

Yet another book on topology I: Point-set topology

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Introduction (to be written)

Part 1

Basic topics

CHAPTER 1

From Euclidean space to metric spaces

We first describe the naive notion of a space. We then make an initial attempt to formalize this via metric spaces and discuss the ways in which this is unsatisfactory.

1.1. Naive spaces

The most familiar spaces are $\mathbb{R}^n$ and its subspaces. Indeed, since we live in $\mathbb{R}^3$ our drawings necessarily lie in $\mathbb{R}^3$. For instance:



We can imagine subspaces of $\mathbb{R}^n$ for $n \geq 4$ by analogy with $\mathbb{R}^3$. These are the geometric objects studied by mathematicians going back to the ancient Greeks.

Modern formalizations of the notion of “space” give a precise language for talking about these spaces and extending our geometric imagination to spaces that are less easily visualized. However, it is important to keep in mind that mathematicians have been studying geometry for thousands of years. The formal language might change and the scope of the field might expand, but it is still the same subject.

1.2. Metric spaces

Perhaps the easiest modern formalization is the notion of a metric space. A *metric space* is a pair $(M, \mathfrak{d})$ where M is a set and $\mathfrak{d}$ is a distance function $\mathfrak{d}: M \times M \rightarrow \mathbb{R}$ such that:

- For all $p, q \in M$, we have $\mathfrak{d}(p, q) \geq 0$ with equality if and only if $p = q$.
- For all $p, q \in M$, we have $\mathfrak{d}(p, q) = \mathfrak{d}(q, p)$.
- For all $p, q, r \in M$, we have the triangle inequality $\mathfrak{d}(p, q) \leq \mathfrak{d}(p, r) + \mathfrak{d}(r, q)$.

Sometimes we will not mention $\mathfrak{d}$ and just say that M is a metric space. Here are some examples:

EXAMPLE 1.2.1. Let $\|\cdot\|$ be the usual norm on $\mathbb{R}^n$:

$$\|(x_1, \dots, x_n)\| = \sqrt{x_1^2 + \dots + x_n^2} \quad \text{for all } (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Define a distance function $\mathfrak{d}: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ via the formula

$$\mathfrak{d}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

The pair $(\mathbb{R}^n, \mathfrak{d})$ is a metric space. We call this the *Euclidean metric* on $\mathbb{R}^n$. □

EXAMPLE 1.2.2. If $(M, \mathfrak{d})$ is a metric space and $M' \subset M$ is a subspace, then $\mathfrak{d}' = \mathfrak{d}|_{M' \times M'}$ is a distance function on M' making $(M', \mathfrak{d}')$ into a metric space. We call M' a *metric subspace* (or sometimes just a *subspace*) of M . □

EXAMPLE 1.2.3. Combining the previous two examples, let $M \subset \mathbb{R}^n$. Endow $\mathbb{R}^n$ with the Euclidean metric. Viewing M as a metric subspace of $\mathbb{R}^n$, we get a metric space $(M, \mathfrak{d})$. We call $\mathfrak{d}$ the *Euclidean metric* on M . The most important spaces in algebraic topology are either subspaces of $\mathbb{R}^n$ themselves or are constructed from subspaces of $\mathbb{R}^n$ by geometric operations (gluing, taking quotients, etc.). □

Spaces of functions provide other important examples. For instance:

EXAMPLE 1.2.4. Let $I = [0, 1]$ be the closed interval and let $\mathcal{C}(I, \mathbb{R})$ be the set of all continuous functions $f: I \rightarrow \mathbb{R}$. Define a metric on $\mathcal{C}(I, \mathbb{R})$ as follows:

$$\mathfrak{d}(f, g) = \max \{|f(x) - g(x)| \mid x \in I\} \quad \text{for all continuous } f, g: I \rightarrow \mathbb{R}.$$

Since I is a closed interval, this maximum makes sense. This makes $\mathcal{C}(I, \mathbb{R})$ into a metric space. $\square$

EXAMPLE 1.2.5. Let $\mathcal{C}(\mathbb{R}, \mathbb{R})$ be the set of all continuous functions $f: \mathbb{R} \rightarrow \mathbb{R}$. Since $\mathbb{R}$ is not a closed interval, constructing a metric on $\mathcal{C}(\mathbb{R}, \mathbb{R})$ is a little subtle and there are several possible choices. One possible metric is as follows:

$$\mathfrak{d}(f, g) = \max \{\min(|f(x) - g(x)|, 1) \mid x \in \mathbb{R}\} \quad \text{for all continuous } f, g: \mathbb{R} \rightarrow \mathbb{R}.$$

We put $\min(|f(x) - g(x)|, 1)$ here to ensure this maximum exists and is finite. See Exercise 1.15 for some pathological properties of this metric and some alternate metrics with better properties. $\square$

1.3. Continuity

Once we have defined metric spaces, we can define continuity by imitating the classical definition from real analysis. Let $(M, \mathfrak{d}_M)$ and $(N, \mathfrak{d}_N)$ be metric spaces and let $f: M \rightarrow N$ be a function. Then:

- f is *continuous at* $p \in M$ if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $q \in M$ with $\mathfrak{d}_M(p, q) < \delta$ we have $\mathfrak{d}_N(f(p), f(q)) < \epsilon$.
- f is *continuous* if it is continuous at all $p \in M$.

Here are some examples:

EXAMPLE 1.3.1. Let $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ be equipped with their Euclidean metrics and let $f: M \rightarrow N$ be a function. In this special case, the above definition of what it means for f to be continuous is exactly the same as the definition from real analysis. $\square$

EXAMPLE 1.3.2. Let $(M, \mathfrak{d})$ be a metric space and let $\mathbb{1}_M: M \rightarrow M$ be the identity function. Then $\mathbb{1}_M$ is continuous. Indeed, consider $p \in M$ and $\epsilon > 0$. Setting $\delta = \epsilon$, if $q \in M$ satisfies $\mathfrak{d}(p, q) < \delta = \epsilon$ then $\mathfrak{d}(\mathbb{1}_M(p), \mathbb{1}_M(q)) = \mathfrak{d}(p, q) < \epsilon$. $\square$

EXAMPLE 1.3.3. Let $(M, \mathfrak{d})$ be a metric space and let $x_0 \in M$ be a point. Define $f: M \rightarrow \mathbb{R}$ via the formula

$$f(p) = \mathfrak{d}(x_0, p) \quad \text{for } p \in M.$$

Equipping $\mathbb{R}$ with the Euclidean metric, the function f is continuous. Indeed, consider $p \in M$ and $\epsilon > 0$. Setting $\delta = \epsilon$, if $q \in M$ satisfies $\mathfrak{d}(p, q) < \delta = \epsilon$ then

$$(1.3.1) \quad |f(p) - f(q)| = |\mathfrak{d}(x_0, p) - \mathfrak{d}(x_0, q)|.$$

We must prove this is less than ϵ . The triangle inequality implies that

$$\mathfrak{d}(x_0, q) \leq \mathfrak{d}(x_0, p) + \mathfrak{d}(p, q) \quad \text{and} \quad \mathfrak{d}(x_0, p) \leq \mathfrak{d}(x_0, q) + \mathfrak{d}(q, p).$$

Rearranging these, we see that

$$\mathfrak{d}(x_0, q) - \mathfrak{d}(x_0, p) \leq \mathfrak{d}(p, q) \quad \text{and} \quad \mathfrak{d}(x_0, p) - \mathfrak{d}(x_0, q) \leq \mathfrak{d}(q, p) = \mathfrak{d}(p, q).$$

These imply that (1.3.1) is at most $\mathfrak{d}(p, q) < \epsilon$, as desired. $\square$

EXAMPLE 1.3.4. Let $(M, \mathfrak{d})$ be a metric space. The collection of continuous maps $f: M \rightarrow \mathbb{R}$ forms an $\mathbb{R}$ -algebra. The nontrivial part of this is that the sum and product of two continuous maps from M to $\mathbb{R}$ is continuous. This can be proved exactly like the corresponding fact from real analysis. Alternatively, see Exercise 1.4. $\square$

1.4. Categories of metric spaces

The collection of metric spaces can be organized using the notion of a category, which we now discuss. A *category* consists of the following data:

- A collection of objects $\mathbf{C}$.
- For all $X, Y \in \mathbf{C}$, a set $\mathbf{C}(X, Y)$ of morphisms between X and Y . We will write $f: X \rightarrow Y$ to indicate that f is a morphism from X to Y .
- For all $X \in \mathbf{C}$, a distinguished unit morphism $\mathbb{1}_X: X \rightarrow X$.
- For all $X, Y, Z \in \mathbf{C}$ a composition map $\mathbf{C}(Y, Z) \times \mathbf{C}(X, Y) \rightarrow \mathbf{C}(X, Z)$. For morphisms $f: X \rightarrow Y$ and $g: Y \rightarrow Z$, we write their composition as $g \circ f: X \rightarrow Z$.

These must satisfy the following axioms:

- For all $f: X \rightarrow Y$, we have $\mathbb{1}_Y \circ f = f$ and $f \circ \mathbb{1}_X = f$.
- For all $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ and $h: Z \rightarrow W$, we have the associative law $(h \circ g) \circ f = h \circ (g \circ f)$.

Here are some familiar examples:

EXAMPLE 1.4.1. There is a category **Set** whose objects are sets X and whose morphisms $f: X \rightarrow Y$ are functions. Here the composition of two functions $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ is the function $g \circ f: X \rightarrow Z$ given by the usual function composition. $\square$

EXAMPLE 1.4.2. Let $\mathbf{k}$ be a field. There is a category $\mathbf{Vect}_{\mathbf{k}}$ whose objects are vector spaces V over $\mathbf{k}$ and whose morphisms $f: V \rightarrow W$ are linear maps. Again, composition is the usual function composition. $\square$

Metric spaces fit into this picture as follows:

EXAMPLE 1.4.3. There is a category **Metric** whose objects are metric space $(M, \mathfrak{d})$ and whose morphisms $f: M \rightarrow N$ are continuous maps. Indeed, we already saw in Example 1.3.2 that the identity map on a metric space is continuous, so we must only prove that the composition of continuous maps is continuous. This is exactly Lemma 1.4.4 below. $\square$

LEMMA 1.4.4. *Let $(M_1, \mathfrak{d}_1)$ and $(M_2, \mathfrak{d}_2)$ and $(M_3, \mathfrak{d}_3)$ be metric spaces. Let $f: M_1 \rightarrow M_2$ and $g: M_2 \rightarrow M_3$ be continuous maps. Then $g \circ f: M_1 \rightarrow M_3$ is continuous.*

PROOF. Let $p \in M_1$ and let $\epsilon > 0$. Since g is continuous at $f(p)$, there is some $\delta' > 0$ such that for all $x \in M_2$ with $\mathfrak{d}_2(f(p), x) < \delta'$ we have $\mathfrak{d}_3(g(f(p)), g(x)) < \epsilon$. Letting $\epsilon' = \delta'$, since f is continuous at p there is some $\delta > 0$ such that for all $q \in M_1$ with $\mathfrak{d}_1(p, q) < \delta$ we have $\mathfrak{d}_2(f(p), f(q)) < \epsilon' = \delta'$. Combining these, for $q \in M_1$ with $\mathfrak{d}_1(p, q) < \delta$ we have $\mathfrak{d}_3(g(f(p)), g(f(q))) < \epsilon$, as desired. $\square$

REMARK 1.4.5. There are other natural classes of maps between metric spaces. Let $(M, \mathfrak{d}_M)$ and $(N, \mathfrak{d}_N)$ be metric spaces and let $f: M \rightarrow N$ be a function. Then:

- The map f is an *isometric embedding* if $\mathfrak{d}_N(f(p), f(q)) = \mathfrak{d}_M(p, q)$ for all $p, q \in M$.
- The map f is *Lipshitz* if there is some $L \geq 0$ with $\mathfrak{d}_N(f(p), f(q)) \leq L \mathfrak{d}_M(p, q)$ for all $p, q \in M$.
- The map f is *bi-Lipshitz* if there is some $L \geq 1$ with

$$\frac{1}{L} \mathfrak{d}_M(p, q) \leq \mathfrak{d}_N(f(p), f(q)) \leq L \mathfrak{d}_M(p, q) \quad \text{for all } p, q \in M.$$

- The map f is a *quasi-isometric embedding* if there is some $L \geq 1$ and $K > 0$ with

$$\frac{1}{L} \mathfrak{d}_M(p, q) - K \leq \mathfrak{d}_N(f(p), f(q)) \leq L \mathfrak{d}_M(p, q) + K \quad \text{for all } p, q \in M.$$

Isometric embeddings and Lipshitz maps and bi-Lipshitz maps are all continuous, while quasi-isometric embeddings need not be continuous (see Exercises 1.12 and 1.13). All of these classes of maps form the morphisms in different categories of metric spaces (see Exercise 1.14). $\square$

1.5. Topology

Ordinary geometry concerns distances, angles, etc. At least for distances, metric spaces are a natural context for this. Topology is a primitive kind of geometry where distances are ignored. Instead, topology focuses on tools for studying continuous functions between spaces.

If $\mathbf{C}$ is a category, then an *isomorphism* in $\mathbf{C}$ is a morphism $f: A \rightarrow B$ such that there exists a morphism $g: B \rightarrow A$ with $g \circ f = \mathbb{1}_A$ and $f \circ g = \mathbb{1}_B$. Note that this implies that g is also an isomorphism. Isomorphisms in the category **Metric** are called *homeomorphisms*. We say that metric spaces M and N are *homeomorphic* if there exists a homeomorphism $f: M \rightarrow N$, and denote this by $M \cong N$. Here are two examples of topological questions one might ask:

QUESTION 1.5.1. Can we classify metric spaces up to homeomorphism? $\square$

QUESTION 1.5.2. Fix metric spaces M and N . An *embedding* of M into N is a continuous injective function $f: M \rightarrow N$ that is a homeomorphism onto its image. Can we determine whether M can be embedded into N ? $\square$

General metric spaces are far too wild for questions like these to have reasonable answers. Typically topologists restrict to classes of spaces like those drawn at the beginning of this chapter. Even with this restriction, this quickly leads to difficult questions. For instance, let $\mathbb{S}^n$ and $\mathbb{D}^n$ be the n -sphere and the n -disk:

$$\begin{aligned}\mathbb{S}^n &= \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}, \\ \mathbb{D}^n &= \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 \leq 1\}.\end{aligned}$$

Consider the following assertions:

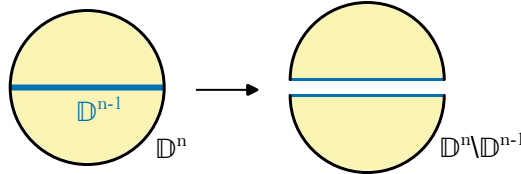
- (i) For $n, m \geq 0$ with $n \neq m$, we have $\mathbb{S}^n \not\cong \mathbb{S}^m$ and $\mathbb{D}^n \not\cong \mathbb{D}^m$.
- (ii) For $m > n$, there do not exist embeddings

$$\begin{aligned}\mathbb{D}^m \rightarrow \mathbb{D}^n, \quad \text{or} \quad \mathbb{D}^m \rightarrow \mathbb{S}^n, \quad \text{or} \\ \mathbb{S}^m \rightarrow \mathbb{D}^n, \quad \text{or} \quad \mathbb{S}^m \rightarrow \mathbb{S}^n.\end{aligned}$$

- (iii) For all $n, m \geq 0$, we have $\mathbb{S}^n \not\cong \mathbb{D}^m$.

Each of these is in fact true, but aside from some degenerate cases none of them have straightforward proofs. Using the point-set topological tools we will develop in this book, we will be able to prove all of them:

- For (i) and (ii), we will develop a notion of “dimension” for spaces in Essay C. The spaces $\mathbb{S}^n$ and $\mathbb{D}^n$ will both have dimension n . Spaces of different dimensions cannot be homeomorphic, and higher-dimensional spaces cannot embed into lower-dimensional ones.
- For (iii), dimension theory will prove everything but the fact that $\mathbb{S}^n \not\cong \mathbb{D}^n$. To tell $\mathbb{S}^n$ and $\mathbb{D}^n$ apart, we must identify a topological property that holds for one but not the other. In Essay E, we will prove that no subspace $X \subset \mathbb{S}^n$ with $X \cong \mathbb{D}^{n-1}$ can separate $\mathbb{S}^n$ into two pieces. On the other hand, the natural subspace $\mathbb{D}^{n-1} \subset \mathbb{D}^n$ does separate it:



This will imply that $\mathbb{S}^n$ and $\mathbb{D}^n$ cannot be homeomorphic.

REMARK 1.5.3. In later volumes of this book, we will develop tools from algebraic topology like homology groups and homotopy groups. Once these tools are in place, results like (i)–(iii) will become routine. $\square$

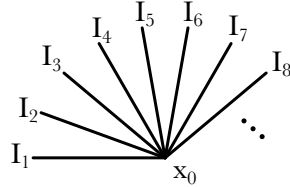
1.6. Downsides of metric spaces

The geometric meaning of the definition of a metric space is easily grasped. However, for topology they have downsides:

- Though continuity is defined in terms of a metric, there are many metrics on a given space that give the same notion of continuity (see Exercise 2.8). In other words, continuity is a more primitive notion than a metric.
- There are many geometric operations one would like to perform on spaces (gluing them together, taking quotients, etc). However, these operations do not always interact well with a metric and often result in “spaces” that are not metric spaces.

Here is an example of this second pathology:

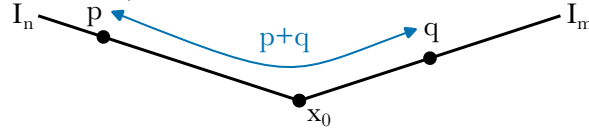
EXAMPLE 1.6.1. For each integer $n \geq 1$, let I_n be a copy of the interval $I = [0, 1]$. Let M be the “space” obtained by identifying the points $0 \in I_n$ all together to a single point x_0 :



Each I_n is a subspace of M , so each $p \in M$ lies in some I_n . This I_n is unique unless $p = x_0$. There is a natural choice of a metric on M :

- Consider $p, q \in M$. If there is some $n \geq 1$ such that $p, q \in I_n = [0, 1]$, define $\mathfrak{d}(p, q) = |p - q|$. Otherwise, if $p \in I_n$ and $q \in I_m$ with $n \neq m$, then define $\mathfrak{d}(p, q) = p + q$.

For an explanation of this formula, see here:



Define a function $f: M \rightarrow \mathbb{R}$ via the formula $f(p) = np$ for $p \in I_n$. This formula makes sense since the map $p \mapsto np$ takes 0 to 0 for all n , so the resulting function f satisfies $f(x_0) = 0$. The restriction of f to each I_n is continuous; however, f itself is not continuous (see Exercise 1.11). $\square$

In this example, it is inconvenient that continuous functions on the I_n do not “glue together” to a continuous function on M . Once we have defined topological spaces, we will be able to turn M into a topological space where this kind of gluing works.

1.7. Open sets and continuity

To give a hint for how to discuss continuity without a metric, we review some other facts about metric spaces. Fix a metric space $(M, \mathfrak{d})$. For $p \in M$ and $r > 0$, let

$$B_r(p) = \{q \in M \mid \mathfrak{d}(p, q) < r\}.$$

This is called the *open ball* of radius r around p . A set $U \subset M$ is *open* if for all $p \in U$, there exists some $r > 0$ such that $B_p(r) \subset U$. With this definition, the set $B_p(r)$ is easily seen to be open (see Exercise 1.2).

More generally, if $A \subset M$ is an arbitrary subset then the *interior* of A , denoted $\text{Int}(A)$, is the set of all $p \in A$ such that there exists some $r > 0$ such that $B_p(r) \subset A$. We have:

LEMMA 1.7.1. *Let $(M, \mathfrak{d})$ be a metric space and let $A \subset M$. Then $\text{Int}(A)$ is an open set contained in A .*

PROOF. We clearly have $\text{Int}(A) \subset A$. To see that $\text{Int}(A)$ is open, consider $p \in \text{Int}(A)$. By definition, there is some $r > 0$ such that $B_r(p) \subset A$. We claim that $B_r(p) \subset \text{Int}(A)$. Indeed, consider $p' \in B_r(p)$. We must prove that $p' \in \text{Int}(A)$. Set $r' = r - \mathfrak{d}(p, p') > 0$. For $q \in B_{r'}(p')$, we have

$$\mathfrak{d}(p, q) \leq \mathfrak{d}(p, p') + \mathfrak{d}(p', q) < \mathfrak{d}(p, p') + (r - \mathfrak{d}(p, p')) = r,$$

so $q \in B_r(p) \subset A$. It follows that $B_{r'}(p') \subset A$, so $p' \in \text{Int}(A)$, as desired. $\square$

EXAMPLE 1.7.2. The open sets in $\mathbb{R}$ are exactly the sets that can be written as the union of countably¹ many disjoint open intervals (a, b) with $a < b$ (see Exercise 1.9). $\square$

Open subsets of metric spaces are closed under arbitrary unions and finite intersections:²

LEMMA 1.7.3. *Let $(M, \mathfrak{d})$ be a metric space. Then:*

- *The empty set $\emptyset$ and the whole set M are open.*
- *If $\{U_i\}_{i \in I}$ is an arbitrary collection of open subsets of M , then $\cup_{i \in I} U_i$ is open.*
- *If $\{U_1, \dots, U_n\}$ is a finite collection of open subsets of M , then $\cap_{i=1}^n U_i$ is open.*

PROOF. See Exercise 1.7. $\square$

We can express continuity in terms of open sets as follows:

LEMMA 1.7.4. *Let $(M_1, \mathfrak{d}_1)$ and $(M_2, \mathfrak{d}_2)$ be metric spaces and let $f: M_1 \rightarrow M_2$ be a function. Then f is continuous if and only if for all $U \subset M_2$ open we have $f^{-1}(U) \subset M_1$ open.*

PROOF. We divide the proof into two steps:

STEP 1. *Assume that f is continuous and $U \subset M_2$ is open. Then $f^{-1}(U) \subset M_1$ is open.*

Let $p \in f^{-1}(U)$. To prove that $f^{-1}(U)$ is open, we must find some $\delta > 0$ such that $B_\delta(p) \subset f^{-1}(U)$. Since U is open, we can find some $\epsilon > 0$ such that $B_\epsilon(f(p)) \subset U$. Since f is continuous at p , there is some $\delta > 0$ such that for $q \in M_1$ with $\mathfrak{d}_1(p, q) < \delta$ we have $\mathfrak{d}_2(f(p), f(q)) < \epsilon$. In other words, f takes all $q \in B_\delta(p)$ to a point of $B_\epsilon(f(p)) \subset U$, so $B_\delta(p) \subset f^{-1}(U)$, as desired.

STEP 2. *Assume that for all open sets $U \subset M_2$ the set $f^{-1}(U) \subset M_1$ is open. Then f is continuous.*

Let $p \in M_1$ and $\epsilon > 0$. Since $B_\epsilon(f(p)) \subset M_2$ is open (see Exercise 1.2), the set $f^{-1}(B_\epsilon(f(p))) \subset M_1$ is open. Since $p \in f^{-1}(B_\epsilon(f(p)))$, there thus exists some $\delta > 0$ such that $B_\delta(p) \subset f^{-1}(B_\epsilon(f(p)))$. In other words, for $q \in M_1$ with $\mathfrak{d}_1(p, q) < \delta$ we have $\mathfrak{d}_2(f(p), f(q)) < \epsilon$, as desired. $\square$

1.8. Limits and continuity

Fix a metric space $(M, \mathfrak{d})$. Consider a sequence $\{x_n\}_{n \geq 1}$ of points of M . We say that the x_n converges to $x \in M$ if for all $\epsilon > 0$, there exists some $N \geq 1$ such that $\mathfrak{d}(x_n, x) < \epsilon$ for all $n \geq N$. In this case, we write $\lim_{n \rightarrow \infty} x_n = x$ and say that $\{x_n\}_{n \geq 1}$ is a *convergent sequence*. The following shows that a sequence $\{x_n\}_{n \geq 1}$ can converge to at most one point of M , so this notation is unambiguous:

LEMMA 1.8.1. *Let $(M, \mathfrak{d})$ be a metric space and let $\{x_n\}_{n \geq 1}$ be a sequence of points of M that converges to points $x \in M$ and $x' \in M$. Then $x = x'$.*

PROOF. Let $\epsilon > 0$. Since $\{x_n\}_{n \geq 1}$ converges to x and x' , we can find $N \geq 1$ such that for $n \geq N$ we have $\mathfrak{d}(x_n, x) < \epsilon/2$ and $\mathfrak{d}(x_n, x') < \epsilon/2$. We thus have

$$\mathfrak{d}(x, x') \leq \mathfrak{d}(x, x_N) + \mathfrak{d}(x_N, x') < \epsilon/2 + \epsilon/2 = \epsilon.$$

Since $\mathfrak{d}(x, x') < \epsilon$ for all $\epsilon > 0$, it follows that $\mathfrak{d}(x, x') = 0$ and thus $x = x'$, as desired. $\square$

The following lemma is a useful restatement of the definition of a limit:

LEMMA 1.8.2. *Let $(M, \mathfrak{d})$ be a metric space and let $\{x_n\}_{n \geq 1}$ be a sequence of points of M . For some $p \in M$, assume that $\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, p) = 0$, where this limit is as a sequence of real numbers. Then $\lim_{n \rightarrow \infty} x_n = p$.*

PROOF. See Exercise 1.5. $\square$

We can express continuity using limits as follows:

LEMMA 1.8.3. *Let $(M_1, \mathfrak{d}_1)$ and $(M_2, \mathfrak{d}_2)$ be metric spaces and let $f: M_1 \rightarrow M_2$ be a function. Then f is continuous if and only if the following holds:*

¹Our convention in this book is a finite set is countable. A *countably infinite* set is a countable set that is infinite.

