $\left\{x_1^{d_1} \cdots x_n^{d_n} \mid d_1, \ldots, d_n \in \mathbb{Z}\right\}$. We must prove that an arbitrary $w \in F(S)$ can be reduced to an element in this normal form. For this, use the relations in R to move the x_1 terms to the left, then the x_2 terms to the left, etc. Here is an example:[2]

$$x_3 x_2^{-1} x_1 x_2^2 x_1^{-3} \equiv x_1 x_1^{-3} x_3 x_2^{-1} x_2^2 \equiv x_1 x_1^{-3} x_2^{-1} x_2^2 x_3 \equiv x_1^{-2} x_2 x_3. \qquad \square$$

Example 13.4.2 (Finite dihedral group). Fix $n \geq 3$. As in Example 13.1.3, let D_{2n} be the dihedral group of order $2n$. This group is generated by the rotation $r \in D_{2n}$ and the reflection $s \in D_{2n}$. Set $S = \{r, s\}$. The elements r and s satisfy the relations

$$R = \{r^n = 1, s^2 = 1, srs^{-1} = r^{-1}\}.$$

We claim that $D_{2n} \cong \langle S \mid R\rangle$. Indeed, as we observed in Example 13.1.3 the group D_{2n} has the normal form $\left\{r^d, sr^d \mid 0 \leq d \leq n-1\right\}$. We must prove that an arbitrary $w \in F(S)$ can be reduced to an element in this normal form. Using the relation $s^2 \equiv 1$, we can replace all s^{-1} terms in w with s. Also using $s^2 \equiv 1$, the relation $srs^{-1} \equiv r^{-1}$ can be rearranged to $rs \equiv sr^{-1}$. This implies that we also have $r^{-1}s \equiv sr$. Applying all of these, we can pull all the s-terms in w to the left and reduce w to $s^e r^d$ as in the following example:

$$sr^{-1}sr^3sr^2 \equiv ssr^{-1}r^3sr^2 \equiv sssr^{-1}r^3r^2 = s^3r^4.$$

Using the fact that $s^2 \equiv 1$ and $r^n \equiv 1$, we see that $s^e r^d$ is equivalent to either r^d or sr^d with $0 \leq d \leq n-1$, as desired. $\square$

Example 13.4.3 (Infinite dihedral group). An argument identical to the one in the previous example shows that $D_\infty \cong \left\langle r, s \mid s^2 = 1, srs^{-1} = r^{-1}\right\rangle$. $\square$

Remark 13.4.4. See Exercise 13.2 for a presentation of the symmetric group $\mathbb{S}_n$. $\square$

13.5. Free products and direct products

Let $G_1 = \langle S_1 \mid R_1\rangle$ and $G_2 = \langle S_2 \mid R_2\rangle$. Recall that the free product Γ of G_1 and G_2 is a group with subgroups $G_1, G_2 < \Gamma$ satisfying the following universal property:

- Let H be a group and let $f_1\colon G_1 \to H$ and $f_2\colon G_2 \to H$ be homomorphisms. Then there is a unique homomorphism $F\colon \Gamma \to H$ whose restrictions to G_1 and G_2 are f_1 and f_2, respectively.

Lemma 13.5.1. *Let $G_1 = \langle S_1 \mid R_1\rangle$ and $G_2 = \langle S_2 \mid R_2\rangle$. Set $\Gamma = \langle S_1 \sqcup S_2 \mid R_1 \sqcup R_2\rangle$. Then Γ is a free product of G_1 and G_2.*

[2] Here we are also using relations like $x_i^{-1}x_j \equiv x_j x_i^{-1}$, which can be obtained from $x_j x_i \equiv x_i x_j$ by multiplying both sides on the left by x_i^{-1} and on the right by x_i^{-1}.

PROOF. The set map

$$S_1 \hookrightarrow S_1 \sqcup S_2 \to \langle S_1 \sqcup S_2 \mid R_1 \sqcup R_2 \rangle = \Gamma$$

induces a homomorphism $G_1 \to \Gamma$. This splits via the map $\Gamma \to G_1$ taking S_1 identically to S_1 and taking S_2 to 1. This makes G_1 a subgroup of Γ. Similarly, G_2 is a subgroup of Γ. We will verify that Γ satisfies the universal property of a free product.

Let H be a group and $f_1 \colon G_1 \to H$ and $f_2 \colon G_2 \to H$ be homomorphisms. For $i = 1, 2$, since $G_i = \langle S_i \mid R_i \rangle$ the map $f_i \colon G_i \to H$ is induced by a homomorphism $g_i \colon F(S_i) \to H$. By restricting g_i to S_i, we get a set map $g_i \colon S_i \to H$. Combining these, we get a set map $g \colon S_1 \sqcup S_2 \to H$ and thus a homomorphism $g \colon F(S_1 \sqcup S_2) \to H$. Since g_i takes each element of R_i to a relation in H, this descends to a homomorphism $F \colon \Gamma \to H$, as desired. □

REMARK 13.5.2. Since all groups have presentations, this proves in particular that free products of groups always exist. □

As far as the direct product goes, we have:

LEMMA 13.5.3. *Let $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$. Let R_C be the following set of relations in $F(S_1 \sqcup S_2)$:*

$$R_C = \{s_1 s_2 = s_2 s_1 \mid s_1 \in S_1 \text{ and } s_2 \in S_2\}.$$

Set $\Gamma = \langle S_1 \sqcup S_2 \mid R_1 \sqcup R_2 \sqcup R_C \rangle$. Then $\Gamma \cong G_1 \times G_2$.

PROOF. This can be proved using normal forms; see Exercise 13.5.3. □

13.6. Abelianization

As we will elaborate on below, it is hard to determine much about a group G from a presentation. The only easy thing to calculate is its abelianization G^{ab}, which we recall is the largest abelian group on which G surjects. Letting $[G, G]$ be the commutator subgroup[3] of G, we have $G^{\mathrm{ab}} = G/[G, G]$. For instance, if S is an n-element set, then $F(S)^{\mathrm{ab}} \cong \mathbb{Z}^n$ (see Exercise 7.12).

Assume that G has a finite presentation $G = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$. Let $V \cong \mathbb{Z}^n$ be the free abelian group with basis $\{X_1, \ldots, X_n\}$. For $1 \leq i \leq m$, let $R_i \in V$ be the following element:

- Write $r_i = x_{j_1}^{e_1} \cdots x_{j_k}^{e_k}$ with $1 \leq j_1, \ldots, j_k \leq n$ and $e_1, \ldots, e_k \in \{\pm 1\}$. Then $R_i = e_1 X_{j_1} + \cdots + e_k X_{j_k} \in V$.

It is immediate from the definitions that G^{ab} is the quotient of V by $\langle R_1, \ldots, R_m \rangle$. As is discussed in most treatements of the classification of finitely generated abelian groups, this quotient can be calculated using tools like Smith normal form. However, it is often easier to work with it directly. Here are some examples.

EXAMPLE 13.6.1 (Infinite dihedral group). As we saw in Example 13.4.3, we have

$$D_\infty \cong \left\langle r, s \mid s^2 = 1, srs^{-1} = r^{-1} \right\rangle.$$

Following the above recipe, the abelianization of D_∞ is the quotient of the free abelian group with basis R and S by the following two relations:[4]

- $2S = 0$ and $S + R - S = -R$. This second relation can be rewritten $2R = 0$.

We thus see that the abelianization of D_∞ is $(\mathbb{Z}/2) \oplus (\mathbb{Z}/2)$. □

EXAMPLE 13.6.2 (Finite dihedral group). Fix $n \geq 3$. As we saw in Example 13.4.2, we have

$$D_{2n} = \left\langle r, s \mid r^n = 1, s^2 = 1, srs^{-1} = r^{-1} \right\rangle.$$

Following the above recipe, the abelianization of D_{2n} is the quotient of the free abelian group with basis R and S by the relations $nR = 0$ and $2S = 0$ and $S + R - S = -R$. This last relation can be rearranged to $2R = 0$. From this, we see that there are two cases:

(a) If n is even, then the relations reduce to $2S = 0$ and $2R = 0$, so the abelianization is $(\mathbb{Z}/2) \oplus (\mathbb{Z}/2)$.

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[3] The commutator subgroup $[G, G]$ is the subgroup generated by commutators $[g, h] = ghg^{-1}h^{-1}$.

[4] Here quotienting an abelian group by a relation $A = B$ should be interpreted as quotienting by $A - B$.