²We point this out since it will later motivate the definition of a topological space.

(♠) For all convergent sequences $\{x_n\}_{n \geq 1}$ of points of M_1 , the sequence $\{f(x_n)\}_{n \geq 1}$ of points of M_2 is convergent and $\lim_{n \rightarrow \infty} f(x_n) = f(\lim_{n \rightarrow \infty} x_n)$.

PROOF. We divide the proof into two steps:

STEP 1. Assume that f is continuous. Then (♠) holds.

Let $\{x_n\}_{n \geq 1}$ be a convergent sequence of points of M_1 . Set $p = \lim_{n \rightarrow \infty} x_n$. Our goal is to prove that $\lim_{n \rightarrow \infty} f(x_n) = f(p)$. Consider some $\epsilon > 0$. We must find some $N \geq 1$ such that for $n \geq N$ we have $\mathfrak{d}_2(f(p), f(x_n)) < \epsilon$. Since f is continuous, there is some $\delta > 0$ such that if $q \in M_1$ satisfies $\mathfrak{d}_1(p, q) < \delta$, then $\mathfrak{d}_2(f(p), f(q)) < \epsilon$. Choose $N \geq 1$ such that for $n \geq N$ we have $\mathfrak{d}_1(p, x_n) < \delta$. It follows that for $n \geq N$ we have $\mathfrak{d}_2(f(p), f(x_n)) < \epsilon$, as desired.

STEP 2. Assume that (♠) holds. Then f is continuous.

Let $p \in M_1$ and $\epsilon > 0$. Assume for the sake of contradiction that there does not exist some $\delta > 0$ such that if $\mathfrak{d}_1(p, q) < \delta$ then $\mathfrak{d}_2(f(p), f(q)) < \epsilon$. For each $n \geq 1$, there therefore exists some $x_n \in M_1$ with $\mathfrak{d}(p, x_n) < 1/n$ and $\mathfrak{d}_2(f(p), f(x_n)) > \epsilon$. Since $\mathfrak{d}(p, x_n) < 1/n$ for all $n \geq 1$, Lemma 1.8.2 implies that the sequence $\{x_n\}_{n \geq 1}$ is convergent and $\lim_{n \rightarrow \infty} x_n = p$. It thus follows from (♠) that $\lim_{n \rightarrow \infty} f(x_n) = f(p)$. In particular, there exists some $N \geq 1$ such that for $n \geq N$ we have $\mathfrak{d}(f(p), f(x_n)) < \epsilon$, contradicting the fact that $\mathfrak{d}(f(p), f(x_n)) > \epsilon$. $\square$

1.9. Limit points and closed sets

Fix a metric space $(M, \mathfrak{d})$. Let $A \subset M$ be a set. A *limit point* of A is a point $p \in M$ such that for all $r > 0$ there exists some $q \in B_r(p)$ with $q \in A$ and $q \neq p$. This can be rephrased using limits; see Exercise 1.6. We also have:

LEMMA 1.9.1. Let $(M, \mathfrak{d})$ be a metric space, let $A \subset M$, and let p be a limit point of A . Then for all $r > 0$, there exists infinitely many $q \in B_r(p)$ with $q \in A$ and $q \neq p$.

PROOF. Set $r_1 = r$. By definition, there exists some $q_1 \in B_{r_1}(p)$ with $q_1 \in A$ and $q_1 \neq p$. Set $r_2 = \mathfrak{d}(p, q_1) < r_1$. There exists some $q_2 \in B_{r_2}(p)$ with $q_2 \in A$ and $q_2 \neq p$. Since $\mathfrak{d}(p, q_2) < \mathfrak{d}(p, q_1)$, we also have $q_2 \neq q_1$. Repeating this over and over again, we get a sequence $\{q_i\}_{i \geq 1}$ of points of A that are all different from p and satisfy

$$r > \mathfrak{d}(p, q_1) > \mathfrak{d}(p, q_2) > \mathfrak{d}(p, q_3) > \cdots.$$

This implies that the q_i are all distinct. The lemma follows. $\square$

We say that A is *closed* if it contains all of its limit points. The *closure* of A , denoted $\overline{A}$, is the union of A with its limit points. We have:

LEMMA 1.9.2. Let $(M, \mathfrak{d})$ be a metric space and let $A \subset M$. Then $\overline{A}$ is a closed set containing A .

PROOF. We have $A \subset \overline{A}$ by definition, so we must prove that $\overline{A}$ is closed. Let $p \in M$ be a limit point of $\overline{A}$. We must prove that $p \in \overline{A}$. In fact, we will prove that p is a limit point of A . Consider some $r > 0$. Our goal is to find some $a \in B_r(p)$ with $a \in A$ and $a \neq p$. Since p is a limit point of $\overline{A}$, there exists a point $q \in B_{r/2}(p)$ such that $q \in \overline{A}$ and $q \neq p$. If $q \in A$, then we can take $a = q$. Otherwise, q must be a limit point of A . By Lemma 1.9.1, there exist infinitely many $a \in B_{r/2}(q)$ with $a \in A$ and $a \neq q$. Since there are infinitely many, we can find one with $a \neq p$ as well. We have

$$\mathfrak{d}(a, p) \leq \mathfrak{d}(a, q) + \mathfrak{d}(q, p) < r/2 + r/2 = r,$$

so $a \in B_r(p)$, as desired. $\square$

EXAMPLE 1.9.3. Unlike open subsets of $\mathbb{R}$, closed subsets of $\mathbb{R}$ do not have a simple description. For instance, the classical *Cantor set* is as follows. Each $x \in I = [0, 1]$ can be written uniquely as

$$x = \sum_{n=1}^{\infty} \frac{x_n}{3^n} \quad \text{with } x_n \in \{0, 1, 2\} \text{ for all } n \geq 1.$$

The Cantor set C consists of all $x \in I$ of the form

$$x = \sum_{n=1}^{\infty} \frac{x_n}{3^n} \quad \text{with } x_n \in \{0, 2\} \text{ for all } n \geq 1.$$

The set C is closed (see Exercise 1.10). □

Closed subsets of metric spaces are closed under arbitrary intersections and finite unions:

LEMMA 1.9.4. *Let M be a metric space. Then:*

- *The empty set $\emptyset$ and the whole set M are closed.*
- *If $\{C_i\}_{i \in I}$ is an arbitrary collection of closed subsets of M , then $\cap_{i \in I} C_i$ is closed.*
- *If $\{C_1, \dots, C_n\}$ is a finite collection of closed subsets of M , then $\cup_{i=1}^n C_i$ is closed.*

PROOF. See Exercise 1.7. □

We close this chapter by relating open and closed sets:

LEMMA 1.9.5. *Let $(M, \mathfrak{d})$ be a metric space and let $A \subset M$ be a set. Then A is open if and only if $M \setminus A$ is closed.*

PROOF. We divide the proof into two steps:

STEP 1. *Assume that A is open. Then $M \setminus A$ is closed.*

Let $p \in M$ be a limit point of $M \setminus A$. We must prove that $p \in M \setminus A$. Assume otherwise. We then have $p \in A$, so since A is open there exists some $r > 0$ such that $B_r(p) \subset A$. Since $B_r(p)$ contains no points of $M \setminus A$, it follows that p is not a limit point of $M \setminus A$, contradicting the fact that it is such a limit point.

STEP 2. *Assume that $M \setminus A$ is closed. Then A is open.*

Consider $a \in A$. We must find some $r > 0$ such that $B_r(a) \subset A$. Since $M \setminus A$ is closed and $a \in A$, it must be the case that a is not a limit point of $M \setminus A$. There therefore exists some $r > 0$ such that $B_r(a)$ contains no points of $M \setminus A$, i.e., such that $B_r(a) \subset A$. □

1.10. Exercises

EXERCISE 1.1. Let M be any set. Define a distance function on M via the formula

$$\mathfrak{d}(p, q) = \begin{cases} 1 & \text{if } p \neq q, \\ 0 & \text{if } p = q. \end{cases}$$

Prove that $(M, \mathfrak{d})$ is a metric space, and identify its open sets and its closed sets. □

EXERCISE 1.2. Let M be a metric space, let $p \in M$, and let $r > 0$. Prove that $B_r(p)$ is open. □

EXERCISE 1.3. Let $(M, \mathfrak{d})$ be a metric space. For $p \in M$ and $r > 0$, define

$$B_r^c(p) = \{q \in M \mid \mathfrak{d}(p, q) \leq r\}.$$

Do the following:

- (a) Prove that $B_r^c(p)$ is closed.
- (b) Give an example to show that $B_r^c(p)$ can be different from the closure $\overline{B_r(p)}$ of the open ball $B_r(p)$. □

EXERCISE 1.4. Let $(M, \mathfrak{d})$ be a metric space and let $f: M \rightarrow \mathbb{R}$ and $g: M \rightarrow \mathbb{R}$ be continuous functions. Prove the following:

- (a) Define a map $\phi: M \rightarrow \mathbb{R}^2$ via the formula

$$\phi(p) = (f(p), g(p)) \quad \text{for all } p \in M.$$

Prove that ϕ is continuous.

- (b) Use part (a) to prove that the sum and product of f and g is continuous. Hint: we already know from real analysis that the maps $s: \mathbb{R}^2 \rightarrow \mathbb{R}$ and $m: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$s(x, y) = s + y \quad \text{and} \quad m(x, y) = xy \quad \text{for all } x, y \in \mathbb{R}.$$

are continuous. Use the fact that the sum and product of f and g are $s \circ \phi$ and $m \circ \phi$. $\square$

EXERCISE 1.5. Let $(M, \mathfrak{d})$ be a metric space and let $\{x_n\}_{n \geq 1}$ be a sequence of points of M . For some $p \in M$, assume that $\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, p) = 0$, where this limit is as a sequence of real numbers. Prove that $\lim_{n \rightarrow \infty} x_n = p$. $\square$

EXERCISE 1.6. Let $(M, \mathfrak{d})$ be a metric space and let $A \subset M$. Prove that $p \in M$ is a limit point of A if and only if there exists a convergent sequence $\{a_n\}_{n \geq 1}$ of points of A such that $\lim_{n \rightarrow \infty} a_n = p$. $\square$

EXERCISE 1.7. Let M be a metric space. Prove the following:

- (a) The empty set $\emptyset$ and the whole set M are both open and closed.
- (b) If $\{U_i\}_{i \in I}$ is an arbitrary collection of open subsets of M , then $\cup_{i \in I} U_i$ is open.
- (c) If $\{C_i\}_{i \in I}$ is an arbitrary collection of closed subsets of M , then $\cap_{i \in I} C_i$ is closed.
- (d) If $\{U_1, \dots, U_n\}$ is a finite collection of open subsets of M , then $\cap_{i=1}^n U_i$ is open.
- (e) If $\{C_1, \dots, C_n\}$ is a finite collection of closed subsets of M , then $\cup_{i=1}^n C_i$ is closed. $\square$

EXERCISE 1.8. Let $I = [0, 1]$ with its standard Euclidean metric. For each $n \geq 1$, construct a closed set $C \subset I$ with exactly n limit points. $\square$

EXERCISE 1.9. Let $U \subset \mathbb{R}$ be an open set. Prove that U is the disjoint union of countably many disjoint open intervals (a, b) with $a < b$. $\square$

EXERCISE 1.10. Let C be the classical Cantor set, i.e., the set of all $x \in I = [0, 1]$ of the form

$$x = \sum_{n=1}^{\infty} \frac{x_n}{3^n} \quad \text{with } x_n \in \{0, 2\} \text{ for all } n \geq 1.$$

Prove that C is closed. $\square$

EXERCISE 1.11. Prove that the function $f: M \rightarrow \mathbb{R}$ from Example 1.6.1 is not continuous. $\square$

EXERCISE 1.12. Let $(M, \mathfrak{d}_M)$ and $(N, \mathfrak{d}_N)$ be metric spaces and let $f: M \rightarrow N$ be a function. Prove that f is continuous if any of the following hold (see Remark 1.4.5 for the definitions):

- (a) f is an isometric embedding.
- (b) f is Lipschitz.
- (c) f is bi-Lipschitz. $\square$

EXERCISE 1.13. Endow $\mathbb{R}$ and $\mathbb{Z}$ with their standard metrics, so $\mathfrak{d}(x, y) = |x - y|$ for x and y in either $\mathbb{R}$ or $\mathbb{Z}$. We defined quasi-isometric embeddings in Remark 1.4.5. Construct a quasi-isometric embedding $f: \mathbb{R} \rightarrow \mathbb{Z}$. This shows that quasi-isometric embeddings need not be continuous. $\square$

EXERCISE 1.14. Prove that there are categories whose objects are metric spaces and whose morphisms are as follows (see Remark 1.4.5 for the definitions):

- (a) Isometric embeddings.
- (b) Lipschitz maps.
- (c) Bi-Lipschitz maps.
- (d) Quasi-isometric embeddings.

In all five cases, the composition is the usual composition of maps. $\square$

EXERCISE 1.15. Let $\mathcal{C}(\mathbb{R}, \mathbb{R})$ be the set of all continuous functions $f: \mathbb{R} \rightarrow \mathbb{R}$. Define the following metric on $\mathcal{C}(\mathbb{R}, \mathbb{R})$:

$$\mathfrak{d}(f, g) = \max \{ \min(|f(x) - g(x)|, 1) \mid x \in \mathbb{R} \} \quad \text{for all continuous } f, g: \mathbb{R} \rightarrow \mathbb{R}.$$

Prove the following:

- (a) For $\lambda \in \mathbb{R}$, let $\mathbf{c}_\lambda: \mathbb{R} \rightarrow \mathbb{R}$ be the constant function $\mathbf{c}_\lambda(x) = \lambda$. Using the metric $\mathfrak{d}$ on $\mathcal{C}(\mathbb{R}, \mathbb{R})$, prove that $\lim_{n \rightarrow \infty} \mathbf{c}_{1/n} = \mathbf{c}_0$.

- (b) For $\lambda \in \mathbb{R}$, let $f_\lambda: \mathbb{R} \rightarrow \mathbb{R}$ be the function $f_\lambda(x) = \lambda x$. Using the metric $\mathfrak{d}$ on $\mathcal{C}(\mathbb{R}, \mathbb{R})$, prove that $\lim_{n \rightarrow \infty} f_{1/n}$ does not exist.
- (c) We would like to have $\lim_{n \rightarrow \infty} f_{1/n} = \mathfrak{c}_0$, which requires a more subtle metric. Define the following:

$$\mathfrak{d}'(f, g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \max \{ \min(|f(x) - g(x)|, 1) \mid x \in [-k, k] \} \quad \text{for all continuous } f, g: \mathbb{R} \rightarrow \mathbb{R}.$$

Prove that $\mathfrak{d}'$ is a metric on $\mathcal{C}(\mathbb{R}, \mathbb{R})$.

- (d) Using the metric $\mathfrak{d}'$ on $\mathcal{C}(\mathbb{R}, \mathbb{R})$, prove that for a sequence $\{f_n\}_{n \geq 1}$ of functions $f_n \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ we have $\lim_{n \rightarrow \infty} f_n = f$ if and only if for all closed intervals $[a, b] \subset \mathbb{R}$ the functions f_n converge uniformly to f on $[a, b]$. In particular, $\lim_{n \rightarrow \infty} f_{1/n} = \mathfrak{c}_0$. $\square$

CHAPTER 2

Topological spaces

Since continuity for metric spaces can be described entirely in terms of open sets, it is natural to abstract the notion of “open sets”.

2.1. Definition of topological space

A *topological space* is a set X equipped with a collection of subsets of X called the *open sets*. These open sets should satisfy the following three properties:

- The whole space X and the empty set $\emptyset$ are both open.
- The collection of open sets is closed under arbitrary unions: if $\{U_i\}_{i \in I}$ is any collection of open sets, then $\cup_{i \in I} U_i$ is open.
- The collection of open sets is closed under finite intersections: if $U_1, \dots, U_n$ are open sets, then $U_1 \cap \dots \cap U_n$ is open.

We call the collection of open sets on X a *topology* on X . A key example is:

EXAMPLE 2.1.1. If $(M, \mathfrak{d})$ is a metric space, then Lemma 1.7.3 implies that the collection of open sets in M makes M into a topological space. We say that this topology on M is *induced* by the metric $\mathfrak{d}$. □

CONVENTION 2.1.2. Whenever we draw a figure in $\mathbb{R}^n$, we give it the topology it inherits as a metric space via the Euclidean metric on $\mathbb{R}^n$ discussed in Example 1.2.3. □

Topologies arising from metrics will play a key role in this book, so we introduce the following terminology. A topological space X is *metrizable* if there exists a distance function $\mathfrak{d}: X \times X \rightarrow \mathbb{R}$ making $(X, \mathfrak{d})$ into a metric space such that the topology on X is the one induced by $\mathfrak{d}$. The metric $\mathfrak{d}$ inducing the topology on X is not unique (see Exercise 2.8).

We will give many more examples of topological spaces in §2.5 and §2.6 below after introducing some more terminology. First, however, we prove an important lemma:

LEMMA 2.1.3. *Let X be a topological space and let $V \subset X$ be a subset. Assume that for all $p \in V$, there exists an open set $U \subset V$ with $p \in U$. Then V is open.*

PROOF. For $p \in V$, let $U_p \subset V$ be an open set with $p \in U_p$. We have $V = \cup_{p \in V} U_p$, so since open sets are closed under arbitrary unions it follows that V is open. □

REMARK 2.1.4. Lemma 2.1.3 allows reasoning like we did for metric spaces where the role of the open set U is played by open balls $B_r(p)$. □

2.2. Closed sets and limit points

Let X be a topological space. A set $C \subset X$ is *closed* if $X \setminus C$ is open. By Lemma 1.9.5, this agrees with our previous definition if X is metrizable. Since the collection of open sets is closed under arbitrary unions and finite intersections, it follows that the collection of closed sets is closed under finite unions and arbitrary intersections. The whole subject could be developed using closed sets instead of open ones (see Exercise 2.6).

Let $A \subset X$ be a set. A *limit point* of A is a point $p \in X$ such that for all open sets U with $p \in U$, there exists some $a \in U \cap A$ with $a \neq p$. Just like for metric spaces, we can characterize closed sets using limit points:

LEMMA 2.2.1. *Let X be a topological space and let $A \subset X$. Then A is closed if and only if all limit points of A lie in A .*

PROOF. We divide the proof into two steps:

STEP 1. Assume that A is closed. Then all limit points of A lie in A .

Consider $p \in X \setminus A$. We must prove that p is not a limit point of A . Since X is closed, $U = X \setminus A$ is open. It follows that U is an open set with $p \in U$ and $U \cap A = \emptyset$, so p is not a limit point of A .

STEP 2. Assume that all limit points of A lie in A . Then A is closed.

We must prove that $X \setminus A$ is open. Consider $p \in X \setminus A$. By Lemma 2.1.3, it is enough to find some open set $U \subset X \setminus A$ with $p \in U$. Since all limit points of A lie in A , the point p is not a limit point of A . There is thus an open set U with $p \in U$ such that $U \cap A$ contains no points of A except for possibly p . Since $p \notin A$, it follows that $U \cap A = \emptyset$, i.e., $U \subset X \setminus A$, as desired. $\square$

2.3. Interiors, closures, and neighborhoods

For a subset $A \subset X$, we define the interior $\text{Int}(A)$ and the closure $\bar{A}$ as follows:

- The interior $\text{Int}(A)$ is the union of all open sets U with $U \subset A$. Since the collection of open sets is closed under arbitrary unions, $\text{Int}(A)$ is open. The set $\text{Int}(A)$ is the largest open set contained in A .
- The closure $\bar{A}$ is the intersection of all closed sets C with $A \subset C$. Since the collection of closed sets is closed under arbitrary intersections, $\bar{A}$ is closed. The set $\bar{A}$ is the smallest closed set containing A .

These agree with our previous definitions if X is metrizable (see Exercise 2.7). For $p \in X$, a *neighborhood* of p is a set A with $p \in \text{Int}(A)$. More generally, for a set $B \subset X$, a *neighborhood* of B is a set A with $B \subset \text{Int}(A)$. The most important special case of this terminology is an open neighborhood of $B \subset X$, which is an open set U with $B \subset U = \text{Int}(U)$.

2.4. Continuity

A map $f: X \rightarrow Y$ between topological spaces is *continuous* if for all $U \subset Y$ open, its preimage $f^{-1}(U) \subset X$ is open. By Lemma 1.7.4, this is equivalent to the usual ϵ - δ definition if X and Y are metric spaces. The following lemma shows that there is a category Top whose objects are topological spaces and whose morphisms are continuous maps. This category is the natural home for topology.

LEMMA 2.4.1. The following hold:

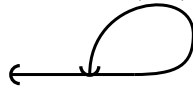
- (i) If X is a topological space, then the identity map $\mathbb{1}_X: X \rightarrow X$ is continuous.
- (ii) If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are continuous maps between topological spaces, then $g \circ f: X \rightarrow Z$ is continuous.

PROOF. Conclusion (i) is obvious. As for conclusion (ii), let $U \subset Z$ be open. Since f is continuous the set $g^{-1}(U) \subset Y$ is open, so since g is continuous the set $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U) \subset X$ is open, as desired. $\square$

If $\mathbf{C}$ is a category, then a morphism $m: A \rightarrow B$ in $\mathbf{C}$ is an *isomorphism* if there exists a morphism $m^{-1}: B \rightarrow A$ such that $m^{-1} \circ m = \mathbb{1}_A$ and $m \circ m^{-1} = \mathbb{1}_B$. If there exists an isomorphism from A to B , then we say that A and B are *isomorphic*. An isomorphism in Top is called a *homeomorphism*. Unwinding the definition, a homeomorphism between topological spaces X and Y is a continuous map $f: X \rightarrow Y$ that is bijective and whose inverse $f^{-1}: Y \rightarrow X$ is continuous. If there exists a homeomorphism from X to Y , then we say that X and Y are *homeomorphic* and write $X \cong Y$.

Here is an example to show that the continuity of f^{-1} is not immediate:

EXAMPLE 2.4.2. Consider the injective map $f: (0, 1) \rightarrow \mathbb{R}^2$ whose image X is as follows:



The map $f: (0, 1) \rightarrow X$ is continuous and bijective, but the inverse map $f^{-1}: X \rightarrow (0, 1)$ is not continuous (see Exercise 2.13). $\square$

CONVENTION 2.4.3. Henceforth, we will use the word “space” as a synonym for “topological space”. Also, unless otherwise specified all maps between spaces are assumed to be continuous. $\square$

REMARK 2.4.4. For metric spaces, we also characterized continuity using limits (see §1.8). The notion of limits can be generalized to topological spaces, though with some subtleties (for instance, limits of sequences need not be unique). However, without some additional assumptions it would give a different notion of continuity. See §5.2 for the definition of a limit in a topological space and Lemma 6.3.1 for the conditions needed for continuity to be described in terms of limits. $\square$

2.5. Other examples

The notion of a topological space is extremely general. Here are a few more examples.

EXAMPLE 2.5.1. Let X be a set. The *discrete topology* on X is the one where all sets are open. The *trivial topology* on X is the one where the only open sets are $\emptyset$ and X . Another topology that can be put on an arbitrary set X is the *cofinite topology* whose open sets are those of the form $X \setminus F$ with F finite. The fact that this is a topology follows from the fact that finite sets are closed under finite unions and arbitrary intersections. $\square$

EXAMPLE 2.5.2. Let $\mathbf{k}$ be a field; for instance, $\mathbf{k}$ might be $\mathbb{C}$ or $\mathbb{R}$. For a polynomial $f \in \mathbf{k}[z_1, \dots, z_n]$, define the *vanishing* and *non-vanishing* loci of f to be

$$V(f) = \{(x_1, \dots, x_n) \in \mathbf{k}^n \mid f(x_1, \dots, x_n) = 0\} \subset \mathbf{k}^n \quad \text{and} \quad NV(f) = \mathbf{k}^n \setminus V(f).$$

The *Zariski topology* on $\mathbf{k}^n$ is the topology whose open sets are the nonvanishing loci $NV(f)$ as f ranges over elements of $\mathbf{k}[z_1, \dots, z_n]$ (see Exercise 2.9). The closed sets are thus the vanishing loci $Z(f)$. For $n = 1$, the vanishing locus of a polynomial in $\mathbf{k}[z_1]$ can be any finite subset of $\mathbf{k}^1$, so the Zariski topology on $\mathbf{k}^1$ is the cofinite topology. $\square$

REMARK 2.5.3. For $\mathbf{k}$ equal to $\mathbb{C}$ or $\mathbb{R}$, we have now seen two topologies on $\mathbf{k}^n$:

- the classical topology obtained by regarding $\mathbf{k}^n$ as a metric space; and
- the Zariski topology.