(b) If n is odd, then $nR = 0$ and $2R = 0$ combine to give that $R = 0$, so the abelianization is $\mathbb{Z}/2$. □

13.7. Some cautionary examples

As we said in the last section, it is not easy to extract information from a group presentation. This section contains a number of cautionary examples. The first is as follows:

EXAMPLE 13.7.1. Consider the group $G = \langle a, b \mid babab = 1\rangle$. What kind of group is this? We first determine its abelianization. Following the recipe in the previous section, G^{ab} is the quotient of the free abelian group with basis $\{A, B\}$ subject to the relation $B + A + B + A + B = 0$, or $2A + 3B = 0$. After making the change of basis $A = X - Y$ and $B = Y$, this relation becomes $2(X - Y) + 3Y = 0$, or $-2X = Y$. Eliminating the variable Y, we conclude that $G^{\mathrm{ab}} \cong \mathbb{Z}$.

In fact, it turns out that G is an abelian group, so $G \cong \mathbb{Z}$. To see this, note that we can conjugate $babab \equiv 1$ by $(bab)^{-1}$ and get

$$1 \equiv (bab)^{-1}(babab)(bab) = abbab.$$

It follows that

$$1 \equiv (abbab)(babab)^{-1} \equiv (abbab)(b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}) = aba^{-1}b^{-1},$$

so a and b commute and G is indeed abelian. □

REMARK 13.7.2. Another way to see that $G = \langle a, b \mid babab = 1\rangle$ is isomorphic to $\mathbb{Z}$ is as follows. Let $H = \langle x, y \mid x^2 y = 1\rangle$. It is clear that $H \cong \mathbb{Z}$ since $y \equiv x^{-2}$. Define $f\colon H \to G$ via the formulas

$$f(x) = ba \quad \text{and} \quad f(y) = b.$$

This works since f takes the relation $x^2y = 1$ to $(ba)^2 b = 1$, which is the relation in G. The map f is an isomorphism with inverse the map $g\colon G \to H$ defined by

$$g(a) = y^{-1}x \quad \text{and} \quad g(b) = y.$$

Again, this works since g takes the relation $babab = 1$ to $y(y^{-1}x)y(y^{-1}x)y = 1$, which reduces to the relation $x^2y = 1$ in H. □

For our next examples, for $n \geq 1$ define

$$G_n = \left\langle x_1, \ldots, x_n \mid x_{i+1}x_i x_{i+1}^{-1} = x_i^2 \text{ for } 1 \leq i \leq n \right\rangle.$$

Here the subscripts should be taken modulo n. These are called the *Higman groups*. All of these groups have trivial abelianizations:

LEMMA 13.7.3. *For all $n \geq 1$, we have $G_n^{\mathrm{ab}} = 0$.*

PROOF. The abelianization G_n^{ab} is the free abelian group with basis $X_1, \ldots, X_n$ modulo the following relations:

- For $1 \leq i \leq n$, we have $X_{i+1} + X_i - X_{i+1} = 2X_i$. This reduces to $X_i = 0$.

The lemma follows. □

The first three G_n are trivial:

LEMMA 13.7.4. *We have $G_1 = 1$.*

PROOF. We have $G_1 \cong \langle x_1 \mid x_1 x_1 x_1^{-1} = x_1^2\rangle = \langle x_1 \mid x_1 = 1\rangle = 1$. □

LEMMA 13.7.5. *We have $G_2 = 1$.*

PROOF. Rewrite the relations in G_2 as $x_2x_1 \equiv x_1^2x_2$ and $x_1x_2 \equiv x_2^2x_1$. These imply that

$$x_1^2x_2^2 \equiv x_2x_1x_2 \equiv x_2^3x_1 \equiv x_1^8x_2^3,$$

so $x_2 \equiv x_1^{-6}$. We conclude that x_1 and x_2 commute, so G_2 is abelian and $G_2 \cong G_2^{\mathrm{ab}} = 1$. □

LEMMA 13.7.6. *We have $G_3 = 1$.*

PROOF. The proof is a more elaborate version of the calculation we used to prove Lemma 13.7.6, so we leave it as Exercise 13.5. □

However, it turns out that G_n is infinite for $n \geq 4$. We will discuss the proof in Essay B. The reason this proof is difficult is that it is hard to map G_n to other well-understood groups. In particular, we have the following theorem of Higman:

THEOREM 13.7.7. *For $n \geq 4$, any homomorphism from G_n to a finite group is trivial.*

PROOF. Let H_n be the image of G_n in some finite group. Each element of H_n has finite order. We can therefore write $H_n = \langle y_1, \ldots, y_n \mid R \rangle$ where R contains the following relations:

- For $1 \leq i \leq n$, the relation $y_{i+1} y_i y_{i+1}^{-1} = y_i^2$. Here the subscripts should be interpreted modulo n.
- For $1 \leq i \leq n$, a relation of the form $y_i^{d_i} = 1$ for some $d_i \geq 1$. Choose the d_i to be as small as possible, so d_i is the order of y_i.

There might also be some other relations. We will prove that these relations force H_n to be trivial.

Assume that H_n is nontrivial. We must have $d_i \geq 2$ for all i. Indeed, if $d_{i_0} = 1$ for some i_0, then

$$y_{i_0-1} = y_{i_0} y_{i_0-1} y_{i_0}^{-1} = y_{i_0-1}^2,$$

so $y_{i_0-1} = 1$ and thus $d_{i_0-1} = 1$. Iterating this, we deduce that $d_i = 1$ for all i and hence that $H_n = 1$, contrary to our assumptions.

Let $p \geq 2$ be the smallest prime dividing some d_i. Since everything is invariant under cyclic permutations of the generators, we can assume without loss of generality that p divides d_1. Since d_2 is the order of y_2, we have

$$y_1 = y_2^{d_2} y_1 y_2^{-d_2} = y_1^{2^{d_2}}$$

and thus $y_1^{2^{d_2}-1} = 1$. Since d_1 is the order of y_1, it follows that d_1 and hence p divides $2^{d_2} - 1$. This implies that p is odd. Since $2^{d_2} \equiv 1 \pmod{p}$, we see that d_2 is a multiple of the order $p-1$ of 2 in $(\mathbb{Z}/p)^\times$. This implies that a prime smaller than p divides d_2, contradicting the fact that p is the smallest prime dividing some d_i. □

REMARK 13.7.8. Malcev proved that if $\mathbf{k}$ is a field and H is a nontrivial finitely generated subgroup of $\mathrm{GL}_m(\mathbf{k})$, then H has many nontrivial finite quotients. In fact, H is *residually finite*: for any $h \in H$ with $h \neq 1$, there exists a finite group F and a homomorphism $f\colon H \to F$ with $f(h) \neq 1$. It therefore follows from Theorem 13.7.7 that all homomorphisms $f\colon G_n \to \mathrm{GL}_m(\mathbf{k})$ are trivial for $n \geq 4$. We outline a simple proof of this for $\mathbf{k} = \mathbb{C}$ in Exercise 13.6. □

13.8. Decision problems for groups

In 1911, Dehn posed the following three problems about group presentations.

PROBLEM 13.8.1 (Word problem). *Let $G = \langle S \mid R \rangle$ be a finitely presented group. Give an algorithm that for $w, v \in F(S)$ determines whether or not $\overline{w} = \overline{v}$ in G.*

PROBLEM 13.8.2 (Conjugacy problem). *Let $G = \langle S \mid R \rangle$ be a finitely presented group. Give an algorithm that for $w, v \in F(S)$ determines whether or not $\overline{w} \in G$ and $\overline{v} \in G$ are conjugate.*[5]

PROBLEM 13.8.3 (Isomorphism problem). *Give an algorithm that for finitely presented group $G = \langle S \mid R \rangle$ and $G' = \langle S' \mid R' \rangle$ determines whether or not G and G' are isomorphic.*

Here are some simple observations about the relationship between these problems.

(a) Let $G = \langle S \mid R \rangle$ be a finitely presented group. To solve the word problem for G, it is enough to give an algorithm that for $w \in F(S)$ determines whether or not $\overline{w} = 1$. Indeed, for $w, v \in F(S)$ such an algorithm allows us to determine whether or not $\overline{wv^{-1}} = 1$, and thus whether or not $\overline{w} = \overline{v}$.
(b) Let $G = \langle S \mid R \rangle$ be a finitely presented group. If we can solve the conjugacy problem for G, then we can also solve the word problem for G. Indeed, an element $w \in F(S)$ satisfies $\overline{w} = 1$ if and only if $\overline{w}$ is conjugate to 1.