Every open set in the Zariski topology is open in the classical topology. We say that the classical topology is *finer* or *stronger* than the Zariski topology, and that the Zariski topology is *coarser* or *weaker* than the classical topology. $\square$

EXAMPLE 2.5.4. A finite set X can be endowed with many topologies. For instance, if $X = \{a, b\}$ then the following are all topologies on X :

- the discrete topology: $\emptyset, \{a\}, \{b\}, \{a, b\}$
- $\emptyset, \{a\}, \{a, b\}$
- $\emptyset, \{b\}, \{a, b\}$
- the topology: $\emptyset, \{a, b\}$

The number of topologies on a finite set X grows very quickly as the size of X grows. Some of these spaces can be given geometric interpretations, but most of them are purely combinatorial objects. $\square$

REMARK 2.5.5. Because the notion of a topological space is so general, there is almost nothing nontrivial that can be said about an arbitrary topological space. They are thus almost never studied for their own sake. Rather, they provide a minimal framework and language for studying continuity as it appears throughout mathematics. $\square$

2.6. Basis for a topology

A *basis* for a topology on a set X consists of a set $\mathfrak{B}$ of subsets of X such that:

- (i) all points of X lie in some $U \in \mathfrak{B}$; and
- (ii) for all $U, V \in \mathfrak{B}$, the intersection $U \cap V$ can be written as a union of sets in $\mathfrak{B}$.

Given such a basis, the corresponding topology is the one where a set $U \subset X$ is open if and only if U is a union of sets in $\mathfrak{B}$. The following shows that this is indeed a topology:

LEMMA 2.6.1. *Let X and $\mathfrak{B}$ be as above. Then the indicated collection of open sets forms a topology on X .*

PROOF. The empty set $\emptyset$ is open since it is the union of no elements of $\mathcal{B}$. The whole set X is open because of (i). It is immediate that the indicated collection of open sets is closed under arbitrary unions, so the only nontrivial thing we must check is that it is closed under finite intersections. By induction, it is enough to prove that if U and V are open then so is $U \cap V$. Write

$$U = \bigcup_{i \in I} U_i \quad \text{and} \quad V = \bigcup_{j \in J} V_j$$

with each U_i and V_j in $\mathcal{B}$. We then have

$$U \cap V = \bigcup_{\substack{i \in I \\ j \in J}} U_i \cap V_j.$$

By (ii), each $U_i \cap V_j$ is the union of sets in $\mathcal{B}$, so $U \cap V$, as desired. $\square$

Here are several examples:

EXAMPLE 2.6.2. Let M be a metric space. The set

$$\mathfrak{B} = \{B_r(p) \mid p \in M \text{ and } r > 0\}$$

of open balls in M is a basis for the metric space topology on M . It is immediate from the definitions that a set $U \subset M$ is open in the metric space topology if and only if it is a union of elements of $\mathfrak{B}$. The only thing we must therefore check is that $\mathfrak{B}$ satisfies the axioms of a basis. For this, the nontrivial thing to verify is that if $B_r(p), B_{r'}(p') \in \mathfrak{B}$, then $B_r(p) \cap B_{r'}(p')$ is a union of elements of $\mathfrak{B}$. Let $x \in B_r(p) \cap B_{r'}(p')$. It is enough to find some $s > 0$ with $B_s(x) \subset B_r(p) \cap B_{r'}(p')$. Set

$$s = \min(r - \mathfrak{d}(x, p), r' - \mathfrak{d}(x, p')) > 0.$$

For $y \in B_s(x)$, we have

$$\begin{aligned} \mathfrak{d}(y, p) &\leq \mathfrak{d}(y, x) + \mathfrak{d}(x, p) < (r - \mathfrak{d}(x, p)) + \mathfrak{d}(x, p) = r, \\ \mathfrak{d}(y, p') &\leq \mathfrak{d}(y, x) + \mathfrak{d}(x, p') < (r' - \mathfrak{d}(x, p')) + \mathfrak{d}(x, p') = r'. \end{aligned}$$

It follows that $y \in B_r(p) \cap B_{r'}(p')$, so $B_s(x) \subset B_r(p) \cap B_{r'}(p')$, as desired. $\square$

EXAMPLE 2.6.3. Let S be a set with a total ordering $\leq$. For $s_1, s_2 \in S$ with $s_1 < s_2$, let $(s_1, s_2) = \{s \in S \mid s_1 < s < s_2\}$. The set

$$\mathfrak{B} = \{(s_1, s_2) \mid s_1, s_2 \in S, s_1 < s_2\}$$

is a basis for a topology on S (see Exercise 2.10) called the *order topology*. For instance, if $S = \mathbb{R}$ with its usual ordering the order topology is the same as the standard topology on $\mathbb{R}$. $\square$

EXAMPLE 2.6.4. Let X and Y be topological spaces. Let $\mathcal{B}$ be the set of subsets of $X \times Y$ of the form $U \times V$ with $U \subset X$ and $V \subset Y$ open. This is a basis for a topology on $X \times Y$ (see Exercise 2.11). We call this the *product topology*, and will discuss it extensively in Chapter 11. $\square$

EXAMPLE 2.6.5. Consider the real line $\mathbb{R}$. Let $\mathcal{B}$ be the set $\{[a, b) \mid a < b\}$ of all half-open intervals in $\mathbb{R}$. This is the basis for a topology on $\mathbb{R}$ called the *lower limit topology* (see Exercise 2.12). The real line $\mathbb{R}$ equipped with the lower limit topology is called the *Sorgenfrey line*. $\square$

REMARK 2.6.6. There is also the weaker notion of a subbasis; see §13.1. $\square$

2.7. Subspaces, embeddings, and open/closed maps

Let X be a space and let $Y \subset X$ be a subset. We would like to make Y into a space. Letting $\iota: Y \rightarrow X$ be the inclusion, the topology we impose on Y should make ι into a continuous function. For an open set $U \subset X$, we therefore need $\iota^{-1}(U) = U \cap Y$ to be open in Y . This suggests the following: the *subspace topology* on Y is the topology whose open sets $V \subset Y$ are the sets of the form $V = U \cap Y$ for an open set $U \subset X$. Unless we say otherwise, all subspaces are given the subspace topology.

Let $f: X \rightarrow Y$ be a continuous map. By definition, $f^{-1}(U) \subset X$ is open if $U \subset Y$ is open, and also $f^{-1}(C) \subset X$ is closed if $C \subset Y$ is closed. We say that the map f is *open* if $f(V) \subset Y$ is

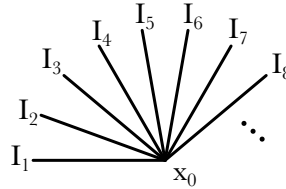
open for all $V \subset X$ open. Similarly, the map f is *closed* if $f(D) \subset Y$ is closed for all $D \subset X$ closed. Homeomorphisms are both open and closed.

A map $f: X \rightarrow Y$ is an *embedding* if it is a homeomorphism onto its image. In other words, an embedding f is a continuous injection onto a subspace $f(X)$ of Y such that the inverse map $f^{-1}: f(X) \rightarrow X$ is continuous. An injection which is either open or closed is automatically an embedding. For a subspace A of X , the inclusion $\iota: A \rightarrow X$ is an embedding. If $A \subset X$ is open (resp. closed), then ι is an open embedding (resp. a closed embedding).

2.8. Gluing intervals

We now return to Example 1.6.1 and explain how the notion of a topological space fixes its pathological behavior.

EXAMPLE 2.8.1. For each integer $n \geq 1$, let I_n be a copy of the interval $I = [0, 1]$. Let M be the set obtained from the disjoint union of all the I_n by identifying the points $0 \in I_n$ together to a single point x_0 :



Endow M with the following topology:

- A set $U \subset M$ is open if and only if $U \cap I_n$ is open for all $n \geq 1$. It is easy to see that this does indeed give a topology (see Exercise 2.4).

It is immediate from this definition that a function $f: M \rightarrow \mathbb{R}$ is continuous if and only if $f|_{I_n}: I_n \rightarrow \mathbb{R}$ is continuous for all $n \geq 1$. In particular, the function $f: M \rightarrow \mathbb{R}$ from Example 1.6.1 defined via the formula $f(p) = np$ for $p \in I_n$ is continuous. We will later see that M is not metrizable (see Exercise 5.4). $\square$

REMARK 2.8.2. The topology we imposed on the space M in Example 2.8.1 is an example of an identification space topology. See Chapter 3 below for more details about this. $\square$

2.9. Exercises

EXERCISE 2.1. Give an example of a topological space X such that there exist open subsets $\{U_i \mid i \geq 1\}$ of X for which $\bigcap_{i=1}^{\infty} U_i$ is not open. $\square$

EXERCISE 2.2. Let X be a topological space and let $A \subset X$. Assume that for each $a \in A$ there exists a neighborhood N of a with $N \subset A$. Note that we are not assuming that N is open. Prove that A is open. $\square$

EXERCISE 2.3. Let X and Y be topological spaces and let $f: X \rightarrow Y$ be a set map. Prove that f is continuous if the following holds:

- For all $x \in X$ and open neighborhoods $V \subset Y$ of $f(x)$, there exists an open neighborhood U of x with $f(U) \subset V$. $\square$

EXERCISE 2.4. Let Y be a set. For each $i \in I$, let X_i be a subset of Y . Assume that X_i is endowed with a topology. Prove that the following gives a topology on Y :

- A set $U \subset Y$ is open if and only if $U \cap X_i$ is open for all $i \in I$. $\square$

EXERCISE 2.5. Let X be a space and $\mathfrak{A}$ be a collection of subsets of X such that $X = \bigcup_{A \in \mathfrak{A}} A$. Let Y be another space and let $f: X \rightarrow Y$ be a map of sets (not necessarily continuous) such that $f|_A$ is continuous for all $A \in \mathfrak{A}$.

- Assume that each $A \in \mathfrak{A}$ is open. Prove that f is continuous.
- Assume that each $A \in \mathfrak{A}$ is closed and that $\mathfrak{A}$ is finite. Prove that f is continuous.
- Construct an example where each $A \in \mathfrak{A}$ is closed but f is *not* continuous. $\square$

EXERCISE 2.6. Formulate a set of axioms for a collection of closed sets to be the closed sets of a topology. Prove that this gives the same notion of a topological space as the axiomization we gave using open sets. $\square$

EXERCISE 2.7. Let M be a metrizable space and let $A \subset M$ be a subset. Prove that the interior $\text{Int}(A)$ and closure $\bar{A}$ defined in §2.3 agrees with the definitions for metric spaces given in §1.7 and §1.9. $\square$

EXERCISE 2.8. Prove the following:

- (a) Let $(M, \mathfrak{d})$ be a metric space. Define $\mathfrak{d}': M \times M \rightarrow \mathbb{R}$ via the formula $\mathfrak{d}'(p, q) = \min\{\mathfrak{d}(p, q), 1\}$. Prove that $\mathfrak{d}'$ is a metric on M that induces the same topology on M that $\mathfrak{d}$ does.
- (b) Let $\| - \|$ be the following standard norm on $\mathbb{R}^n$:

$$\|(x_1, \dots, x_n)\| = \sqrt{x_1^2 + \dots + x_n^2} \quad \text{for all } (x_1, \dots, x_n) \in \mathbb{R}^n.$$

This induces the metric $\mathfrak{d}(p, q) = \|p - q\|$ on $\mathbb{R}^n$. Now let $\| - \|'$ be an arbitrary norm on the vector space $\mathbb{R}^n$. Define a function $\mathfrak{d}': \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ via the formula

$$\mathfrak{d}'(p, q) = \|p - q\|' \quad \text{for } p, q \in \mathbb{R}^n.$$

Prove that $\mathfrak{d}'$ is a metric on $\mathbb{R}^n$ and that $\mathfrak{d}'$ induces the same topology on $\mathbb{R}^n$ as $\mathfrak{d}$. $\square$

EXERCISE 2.9. Let $\mathbf{k}$ be a field. Prove that the Zariski topology on $\mathbf{k}^n$ described in Example 2.5.2 is a topology. $\square$

EXERCISE 2.10. Let S be a set with a total ordering $\leq$. For $s_1, s_2 \in S$ with $s_1 < s_2$, let $(s_1, s_2) = \{s \in S \mid s_1 < s < s_2\}$. Prove that the collection of all sets of the form (s_1, s_2) forms a basis for a topology on S . $\square$

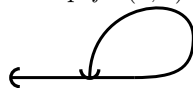
EXERCISE 2.11. Let X and Y be topological spaces. Let $\mathcal{B}$ be the set of subsets of $X \times Y$ of the form $U \times V$ with $U \subset X$ and $V \subset Y$ open. Prove that $\mathcal{B}$ is the basis for a topology on $X \times Y$. This is called the *product topology*. $\square$

EXERCISE 2.12. Consider the real line $\mathbb{R}$.

- (a) Let $\mathcal{B}$ be the set $\{(a, b) \mid a < b\}$ of all half-open intervals in $\mathbb{R}$. Prove that $\mathcal{B}$ is the basis for a topology on $\mathbb{R}$ called the *lower limit topology*.
- (b) Prove that the lower limit topology on $\mathbb{R}$ is finer than the standard topology.

The real line $\mathbb{R}$ equipped with the lower limit topology is called the *Sorgenfrey line*. $\square$

EXERCISE 2.13. Consider the injective map $f: (0, 1) \rightarrow \mathbb{R}^2$ whose image X is as follows:



Regard f as a bijective map $f: (0, 1) \rightarrow X$. Prove that $f^{-1}: X \rightarrow (0, 1)$ is not continuous. $\square$

Identification spaces and the quotient topology

We now explain how to construct a new space from a collection of existing ones by identifying certain points together. This generalizes the construction we gave in Example 2.8.1.

3.1. Identification spaces

Let $\{X_i\}_{i \in I}$ be a collection of spaces. An *identification space* is a topological space Y equipped with maps $f_i: X_i \rightarrow Y$ for each $i \in I$ such that:

- each $y \in Y$ is in the image of some f_i ; and
- a set $U \subset Y$ is open if and only if $f_i^{-1}(U) \subset X_i$ is open for all $i \in I$.

The second condition ensures that each $f_i: X_i \rightarrow Y$ is continuous. It also ensures that for a space Z a map of sets $\phi: Y \rightarrow Z$ is continuous if and only if $\phi \circ f_i: X_i \rightarrow Z$ is continuous for all $i \in I$ (we will say more about this in §3.4 below).

EXAMPLE 3.1.1. Let $\{X_i\}_{i \in I}$ be a collection of spaces. As a set, let Y be the disjoint union $\sqcup_{i \in I} X_i$ of the X_i . Topologize Y by letting $U \subset Y$ be open if and only if $U \cap X_i$ is open for all $i \in I$ (see Exercise 2.4). The space Y is an identification space under the inclusion maps $f_i: X_i \rightarrow Y$. $\square$

Now assume that the $\{X_i\}_{i \in I}$ are spaces and $\sim$ is an equivalence relation on the disjoint union $\sqcup_{i \in I} X_i$. Define

$$Y = \sqcup_{i \in I} X_i / \sim.$$

Informally, Y is obtained from the X_i by identifying some of their points together. Let $f_i: X_i \rightarrow Y$ be the projections. Give Y the following topology:

- A set $U \subset Y$ is open if and only if $f_i^{-1}(U) \subset X_i$ is open for all $i \in I$. This is easily seen to give a topology (see Exercise 3.1).

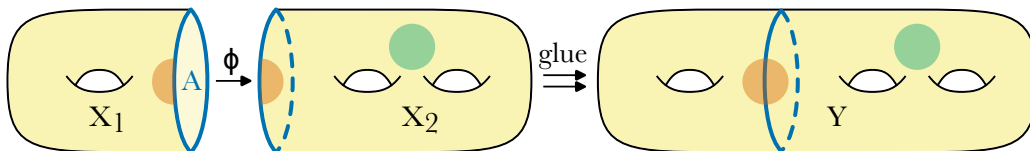
We call this the *identification space topology* on Y . It makes Y into an identification space. If we have a construction of a purported “space” from the points of the X_i , then this gives a canonical way of turning our purported “space” into a topological space.

3.2. Examples

The above discussion is a little abstract. Here are some examples.

EXAMPLE 3.2.1 (Gluing). Let X_1 and X_2 be spaces, $A \subset X_1$ be a subspace, and $\phi: A \rightarrow X_2$ be a map. As a set, let Y be the disjoint union of X_1 and X_2 modulo the equivalence relation that identifies each $a \in A \subset X_1$ with $\phi(a) \in X_2$. There are natural maps $f_1: X_1 \rightarrow Y$ and $f_2: X_2 \rightarrow Y$, and we give Y the identification space topology. We call Y the space obtained by *gluing* X_1 to X_2 via the gluing map ϕ . If $A \subset X_1$ is closed, then $f_2: X_2 \rightarrow Y$ is a closed embedding and $f_1|_{X_1 \setminus A}: X_1 \setminus A \rightarrow Y$ is an open embedding (see Exercise 3.2). If A is not closed, then things can be much more complicated. $\square$

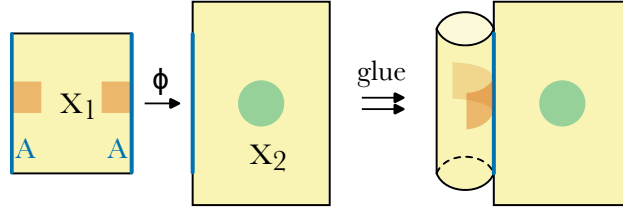
EXAMPLE 3.2.2. Here is one easy example of gluing. Let X_1 and X_2 and Y be the following surfaces:



As is shown in this figure, Y is obtained by gluing the boundary component $A \cong \mathbb{S}^1$ of X_1 to the

boundary component of X_2 via a homeomorphism. Two open sets on Y are drawn together with their preimages in X_1 and X_2 . In this example, ϕ is a homeomorphism onto its image and both X_1 and X_2 are subspaces of Y . $\square$

EXAMPLE 3.2.3. Here is another example of gluing where the gluing map is not injective:

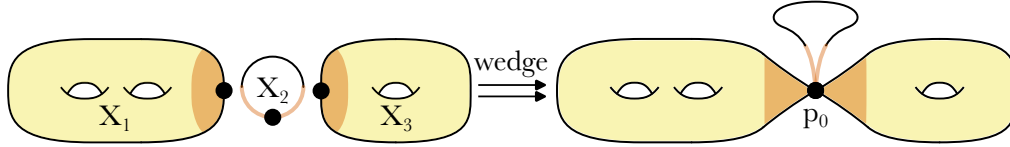


In this figure, ϕ identifies the two blue vertical edges of the rectangle X_1 with a single segment in the left-hand vertical edge of X_2 . $\square$

REMARK 3.2.4. The most important class of spaces in algebraic topology are CW complexes, which are constructed from collections of disks $\mathbb{D}^n$ of various dimensions n by gluing their boundaries together. See Essay G for more details. $\square$

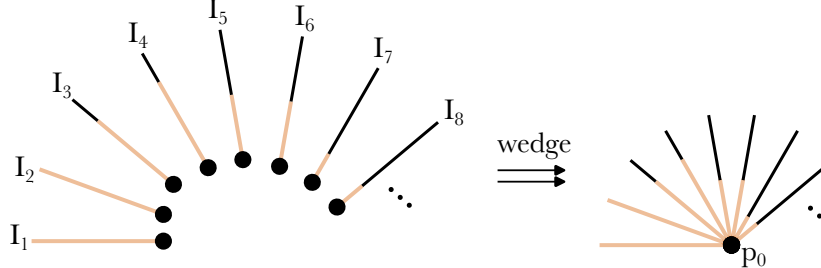
EXAMPLE 3.2.5 (Wedge product). Let $\{X_i\}_{i \in I}$ be a collection of topological spaces. Assume that each X_i has a distinguished basepoint $x_i \in X_i$. The *wedge product* of the X_i , denoted $\vee_{i \in I} X_i$, is the space obtained by identifying all the x_i together to a single point p_0 . There are inclusions $f_i: X_i \rightarrow \vee_{i \in I} X_i$, and we give $\vee_{i \in I} X_i$ the identification space topology. The maps f_i are all embeddings (see Exercise 3.3). $\square$

EXAMPLE 3.2.6. Here is an example of a wedge product:



This figure shows an open neighborhood of p_0 together with its preimage in the X_i . $\square$

EXAMPLE 3.2.7. Example 2.8.1 is the wedge product of countably many intervals $I_n = I$ equipped with the basepoints $0 \in I_n$. Here is a picture of this, with an open neighborhood p_0 together with its preimage in the I_n indicated:

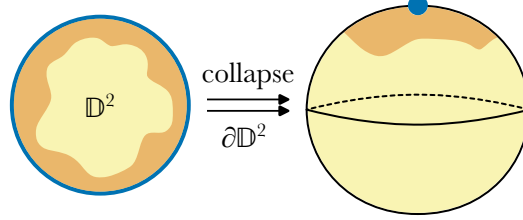


See Exercise 5.4 for an explanation of why this topology does not come from a metric. $\square$

EXAMPLE 3.2.8 (Collapsing subspace). In an identification space, we allow there to only be a single space X . As an example of this, let X be a space and let $A \subset X$ be a subspace. Denote by X/A the result of collapsing A to a single point. The points of X/A are thus the points of $X \setminus A$ together with a single point $[A]$ corresponding to A . Letting $f: X \rightarrow X/A$ be the projection, we can endow X/A with the identification space topology. If A is a closed (resp. open) set, then the restriction of f to $X \setminus A$ is an open (resp. closed) embedding (see Exercise 3.4). $\square$

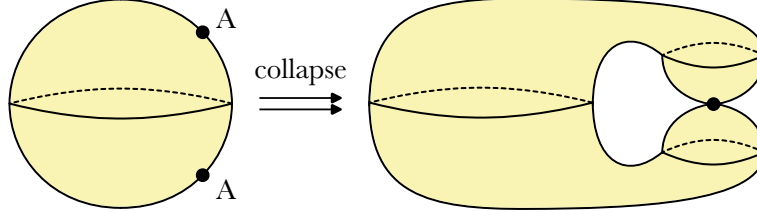
REMARK 3.2.9. Collapsing a subspace $A \subset X$ will typically yield a pathological space X/A if A is not closed. For instance, in most natural spaces single points are closed, and this will only hold for the point $[A]$ of X/A if $A \subset X$ is closed. As an example of how pathological this can be, collapsing the subspace $\mathbb{Q}$ of $\mathbb{R}$ gives a terrible space $\mathbb{R}/\mathbb{Q}$. $\square$

EXAMPLE 3.2.10. Consider the boundary $\partial\mathbb{D}^n = \mathbb{S}^{n-1}$. As the following shows, $\mathbb{D}^n/\partial\mathbb{D}^n \cong \mathbb{S}^n$, with the blue $\partial\mathbb{D}^n$ mapping to the north pole of $\mathbb{S}^n$:



As this figure shows, a neighborhood of the north pole in $\mathbb{S}^n$ lifts to a neighborhood of $\partial\mathbb{D}^n$. □

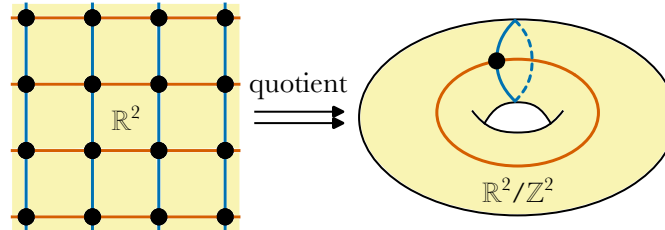
EXAMPLE 3.2.11. The set A need not be connected. Here is an example:



In this example, $X = \mathbb{S}^2$ and A is two points on X . □

EXAMPLE 3.2.12 (Quotienting by group action). Let X be a space and let G be a group acting on X . As a set, X/G consists of the orbits of X under the action of G . Letting $f: X \rightarrow X/G$ be the quotient map, we endow X/G with the identification space topology. The point-set topological properties of X/G depend both on the properties of X and the qualities of the group action G . We explore this in Essay I. □

EXAMPLE 3.2.13. Let the group $\mathbb{Z}^2$ act on $\mathbb{R}^2$ by translations. The quotient $\mathbb{R}^2/\mathbb{Z}^2$ is homeomorphic to the 2-torus:



The orange and blue loops on $\mathbb{R}^2/\mathbb{Z}^2$ lift to the orange and blue parallel lines on $\mathbb{R}^2$. □

3.3. Quotient topology

A map $f: X \rightarrow Y$ is a *quotient map* if f is surjective and $U \subset Y$ is open if and only if $f^{-1}(U) \subset X$ is open. Given a space X and a surjection of sets $f: X \rightarrow Y$, the *quotient topology* on Y is the topology making $f: X \rightarrow Y$ a quotient map. We call Y a *quotient space* of X .