[5] Recall that $g, h \in G$ are conjugate if there exists some $k \in G$ such that $g = khk^{-1}$.

(c) A special case of the isomorphism problem is the triviality problem: give an algorithm that for a finitely presented group $G = \langle S \mid R \rangle$ determines whether or not $G = 1$. The group satisfies $G = 1$ if and only if for all $s \in S$ we have $\overline{s} = 1$, so a solution to the word problem gives a solution to the triviality problem.

While these problems can be solved in many special cases, they are quite difficult and our knowledge about them is limited. For instance, consider a one-relator group G, that is, a group of the form $G = \langle S \mid r \rangle$ where S is finite and $r \in F(S)$. In 1932, Magnus showed how to solve the word problem for one-relator groups. However, we still do not know how to solve the conjugacy problem for one-relator groups. We also do not know an algorithm to determine if two one-relator groups are isomorphic. For groups with two relations, even the word problem is open.

One reason that these problems are difficult is that they are unsolvable in general. Indeed, in the 1950's Novikov and Boone independently proved the following theorem:

THEOREM 13.8.4 (Novikov–Boone). *There exists a finitely presented group $G = \langle S \mid R \rangle$ for which there does not exist an algorithm solving the word problem.*

REMARK 13.8.5. To make sense of Theorem 13.8.4, the notion of an "algorithm" must be formally defined. Roughly speaking, an "algorithm" here means a program in any standard computer language (C, Python, LISP, etc.) run on a computer with unlimited memory. This algorithm must terminate in finite time for any input. The formal definition involves the notion of a Turing machine, and is discussed in any book on computability theory. □

One observation about Theorem 13.8.4 is that the hard part in it is checking that some $w \in F(S)$ does not represent the identity. More precisely, we have the following:

LEMMA 13.8.6. *Let $G = \langle S \mid R \rangle$ be a finitely presentable group. There exists an algorithm that takes as input $w \in F(S)$ and does the following:*

- *If $\overline{w} = 1$, then the algorithm terminates and certifies that $\overline{w} = 1$.*
- *If $\overline{w} \neq 1$, then the algorithm does not terminate.*

PROOF. Each $w \in F(S)$ can be uniquely be written as a reduced word $w = s_1^{\epsilon_1} \cdots s_n^{\epsilon_n}$ with $s_i \in S$ and $\epsilon_i \in \{\pm 1\}$ for all $1 \leq i \leq n$, and we define the *length* of w to be $\ell(w) = n$. Set $B_n(S) = \{w \in F(S) \mid \ell(w) \leq n\}$. Define $B_n(R)$ to be the set of all elements of $F(S)$ that for some $m \leq n$ can be expressed as

$$(w_1 r_1 w_1^{-1})^{\epsilon_1} \cdots (w_m r_m w_m^{-1})^{\epsilon_m}$$

with $r_i \in R$ and $w_i \in B_n(S)$ and $\epsilon_i \in \{\pm 1\}$ for $1 \leq i \leq m$. Note that the above is not necessarily a reduced word. The set $B_n(R)$ is a finite set, and can be effectively enumerated on a computer. We have

$$B_1(R) \subset B_2(R) \subset B_3(R) \subset \cdots,$$

and the normal closure $\langle\langle R \rangle\rangle$ is $\cup_{n=1}^{\infty} B_n(R)$. Our algorithm is as follows:

- Start at Step $n = 1$.
- At Step n, enumerate $B_n(R)$ and check whether or not $w \in B_n(R)$. If it does, terminate. Otherwise, go on to Step $n + 1$.

This terminates if and only if $w \in \langle\langle R \rangle\rangle$, i.e., if and only if $\overline{w} = 1$. □

We close with the following observation whose proof illustrates the fact that Theorem 13.8.4 touches on deep logical issues that seemingly have nothing to do with computers:

THEOREM 13.8.7. *Let $G = \langle S \mid R \rangle$ be the group with an unsolvable word problem from Theorem 13.8.4. There exists some $w \in F(S)$ with the following two properties:*

- *We have $\overline{w} \neq 1$ in G.*
- *There is no proof in ZFC[6] that $\overline{w} \neq 1$ in G.*

[6] There is nothing special about ZFC, and the same argument works in other foundational systems that are rich enough to express facts about group theory and unsolvability. We focus on ZFC since that is the set of foundations we are assuming in this book.