Of course, this is a special case of an identification space. Moreover, given a collection of spaces $\{X_i\}_{i \in I}$ and an identification space Y of the X_i with maps $f_i: X_i \rightarrow Y$, we can realize Y as a quotient space in the following way. Let $\sqcup_{i \in I} X_i$ be the disjoint union of the X_i (see Example 3.1.1). The maps $f_i: X_i \rightarrow Y$ then assemble to a quotient map $F: \sqcup_{i \in I} X_i \rightarrow Y$.

REMARK 3.3.1. Many treatments of point-set topology only talk about quotient spaces, but we find it convenient to use the slightly more general notion of identification spaces since like in the examples from earlier in this chapter, we often use them to build a space out of several spaces, not just one. □

3.4. Universal mapping property

Let $f: X \rightarrow Y$ be a quotient map. Let $\sim$ be the equivalence relation on the set X where $p \sim q$ if and only if $f(p) = f(q)$. The equivalence classes of $\sim$ are the fibers $f^{-1}(y)$ for $y \in Y$. Letting Z be

another space and $\phi: Y \rightarrow Z$ be a continuous map, the composition $\Phi = \phi \circ f$ is a continuous map $\Phi: X \rightarrow Z$ that is $\sim$ -invariant, i.e., that has $\Phi(p) = \Phi(q)$ whenever $p \sim q$. Conversely, if $\Phi: X \rightarrow Z$ is a continuous $\sim$ -invariant map, then there is a set map $\phi: Y \rightarrow Z$ such that $\Phi = \phi \circ f$ and the quotient topology on Y is set up to ensure that ϕ is continuous.

The above discussion shows that composition with f gives a bijection between continuous maps $\phi: Y \rightarrow Z$ and $\sim$ -invariant continuous maps $\Phi: X \rightarrow Z$. This is an example of a *universal mapping property*, and we will describe it informally by saying that a map $\phi: Y \rightarrow Z$ is the same as a $\sim$ -invariant map $\Phi: X \rightarrow Z$. This universal mapping property characterizes quotient spaces (see Exercise 3.8).

EXAMPLE 3.4.1 (Disjoint union). Let $\{X_i\}_{i \in I}$ be a collection of spaces. Let $Y = \sqcup_{i \in I} X_i$ with the disjoint union topology discussed in Example 3.1.1. For a space Z , maps $\phi: \sqcup_{i \in I} X_i \rightarrow Z$ are the same as collections of maps $\Phi_i: X_i \rightarrow Z$ for each $i \in I$. $\square$

REMARK 3.4.2. If $\mathbf{C}$ is a category and $\{C_i\}_{i \in I}$ are objects of $\mathbf{C}$, then a *categorical sum* of the C_i is an object D of $\mathbf{C}$ together with morphisms $\{f_i: C_i \rightarrow D\}_{i \in I}$ such that the following holds:

- For all objects E of $\mathbf{C}$, there is a bijection between morphisms $\phi: D \rightarrow E$ and collections of morphisms $\{\Phi_i: C_i \rightarrow E\}_{i \in I}$ taking a morphism $\phi: D \rightarrow E$ to the collection of morphisms $\{\phi \circ f_i: C_i \rightarrow E\}_{i \in I}$.

Categorical sums might or might not exist, but if they do exist they are unique up to isomorphism (see Exercise 3.9). For spaces $\{X_i\}_{i \in I}$, the disjoint union $U = \sqcup_{i \in I} X_i$ together with the natural inclusions $\{\iota_i: X_i \hookrightarrow U\}_{i \in I}$ is therefore the categorical sum of the $\{X_i\}_{i \in I}$ in the category $\mathbf{Top}$. See Exercise 3.10 for what the categorical sum means in the category of abelian groups. $\square$

REMARK 3.4.3. There are also categorical products, which we will discuss in Chapter 11. $\square$

EXAMPLE 3.4.4 (Wedge product). Let $\{X_i\}_{i \in I}$ be a collection of topological spaces. Assume that each X_i has a distinguished basepoint $x_i \in X_i$. For a space Z , maps $\phi: \sqcup_{i \in I} X_i \rightarrow Z$ are the same as collections of maps $\Phi_i: X_i \rightarrow Z$ such that $\Phi_i(x_i) = \Phi_j(x_j)$ for all $i, j \in I$. In particular, this explains why the quotient topology is the right one to ensure the real-valued function in Example 2.8.1 is continuous. $\square$

REMARK 3.4.5. Let $\mathbf{Top}_*$ be the following category:

- The objects of $\mathbf{Top}_*$ are pairs (X, x_0) with X a space and $x_0 \in X$ a basepoint.
- The morphism in $\mathbf{Top}_*$ from (X, x_0) to (Y, y_0) are continuous maps $f: X \rightarrow Y$ with $f(x_0) = y_0$.

The universal mapping property from Example 3.4.4 says that the wedge product is the categorical sum in the category $\mathbf{Top}_*$. $\square$

EXAMPLE 3.4.6 (Collapsing subspace). Let X be a space and let $A \subset X$ be a subspace. For a space Z , maps $\phi: X/A \rightarrow Z$ are the same as maps $\Phi: X \rightarrow Z$ such that $\Phi(A)$ is a single point. $\square$

EXAMPLE 3.4.7 (Quotienting by group action). Let X be a space and let G be a group acting on X . For a space Z , maps $\phi: X/G \rightarrow Z$ are the same as maps $\Phi: X \rightarrow Z$ that are G -invariant in the sense that $\Phi(g \cdot x) = \Phi(x)$ for all $x \in X$ and $g \in G$. $\square$

3.5. Exercises

EXERCISE 3.1. Let $\{X_i\}_{i \in I}$ be a collection of spaces. Let Y be a set, and for all $i \in I$ let $f_i: X_i \rightarrow Y$ be a set map. Prove that the following gives a topology on Y :

- A set $U \subset Y$ is open if and only if $f_i^{-1}(U) \cap X_i$ is open for all $i \in I$. $\square$

EXERCISE 3.2. Let X_1 and X_2 be spaces, let $A \subset X_1$ be a closed subspace, and let $\phi: A \rightarrow X_2$ is a map. Let Y be the space obtained by gluing X_1 to X_2 via the gluing map ϕ , and let $f_1: X_1 \rightarrow Y$ and $f_2: X_2 \rightarrow Y$ be the natural maps.

- Prove that $f_2: X_2 \rightarrow Y$ is a closed embedding.
- Prove that $f_1|_{X_1 \setminus A}: X_1 \setminus A \rightarrow Y$ is an open embedding. $\square$

EXERCISE 3.3. Let $\{X_i\}_{i \in I}$ be a collection of topological spaces. Assume that each X_i has a distinguished basepoint $x_i \in X_i$, and consider the edge product $\vee_{i \in I} X_i$. For each $i \in I$, let $f_i: X_i \rightarrow \vee_{i \in I} X_i$ be the inclusion.

- (a) For each $i \in I$, prove that the map f_i is an embedding.
- (b) If each basepoint $x_i \in X_i$ is closed (as a one-point subspace of X_i) then prove that each f_i is a closed embedding. $\square$

EXERCISE 3.4. Let X be a space, let $A \subset X$ be a subspace, and let $f: X \rightarrow X/A$ be the projection.

- (a) If A is closed, then prove that the restriction of f to $X \setminus A$ is an open embedding.
- (b) If A is open, then prove that the restriction of f to $X \setminus A$ is a closed embedding. $\square$

EXERCISE 3.5. Let $f: X \rightarrow Y$ be a quotient map. Let $B \subset Y$ be either open or closed and let $A = f^{-1}(B)$. Prove that $f|_A: A \rightarrow B$ is a quotient map. $\square$

EXERCISE 3.6. Let $f: X \rightarrow Y$ be a surjective continuous map that is either open or closed. Prove that f is a quotient map. $\square$

EXERCISE 3.7. Let $\sim_1$ and $\sim_2$ be the following equivalence relations on $\mathbb{R}^2$:

$$\begin{aligned} (x, y) \sim_1 (z, w) & \text{ if } x + y^2 = z + w^2, \\ (x, y) \sim_2 (z, w) & \text{ if } x^2 + y^2 = z^2 + w^2. \end{aligned}$$

For $i = 1, 2$, let X_i be the quotient of $\mathbb{R}^2$ by $\sim_i$. Identify the spaces X_1 and X_2 . $\square$

EXERCISE 3.8. Let X be a space and let $\sim$ be an equivalence relation on X . As a set, let $Y = X/\sim$ and let $f: X \rightarrow Y$ be the projection. Endow Y with the quotient topology, so $f: X \rightarrow Y$ is a quotient map. Let Y' be another space and let $f': X \rightarrow Y'$ be a map such that the following holds:

- For all spaces Z , composition with f' gives a bijection between continuous maps $\phi: Y' \rightarrow Z$ and $\sim$ -invariant continuous maps $\Phi: X \rightarrow Z$.

Prove that there is a homeomorphism $g: Y' \rightarrow Y$ such that $f' = g \circ f$. In other words, the above universal mapping property characterizes the quotient space Y . $\square$

EXERCISE 3.9. Let $\mathbf{C}$ be a category and let $\{C_i\}_{i \in I}$ be objects of $\mathbf{C}$. For $k = 1, 2$, let $D(k)$ be a categorical sum of the $\{C_i\}_{i \in I}$ with morphisms $\{f_i(k): C_i \rightarrow D(k)\}_{i \in I}$. Prove that there exists an isomorphism $\lambda: D(1) \rightarrow D(2)$ such that $f_i(2) = \lambda \circ f_i(1)$ for all $i \in I$. This can be interpreted as saying that categorical sums are unique up to isomorphism. $\square$

EXERCISE 3.10. Let $\mathbf{AbGrp}$ be the category of abelian groups and group homomorphisms. Let $\{A_i\}_{i \in I}$ be a collection of abelian groups. Let $\oplus_{i \in I} A_i$ be the sum of the A_i , so

$$\bigoplus_{i \in I} A_i = \left\{ (a_i)_{i \in I} \in \prod_{i \in I} A_i \mid a_i = 0 \text{ for all but finitely many } i \in I \right\}.$$

Prove that $S = \oplus_{i \in I} A_i$ together with the natural inclusions $\{\iota_i: A_i \hookrightarrow S\}_{i \in I}$ is the categorical sum of the $\{A_i\}_{i \in I}$ in the category $\mathbf{AbGrp}$. We remark that in the exercises to Chapter 11 we will prove that $\prod_{i \in I} A_i$ is the categorical product of the $\{A_i\}_{i \in I}$ (which in particular requires defining the categorical product). $\square$

CHAPTER 4

Connectivity properties

Our next topic is connectivity and path connectivity.

4.1. Path connectivity

Recall that $I = [0, 1]$. A *path* in a space X from $p \in X$ to $q \in X$ is a map $\gamma: I \rightarrow X$ with $\gamma(0) = p$ and $\gamma(1) = q$:



The space X is *path connected* if for all $p, q \in X$ there exists a path in X from p to q .

EXAMPLE 4.1.1. The space $\mathbb{R}^n$ is path connected. Indeed, for $p, q \in \mathbb{R}^n$ the map $\gamma: I \rightarrow \mathbb{R}^n$ defined by

$$\gamma(t) = (1 - t)p + tq \quad \text{for } t \in I$$

is a path from p to q . □

4.2. Connectivity

We now turn to connectivity. It is easier to say what it means for a space to be disconnected. A space X is *disconnected* if we can write $X = U \sqcup V$ with $U, V \subset X$ disjoint nonempty open subsets of X . Since $X \setminus U = V$ and $X \setminus V = U$, the sets U and V are necessarily closed as well as open. Sets that are both open and closed are called *clopen sets*.¹

EXAMPLE 4.2.1. The space $\mathbb{R} \setminus 0$ is disconnected. Indeed, $\mathbb{R} \setminus 0 = (-\infty, 0) \sqcup (0, \infty)$. □

EXAMPLE 4.2.2. The space $\mathbb{Q}$ is disconnected. Indeed, for $U \subset \mathbb{R}$ let $U_{\mathbb{Q}} = U \cap \mathbb{Q}$. We then have $\mathbb{Q} = (-\infty, \sqrt{2})_{\mathbb{Q}} \sqcup (\sqrt{2}, \infty)_{\mathbb{Q}}$. □

A space X is *connected* if it is not disconnected. Another way of saying this is that X is connected if whenever $U, V \subset X$ are disjoint open sets with $X = U \sqcup V$, then either $U = X$ or $V = X$ (see Exercise 4.1).

REMARK 4.2.3. It is a little annoying that $\emptyset$ is connected and path connected. In later volumes when we talk about things like homology, there will be many places where the hypotheses of results will include the phrase “path connected and nonempty”. I was tempted to insist that connected and path connected spaces be nonempty, but decided that it was better to retain the traditional terminology. □

4.3. Basic properties of connectivity

Here are some basic properties of connected spaces:

LEMMA 4.3.1. *Let X be a space and let $Y \subset X$ be a connected subspace. Then the closure $\overline{Y}$ of Y is connected.*

¹This is a terrible term, but is the standard word for this.

PROOF. Replacing X with $\bar{Y}$, we can assume that $\bar{Y} = X$. Our goal now is to prove that X is connected. Let $U, V \subset X$ be disjoint open subsets such that $X = U \sqcup V$. We must prove that either $U = X$ or $V = X$. Since Y is connected and $Y = (U \cap Y) \sqcup (V \cap Y)$, we must have either $U \cap Y = Y$ or $V \cap Y = Y$. Reordering U and V , we can assume that $U \cap Y = Y$, and hence $Y \subset X \setminus V$. Since $X \setminus V$ is closed, we have $\bar{Y} \subset X \setminus V$. Since $\bar{Y} = X$ and $X = U \sqcup V$, this implies that $U = X$, as desired. $\square$

LEMMA 4.3.2. *Let $f: X \rightarrow Y$ be a continuous map from a connected space X to a space Y . Then $f(X)$ is connected.*

PROOF. Replacing Y with $f(X)$, we can assume that $f(X) = Y$. Our goal now is to prove that Y is connected. Assume that $U, V \subset Y$ are disjoint open sets such that $Y = U \sqcup V$. We must prove that either $U = Y$ or $V = Y$. Since $f(X) = Y$, this is equivalent to showing that either $f^{-1}(U) = X$ or $f^{-1}(V) = X$. Since $Y = U \sqcup V$, we have that $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint open subsets of X such that $X = f^{-1}(U) \sqcup f^{-1}(V)$. Since X is connected, we must therefore have either $f^{-1}(U) = X$ or $f^{-1}(V) = X$, as desired. $\square$

LEMMA 4.3.3. *Let X be a space and let $\{Y_i\}_{i \in I}$ be a collection of subspaces of X . Assume that:*

- *each Y_i is connected; and*
- *for all $i, j \in I$, the space $Y_i \cap Y_j$ is nonempty; and*
- *$X = \cup_{i \in I} Y_i$.*

Then X is connected.

PROOF. This is trivial if $X = \emptyset$, so assume that $X \neq \emptyset$. Assume that $U, V \subset X$ are disjoint open sets such that $X = U \sqcup V$. We prove that either $U = X$ or $V = X$. Pick some $i_0 \in I$. Since Y_{i_0} is connected, either $U \cap Y_{i_0} = Y_{i_0}$ or $V \cap Y_{i_0} = Y_{i_0}$. Reordering U and V , assume that $U \cap Y_{i_0} = Y_{i_0}$. Now consider an arbitrary $i \in I$. Since $Y_i \cap Y_{i_0}$ is nonempty, we have $U \cap Y_i \neq \emptyset$. Since Y_i is connected, this implies that $U \cap Y_i = Y_i$. Since $U \cap Y_i = Y_i$ for all $i \in I$, we conclude that $U = X$, as desired. $\square$

4.4. Path connected spaces are connected

We now prove:

LEMMA 4.4.1. *Let X be a path connected space. Then X is connected.*

PROOF. This is trivial if $X = \emptyset$, so assume that $X \neq \emptyset$. Fix a point $p \in X$. For each $q \in X$, pick a path $\gamma_q: I \rightarrow X$ from p to q . Set $Y_q = \gamma_q(I)$. Below in Lemma 4.4.2 we will prove that I is connected. Lemma 4.3.2 therefore implies that Y_q is connected. The space X is the union of the Y_q , and for $q, q' \in X$ we have $p \in Y_q \cap Y_{q'}$. By Lemma 4.3.3, we conclude that X is connected. $\square$

The above proof required:

LEMMA 4.4.2. *The space $I = [0, 1]$ is connected.*

PROOF. Assume for the sake of contradiction that I is disconnected. We can therefore write $I = U \sqcup V$ with $U, V \subset I$ disjoint nonempty open sets. Define a set map $f: I \rightarrow \mathbb{R}$ via the formula

$$f(x) = \begin{cases} 0 & \text{if } x \in U, \\ 1 & \text{if } x \in V. \end{cases}$$

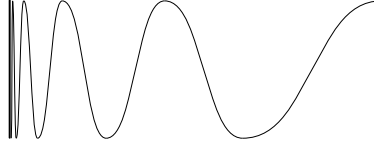
This is continuous; indeed, if $W \subset \mathbb{R}$ is any open set then $f^{-1}(W)$ is either $\emptyset$ or U or V or $U \cup V = I$. Since the image of f contains both 0 and 1, the intermediate value theorem implies that the image of f contains $1/2$. Since the image of f is $\{0, 1\}$, this is a contradiction. $\square$

The converse of Lemma 4.4.1 is not true:

EXAMPLE 4.4.3. Let X be the following subset of $\mathbb{R}^2$:

$$X = \{(0, y) \mid -1 \leq y \leq 1\} \cup \{(x, \sin(1/x)) \mid x > 0\}.$$

This is a closed subset of $\mathbb{R}^2$ that is often called the *topologist's sine-curve*:



The space X is *not* path connected; indeed, there is no path connecting $(0,0)$ and $(x, \sin(1/x))$ for any $x > 0$ (see Exercise 4.4). However, X is connected (see Exercise 4.4). $\square$

4.5. Path components

Let X be a space. Let $\sim$ be the following relation on the points of X : for $p, q \in X$, we have $p \sim q$ if there is a path in X from p to q . This is an equivalence relation on the points of X (see Exercise 4.2), and its equivalence classes are the *path components* of X . It is immediate from the definition that the path components of X are path connected and that X is the disjoint union of its path components.

EXAMPLE 4.5.1. Let X be the topologist's sine-curve from Example 4.4.3. The path components of X are as follows (see Exercise 4.4):

$$\begin{aligned} X_1 &= \{(0, y) \mid -1 \leq y \leq 1\}, \\ X_2 &= \{(x, \sin(1/x)) \mid x > 0\}. \end{aligned}$$

$\square$

4.6. Connected components

Continue to let X be a space. Now let $\sim$ be the following relation on the points of X : for $p, q \in X$, we have $p \sim q$ if there is a connected subspace $Y \subset X$ with $p, q \in Y$. Just like for path connectedness, we have:

LEMMA 4.6.1. *Letting the notation be as above, $\sim$ is an equivalence relation on the points of X .*

PROOF. It is clear that $\sim$ is reflexive, i.e., that $p \sim p$ for all $p \in X$. It is also clear that $\sim$ is symmetric, i.e., that if $p \sim q$ for some $p, q \in X$ then $q \sim p$. We must therefore only prove that $\sim$ is transitive. Consider $p, q, r \in X$ such that $p \sim q$ and $q \sim r$. We must prove that $p \sim r$. Since $p \sim q$ and $q \sim r$, there are connected subspaces $Y, Z \subset X$ with $p, q \in Y$ and $q, r \in Z$. Since Y and Z are connected subspaces with $q \in Y \cap Z$, it follows from Lemma 4.3.3 that $Y \cup Z$ is connected. Since $p, r \in Y \cup Z$, we conclude that $p \sim r$, as desired. $\square$

The equivalence classes of $\sim$ are the *connected components* of X . The space X is the disjoint union of its connected components. They have the following properties:

LEMMA 4.6.2. *Let X be a space and let A be a connected component of X . Then A is closed and connected.*

PROOF. We first prove that A is connected. Fix some $a \in A$. For $p \in A$, there exists a connected subspace Y_p of X with $a, p \in Y_p$. Since Y_p is connected we have $a \sim q$ for all $q \in Y_p$, so $Y_p \subset A$. Since $A = \cup_{p \in A} Y_p$ and each Y_p contains the point a , Lemma 4.3.3 implies that A is connected, as desired. Since A is connected, Lemma 4.3.1 implies that $\bar{A}$ is connected. It follows that $A = \bar{A}$, i.e., that A is closed. $\square$

REMARK 4.6.3. Example 4.5.1 shows that path components need not be closed. $\square$

4.7. Local connectivity

Since a space X is disconnected if we can write $X = U \sqcup V$ with $U, V \subset X$ disjoint nonempty clopen subsets, it is natural to hope that the connected components of X are clopen. Unfortunately, this need not hold:

EXAMPLE 4.7.1. Let $X = \mathbb{Q}$. The connected components of X and the path components of X both consist of the one-point sets $\{q\}$ for $q \in \mathbb{Q}$. $\square$

REMARK 4.7.2. A space like $\mathbb{Q}$ whose connected components consist of single points is called *totally disconnected*. $\square$

As this example suggests, the cause of this is pathological local behavior. A space X is *locally connected* at $p \in X$ if for all open neighborhoods U of p , there is a connected open neighborhood V of p with $V \subset U$. The space X is *locally connected* if it is locally connected at all $p \in X$. Similarly, a space X is *locally path connected* at $p \in X$ if for all open neighborhoods U of p , there is a path connected open neighborhood V of p with $V \subset U$. The space X is *locally path connected* if it is locally path connected at all $p \in X$. We then have:

LEMMA 4.7.3. *Let X be a space. Then:*

- *If X is locally connected, then all connected components of X are clopen.*
- *If X is locally path connected, then all path components of X are clopen.*

PROOF. The proofs for components and path components are similar, so we will prove it for path components.² Assume that X is locally path connected. Let Y be a path component of X . For $p \in Y$, since X is locally connected we can find a path connected open neighborhood V of p . By definition, p is in the same path component as all the points of V , so $V \subset Y$. We deduce that Y is open. Since $X \setminus Y$ is the union of the path components of X that are different from Y and all those path components are open, it follows that $X \setminus Y$ is open and hence that Y is closed. $\square$

COROLLARY 4.7.4. *Let X be a locally path connected space. Then the connected components and path components of X coincide.*

PROOF. Let Y be a connected component of X . The subspace Y is the disjoint union of a collection of path components. To prove that it is actually a path component, it is enough to prove that Y is path connected. Assume otherwise. We can then write $Y = Y_1 \sqcup Y_2$ with each Y_i a nonempty union of path components and $Y_1 \cap Y_2 = \emptyset$. Lemma 4.7.3 implies that each path component is clopen, so both Y_1 and Y_2 are also clopen. Since $Y = Y_1 \sqcup Y_2$, we deduce that Y is disconnected, contradicting the fact that it is connected. $\square$

REMARK 4.7.5. As our examples show, not all metric spaces (or even subspaces of $\mathbb{R}^n$) are locally connected or locally path connected. However, most spaces that appear in algebraic topology are locally path connected. $\square$

4.8. Exercises

EXERCISE 4.1. Let X be a space. Prove the following:

- (a) The space X is connected if whenever $U, V \subset X$ are disjoint open sets with $X = U \sqcup V$, then either $U = X$ or $V = X$.
- (b) The space X is connected if whenever $U, V \subset X$ are disjoint open sets with $X = U \sqcup V$, then either $U = \emptyset$ or $V = \emptyset$.
- (c) The space X is connected if whenever $U, V \subset X$ are open sets with $X = U \sqcup V$, then $U \cap V \neq \emptyset$. $\square$

EXERCISE 4.2. Let X be a space. Prove that the following is an equivalence relation on the points of X :

- (a) For $p, q \in X$, the relation where p is equivalent to q if there is a path in X from p to q . $\square$

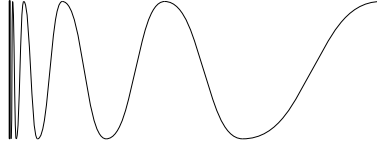
EXERCISE 4.3. Let X and Y be spaces. Give $X \times Y$ the product topology (see Example 2.6.4).

- (a) If X and Y are connected, prove that $X \times Y$ is connected.
- (b) If X and Y are path connected, prove that $X \times Y$ is path connected. Be careful to prove that your paths are continuous! $\square$

EXERCISE 4.4. Let X be the topologist's sine curve:

$$X = \{(0, y) \mid -1 \leq y \leq 1\} \cup \{(x, \sin(1/x)) \mid x > 0\} \subset \mathbb{R}^2.$$

²In fact, this case is slightly harder since in general path components need not be closed.



Prove that X is connected but not path connected. Also, prove that its path components are

$$X_1 = \{(0, y) \mid -1 \leq y \leq 1\},$$

$$X_2 = \{(x, \sin(1/x)) \mid x > 0\}.$$

□

EXERCISE 4.5. Let $I = [0, 1]$ and let $\leq$ be the dictionary order topology on I^2 , so $(x, y) \leq (z, w)$ if $x \leq z$ or if $x = z$ and $y \leq w$. Let X the topological space on the set I^2 with the corresponding order topology (see Example 2.6.3). Prove the following:

(a) X is connected.

(b) X is not path connected.

□

EXERCISE 4.6. Let C be the classical Cantor set (see Example 1.9.3), so C consists of all $x \in I = [0, 1]$ of the form

$$x = \sum_{n=1}^{\infty} \frac{x_n}{3^n} \quad \text{with } x_n \in \{0, 2\} \text{ for all } n \geq 1.$$

Prove that C is totally disconnected.

□

EXERCISE 4.7. Let S be the Sorgenfrey line (see Example 2.6.5), so S is $\mathbb{R}$ with the topology given by the basis $\{[a, b) \mid a < b\}$ of all half-open intervals in $\mathbb{R}$. Prove that S is totally disconnected. □

EXERCISE 4.8. Let X and Y be spaces such that there exists a homeomorphism $f: \mathbb{R} \rightarrow X \times Y$. Here we topologize $X \times Y$ using the product topology as in Example 2.6.4. Prove that either X or Y is a point. □

Countability properties

This chapter discussed properties that ensure a topological space is not “too large”.

5.1. First countability

Let X be a space. A *neighborhood basis* for X at a point $p \in X$ is a collection $\mathfrak{B}_p$ of open neighborhoods of p such that:

- For all open neighborhoods V of p , we have $U \subset V$ for some $U \in \mathfrak{B}_p$.

The space X is *first countable* if it has a countable neighborhood basis at each point $p \in X$. All metrizable spaces have this property:

LEMMA 5.1.1. *Let M be a metrizable space. Then M is first countable.*

PROOF. Fix a metric on M inducing its topology. Recall that $B_r(p)$ is the open ball of radius $r > 0$ around $p \in M$. For $p \in M$, the set $\{B_r(p) \mid r > 0 \text{ rational}\}$ is a countable neighborhood basis for X at p . $\square$

5.2. Sequences

Let X be a space. A *sequence* in X is an ordered collection $\{x_n\}_{n \geq 1}$ of points of X . Given such a sequence, a point $y \in X$ is its *limit* if for all open neighborhoods U of y there is some $N \geq 1$ such that $x_n \in U$ for $n \geq N$. If y is a limit of $\{x_n\}_{n \in \mathbb{N}}$, then we write $\lim_{n \rightarrow \infty} x_n = y$ and say that $\{x_n\}_{n \geq 1}$ *converges* to y . If such a y exists, then we say that $\{x_n\}_{n \geq 1}$ is a *convergent sequence*. If X is metrizable, then this agrees with our previous definition (see Exercise 5.1).

REMARK 5.2.1. Be warned that a sequence can have multiple distinct limits. This only happens for fairly pathological spaces. In the next chapter, we introduce a property of spaces called being Hausdorff that forces convergent sequences to have unique limits. $\square$

5.3. Closure

Sequences are most useful for first countable spaces. For instance, if X is first countable, then for $A \subset X$ we can construct the closure $\bar{A}$ using limits:

LEMMA 5.3.1. *Let X be a first countable space and let $A \subset X$. Then $\bar{A}$ is the set of all $y \in A$ such that there exists a sequence $\{a_n\}_{n \geq 1}$ of points of A such that $\lim_{n \rightarrow \infty} a_n = y$.*

PROOF. Let B be the set of limits of sequences of points of A . We first prove that $B \subset \bar{A}$. Let $b \in B$ and let $C \subset X$ be a closed set with $A \subset C$. We must prove that $b \in C$. Indeed, if $b \notin C$ then we can find an open neighborhood U of b such that $U \subset X \setminus C$. However, since $b \in B$ there must exist points of $A \subset C$ in U , contradicting the fact that U is disjoint from C .

We next prove that $\bar{A} \subset B$. This uses first countability. Consider a point $p \in \bar{A}$. Each open neighborhood V of p must contain a point of A . Let $\mathfrak{B}_p = \{U_1, U_2, \dots\}$ be a countable neighborhood basis at p . For each $n \geq 1$, choose $x_n \in U_n$ with $x_n \in A$. We then have $\lim_{n \rightarrow \infty} x_n = p$, so $p \in B$. $\square$

REMARK 5.3.2. Though metrizable spaces are first countable, not all spaces that appear in algebraic topology are first countable. This is why arguments using limits are mostly avoided in this book. There are generalizations of sequences and limits (nets, filters, etc.) that work for spaces that are not first countable (see [1]), but in practice they do not simplify arguments in algebraic topology. $\square$

5.4. Second countability

A space X is *second countable* if there is a countable basis for its topology. It is clear that all second countable spaces are first countable. It is not true that all metrizable spaces are second countable, but all subspaces of $\mathbb{R}^n$ are second countable:

LEMMA 5.4.1. *Let X be a subspace of $\mathbb{R}^n$. Then X is second countable.*

PROOF. For all $p \in \mathbb{R}^n$ and $r > 0$, let $B_r(p) \subset \mathbb{R}^n$ be the open ball around p . Then X has the countable basis $\{B_r(p) \cap X \mid p \in \mathbb{Q}^n \text{ and } r > 0 \text{ rational}\}$. $\square$

5.5. Separability

There is one further countability condition that occasionally shows up. For a space X , a set $A \subset X$ is *dense* if its closure $\overline{A}$ equals X . The space X is *separable* if X has a countable dense subset. This is slightly weaker than second countability:

LEMMA 5.5.1. *Let X be a second countable space. Then X is separable.*

PROOF. Let $\mathfrak{B} = \{U_1, U_2, \dots\}$ be a countable basis for the topology of X . Pick $x_n \in U_n$. Then the set $\{x_n \mid n \geq 1\}$ is a countable dense set in X . $\square$

For metrizable spaces, these two notions coincide:

LEMMA 5.5.2. *Let M be a separable metrizable space. Then M is second countable.*

PROOF. Fixing a metric on M inducing its topology, the proof is similar to that of Lemma 5.4.1: if $A \subset M$ is a countable dense set, then $\{B_{1/n}(a) \mid a \in A, n \geq 1\}$ is a countable basis for the topology on M . $\square$

5.6. Exercises

EXERCISE 5.1. Let M be a metric space. Prove that the notion of limit from §5.2 agrees with the notion of limit for metric spaces from §1.8. $\square$

EXERCISE 5.2. Let X be second countable. Prove the following:

- (a) Let $\mathfrak{U}$ be a collection of open subsets of X with $X = \cup_{U \in \mathfrak{U}} U$. Prove that there is a countable subset $\mathfrak{U}' \subset \mathfrak{U}$ such that $X = \cup_{U \in \mathfrak{U}'} U$.
- (b) Let $\mathfrak{B}$ be any basis for the topology on X . Prove that there is a countable subset $\mathfrak{B}' \subset \mathfrak{B}$ that is also a basis for the topology on X . $\square$

EXERCISE 5.3. Let X be second countable and let $\mathfrak{U}$ be a collection of disjoint open subsets of X . Prove that $\mathfrak{U}$ is countable. $\square$

EXERCISE 5.4. As in Example 3.2.7, let X be the wedge product of countably many intervals $I_n = I$ equipped with the basepoints $0 \in I_n$. Prove that X is not first countable, and deduce that X is not metrizable. $\square$

EXERCISE 5.5. Let $I = [0, 1]$ and let $\leq$ be the dictionary order topology on I^2 , so $(x, y) \leq (z, w)$ if $x \leq z$ or if $x = z$ and $y \leq w$. Let X the topological space on the set I^2 with the corresponding order topology (see Example 2.6.3). Prove the following:

- (a) X is first countable.
- (b) X is not second countable. $\square$

EXERCISE 5.6. Let S be the Sorgenfrey line (see Example 2.6.5), so S is $\mathbb{R}$ with the topology given by the basis $\{[a, b) \mid a < b\}$ of all half-open intervals in $\mathbb{R}$. Prove the following:

- (a) S is first countable.
- (b) S is separable.
- (c) S is not second countable. $\square$

Bibliography

- [1] J. Kelley, *General topology*, D. Van Nostrand Co., Inc., Toronto-New York-London, 1955.

Separation properties and the Tietze extension theorem

This chapter discusses properties that are necessary to ensure that continuous functions have the properties one would expect.

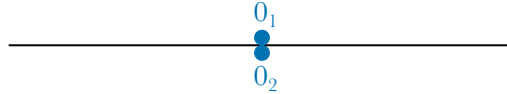
6.1. Pathology

Consider maps $f, g: X \rightarrow Y$. If $A \subset X$ is dense and $f|_A = g|_A$, then it is natural to expect that $f = g$. Unfortunately, this need not hold:

EXAMPLE 6.1.1 (Line with two origins). As a set, let $Y = (\mathbb{R} \setminus \{0\}) \sqcup \{0_1, 0_2\}$. For $i = 1, 2$, let $f_i: \mathbb{R} \rightarrow Y$ be the map defined by $f_i(x) = x$ for $x \in \mathbb{R} \setminus \{0\}$ and $f_i(0) = 0_i$. Give Y the identification space topology, so:

- a set $U \subset Y$ is open if and only if $f_1^{-1}(U)$ and $f_2^{-1}(U)$ are open in $\mathbb{R}$.

Here is a picture of Y :



With this topology, the subspaces $Y \setminus \{0_2\} = f_1(\mathbb{R})$ and $Y \setminus \{0_1\} = f_2(\mathbb{R})$ are both homeomorphic to $\mathbb{R}$. The maps $f_1, f_2: \mathbb{R} \rightarrow Y$ are continuous and agree on the dense set $\mathbb{R} \setminus \{0\}$. However, $f_1 \neq f_2$. $\square$

6.2. Hausdorff spaces

The issue with the line with two origins from Example 6.1.1 is that the points 0_1 and 0_2 do not have disjoint open neighborhoods. To rule this out, say that a space X is *Hausdorff* if for all distinct points $p, q \in X$, there are open neighborhoods U of p and V of q with $U \cap V = \emptyset$. This has a number of nice consequences (see Exercise 6.3):

- All points in X are closed, i.e., for all $p \in X$ the one-point set $\{p\}$ is closed.
- If Z is another space and $f, g: Z \rightarrow X$ are two maps, then the subset $\{z \in Z \mid f(z) = g(z)\}$ of points in Z where f and g are equal is closed. In particular, if f and g agree on a dense subset of Z , then $f = g$.
- Limits in X are unique in the following sense. Let $\{x_n\}_{n \geq 1}$ be a sequence in X and let $y_1, y_2 \in X$ be such that $\lim_{n \rightarrow \infty} x_n = y_1$ and $\lim_{n \rightarrow \infty} x_n = y_2$. Then $y_1 = y_2$.

Most geometrically natural spaces are Hausdorff. In particular:

LEMMA 6.2.1. *Let M be a metrizable space. Then M is Hausdorff.*

PROOF. Fix a metric on M inducing its topology. For distinct $p, q \in M$, let $\epsilon = \mathfrak{d}(p, q)/2$. The open balls $B_\epsilon(p)$ and $B_\epsilon(q)$ are disjoint. $\square$

REMARK 6.2.2. For an infinite field $\mathbf{k}$, an important non-example is given by the Zariski topology on $\mathbf{k}^n$. See Exercise 6.4. $\square$

6.3. Continuity

For first countable Hausdorff spaces, we can characterize continuity with sequences:

LEMMA 6.3.1. *Let X be a first countable Hausdorff space, let Y be a Hausdorff space, and let $f: X \rightarrow Y$ be a map of sets. Then f is continuous if and only if the following holds.¹*

¹There is a version of this result that is true without the Hausdorff assumption, but it is awkward to state since in non-Hausdorff spaces limits need not be unique.

- Let $\{x_n\}_{n \geq 1}$ be a convergent sequence in X . Then $\{f(x_n)\}_{n \geq 1}$ is a convergent sequence in Y and $\lim_{n \rightarrow \infty} f(x_n) = f(\lim_{n \rightarrow \infty} x_n)$.

PROOF. See Exercise 6.2. □

6.4. Normal spaces

In fact, most geometrically natural spaces have even stronger separation properties. A space X is *normal* if it satisfies the following two conditions:

- for all disjoint closed sets $C, D \subset X$, there exist open neighborhoods U of C and V of D with $U \cap V = \emptyset$; and
- all points in X are closed.²

All normal spaces are Hausdorff. The key example is:

LEMMA 6.4.1. *Let M be a metrizable space. Then M is normal.*

PROOF. Fix a metric $\mathfrak{d}$ on M inducing its topology. Since M is Hausdorff, all points in M are closed. Consider disjoint closed sets $C, D \subset M$. For $z \in M$, let

$$r(z) = \inf \{\mathfrak{d}(z, c) \mid c \in C\} \quad \text{and} \quad s(z) = \inf \{\mathfrak{d}(z, d) \mid d \in D\}.$$

Since C and D are disjoint closed sets, we have $r(d) > 0$ for $d \in D$ and $s(c) > 0$ for $c \in C$. Define

$$U = \bigcup_{c \in C} B_{s(c)/3}(c) \quad \text{and} \quad V = \bigcup_{d \in D} B_{r(d)/3}(d).$$

The sets U and V are open, and $C \subset U$ and $D \subset V$. To prove the lemma, it is enough to show that $U \cap V = \emptyset$. Assume this is false, and let $x \in U \cap V$. We can therefore find $c_0 \in C$ and $d_0 \in D$ such that $\mathfrak{d}(c_0, x) < s(c_0)/3$ and $\mathfrak{d}(d_0, x) < r(d_0)/3$. We either have $s(c_0) \leq r(d_0)$ or $r(d_0) \leq s(c_0)$. Both cases lead to a similar contradiction, so we will give the details for $s(c_0) \leq r(d_0)$. This implies that

$$\mathfrak{d}(c_0, d_0) \leq \mathfrak{d}(c_0, x) + \mathfrak{d}(x, d_0) < s(c_0)/3 + r(d_0)/3 \leq r(d_0)/3 + r(d_0)/3 = \frac{2}{3}r(d_0).$$

However, we also have $\mathfrak{d}(c_0, d_0) \geq \inf \{\mathfrak{d}(d_0, c) \mid c \in C\} = r(d_0)$, a contradiction. □

The following characterization of normality is often useful. Recall that $\overline{V}$ is the closure of V .

LEMMA 6.4.2. *A space X is normal if and only if all points in X are closed and:*

- (♠) *For all closed sets $C \subset X$ and all open neighborhoods U of C , there exists an open neighborhood V of C with $\overline{V} \subset U$.*

PROOF. Assume first that X is normal. To verify (♠), let $C \subset X$ be closed and let U be an open neighborhood of C . The set $D = X \setminus U$ is then a closed set that is disjoint from C , so by normality there exist disjoint open neighborhoods V and W of C and D . Since $X \setminus W$ is a closed subset of U containing V , it follows that $\overline{V} \subset U$.

Assume now that all points in X are closed and (♠) holds. To verify normality, let $C, D \subset X$ be disjoint closed sets. Applying (♠) to the open neighborhood $U = X \setminus D$ of C , we obtain an open neighborhood V of C with $\overline{V} \subset U$. It follows that V and $W = X \setminus \overline{V}$ are disjoint open neighborhoods of C and D . □

6.5. Urysohn's Lemma

A key feature of normal spaces is that they have a rich supply of continuous real-valued functions. For our first example of this, we need a definition. The *support* of a function $f: X \rightarrow \mathbb{R}$, denoted $\text{supp}(f)$, is the closure of the set $\{p \in X \mid f(p) \neq 0\}$. We then have:

THEOREM 6.5.1 (Urysohn's Lemma). *Let X be a normal space, let $C \subset X$ be closed, and let $U \subset X$ be an open neighborhood of C . Then there exists a map $f: X \rightarrow I = [0, 1]$ such that $f|_C = 1$ and $\text{supp}(f) \subset U$.*

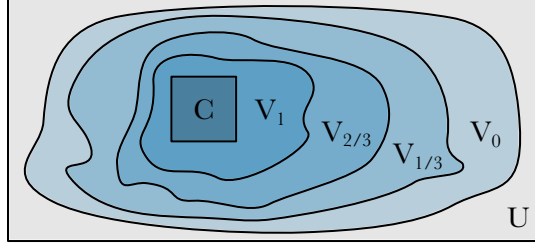
PROOF. We must use the open sets provided by normality to construct f . The key is:

²This is not always included in the definition of normality, but it ensures that normal spaces are Hausdorff.

CLAIM. *There exist open sets V_α for all $\alpha \in \mathbb{Q}$ with the following properties:*

- (i) *For rational $r < s$, we have $\overline{V}_s \subset V_r$.*
- (ii) *$C \subset V_1$ and $\overline{V}_0 \subset U$.*
- (iii) *$V_\alpha = \emptyset$ for $\alpha > 1$ and $V_\alpha = X$ for $\alpha < 0$.*

PROOF OF CLAIM. The picture is as follows:



Define V_α for $\alpha > 1$ and $\alpha < 0$ as in (iii). Next, using Lemma 6.4.2 choose an open neighborhood V_0 of C with $\overline{V}_0 \subset U$ and an open neighborhood V_1 of C with $\overline{V}_1 \subset \overline{V}_0$. Conditions (ii) and (iii) hold, and we inductively construct the remaining V_α satisfying (i) as follows. Enumerate the rational numbers in I as $\{\alpha_0, \alpha_1, \dots\}$ with $\alpha_0 = 0$ and $\alpha_1 = 1$. We have already constructed V_0 and V_1 , so assume that $n \geq 2$ and that we have constructed V_{α_m} for $0 \leq m \leq n-1$ satisfying (i). We construct V_{α_n} as follows. Let

$$r = \max \{ \alpha_m \mid 0 \leq m \leq n-1, \alpha_m < \alpha_n \} \quad \text{and} \quad s = \min \{ \alpha_m \mid 0 \leq m \leq n-1, \alpha_m > \alpha_n \}.$$

We thus have $r < \alpha_n < s$, and by (iii) we have $\overline{V}_s \subset V_r$. Using Lemma 6.4.2, we can then find an open neighborhood V_{α_n} of $\overline{V}_s$ such that $\overline{V}_{\alpha_n} \subset V_r$. $\square$

We now define a set map $f: X \rightarrow \mathbb{R}$ via the formula

$$f(p) = \sup \{ \alpha \in \mathbb{Q} \mid p \in V_\alpha \}.$$

By (iii) we have $f(p) \in I$ for all $p \in X$. Also, by (ii) we have $f(p) = 1$ for $p \in C$ and $\text{supp}(f) \subset U$. All that remains is to check that f is continuous.

Let $W \subset \mathbb{R}$ be open. We must prove that $f^{-1}(W)$ is open. Consider $p \in f^{-1}(W)$. Choose rational $r < s$ such that $[r, s] \subset W$ and $f(p) \in [r, s]$. By (iii), we have $\overline{V}_s \subset V_r$. To prove that $f^{-1}(W)$ is open, it is enough to prove that the open set $V_r \setminus \overline{V}_s$ is contained in $f^{-1}(W)$. To do this, it is enough to prove that f maps $V_r \setminus \overline{V}_s$ into $[r, s] \subset W$. This follows from the following two facts, both of which are immediate from (iii):

- for $q \in V_r$, we have $f(q) \geq r$; and
- for $q \notin V_s$, we have $f(q) \leq s$.

$\square$

6.6. Converse to Urysohn

The following lemma shows that the conclusion of Urysohn's lemma characterizes normality:

LEMMA 6.6.1. *Let X be a space such that all points in X are closed. For every closed $C \subset X$ and every neighborhood U of C , assume that there exists a continuous map³ $f: X \rightarrow \mathbb{R}$ with $f|_C = 1$ and $\text{supp}(f) \subset U$. Then X is normal.*

PROOF. Let C and D be disjoint closed sets in X . By assumption, there is a continuous map $f: X \rightarrow \mathbb{R}$ with $f|_C = 1$ and $\text{supp}(f) \subset X \setminus D$. The sets $U = f^{-1}((1/2, \infty))$ and $V = X \setminus \text{supp}(f)$ are then disjoint open neighborhoods of C and D . $\square$

³In Urysohn's lemma, the target of f is $I = [0, 1]$. Here we relax this.

6.7. Perfectly normal spaces

Say that a space X is *perfectly normal* if points in X are closed and for all closed $C \subset X$ and all open neighborhoods U of C , there exists a continuous map $f: X \rightarrow I = [0, 1]$ such that $f^{-1}(1) = C$ and $\text{supp}(f) \subset U$. Lemma 6.6.1 implies that perfectly normal spaces are normal.

The definition of a perfectly normal space resembles the conclusion of Urysohn's lemma, but there is a small difference: in a perfectly normal space we have $f^{-1}(1) = C$, while in the conclusion of Urysohn's lemma we only have $C \subset f^{-1}(1)$. Most geometrically natural spaces are perfectly normal. In particular:

LEMMA 6.7.1. *Let M be a metrizable space. Then M is perfectly normal.*

PROOF. Fix a metric $\mathfrak{d}$ on M inducing its topology. Lemma 6.4.1 implies that M is normal, and in particular points are closed. Consider $C \subset X$ closed and U an open neighborhood of C . By Urysohn's Lemma, there exists a continuous map $f: X \rightarrow I$ such that $f|_C = 1$ and $\text{supp}(f) \subset U$. We want to modify f to ensure it is less than 1 at all points that do not lie in C . Let $g: X \rightarrow \mathbb{R}$ be the function

$$g(p) = \inf \{ \mathfrak{d}(p, c) \mid c \in C \} \quad \text{for } p \in X$$

and let $h: X \rightarrow I$ be the function

$$h(p) = \min(g(p), 1) \quad \text{for } p \in X.$$

Both g and h are continuous and satisfy $g^{-1}(0) = h^{-1}(0) = C$. The function $f': X \rightarrow I$ defined by

$$f'(p) = (1 - h(p)) \cdot f(p) \quad \text{for all } p \in M$$

then satisfies $(f')^{-1}(1) = C$ and $\text{supp}(f') \subset U$. $\square$

REMARK 6.7.2. We have introduced the notion of a space being Hausdorff, being normal, and being perfectly normal. These are called *separation axioms*. It is common to call a Hausdorff space a T_2 -space, a normal space a T_4 -space, and a perfectly normal space a T_6 -space. As this terminology suggests, there are many other separation axioms as well.⁴ For instance, we will discuss regular spaces (i.e., T_3 -spaces) below in §6.10. The vast majority of spaces considered in algebraic topology are perfectly normal. $\square$

6.8. Uniform limits of functions

Our next goal is to prove the Tietze extension theorem, which says that continuous real-valued functions on closed subsets of normal spaces can be extended to the whole space. The extension we construct will be a limit of functions constructed using Urysohn's Lemma. We therefore need a way to certify that such functions are continuous.

Let X be a space. A sequence of functions $f_n: X \rightarrow \mathbb{R}$ is said to *converge uniformly* to a function $f: X \rightarrow \mathbb{R}$ if the following holds:

- for all $\epsilon > 0$, there exists some $N \geq 1$ such that $|f(p) - f_n(p)| < \epsilon$ for all $n \geq N$ and $p \in X$.

We then have the following, which generalizes a familiar fact from real analysis:

LEMMA 6.8.1. *Let X be a space and let $f_n: X \rightarrow \mathbb{R}$ be a sequence of continuous functions converging uniformly to a function $f: X \rightarrow \mathbb{R}$. Then f is continuous.*

PROOF. This can be proved using an argument similar to the one used to prove the analogous fact for functions defined on $X = \mathbb{R}$. See Exercise 6.10. $\square$

6.9. Tietze Extension Theorem

We can now prove the Tietze Extension Theorem:

THEOREM 6.9.1 (Tietze Extension Theorem). *Let X be a normal space, let $C \subset X$ be closed, and let $f: C \rightarrow \mathbb{R}$ be a continuous function. Then f can be extended to a continuous function $F: X \rightarrow \mathbb{R}$. Moreover, if the image of f lies in a closed interval $[a, b]$ then F can be chosen such that its image also lies in $[a, b]$.*

⁴In fact, not only are there T_k -spaces for $0 \leq k \leq 6$, but there are even $T_{2.5}$ -spaces and $T_{3.5}$ -spaces.