PROOF. Consider $w \in F(S)$. Using two computers (or two threads on one computer), run the following two algorithms on w at the same time, terminating if either of the two terminate:

- Computer one runs the algorithm from Lemma 13.8.6, and thus terminates if and only if $\overline{w} = 1$ in G.
- Computer two systematically enumerates all possible proofs in ZFC. For each proof, it first determines whether or not it is a valid proof. If it is, it checks whether it proves that $\overline{w} \neq 1$ in G. If it does, then this algorithm terminates. Otherwise, it keeps going.

This algorithm will terminate if $\overline{w} = 1$ in G or if there is a proof in ZFC that $\overline{w} \neq 1$ in G. Since there is no algorithm to solve the word problem in G, there must be some $w \in F(S)$ for which this algorithm does not terminate. The theorem follows. $\square$

REMARK 13.8.8. Theorem 13.8.7 is not effective, and there is no way to determine which $w \in F(S)$ satisfies its conclusions since doing so would in particular provide a proof that $\overline{w} \neq 1$. $\square$

13.9. Exercises

EXERCISE 13.1. Let

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}.$$

The group H is often called the integer Heisenberg group.

(a) Set

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

As notation, set $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$. Prove that $H = \langle x, y \mid [[x, y], x] = 1, [[x, y], y] = 1 \rangle$. Hint: work out a normal form.

(b) Calculate the abelianization of H. $\square$

EXERCISE 13.2. Fix $n \geq 2$. Let $\mathfrak{S}_n$ be the symmetric group on $[n] = \{1, \ldots, n\}$. For $1 \leq i \leq n-1$, let $\tau_i \in \mathfrak{S}_n$ be the transposition $(i, i+1)$. Let

$$\begin{aligned} S &= \{\tau_i \mid 1 \leq i \leq n-1\}, \\ R &= \left\{\tau_i^2 = 1 \mid 1 \leq i \leq n-1\right\} \cup \{\tau_i \tau_j = \tau_j \tau_i \mid 1 \leq i, j \leq n,\ |i-j| \geq 2\} \\ &\quad \cup \{\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \mid 1 \leq i \leq n-2\}. \end{aligned}$$

Prove the following:

(a) The group $\mathfrak{S}_n$ is generated by S.
(b) Each element of R is a relation in $\mathfrak{S}_n$.
(c) We have $\mathfrak{S}_n = \langle S \mid R \rangle$. Hint: For $1 \leq i \leq j \leq n$, let $\sigma_{i,j} = \tau_{j-1} \tau_{j-1} \cdots \tau_i$. For $i = j$, our convention is that $\sigma_{i,j} = 1$. The key feature of $\sigma_{i,j}$ is that as an element of $\mathfrak{S}_n$, it takes $i \in [n]$ to $j \in [n]$. Prove that $\mathfrak{S}_n$ has the normal form

$$\left\{\sigma_{n-1, i_{n-1}} \sigma_{n-2, i_{n-2}} \cdots \sigma_{1, i_1} \mid i_j \geq j \text{ for } 1 \leq j \leq n-1\right\}.$$

Use this normal form to prove the result.
(d) Calculate the abelianization of $\mathfrak{S}_n$. $\square$

EXERCISE 13.3. Let $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$. Let R_C be the following set of relations in $F(S_1 \sqcup S_2)$:

$$R_C = \{s_1 s_2 = s_2 s_1 \mid s_1 \in S_1 \text{ and } s_2 \in S_2\}.$$

Set $\Gamma = \langle S_1 \sqcup S_2 \mid R_1 \sqcup R_2 \sqcup R_C \rangle$. Prove that $\Gamma \cong G_1 \times G_2$. Hint: use normal forms. $\square$

EXERCISE 13.4. Let $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$. Assume that G_2 acts on G_1 on the left. Determine a presentation for the resulting semidirect product $G_1 \rtimes G_2$. Hint: For each $s \in S_1$ and $t \in S_2$, start by writing ${}^t s$ as a word $w_{s,t}$ in S_1. $\square$

Exercise 13.5. Recall that the third Higman group is

$$G_3 = \left\langle x_1, x_2, x_3 \mid x_2 x_1 x_2^{-1} = x_1^2, x_3 x_2 x_3^{-1} = x_2^2, x_1 x_3 x_1^{-1} = x_3^2 \right\rangle.$$

Prove that $G_3 = 1$. Hint: Start by applying the relations to $x_1^2 x_2^2 x_3$ to find an identity that lets you write x_3 in terms of x_1 and x_2. Plug this into $x_3 x_2 x_3^{-1} \equiv x_2^2$ and manipulate the result to show that x_2 equals a power of x_1, and hence that x_2 and x_1 commute. $\square$

Exercise 13.6. Recall from Remark 13.7.8 that if $\mathbf{k}$ is a field and G_n is the Higman group, then all homomorphisms $f\colon G_n \to \mathrm{GL}_m(\mathbf{k})$ are trivial. This exercise explains how to prove this for $\mathbf{k} = \mathbb{C}$. Let $H < \mathrm{GL}_m(\mathbb{C})$ be the image of such a homomorphism. Let $y_1, \ldots, y_n \in H$ be the images of the generators $x_1, \ldots, x_n$ of G_n. By Theorem 13.7.7, it is enough to prove that y_i has finite order for all $1 \leq i \leq n$. Do the following:

(a) Prove that all the eigenvalues of each y_i are roots of unity. Hint: use the relation $y_{i+1} y_i y_{i+1} = y_i^2$.
(b) Prove that there is some polynomial $p \in \mathbb{C}[z]$ such that for all $1 \leq i \leq n$ and all $k \geq 1$, if $c \in \mathbb{C}$ is one of the entries of the matrix y_i^k then $|c| \leq p(k)$. Hint: use Jordan normal form along with (a).
(c) Prove that the matrix y_i is diagonalizable for $1 \leq i \leq n$. Hint: think about the Jordan normal form of y_i, and consider the identity $p_{i+1}^k p_i p_{i+1}^{-k} = p_i^{2^k}$. You will use part (b).
(d) Prove that the matrix y_i has finite order for $1 \leq i \leq n$. $\square$

Exercise 13.7. Recall that a group G is residually finite if for all $g \in G$ with $g \neq 1$, there exists a finite group F and a homomorphism $f\colon G \to F$ with $f(g) \neq 1$. Prove the following:

(a) The group $G = \mathrm{GL}_n(\mathbb{Z})$ is residually finite. Hint: think about reducing matrices modulo p for various primes p.
(b) Let G be a residually finite group and let $G' < G$ be a subgroup. Prove that G' is residually finite.
(c) Let G be a finitely generated group. For all $g \in G$ with $g \neq 1$, assume that there exists a finite-index subgroup $H < G$ with $g \notin H$. Prove that G is residually finite. Hint: the problem is that H might not be normal, so G/H might not be a group. Try to intersect conjugates of H to get a smaller finite-index normal subgroup.
(d) Let $F(S)$ be a free group on a finite set S. Prove that $F(S)$ is residually finite. Hint: it might be easier to first use part (b) to reduce to the case of the free group $F(a, b)$ on a and b. For $w \in F(a, b)$ with $w \neq 1$, try to use covering spaces to produce a finite-index subgroup H of $F(a, b)$ as in part (c).
(e) Let $G = \langle S \mid R \rangle$ be a finitely generated residually finite group. Prove that there is an algorithm to solve the word problem in G. Hint: just like in the proof of Theorem 13.8.7, it is enough to produce an algorithm that terminates for $w \in F(S)$ with $w \neq 1$. Try to systematically list all finite groups F and all homomorphisms $f\colon G \to F$. One key observation is that a group structure on a finite set X is determined by its multiplication table. $\square$

CHAPTER 14

Calculating fundamental groups: the Seifert–van Kampen theorem

Part 2

Essays on applications

ESSAY A

The unsolvability of the quintic (to be written)

ESSAY B

Topological methods in combinatorial group theory (to be written)

This essay will give a number of applications of topological reasoning in low-dimensional topology, including:

- HNN extensions and amalgamated free products, including Britton's lemma and the fact that the factors of these groups embed as subgroups. As an interesting example of these, I will talk about the Baumslag–Solitar groups and the Higman groups.
- A bunch of applications of the previous section to embedding problems.
- Stallings's proofs of the Kurosh subgroup theorem and Grushko's theorem.

ESSAY C

The standard construction of universal covers (to be written)