PROOF. We first prove the case where f is bounded, and then derive the unbounded case.

CASE 1. *The theorem holds if the image of f lies in a closed interval $[a, b]$.*

Since $[a, b] \cong [-1, 1]$, we can assume without loss of generality that $[a, b] = [-1, 1]$. For $n \geq 1$, we will construct continuous functions $G_n: X \rightarrow \mathbb{R}$ such that letting $F_n = G_1 + \cdots + G_n$, we have:

- (i) The function F_n satisfies $|f(p) - F_n(p)| \leq (2/3)^n$ for all $p \in C$.
- (ii) The function G_n satisfies $|G_n(p)| \leq (1/3)(2/3)^{n-1}$ for all $p \in X$.

Condition (ii) will imply that the functions $F_n = G_1 + \cdots + G_n$ converge uniformly to a function F such that

$$|F(p)| \leq \frac{1}{3} (1 + (2/3) + (2/3)^2 + \cdots) = \frac{1}{3} \left(\frac{1}{1 - 2/3} \right) = 1 \quad \text{for all } p \in X.$$

Lemma 6.8.1 implies that $F: X \rightarrow [-1, 1]$ is continuous, and condition (i) implies that $F|_C = f$.

It remains to construct the G_n . Assume that $n \geq 1$ and we have constructed $G_1, \dots, G_{n-1}$ satisfying (ii) such that letting $F_{n-1} = G_1 + \cdots + G_{n-1}$, we have

$$(6.9.1) \quad |f(p) - F_{n-1}(p)| \leq (2/3)^{n-1} \quad \text{for all } p \in C.$$

This is vacuous for $n = 1$. We will construct G_n as follows. Let

$$\begin{aligned} L &= \{p \in C \mid f(p) - F_{n-1}(p) \leq -(1/3)(2/3)^{n-1}\} \\ R &= \{p \in C \mid f(p) - F_{n-1}(p) \geq (1/3)(2/3)^{n-1}\}. \end{aligned}$$

The sets L and R are disjoint closed sets. Using Urysohn's lemma, we can find:

- a continuous map $h_L: X \rightarrow I$ with $h_L|_L = 1$ and $\text{supp}(h_L) \subset X \setminus R$; and
- a continuous map $h_R: X \rightarrow I$ with $h_R|_R = 1$ and $\text{supp}(h_R) \subset X \setminus L$.

Let $G_n: X \rightarrow [-(1/3)(2/3)^{n-1}, (1/3)(2/3)^{n-1}]$ be the map

$$G_n = -(1/3)(2/3)^{n-1}h_L + (1/3)(2/3)^{n-1}h_R.$$

By construction, G_n satisfies (ii). To show that $F_n = F_{n-1} + G_n$ satisfies (i), consider some $p \in C$. There are three cases:

- If $p \in L$, then by (6.9.1) we have

$$|f(p) - F_n(p)| = |f(p) - F_{n-1}(p) + (1/3)(2/3)^{n-1}| \leq (2/3)^{n-1} - (1/3)(2/3)^{n-1} = (2/3)^n.$$

- If $p \in R$, then by (6.9.1) we have

$$|f(p) - F_n(p)| = |f(p) - F_{n-1}(p) - (1/3)(2/3)^{n-1}| \leq (2/3)^{n-1} - (1/3)(2/3)^{n-1} = (2/3)^n.$$

- If $p \notin L \cup R$, then by definition we have $|f(p) - F_{n-1}(p)| \leq (1/3)(2/3)^{n-1}$, so since $|G_n(p)| \leq (1/3)(2/3)^{n-1}$ we have

$$|f(p) - F_n(p)| = |f(p) - F_{n-1}(p) - G_n(p)| \leq (1/3)(2/3)^{n-1} + (1/3)(2/3)^{n-1} = (2/3)^n.$$

In all three cases, (i) is satisfied. The theorem follows.

CASE 2. *The theorem holds in general.*

Since $\mathbb{R} \cong (-1, 1)$, it is enough to prove that every continuous function $f: C \rightarrow (-1, 1)$ can be extended to a continuous function $F: X \rightarrow (-1, 1)$. By Case 1, we can extend f to a continuous function $F': X \rightarrow [-1, 1]$. Our goal is to modify F' such that its image does not contain -1 or 1 . Set $U = (F')^{-1}((-1, 1))$. Applying Urysohn's Lemma (Theorem 6.5.1), there exists a continuous function $g: X \rightarrow I$ with $g|_C = 1$ and $\text{supp}(g) \subset U$. The product $F = g \cdot F'$ then still extends f and satisfies $F(X) \subset (-1, 1)$. $\square$

6.10. Regular spaces

We close this chapter by discussing one further separation property that will play a small technical role at several points throughout the rest of this book. A space X is *regular* if it satisfies the following two conditions:

- for all points $p \in X$ and closed sets $C \subset X$ with $p \notin C$, there exist open neighborhoods U of p and V of C with $U \cap V = \emptyset$; and
- all points in X are closed.⁵

All normal spaces are regular and all regular spaces are Hausdorff. There exist Hausdorff spaces all of whose points are closed that are not regular (see Exercise 6.5). There also exist regular spaces that are not normal, but these are fairly pathological; see [1]. The following lemma shows that many regular spaces are normal:

LEMMA 6.10.1. *Let X be a space that is regular and second countable. Then X is normal.*

PROOF. Let $A, B \subset X$ be disjoint closed sets. To prove that X is normal, we must find disjoint open neighborhoods U and V of A and B . The key to this is:

CLAIM. *There exist countable collections of open sets $\{U_n\}_{n \geq 1}$ and $\{V_n\}_{n \geq 1}$ such that:*

- $A \subset \bigcup_{n=1}^{\infty} U_n$ and $B \subset \bigcup_{n=1}^{\infty} V_n$; and
- For all $n \geq 1$, we have $\overline{U}_n \cap B = \emptyset$ and $\overline{V}_n \cap A = \emptyset$.

PROOF OF CLAIM. The constructions of the $\{U_n\}_{n \geq 1}$ and the $\{V_n\}_{n \geq 1}$ are identical, so we will explain how to construct the $\{U_n\}_{n \geq 1}$. Let $\mathcal{B}$ be a countable basis for X . For $a \in A$, since X is regular we can find disjoint open neighborhoods U'_a of a and V'_a of B . Choose $U_a \in \mathcal{B}$ such that U_a is an open neighborhood of a with $U_a \subset U'_a$. We then have $\overline{U}_a \subset \overline{U}'_a \subset B \setminus V'_a$, so $\overline{U}_a \cap B = \emptyset$. For the $\{U_n\}_{n \geq 1}$ we can then take an enumeration of the countable set $\{U_a \mid a \in A\} \subset \mathcal{B}$. $\square$

The naive thing to do now would be to take our open neighborhoods of A and B to be $\bigcup_{n=1}^{\infty} U_n$ and $\bigcup_{n=1}^{\infty} V_n$, but these would not necessarily be disjoint. To fix this, for $n \geq 1$ define

$$U'_n = U_n \setminus \bigcup_{k=1}^n \overline{V}_k \quad \text{and} \quad V'_n = V_n \setminus \bigcup_{k=1}^n \overline{U}_k.$$

Since each $\overline{V}_k$ is disjoint from A , we still have $A \subset \bigcup_{n=1}^{\infty} U'_n$. Similarly, we still have $B \subset \bigcup_{n=1}^{\infty} V'_n$. Now, however, we have $U'_n \cap V'_m = \emptyset$ for all $n, m \geq 1$. It follows that $U = \bigcup_{n=1}^{\infty} U'_n$ and $V = \bigcup_{n=1}^{\infty} V'_n$ are disjoint open neighborhoods of A and B , as desired. $\square$

6.11. Exercises

EXERCISE 6.1. Let X be Hausdorff and let $x_1, \dots, x_n \in X$ be distinct points. Prove that there exist open neighborhoods U_i of the x_i such that $U_i \cap U_j = \emptyset$ for all $1 \leq i, j \leq n$ distinct. $\square$

EXERCISE 6.2. Let X be a first countable Hausdorff space, let Y be a Hausdorff space, and let $f: X \rightarrow Y$ be a map of sets. Then f is continuous if and only if the following holds:

- Let $\{x_n\}_{n \geq 1}$ be a convergent sequence in X . Then $\{f(x_n)\}_{n \geq 1}$ is a convergent sequence in Y and $\lim_{n \rightarrow \infty} f(x_n) = f(\lim_{n \rightarrow \infty} x_n)$. $\square$

EXERCISE 6.3. Let X be a Hausdorff space. Prove the following:

- All points in X are closed, i.e., for all $p \in X$ the one-point set $\{p\}$ is closed.
- If Z is another space and $f, g: Z \rightarrow X$ are two maps, then the subset $\{z \in Z \mid f(z) = g(z)\}$ of points in Z where f and g are equal is closed. In particular, if f and g agree on a dense subset of Z , then $f = g$.
- Let $\{x_n\}_{n \geq 1}$ be a sequence in X and let $y_1, y_2 \in X$ be such that $\lim_{n \rightarrow \infty} x_n = y_1$ and $\lim_{n \rightarrow \infty} x_n = y_2$. Then $y_1 = y_2$. $\square$

EXERCISE 6.4. Let $\mathbf{k}$ be a field. Prove that the Zariski topology on $\mathbf{k}^n$ described in Example 2.5.2 is Hausdorff if and only if $\mathbf{k}$ is a finite field. $\square$

⁵This is not always included in the definition of regularity, but it ensures that regular spaces are Hausdorff.

EXERCISE 6.5. As a set, let $X = \mathbb{R}$. Let $\mathcal{B}$ be the set of open subsets of $X = \mathbb{R}$ in the standard topology on $\mathbb{R}$ and let

$$\mathcal{B}' = \mathcal{B} \cup \{U \cap \mathbb{Q} \mid U \in \mathcal{B}\}.$$

Prove the following:

- The set $\mathcal{B}'$ is the basis for a topology on X that we will call the $\mathcal{B}'$ -topology.
- The $\mathcal{B}'$ -topology on X is Hausdorff and all points are closed.
- The $\mathcal{B}'$ -topology on X is not regular. □

EXERCISE 6.6. Let X be a Hausdorff space and let $Y \subset X$ be a subspace. Prove that Y is Hausdorff. □

EXERCISE 6.7. Let X be a normal space and let $Y \subset X$ be a closed subspace. Prove that Y is normal. We remark that this need not hold if Y is not closed. See [1] for examples. □

EXERCISE 6.8. Say that a space X is *completely normal* if every subspace of X is normal. Prove that X is completely normal if and only if for every pair of subsets $A, B \subset X$ with $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$, there exist disjoint open neighborhoods U and V of A and B . □

EXERCISE 6.9. Let $f: X \rightarrow Y$ be a quotient map that is also closed. Assume that X is normal. Prove that Y is normal. □

EXERCISE 6.10. Let X be a space and let $f_n: X \rightarrow \mathbb{R}$ be a sequence of continuous functions converging uniformly to a function $f: X \rightarrow \mathbb{R}$. Prove that f is continuous. □

EXERCISE 6.11. Let X be a connected normal space containing more than one point. Prove that X has uncountably many points. □

EXERCISE 6.12. Let S be the Sorgenfrey line (see Example 2.6.5), so S is $\mathbb{R}$ with the topology given by the basis $\{[a, b) \mid a < b\}$ of all half-open intervals in $\mathbb{R}$. Prove that S is normal. □

Bibliography

- [1] L. Steen & J. Seebach Jr., *Counterexamples in topology*, second edition, Springer, New York-Heidelberg, 1978.

Compactness and the Heine–Borel theorem

We now introduce the key concept of compactness, which generalizes the notion of compactness for subsets of $\mathbb{R}$ and $\mathbb{R}^n$ from real analysis.

7.1. Compactness

Let X be a space and let $K \subset X$. An *open cover* of K is a collection $\mathcal{U}$ of open sets in X such that $K \subset \bigcup_{U \in \mathcal{U}} U$. The open cover $\mathcal{U}$ is *finite* if it consists of finitely many open sets. A *subcover* of an open cover $\mathcal{U}$ is a subset $\mathcal{U}' \subset \mathcal{U}$ that is still a cover. The subspace K is *compact* if every open cover of K has a finite subcover. In particular, X itself is compact if every open cover of X has a finite subcover.

7.2. Compactness and closed sets

Compactness behaves best for Hausdorff spaces. In fact, in some treatments of point-set topology a space is said to be quasi-compact if each open cover has a finite subcover, and a compact space is a space that is Hausdorff and quasi-compact. For Hausdorff spaces, we have:

LEMMA 7.2.1. *Let X be a Hausdorff space and let $K \subset X$ be compact. Then K is closed.*

PROOF. We must prove that $X \setminus K$ is open. Consider $p \in X \setminus K$. Since X is Hausdorff, for each $k \in K$ there are disjoint open neighborhoods U_k and V_k of p and k . Since K is compact, we can find finitely many points $k_1, \dots, k_n \in K$ such that $\{V_{k_1}, \dots, V_{k_n}\}$ is an open cover of K . Letting $U = U_{k_1} \cap \dots \cap U_{k_n}$, the set U is an open neighborhood of p that is disjoint from K , as desired. $\square$

For all spaces, we have:

LEMMA 7.2.2. *Let X be a space, let $K \subset X$ be compact, and let C be a closed subset of X with $C \subset K$. Then C is compact.*

PROOF. Let $\mathcal{U}$ be an open cover of $C \subset X$. The set $\{X \setminus C\} \cup \mathcal{U}$ is an open cover of K . Since K is compact, it has a finite subcover. Removing $X \setminus C$ from this finite subcover if necessary, we obtain a finite subcover of $\mathcal{U}$. $\square$

As another indication of how strong an assumption being compact Hausdorff is, we have:

LEMMA 7.2.3. *Let X be a compact Hausdorff space. Then X is normal.*

PROOF. See Exercise 7.2. $\square$

7.3. Compactness and functions

Continuous maps take compact sets to compact sets:

LEMMA 7.3.1. *Let $f: X \rightarrow Y$ be a map of spaces and let $K \subset X$ be compact. Then $f(K)$ is compact.*

PROOF. See Exercise 7.4 $\square$

This has the following corollary:

COROLLARY 7.3.2. *Let $f: X \rightarrow Y$ be a map of spaces with X compact and Y Hausdorff. Then f is a closed map.*

PROOF. Let $C \subset X$ be closed. Since X is compact, C is compact. It follows that $f(C)$ is compact, so since Y is Hausdorff $f(C)$ is closed. $\square$

Another important property of compact sets is that real-valued functions on them are bounded and attain maximum and minimum values:

LEMMA 7.3.3. *Let X be a compact space and let $f: X \rightarrow \mathbb{R}$ be a map. Then there exist real numbers $m \leq M$ such that:*

- for all $p \in X$, we have $m \leq f(p) \leq M$; and
- there exists $p_0, q_0 \in X$ such that $m = f(p_0)$ and $M = f(q_0)$.

PROOF. By Lemma 7.3.1, the image $K = f(X)$ is a compact subset of $\mathbb{R}$. The lemma now follows from the following standard fact about compact subsets of $\mathbb{R}$: there exist $m, M \in K$ such that $m \leq k \leq M$ for all $k \in K$ (see Exercise 7.5). $\square$

7.4. Compactness and injective maps

For general spaces X and Y , an injective map $f: X \rightarrow Y$ need not be an embedding, i.e., a homeomorphism onto its image. Here is an example:

EXAMPLE 7.4.1. Consider the injective map $f: (0, 1) \rightarrow \mathbb{R}^2$ whose image X is as follows:



This is not an embedding; indeed, for every $p \in (0, 1)$ the space $(0, 1) \setminus \{p\}$ is disconnected but for the indicated point $p_0 \in X$ we have $X \setminus \{p_0\}$ connected. $\square$

However, if X is compact and Y is Hausdorff this pathology does not occur:

LEMMA 7.4.2. *Let X be a compact space, let Y be a Hausdorff space, and let $f: X \rightarrow Y$ be an injective map. Then f is a closed embedding.*

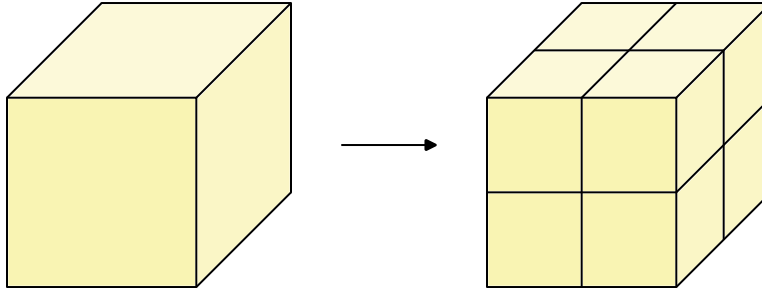
PROOF. Immediate from the fact that f is injective and closed (see Corollary 7.3.2). $\square$

7.5. Heine–Borel Theorem

Let $(M, \mathfrak{d})$ be a metric space. A subset $K \subset M$ is *bounded* if there is some $R \geq 0$ such that $\mathfrak{d}(p, q) \leq R$ for all $p, q \in K$. The following theorem gives a large supply of compact spaces:

THEOREM 7.5.1 (Heine–Borel Theorem). *Let $K \subset \mathbb{R}^n$ be closed and bounded. Then K is compact.*

PROOF. For some $D \gg 0$, the set K is contained in the cube $[-D, D]^n$. By Lemma 7.2.2, it is enough to prove that $[-D, D]^n$ is compact. Since all cubes in $\mathbb{R}^n$ are homeomorphic, it is actually enough to prove that the unit cube $C_1 = I^n$ is compact. Let $\mathfrak{U}$ be an open cover of C_1 . For the sake of contradiction, assume that it has no finite subcover. Divide C_1 into 2^n subcubes with side lengths $1/2$:



The cover $\mathfrak{U}$ is a cover of each of these subcubes. Since no finite subset of $\mathfrak{U}$ covers C_1 , it must be the case that among these 2^n subcubes there is a subcube C_2 such that no finite subset of $\mathfrak{U}$ covers C_2 . This process can then be repeated: C_2 can be divided into 2^n subcubes with side length $1/2^2$, and

there among these there must exist a subcube C_3 such that no finite subset of $\mathfrak{U}$ covers it. We then divide C_3 into 2^n subcubes with side lengths $1/2^3$, etc. This procedure gives a nested sequence

$$C_1 \supset C_2 \supset C_3 \supset \cdots$$

of cubes with the following properties:

- the cube C_n has side lengths $1/2^n$; and
- no finite subset of $\mathfrak{U}$ covers any of the C_n .

By the completeness of $\mathbb{R}$, the intersection $\bigcap_{n=1}^{\infty} C_n$ must consist of a single point p . Pick $U \in \mathfrak{U}$ such that $p \in U$. Since U is open, for some $\epsilon > 0$ the ϵ -ball around p must be contained in U . This implies that for $n \gg 0$ we have $C_n \subset U$, contradicting the fact that no finite subset of $\mathfrak{U}$ covers any C_n . $\square$

REMARK 7.5.2. A metric space in which closed and bounded subsets are compact is called a *proper* metric space. This is a property of the metric, not just of the topology (see Exercise 7.6). $\square$

7.6. Compactness and intersections of closed sets

The following is a useful rephrasing of the definition of compactness:

LEMMA 7.6.1. *Let X be a space. The X is compact if and only if the following holds for all sets $\mathfrak{C}$ of closed subsets of X :*

(*) *If for all finite subsets $\mathfrak{C}' \subset \mathfrak{C}$ we have $\bigcap_{C \in \mathfrak{C}'} C \neq \emptyset$, then $\bigcap_{C \in \mathfrak{C}} C \neq \emptyset$.*

PROOF. The condition (*) is equivalent to:

(*)' *If $\bigcap_{C \in \mathfrak{C}} C = \emptyset$, then there exists a finite subset $\mathfrak{C}' \subset \mathfrak{C}$ such that $\bigcap_{C \in \mathfrak{C}'} C = \emptyset$.*

There is a bijection between sets of closed subsets of X and sets of open subsets of X taking a set $\mathfrak{C}$ of closed subsets to $\mathfrak{U}(\mathfrak{C}) = \{X \setminus C \mid C \in \mathfrak{C}\}$. A set $\mathfrak{C}$ of closed subsets of X has empty intersection exactly when $\mathfrak{U}(\mathfrak{C})$ covers X . It follows (*)' is equivalent to saying that if $\mathfrak{U}(\mathfrak{C})$ is a cover of X , then $\mathfrak{U}(\mathfrak{C})$ has a finite subcover. $\square$

This has the following immediate corollary:

COROLLARY 7.6.2. *Let X be a space and let $C_1 \supset C_2 \supset \cdots$ be a nested sequence of nonempty compact subspaces of X . Then $\bigcap_{n \geq 1} C_n \neq \emptyset$.*

7.7. Lebesgue number

If M is a metric space and $\mathfrak{U}$ is an open cover of M , then a *Lebesgue number* for $\mathfrak{U}$ is an $\epsilon > 0$ such that for all $p \in M$ there exists some $U \in \mathfrak{U}$ such that the ϵ -ball $B_\epsilon(p)$ is contained in U . The following basic result shows that these always exist if M is compact:

LEMMA 7.7.1 (Lebesgue number lemma). *Let M be a compact metric space and let $\mathfrak{U}$ be an open cover of M . Then $\mathfrak{U}$ has a Lebesgue number.*

PROOF. Since M is compact, we can write M as

$$M = B_{\epsilon_1}(p_1) \cup \cdots \cup B_{\epsilon_n}(p_n) \quad \text{for some } p_1, \dots, p_n \in M \text{ and } \epsilon_1, \dots, \epsilon_n > 0$$

such that for each $1 \leq i \leq n$ there is some $U \in \mathfrak{U}$ with $B_{2\epsilon_i}(p_i) \subset U$. Set $\epsilon = \min(\epsilon_1, \dots, \epsilon_n)$, and consider $p \in M$. We have $p \in B_{\epsilon_i}(p_i)$ for some $1 \leq i \leq n$. By assumption, there is some $U \in \mathfrak{U}$ with $B_{2\epsilon_i}(p_i) \subset U$. The triangle inequality implies that $B_\epsilon(p) \subset B_{2\epsilon_i}(p_i)$ and thus $B_\epsilon(p) \subset U$. $\square$

7.8. Compactness and limits

If X is a space and $\{x_n\}_{n \geq 1}$ is a sequence in X , then a *subsequence* of $\{x_n\}_{n \geq 1}$ is a sequence of the form $\{x_{n_i}\}_{i \geq 1}$ with $n_1 < n_2 < \cdots$ a strictly increasing sequence of natural numbers. A subspace $K \subset X$ is *sequentially compact* if every sequence in K has a subsequence that converges to a point of K . With appropriate countability assumptions, this is equivalent to compactness. We divide this into two results:

LEMMA 7.8.1. *Let X be a first countable space and let $K \subset X$ be compact. Then K is sequentially compact.*

PROOF. See Exercise 7.3. $\square$

LEMMA 7.8.2. *Let X be a second countable space and let $K \subset X$ be sequentially compact. Then K is compact.*

PROOF. See Exercise 7.3. $\square$

Similarly, for metrizable spaces compactness and sequential compactness are the same:

LEMMA 7.8.3. *Let M be a metrizable space and let $K \subset X$. Then K is compact if and only if K is sequentially compact.*

PROOF. Fix a metric $\mathfrak{d}$ on M inducing its topology. Since M is first countable, Lemma 7.8.1 implies that compact subsets of M are sequentially compact. For the converse, we can replace M by the subspace in question and prove that if M is sequentially compact, then M is compact. By Lemma 7.8.2 it is enough to prove that M is second countable, which by Lemma 5.5.2 is equivalent to proving that M is separable, i.e., that M has a countable dense subset.

Since M is sequentially compact, it cannot contain an infinite discrete subspace. In particular, for each $n \geq 1$ there does not exist an infinite subset $T \subset M$ with $\mathfrak{d}(t_1, t_2) \geq 1/n$ for all distinct $t_1, t_2 \in T$. For each $n \geq 1$, we can therefore find a finite set S_n such that for all $p \in M$ there exists some $s \in S_n$ with $\mathfrak{d}(p, s) < 1/n$. The set $\cup_{n \geq 1} S_n$ is then a countable dense subset of M . $\square$

7.9. Exercises

EXERCISE 7.1. Let X be a space and let $C_1, \dots, C_n \subset X$ be compact subspaces. Prove that $C_1 \cup \dots \cup C_n$ is compact. $\square$

EXERCISE 7.2. Let X be a compact Hausdorff space. Prove that X is normal. $\square$

EXERCISE 7.3. Let X be a space and $K \subset X$ be a subspace. Prove:

- (a) If X is first countable and K is compact, then K is sequentially compact.
- (b) If X is second countable and K is sequentially compact, then K is compact. $\square$

EXERCISE 7.4. Let $f: X \rightarrow Y$ be a map of spaces and let $K \subset X$ be compact. Prove that $f(K)$ is compact. $\square$

EXERCISE 7.5. Let $K \subset \mathbb{R}$ be compact. Prove that there exist $m, M \in K$ such that $m \leq k \leq M$ for all $k \in K$. $\square$

EXERCISE 7.6. Construct a metric on $\mathbb{R}^n$ inducing its usual topology such that the closed 1-ball around the origin is not compact. In the terminology of Remark 7.5.2, with this metric $\mathbb{R}^n$ is not a proper metric space. $\square$

EXERCISE 7.7. Let $(M, \mathfrak{d}_M)$ and $(N, \mathfrak{d}_N)$ be metric spaces and let $f: M \rightarrow N$ be a map. We say that f is *uniformly continuous* if for all $\epsilon > 0$ there exists some $\delta > 0$ such that if $p, q \in M$ satisfy $\mathfrak{d}_M(p, q) < \delta$ then $\mathfrak{d}_N(f(p), f(q)) < \epsilon$. Use the Lebesgue number lemma to prove that if M is compact then all continuous maps $f: M \rightarrow N$ are uniformly continuous. $\square$

EXERCISE 7.8. Let X and Y be spaces. Give $X \times Y$ the product topology (see Example 2.6.4). Let $U \subset X \times Y$ be open. Let $A \subset X$ and $K \subset Y$ be such that $A \times K \subset U$. Assume that K is compact. Prove that there exists an open neighborhood V of A such that $V \times K \subset U$. We remark that this is often called the “tube lemma”. $\square$

EXERCISE 7.9. Let $f: X \rightarrow Y$ be a map of spaces (not necessarily continuous) with Y compact Hausdorff. Give $X \times Y$ the product topology (see Example 2.6.4). Define the *graph* of f to be

$$\Gamma_f = \{(x, f(x)) \in X \times Y \mid x \in X\}.$$

Prove that f is continuous if and only if Γ_f is a closed subset of $X \times Y$. $\square$

EXERCISE 7.10. Let X and Y be spaces with Y compact. Give $X \times Y$ the product topology (see Example 2.6.4). Let $\pi: X \times Y \rightarrow X$ be the projection onto the first factor. Prove that π is a closed map. $\square$

EXERCISE 7.11. Let S be the Sorgenfrey line (see Example 2.6.5), so S is $\mathbb{R}$ with the topology given by the basis $\{[a, b) \mid a < b\}$ of all half-open intervals in $\mathbb{R}$. Prove the following:

- (a) Let C be a compact subset of S . Prove that C is countable. Hint: first prove that C is compact in the classical topology on $\mathbb{R}$, then prove that there is no strictly increasing sequence $x_1 < x_2 < \cdots$ of points of C , and then prove that C is countable.
- (b) Prove that every open cover of S has a countable subcover (spaces with this property are called *Lindelöf space*). □

Local compactness and the Baire category theorem

We now turn to the local version of compactness.

8.1. Local compactness

Let X be a space. Recall that a general neighborhood of $p \in X$ is a set $Z \subset X$ with $p \in \text{Int}(Z)$. The space X is *locally compact* if the following holds for all $p \in X$:

- For all open neighborhoods U of p , there exists a compact neighborhood K of p with $K \subset U$.

For Hausdorff spaces, this is easier to understand:

LEMMA 8.1.1. *Let X be a Hausdorff space. Then X is locally compact if and only if for all $p \in X$, there exists a compact neighborhood K of p . In particular, if X is compact then X is locally compact.*

PROOF. See Exercise 8.2. □

REMARK 8.1.2. Local compactness is poorly behaved for non-Hausdorff spaces, and not all sources agree on the right definition for non-Hausdorff spaces. □

EXAMPLE 8.1.3. If X is either an open or a closed subspace of $\mathbb{R}^n$, then the Heine–Borel Theorem (Theorem 7.5.1) implies that X is locally compact. □

8.2. Regularity and normality

We now prove that locally compact Hausdorff spaces are regular:

LEMMA 8.2.1. *Let X be a locally compact Hausdorff space. Then X is regular.*

PROOF. Consider a point $p \in X$ and a closed set $C \subset X$ with $p \notin C$. We must find disjoint open neighborhoods of p and C . Since X is locally compact, there exists a compact neighborhood K of p with $K \subset X \setminus C$. Since X is Hausdorff the compact set K is closed. The desired open neighborhoods are then $\text{Int}(K)$ and $X \setminus K$. □

Since second countable regular spaces are normal (Lemma 6.10.1), this has the following corollary:

COROLLARY 8.2.2. *Let X be a locally compact Hausdorff space that is second countable. Then X is normal.*

8.3. One-point compactification

Let X be a space. A *compactification* of X is a compact space $\hat{X}$ containing X as an open dense subspace.

EXAMPLE 8.3.1. The space $\mathbb{S}^n$ is a compactification of $\mathbb{R}^n$. Indeed, for all $p_0 \in \mathbb{S}^n$ we have $\mathbb{S}^n \setminus p_0 \cong \mathbb{R}^n$ (see Exercise 8.6). □

If X is locally compact Hausdorff, there is a natural way to compactify X that generalizes the compactification $\mathbb{S}^n$ of $\mathbb{R}^n$. As a set, let $X^* = X \sqcup \{\infty\}$ with ∞ a formal symbol that does not lie in X . Say that $U \subset X^*$ is open if either:

- U is an open subset of X ; or
- $U = (X \setminus C) \cup \{\infty\}$, where $C \subset X$ is compact.

This is a topology (see Exercise 8.5), and the space X^* is called the *one-point compactification* of X . The following shows that it is indeed a compactification:

LEMMA 8.3.2. *Let X be locally compact Hausdorff and let X^* be the one-point compactification of X . Then X^* is compact Hausdorff, and X^* is a compactification of X .*

PROOF. See Exercise 8.5. □

8.4. σ -compactness

A space X is σ -compact if it is the union of countably many compact subspaces. This condition will be important in the next chapter when we discuss paracompactness and partitions of unity. Here we prove:

LEMMA 8.4.1. *Let X be a Hausdorff space that is second countable and locally compact. Then X is σ -compact.*

PROOF. Let $\mathfrak{B}$ be a countable basis for the topology of X . Set $\mathfrak{U} = \{U \in \mathfrak{B} \mid \overline{U} \text{ is compact}\}$, so $\mathfrak{U}$ is a countable collection of open sets of X . It is enough to prove that $\mathfrak{U}$ covers X . Indeed, consider $p \in X$. We must find some $U \in \mathfrak{U}$ with $p \in U$. By Lemma 8.1.1, there is a compact neighborhood K of p . Since $p \in \text{Int}(K)$, we can find $U \in \mathfrak{B}$ such that $p \in U$ and $U \subset K$. Since X is Hausdorff the compact set K is closed, so $\overline{U} \subset K$. Since $\overline{U}$ is a closed subset of the compact set K , it follows that $\overline{U}$ is compact and $U \in \mathfrak{U}$, as desired. □

EXAMPLE 8.4.2. If X is either an open or a closed subspace of $\mathbb{R}^n$, then the Heine–Borel Theorem (Theorem 7.5.1) implies that X is σ -compact. □

8.5. Baire category theorem

The following is a surprisingly powerful tool for proving existence theorems:

THEOREM 8.5.1 (Baire category theorem). *Let X be a locally compact Hausdorff space and let $\{U_n\}_{n \geq 1}$ be a collection of open dense subsets of X . Then $\cap_{n \geq 1} U_n$ is dense.*

PROOF. Let $V_0 \subset X$ be a nonempty open set. We must prove that V_0 intersects $\cap_{n \geq 1} U_n$. Since U_1 is open and dense, the set $V_0 \cap U_1$ is open and nonempty. Since X is locally compact and Hausdorff, we can find a nonempty open set V_1 with $\overline{V_1}$ compact such that $\overline{V_1} \subset V_0 \cap U_1$. The same argument shows that there exists a nonempty open set V_2 with $\overline{V_2}$ compact such that $\overline{V_2} \subset V_1 \cap U_2$. Repeating this over and over, we find nonempty open sets $\{V_n\}_{n \geq 1}$ with the following property for all $n \geq 1$:

- $\overline{V_n}$ is compact and $\overline{V_{n+1}} \subset V_n \cap U_{n+1}$.

Applying Corollary 7.6.2 to the nested sequence $\overline{V_1} \supset \overline{V_2} \supset \overline{V_3} \supset \cdots$ of nonempty compact subspaces of X , we see that their intersection must be nonempty, i.e., there exists some p with $p \in \overline{V_n}$ for all $n \geq 1$. By construction, p lies in both V_0 and $\cap_{n \geq 1} U_n$, as desired. □

REMARK 8.5.2. The word “category” in the Baire category theorem has nothing to do with category theory. Instead, it refers to the following archaic terminology: a space X is of the *first category* if it is the union of countably many nowhere dense¹ sets, and is of the *second category* otherwise. The conclusion of the Baire category theorem then is equivalent to saying that every nonempty open set in X is of the second category. □

8.6. Complete metric spaces

A space X is a *Baire space* if all countable intersections of open dense subsets of X are dense. Theorem 8.5.1 says that locally compact Hausdorff spaces are Baire spaces. For another useful class of such spaces, consider a metric space $(M, \mathfrak{d})$. A *Cauchy sequence* in M is a sequence $\{p_n\}_{n \geq 1}$ such that for all $\epsilon > 0$ there exists some $N \geq 1$ such that $\mathfrak{d}(p_n, p_m) < \epsilon$ for all $n, m \geq N$. The metric space M is *complete* if all Cauchy sequences in M have limits. For instance, $\mathbb{R}^n$ is complete (Exercise 8.8). We have:

¹A subset A of a topological space is *nowhere dense* if $\overline{A}$ contain no nonempty open sets, i.e., if $\text{Int}(\overline{A}) = \emptyset$.

THEOREM 8.6.1 (Baire category theorem'). *Let M be a complete metric space. Then M is a Baire space.*

PROOF. This is similar to the proof of Theorem 8.5.1, so we leave it as Exercise 8.9. $\square$

8.7. Application: nowhere differentiable functions

To illustrate how the Baire category theorem can be used, we prove the following classic result:

THEOREM 8.7.1. *Let $\mathcal{C}(I, \mathbb{R})$ be the set of continuous functions $f: I \rightarrow \mathbb{R}$. Let $\mathfrak{d}(f, g) = \max\{|f(x) - g(x)| \mid x \in I\}$ be the standard metric on $\mathcal{C}(I, \mathbb{R})$. Then the set of nowhere-differentiable functions on I is dense in $\mathcal{C}(I, \mathbb{R})$.*

PROOF. For each $n \geq 1$, let U_n be the set of all continuous functions $f: I \rightarrow \mathbb{R}$ satisfying:

(♠) There exists $0 < \delta < 1/n$ and $\lambda > 0$ such that for all $x \in I$, there exists some $y \in I$ with $\delta < |x - y| < 1/n$ and $\left| \frac{f(x) - f(y)}{x - y} \right| > n + \lambda$.

In the three steps below, we will prove that U_n is open (Step 1), we will construct a family of function in U_n (Step 2), and we will show that U_n is dense (Step 3). Since $\mathcal{C}(I, \mathbb{R})$ is a complete metric space, Theorem 8.6.1 will then apply and show that $\Lambda = \bigcap_{n \geq 1} U_n$ is dense in $\mathcal{C}(I, \mathbb{R})$. Each $f \in \Lambda$ is nowhere differentiable; indeed, for $x \in I$ the condition (♠) forces $\lim_{y \rightarrow x} \frac{f(x) - f(y)}{x - y}$ to not exist.

STEP 1. *For all $n \geq 1$, the set U_n is open in $\mathcal{C}(I, \mathbb{R})$.*

Consider $f \in U_n$. Let $0 < \delta < 1/n$ and $\lambda > 0$ be the constants for f from (♠). Let $g \in \mathcal{C}(I, \mathbb{R})$ be such that $\mathfrak{d}(f, g) < \lambda\delta/4$. We claim that $g \in U_n$. Indeed, consider $x \in I$. Choose $y \in I$ such that $\delta < |x - y| < 1/n$ and $\left| \frac{f(x) - f(y)}{x - y} \right| > n + \lambda$. We then have

$$\begin{aligned} \left| \frac{g(x) - g(y)}{x - y} \right| &\geq \left| \frac{f(x) - f(y)}{x - y} \right| - \left| \frac{g(x) - f(x)}{x - y} \right| - \left| \frac{g(y) - f(y)}{x - y} \right| \\ &> (n + \lambda) - 2 \frac{\lambda\delta/4}{\delta} = n + \lambda/2. \end{aligned}$$

It follows that g satisfies (♠) with the constants δ and $\lambda/2$, so $g \in U_n$.

STEP 2. *For some $n \geq 1$, let $g: I \rightarrow \mathbb{R}$ be a piecewise-linear continuous function such that $|g'(x)| > n$ for all $x \in I$ where g is differentiable. Then $g \in U_n$.*

Let $0 = a_0 < a_1 < \dots < a_m = 1$ be a partition of I such that $g|_{[a_i, a_{i+1}]}$ is linear for all $0 \leq i < m$. For each $0 \leq i < m$, let $c_i, d_i \in \mathbb{R}$ be the constants such that $g(x) = c_i x + d_i$ for all $x \in [a_i, a_{i+1}]$. By assumption, $|c_i| > n$ for all $0 \leq i < m$. Pick $\lambda > 0$ such that $|c_i| > n + \lambda$ for all $0 \leq i < m$. Also, pick $0 < \delta < 1/n$ such that $\delta < (a_{i+1} - a_i)/2$ for all $0 \leq i < m$. Consider some $x \in I$. We have $x \in [a_{i_0}, a_{i_0+1}]$ for some $0 \leq i_0 < m$. Since $0 < \delta < (a_{i_0+1} - a_{i_0})/2$, we can choose some $y \in [a_{i_0}, a_{i_0+1}]$ such that $\delta < |x - y| < 1/n$. It follows that

$$\left| \frac{g(x) - g(y)}{x - y} \right| = \left| \frac{(c_{i_0}x + d_{i_0}) - (c_{i_0}y + d_{i_0})}{x - y} \right| = |c_{i_0}| > n + \lambda,$$

proving that g satisfies (♠) and thus $g \in U_n$.

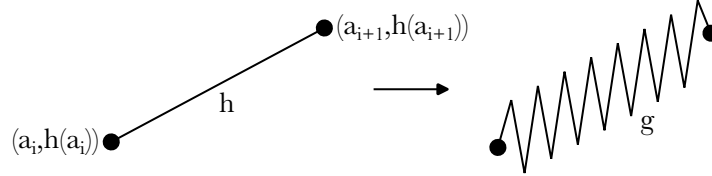
STEP 3. *For all $n \geq 1$, the set U_n is dense in $\mathcal{C}(I, \mathbb{R})$.*

Consider $f \in \mathcal{C}(I, \mathbb{R})$ and $\epsilon > 0$. We must find some $g \in U_n$ such that $\mathfrak{d}(f, g) < \epsilon$. Since f is uniformly continuous on I , we can choose a partition $0 = a_0 < a_1 < \dots < a_m = 1$ of I such that for all $0 \leq i < m$ and $x \in [a_i, a_{i+1}]$ we have $|f(x) - f(a_i)| < \epsilon/4$. Let $h: I \rightarrow \mathbb{R}$ be the piecewise-linear continuous function that is linear on each $[a_i, a_{i+1}]$ and satisfies $h(a_i) = f(a_i)$ for $0 \leq i \leq m$. For

$x \in [a_i, a_{i+1}]$, we therefore have

$$\begin{aligned} |h(x) - f(x)| &= \left| \frac{f(a_{i+1}) - f(a_i)}{a_{i+1} - a_i} (x - a_i) + f(a_i) - f(x) \right| \\ &\leq \left| \frac{f(a_{i+1}) - f(a_i)}{a_{i+1} - a_i} \right| |x - a_i| + |f(a_i) - f(x)| \\ &\leq |f(a_{i+1}) - f(a_i)| + |f(a_i) - f(x)| < \epsilon/4 + \epsilon/4 = \epsilon/2. \end{aligned}$$

It follows that $\mathfrak{d}(f, h) < \epsilon/2$. As in the following figure, we can find a piecewise-linear continuous function $g: I \rightarrow \mathbb{R}$ with $\mathfrak{d}(g, h) < \epsilon/2$ and $|g'(x)| > n$ for all $x \in I$ where g is differentiable by changing h on each interval $[a_i, a_{i+1}]$ to a function whose graph is a rapidly oscillating sawtooth:



We have $\mathfrak{d}(f, g) \leq \mathfrak{d}(f, h) + \mathfrak{d}(h, g) < \epsilon$, and by Step 2 we have $g \in U_n$. $\square$

8.8. Exercises

EXERCISE 8.1. Find an example of a locally compact Hausdorff space X that has a subspace $Y \subset X$ that is not locally compact. $\square$

EXERCISE 8.2. Let X be a Hausdorff space. Prove that X is locally compact if and only if for all $p \in X$, there exists a compact neighborhood K of p . $\square$

EXERCISE 8.3. Let X be a locally compact space. Let $K \subset X$ be compact and $U \subset X$ be open with $K \subset U$. Prove that there is a compact neighborhood L of K with $L \subset U$. $\square$

EXERCISE 8.4. Let X be a locally compact Hausdorff space, let $K \subset X$ be a compact subspace, and let $f: K \rightarrow \mathbb{R}$ be a continuous function. Prove that f can be extended to a continuous function $F: X \rightarrow \mathbb{R}$ with $\text{supp}(F)$ compact. We remark that X might not be normal, so you can't just apply the Tietze extension theorem. $\square$

EXERCISE 8.5. Let X be a locally compact Hausdorff space. Recall that the one-point compactification of X is the set $X^* = X \sqcup \{\infty\}$ with the following topology: a set $U \subset X^*$ is open if either:

- U is an open subset of X ; or
- $U = (X \setminus C) \cup \{\infty\}$, where $C \subset X$ is compact.

Prove the following:

- (a) This is a topology.
- (b) The space X^* is compact Hausdorff.
- (c) The space X^* is a compactification of X . $\square$

EXERCISE 8.6. Prove the following:

- (a) For all $p_0 \in \mathbb{S}^n$ we have $\mathbb{S}^n \setminus p_0 \cong \mathbb{R}^n$.
- (b) The space $\mathbb{S}^n$ is a compactification of $\mathbb{R}^n$.
- (c) The space $\mathbb{S}^n$ is homeomorphic to the one-point compactification of $\mathbb{R}^n$. $\square$

EXERCISE 8.7. Let X be the one-point compactification of $\mathbb{Z}$. Prove that X is homeomorphic to $\{0\} \cup \{1/n \mid n \in \mathbb{Z} \text{ nonzero}\} \subset \mathbb{R}$. $\square$

EXERCISE 8.8. Prove that $\mathbb{R}^n$ with its standard metric is complete, i.e., that all Cauchy sequences in $\mathbb{R}^n$ have limits. $\square$

EXERCISE 8.9. Let M be a complete metric space. Prove that M is a Baire space, i.e., that the following holds. Let $\{U_n\}_{n \geq 1}$ be a collection of open dense subsets of X . Then $\bigcap_{n \geq 1} U_n$ is dense. $\square$

EXERCISE 8.10. Use the Baire category theorem to prove that if $S \subset \mathbb{R}^n$ is a countable collection of points, then $\mathbb{R}^n \setminus S$ is path-connected. $\square$

EXERCISE 8.11. Recall that a subset A of a space is nowhere dense if $\overline{A}$ contain no nonempty open sets, i.e., if $\text{Int}(\overline{A}) = \emptyset$. Prove that $\mathbb{R} \setminus \mathbb{Q}$ cannot be written as a countable union of nowhere dense sets. $\square$

EXERCISE 8.12. Let S be the Sorgenfrey line (see Example 2.6.5), so S is $\mathbb{R}$ with the topology given by the basis $\{[a, b) \mid a < b\}$ of all half-open intervals in $\mathbb{R}$. Prove the following:

- (a) S is not locally compact.
- (b) S is not σ -compact.
- (c) S is a Baire space.

Hint for (a) and (b): use Exercise 7.11. $\square$

CHAPTER 9

Proper maps

This brief chapter covers proper maps, which play an important role in both group actions and the study of embeddings.

9.1. Definition and examples

A map $f: X \rightarrow Y$ is *proper* if it is closed and $f^{-1}(y)$ is compact for all $y \in Y$. The topological meaning of this is a little subtle, and will become more clear as we give examples and alternate characterizations of these maps.

EXAMPLE 9.1.1. A closed embedding $f: X \rightarrow Y$ is trivially proper. □

EXAMPLE 9.1.2. If X is compact and Y is Hausdorff, then any map $f: X \rightarrow Y$ is proper (see Exercise 9.1). □

9.2. Preimages of compact sets

The following is a key property of proper maps:

LEMMA 9.2.1. *Let $f: X \rightarrow Y$ be a proper map and let $K \subset Y$ be compact. Then $f^{-1}(K)$ is compact.*

PROOF. Set $L = f^{-1}(K)$ and let $\mathfrak{U}$ be an open cover of L . Consider some $k \in K$. Set $L_k = f^{-1}(k)$. By assumption, L_k is compact. We can therefore find a finite subset $\mathfrak{U}_k \subset \mathfrak{U}$ that covers L_k . Let $U_k = \cup_{U \in \mathfrak{U}_k} U$. Since f is a closed map, the set $f(X \setminus U_k)$ is closed. We can therefore find an open neighborhood V_k of k with $V_k \subset Y \setminus f(X \setminus U_k)$, i.e., with $f^{-1}(V_k) \subset U_k$. Since K is compact, we can find $k_1, \dots, k_n \in K$ such that $K \subset V_{k_1} \cup \dots \cup V_{k_n}$. We therefore have $L \subset f^{-1}(V_{k_1}) \cup \dots \cup f^{-1}(V_{k_n})$. The finite subset $\mathfrak{U}_{k_1} \cup \dots \cup \mathfrak{U}_{k_n}$ of $\mathfrak{U}$ therefore covers $L = f^{-1}(K)$. □

The conclusion of Lemma 9.2.1 is often taken as the definition of a proper map. The following shows that this alternate definition is equivalent to the one we gave for maps between spaces that are reasonable. For instance, it applies if the spaces in question are metrizable.

LEMMA 9.2.2. *Let $f: X \rightarrow Y$ be a continuous map satisfying the following:*

- *For all compact subsets $K \subset Y$, the preimage $f^{-1}(K) \subset X$ is compact.*

Assume that X and Y are Hausdorff and both are either first countable or locally compact. Then f is proper.

PROOF. We must prove that f is a closed map. We will prove this when X and Y are first countable; see Exercise 9.2 for the case where they are locally compact. Assume that X and Y are first countable and that $C \subset X$ is closed. We must prove that $f(C)$ is closed, i.e., that $\overline{f(C)} = f(C)$. Since Y is first countable, we can calculate the closure $\overline{f(C)}$ using limits; see Lemma 5.3.1. Letting $y \in \overline{f(C)}$, we can therefore find a sequence $\{c_n\}_{n \geq 1}$ of points of C such that $\lim_{n \rightarrow \infty} f(c_n) = y$. Set

$$K = \{y\} \cup \{f(c_n) \mid n \geq 1\}.$$

Since $\lim_{n \rightarrow \infty} f(c_n) = y$, the set K is compact. By assumption, $f^{-1}(K)$ is compact. Since X is first countable, it follows that $f^{-1}(K)$ is sequentially compact; see Lemma 7.8.1. After possibly replacing $\{c_n\}_{n \geq 1}$ by a subsequence, we can therefore assume that there is some $x \in f^{-1}(K)$ with $\lim_{n \rightarrow \infty} c_n = x$. Since Y is Hausdorff, we have $f(x) = f(\lim_{n \rightarrow \infty} c_n) = \lim_{n \rightarrow \infty} f(c_n) = y$. □

9.3. Properness and strongly divergent sequences

Let X be a space and let $\{x_n\}_{n \geq 1}$ be a sequence of points of X . We say that $\{x_n\}_{n \geq 1}$ is *strongly divergent* if for all compact $K \subset X$ there exists some $N \geq 1$ such that $x_n \notin K$ for all $n \geq N$. This implies that $\{x_n\}_{n \geq 1}$ has no convergent subsequence (see Exercise 9.3).

EXAMPLE 9.3.1. Let $(M, \mathfrak{d})$ be a metric space. Assume that M is a proper metric space, which we recall means that closed and bounded subsets of M are compact. A sequence $\{x_n\}_{n \geq 1}$ in M is strongly divergent if and only if for all $p \in M$ we have $\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, p) = \infty$ (see Exercise 9.4). $\square$

Proper maps take strongly divergent sequences to strongly divergent sequences:

LEMMA 9.3.2. *Let $f: X \rightarrow Y$ be a proper map and let $\{x_n\}_{n \geq 1}$ be a strongly divergent sequence in X . Then $\{f(x_n)\}_{n \geq 1}$ is a strongly divergent sequence in Y .*

PROOF. See Exercise 9.5. $\square$

For metrizable spaces, the following lemma shows that the conclusion of Lemma 9.3.2 characterizes proper maps. This gives an important source of intuition about the meaning of properness.

LEMMA 9.3.3. *Let $f: X \rightarrow Y$ be a map between metrizable spaces. Assume that the following holds:*

- *For all strongly divergent sequences $\{x_n\}_{n \geq 1}$ in X , the sequence $\{f(x_n)\}_{n \geq 1}$ in Y is strongly divergent.*

Then f is proper.

PROOF. Let $K \subset Y$ be compact. By Lemma 9.2.2, it is enough to show that $L = f^{-1}(K)$ is compact. Since X is metrizable, to prove that L is compact it suffices to prove that L is sequentially compact. Let $\{x_n\}_{n \geq 1}$ be a sequence of points of L . Since all points of $\{f(x_n)\}_{n \geq 1}$ lie in the compact set K , it follows that $\{f(x_n)\}_{n \geq 1}$ is not strongly divergent. By assumption, it follows that $\{x_n\}_{n \geq 1}$ is not strongly divergent.

Passing to a subsequence, we can therefore assume that there is some compact $C \subset X$ such that $x_n \in C$ for all $n \geq 1$. The sequence $\{x_n\}_{n \geq 1}$ thus lies in $C \cap L$. Since $C \subset X$ and $K \subset Y$ are compact it follows that they are closed, so $L = f^{-1}(K)$ and $C \cap L$ are closed. The closed subset $C \cap L$ of the compact set C is compact and hence sequentially compact, so we conclude that $\{x_n\}_{n \geq 1}$ has a subsequence that converges to a point of $C \cap L \subset L$, as desired. $\square$

9.4. Application: fundamental theorem of algebra

To illustrate the geometric meaning of proper maps, we use them to prove the fundamental theorem of algebra. One way of stating this theorem is that every degree- n complex polynomial has n roots, at least if you count these roots with multiplicity. The nontrivial part of this is that every nonconstant polynomial has a root, so this is what we will prove:

THEOREM 9.4.1 (Fundamental theorem of algebra). *Let $f(z) \in \mathbb{C}[z]$ be a nonconstant polynomial. Then there exists some $z_0 \in \mathbb{C}$ such that $f(z_0) = 0$.*

PROOF. We will prove more generally that regarded as a map $f: \mathbb{C} \rightarrow \mathbb{C}$, the polynomial $f(z)$ is surjective. In fact, it is slightly easier to prove a variant of this. Let $f'(z)$ be the derivative of $f(z)$ and $C = \{z \in \mathbb{C} \mid f'(z) = 0\}$. Define $B = f(C)$ and $A = f^{-1}(X)$. Both A and B are finite sets, and f restricts to a map $F: \mathbb{C} \setminus A \rightarrow \mathbb{C} \setminus B$. To prove that f is surjective, it is enough to prove that F is surjective. Since B is a finite set, its complement $\mathbb{C} \setminus B$ is connected. This reduces us to showing that $F(\mathbb{C} \setminus B)$ is both open and closed. We will prove that F is both an open map and a closed map.¹

Since $\lim_{z \rightarrow \infty} f(z) = \infty$, the map f takes stronger divergent sequences to strongly divergent sequences. By Lemma 9.3.2, the map f is proper and thus so is F (see Exercise 9.6). In particular, F is a closed map. For $z \in \mathbb{C} \setminus A$, since $f'(z) \neq 0$ the inverse function theorem implies that f is a *local homeomorphism* at z , i.e., there is an open neighborhood U of z such that $f|_U: U \rightarrow \mathbb{C}$ is an open embedding. This implies that F is also a local homeomorphism at z . Since F is a local homeomorphism at all points in its domain, it follows that F is an open map (see Exercise 9.7). $\square$

¹In fact, f itself is both an open and closed map, but it is a little easier to prove this for F .

9.5. Exercises

EXERCISE 9.1. Let $f: X \rightarrow Y$ be a map from a compact space X to a Hausdorff space Y . Prove that f is proper. $\square$

EXERCISE 9.2. Let $f: X \rightarrow Y$ be a continuous map satisfying the following:

- For all compact subsets $K \subset Y$, the preimage $f^{-1}(K) \subset X$ is compact.

Assume that X and Y are Hausdorff and locally compact. Prove that f is a closed map, and hence is proper. $\square$

EXERCISE 9.3. Let X be a space and let $\{x_n\}_{n \geq 1}$ be a strongly divergent sequence of points of X . Prove that $\{x_n\}_{n \geq 1}$ has no convergent subsequence. $\square$

EXERCISE 9.4. Let $(M, \mathfrak{d})$ be a proper metric space. Prove that a sequence $\{x_n\}_{n \geq 1}$ in M is strongly divergent if and only if for all $p \in M$ we have $\lim_{n \rightarrow \infty} \mathfrak{d}(x_n, p) = \infty$. $\square$

EXERCISE 9.5. Let $f: X \rightarrow Y$ be a proper map and let $\{x_n\}_{n \geq 1}$ be a strongly divergent sequence in X . Prove that $\{f(x_n)\}_{n \geq 1}$ is a strongly divergent sequence in Y . $\square$

EXERCISE 9.6. Let $f: X \rightarrow Y$ be a proper map, let $Y' \subset Y$ be any subspace, and let $X' = f^{-1}(Y')$. Prove that $f|_{X'}: X' \rightarrow Y'$ is a proper map. $\square$

EXERCISE 9.7. Let $f: X \rightarrow Y$ be a map. For each $x \in X$, assume that there is an open neighborhood U of x such that $f|_U: U \rightarrow Y$ is an open map. Prove that f is an open map. $\square$

Paracompactness and partitions of unity

We now turn to paracompactness, which is a condition that ensure the existence of what are called partitions of unity. These play a basic role in algebraic topology, especially in the theory of manifolds.

10.1. Paracompactness

Let X be a space and let $\mathfrak{Z}$ be a collection of subsets X . We say that $\mathfrak{Z}$ is *locally finite* if for all $p \in X$, there exists a neighborhood U such that only finitely many $Z \in \mathfrak{Z}$ satisfy $U \cap Z \neq \emptyset$. This implies in particular that for all $p \in X$ there are only finitely many $Z \in \mathfrak{Z}$ with $p \in Z$, but is stronger (see Exercise 10.1). One nice property of locally finite collections of subset is:

LEMMA 10.1.1. *Let X be a space and let $\mathfrak{Z}$ be a locally finite collection of subsets of X . Then*

$$\overline{\bigcup_{Z \in \mathfrak{Z}} Z} = \bigcup_{Z \in \mathfrak{Z}} \overline{Z}.$$

PROOF. See Exercise 10.2. In that exercise, you will also show that this is false without the local finiteness assumption. $\square$

Now let $\mathfrak{U}$ be an open cover of X . A *refinement* of $\mathfrak{U}$ is an open cover $\mathfrak{V}$ such that for all $V \in \mathfrak{V}$, there exists some $U \in \mathfrak{U}$ with $V \subset U$. A space X is *paracompact* if it is Hausdorff and every open cover of X admits a locally finite refinement. We will prove that this has strong consequences for the topology of X . In particular, X must be normal (see Lemma 10.4.1).

10.2. Locally compact Hausdorff spaces that are σ -compact are paracompact

The easiest examples of paracompact spaces are compact Hausdorff spaces, where every open cover admits a finite cover (not just a locally finite one). Our next goal is to prove the following generalization of this:

THEOREM 10.2.1. *Let X be a locally compact Hausdorff space that is σ -compact. Then X is paracompact.*

Before we prove this, we note that in light of Lemma 8.4.1 it implies:

COROLLARY 10.2.2. *Let X be a locally compact Hausdorff space that is second countable. Then X is paracompact. In particular, both open and closed subspaces of $\mathbb{R}^n$ are paracompact.*

PROOF OF THEOREM 10.2.1. We start by proving:

CLAIM. *There exists a countable open cover $\{W_1, W_2, \dots\}$ of X such that for all $n \geq 1$ the set $\overline{W}_n$ is compact and satisfies $\overline{W}_n \subset W_{n+1}$.*

PROOF OF CLAIM. Since X is σ -compact, we can write $X = \bigcup_{n \geq 1} K_n$ with K_n compact. We will inductively construct open sets W_n of X such that $W_0 = \emptyset$ and for all $n \geq 0$ we have:

- $\overline{W}_n$ is compact; and
- W_{n+1} contains $\overline{W}_n \cup K_{n+1}$.

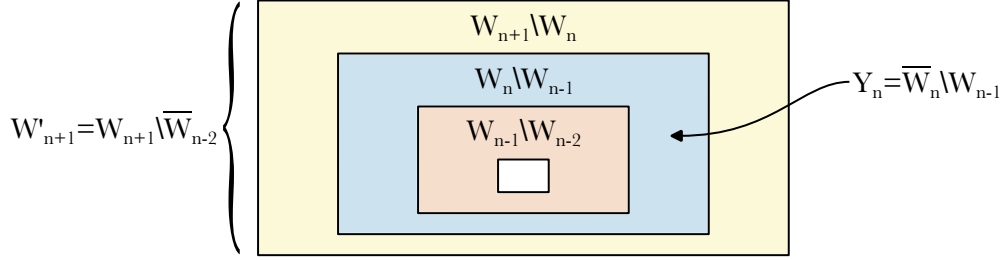
Since $X = \bigcup_{n \geq 1} K_n$, this will be an open cover of X with the properties indicated in the claim. Start by setting $W_0 = \emptyset$, and assume we have constructed $W_0, \dots, W_n$. For $p \in \overline{W}_n \cup K_{n+1}$, local compactness gives an open neighborhood $W_{n+1}(p)$ of p with $\overline{W}_{n+1}(p)$ compact. Since $\overline{W}_n \cup K_{n+1}$ is compact, we can find $p_1, \dots, p_m \in \overline{W}_n \cup K_{n+1}$ such that $\{W_{n+1}(p_1), \dots, W_{n+1}(p_m)\}$ covers

$\overline{W}_n \cup K_{n+1}$. We can then define $W_{n+1} = W_{n+1}(p_1) \cup \cdots \cup W_{n+1}(p_m)$. The set $\overline{W}_{n+1}$ is compact since $\overline{W}_{n+1} = \overline{W}_{n+1}(p_1) \cup \cdots \cup \overline{W}_{n+1}(p_m)$ (see Lemma 10.1.1). $\square$

We now prove that X is paracompact. Let $\mathfrak{U}$ be an open cover of X . Let $\{W_n\}_{n \geq 1}$ be as in the claim. Set $W_m = \emptyset$ for $m \leq 0$. For $n \in \mathbb{Z}$, define $Y_n = \overline{W}_n \setminus W_{n-1}$ and $W'_{n+1} = W_{n+1} \setminus \overline{W}_{n-2}$. These satisfy:

- (i) Y_n is a compact subset of the open set W'_{n+1} ; and
- (ii) $X = \bigcup_{n=1}^{\infty} Y_n$; and
- (iii) $W'_{n_1} \cap W'_{n_2} = \emptyset$ whenever $|n_1 - n_2| \geq 3$.

See here:



For each $n \geq 1$, the set $\{U \cap W'_{n+1} \mid U \in \mathfrak{U}\}$ is an open cover of compact set Y_n , so there is a finite subset $\mathfrak{U}(n) \subset \mathfrak{U}$ such that $\{U \cap W'_{n+1} \mid U \in \mathfrak{U}(n)\}$ covers Y_n . Let

$$\mathfrak{V} = \{U \cap W'_{n+1} \mid n \geq 1 \text{ and } U \in \mathfrak{U}(n)\}.$$

The set $\mathfrak{V}$ is an open cover of each Y_n , so by (ii) it follows that $\mathfrak{V}$ is an open cover of X . By construction, $\mathfrak{V}$ refines $\mathfrak{U}$. Using (iii) together with the fact that only finitely many $V \in \mathfrak{V}$ are contained in each W'_n , the open cover $\mathfrak{V}$ is locally finite. The theorem follows. $\square$

10.3. Metrizable spaces are paracompact

We next prove that metrizable spaces are paracompact. This was originally proved by Stone [2], but the proof we give is a much easier argument of Rudin [1]. The proof uses some set-theoretic technology. A *well-ordered set* is a set I equipped with a total ordering $\leq$ such that every nonempty subset $S \subset I$ has a minimal element. We call $\leq$ a *well-ordering*. A remarkable consequence of the axiom of choice is that every set can be equipped with a well-ordering.

EXAMPLE 10.3.1. All total orderings on finite sets are well-orderings, and the standard ordering on $\mathbb{N}$ is a well-ordering. $\square$

We will develop the theory of well-ordered sets more in Chapter 11 when we introduce transfinite induction, which is needed to prove that arbitrary products of compact spaces are compact (Tychonoff's Theorem). Here we use them to prove the following:

LEMMA 10.3.2. *Let M be a metrizable space. Then M is paracompact.*

PROOF. Let $\mathfrak{d}$ be a distance function on M inducing its topology. Consider an open cover $\mathfrak{U}$ of M . Choose a well-ordering on $\mathfrak{U}$. For each $x \in M$, let U_x be the minimal element of the set $\{U \in \mathfrak{U} \mid x \in U\}$. For each $U \in \mathfrak{U}$ and $n \geq 1$, we inductively construct a subset $U[n] \subset U$ as follows. Assume that $n \geq 1$ and that for all $1 \leq m < n$ we have constructed $V[m] \subset V$ for all $V \in \mathfrak{U}$ (a vacuous assumption if $n = 1$). Consider some $U \in \mathfrak{U}$. We define $U[n]$ to be the union of all open balls $B_x(1/2^n)$ where $x \in M$ satisfies the following three conditions:

- (i) $U_x = U$; and
- (ii) $x \notin V[m]$ for any $V \in \mathfrak{U}$ and $1 \leq m < n$; and
- (iii) $B_x(3/2^n) \subset U$.

By (iii) we have $U[n] \subset U$. Define

$$\mathfrak{V} = \{U[n] \mid U \in \mathfrak{U} \text{ and } n \geq 1\}.$$

This clearly refines $\mathfrak{U}$. We will prove that it covers M and is locally finite:

CLAIM 1. *The set $\mathfrak{V}$ covers M .*

Consider some $x \in M$. Set $U = U_x$. Since U is an open neighborhood of x , there is some $n \geq 1$ such that $B_x(3/2^n) \subset U$. We then either have $x \in V[m]$ for some $V \in \mathfrak{U}$ and $1 \leq m < n$ or we have $B_x(1/2^n) \subset U[n]$ and thus $x \in U[n]$. In either case, x lies in some element of $\mathfrak{V}$.

CLAIM 2. *The set $\mathfrak{V}$ is locally finite.*

Consider some $x \in M$. Let U be the smallest element of $\mathfrak{U}$ such that there exists some $n \geq 1$ with $x \in U[n]$. Letting $n \geq 1$ be the minimal element such that $x \in U[n]$, choose $d \geq 1$ such that $B_x(1/2^d) \subset U[n]$. We will prove that the open neighborhood $B_x(1/2^{n+d})$ of x only intersects finitely many elements of $\mathfrak{V}$. In fact, we will prove the following:

- (a) For all $V \in \mathfrak{U}$ and $m \geq n + d$, the element $V[m]$ of $\mathfrak{V}$ does not intersect $B_x(1/2^{n+d})$.
- (b) For all $1 \leq \ell < n + d$, at most one element of $\mathfrak{V}$ of the form $W[\ell]$ can intersect $B_x(1/2^{n+d})$.

Together these will imply that at most $n + d - 1$ elements of $\mathfrak{V}$ can intersect $B_x(1/2^{n+d})$.

We start with (a). Consider some $V \in \mathfrak{U}$ and $m \geq n + d$. We wish to prove that $V[m]$ is disjoint from $B_x(1/2^{n+d})$. The open set $V[m]$ is the union of open balls $B_y(1/2^m)$ where $y \in M$ satisfies:

- (i) $U_y = V$; and
- (ii) $y \notin W[k]$ for any $W \in \mathfrak{U}$ and $1 \leq k < m$.

We do not include condition (iii) since it is not needed for the proof. Letting $y \in M$ satisfy these conditions, we must show that $B_y(1/2^m)$ is disjoint from $B_x(1/2^{n+d})$, i.e., that $\mathfrak{d}(x, y) \geq 1/2^{n+d} + 1/2^m$. Since $m \geq n + d > n$, condition (ii) implies that $y \notin U[n]$. Since $B_x(1/2^d) \subset U[n]$, this implies that

$$\mathfrak{d}(x, y) \geq 1/2^d = 1/2^{d+1} + 1/2^{d+1} \geq 1/2^{n+d} + 1/2^m,$$

as desired.

We now prove (b). Fix some $1 \leq \ell < n + d$, and let $W, W' \in \mathfrak{U}$ be such that $W[\ell]$ and $W'[\ell]$ both intersect $B_x(1/2^{n+d})$. We must prove that $W = W'$. The open set $W[\ell]$ is the union of open balls $B_z(1/2^\ell)$ where $z \in M$ satisfies:

- (i) $U_z = W$; and
- (iii) $B_z(3/2^\ell) \subset W$.

We do not include condition (ii) since it is not needed for the proof. We can therefore find some $z \in M$ satisfying (i) and (iii) such that $B_z(1/2^\ell)$ intersects $B_x(1/2^{n+d})$. Choose a point p in this intersection. Similarly, we can find $z' \in M$ satisfying the analogues of (i) and (iii) for W' such that $B_{z'}(1/2^\ell)$ intersects $B_x(1/2^{n+d})$. Choose a point p' in this intersection. We have

$$\mathfrak{d}(z, z') \leq \mathfrak{d}(z, p) + \mathfrak{d}(p, p') + \mathfrak{d}(p, z') < 1/2^\ell + 2/2^{n+d} + 1/2^\ell \leq 1/2^\ell + 1/2^\ell + 1/2^\ell = 3/2^\ell.$$

Applying (iii), we deduce that

$$z \in B_{z'}(3/2^\ell) \subset W' \quad \text{and} \quad z' \in B_z(3/2^\ell) \subset W.$$

Condition (i) says that W is the minimal element of $\mathfrak{U}$ with $z \in W$, so $W \leq W'$. Similarly, W' is the minimal element of $\mathfrak{U}$ with $z' \in W'$, so $W' \leq W$. We conclude that $W = W'$, as desired. $\square$

10.4. Normality

Our next goal is to prove that paracompact spaces are normal:

LEMMA 10.4.1. *Let X be a paracompact space. Then X is normal.*

PROOF. Recall that paracompact spaces are assumed to be Hausdorff. We start by proving that X is regular (see §6.10), i.e., that the following holds:

CLAIM. *For $p \in X$ and $C \subset X$ closed with $p \notin C$, there exist disjoint open neighborhoods of p and C .*

PROOF OF CLAIM. For each $q \in C$, since X is Hausdorff there exist open neighborhoods U_{qp} of q and U'_{qp} of p such that $U_{qp} \cap U'_{qp} = \emptyset$. Since X is paracompact, the open cover $\{X \setminus C\} \cup \{U_{qp} \mid q \in C\}$ admits a locally finite refinement. Let $\mathfrak{V}$ be the open sets in this locally finite refinement that are not contained in $X \setminus C$. For each $V \in \mathfrak{V}$, there is some $q \in C$ such that $V \subset U_{qp}$. Since U'_{qp} is an open

neighborhood of p that is disjoint from U_{qp} , we deduce that $p \notin \bar{V}$ for all $V \in \mathfrak{V}$. Set $W = \cup_{V \in \mathfrak{V}} V$. The set W is an open neighborhood of C , and by local finiteness and Lemma 10.1.1 we have

$$\bar{W} = \bigcup_{V \in \mathfrak{V}} \bar{V}.$$

Since $p \notin \bar{V}$ for all $V \in \mathfrak{V}$, we deduce that $p \notin \bar{W}$. It follows that $X \setminus \bar{W}$ and W are disjoint open neighborhoods of p and C . $\square$

To prove that X is normal, let C and D be disjoint closed subsets of X . We can find disjoint open neighborhoods of C and D by the same argument we used to prove the above claim. Simply substitute the above claim for X being Hausdorff and replace every occurrence of the point p by the closed set D . $\square$

10.5. Strong refinements

Let $\mathfrak{U}$ be an open cover of a space X . Enumerate $\mathfrak{U}$ as $\mathfrak{U} = \{U_i\}_{i \in I}$. A *strong refinement* of $\mathfrak{U}$ consists of an open cover $\{V_i\}_{i \in I}$ such that $\bar{V}_i \subset U_i$ for all $i \in I$. We have:

LEMMA 10.5.1. *Let X be a paracompact space and let $\mathfrak{U}$ be an open cover of X . Then there exists a locally finite strong refinement of $\mathfrak{U}$.*

PROOF. Enumerate $\mathfrak{U}$ as $\mathfrak{U} = \{U_i\}_{i \in I}$. Let

$$\mathfrak{W}' = \left\{ W' \mid W' \text{ open set with } \bar{W}' \subset U_i \text{ for some } i \in I \right\}.$$

The set $\mathfrak{W}'$ is an open cover of X ; indeed, since X is normal for all $p \in X$ and all $i \in I$ with $p \in U_i$ we can find an open neighborhood W' of p with $\bar{W}' \subset U_i$. Since X is paracompact, we can find a locally finite refinement $\mathfrak{W}$ of $\mathfrak{W}'$. For each $W \in \mathfrak{W}$, there is some $i \in I$ with $\bar{W} \subset U_i$. For $i \in I$, let $\mathfrak{W}(i) = \{W \in \mathfrak{W} \mid \bar{W} \subset U_i\}$ and $V_i = \cup_{W \in \mathfrak{W}(i)} W$. Since $\mathfrak{W}(i)$ is a locally finite collection of open sets, Lemma 10.1.1 implies that

$$\bar{V}_i = \bigcup_{W \in \mathfrak{W}(i)} \bar{W} \subset U_i.$$

The open cover $\mathfrak{V} = \{V_i\}_{i \in I}$ is thus a locally finite strong refinement of $\mathfrak{U} = \{U_i\}_{i \in I}$. $\square$

10.6. Partitions of unity

We now come to the most important property of paracompact spaces. Let X be a space. Recall that for a continuous function $f: X \rightarrow \mathbb{R}$, the support of f is $\text{supp}(f) = \overline{\{p \in X \mid f(p) \neq 0\}}$. A *partition of unity* subordinate to an open cover $\mathfrak{U}$ of X consists of continuous functions $f_U: X \rightarrow I = [0, 1]$ for each $U \in \mathfrak{U}$ satisfying the following three conditions:

- (a) For all $U \in \mathfrak{U}$, we have $\text{supp}(f_U) \subset U$.
- (b) The set $\{\text{supp}(f_U) \mid U \in \mathfrak{U}\}$ is locally finite.
- (c) For all $p \in X$, we have $\sum_{U \in \mathfrak{U}} f_U(p) = 1$. Note that (b) implies that only finitely many terms of this sum are nonzero, so this sum makes sense.

We have:

THEOREM 10.6.1. *Let X be a paracompact space and let $\mathfrak{U}$ be an open cover of X . Then there exists a partition of unity subordinate to $\mathfrak{U}$.*

PROOF. Enumerate $\mathfrak{U}$ as $\mathfrak{U} = \{U_i\}_{i \in I}$. By Lemma 10.5.1, we can find a locally finite strong refinement $\{V_i\}_{i \in I}$ of $\{U_i\}_{i \in I}$. Applying this lemma again, we obtain a locally finite strong refinement $\{W_i\}_{i \in I}$ of $\{V_i\}_{i \in I}$. Lemma 10.4.1 says that X is normal, so we can apply Urysohn's Lemma (Theorem 6.5.1) to X . For $i \in I$, since $\bar{W}_i \subset V_i$ Urysohn's Lemma (Theorem 6.5.1) implies that there is a continuous function $f'_i: X \rightarrow I$ such that $f'_i|_{\bar{W}_i} = 1$ and $\text{supp}(f'_i) \subset V_i$. Since $\{V_i\}_{i \in I}$ is locally finite and $\text{supp}(f'_i) \subset \bar{W}_i \subset V_i$ for each $i \in I$, we can define $g: X \rightarrow [0, \infty)$ via the formula

$$g(p) = \sum_{i \in I} f'_i(p) \quad \text{for } p \in X.$$

The function $g: X \rightarrow [0, \infty)$ is continuous (see Exercise 10.6). Each $p \in X$ lies in some W_i , so since $f'_i|_{\overline{W_i}} = 1$ it follows that $g(p) > 0$ for all $p \in X$. For $i \in I$, we can therefore define $f_i: X \rightarrow [0, \infty)$ via the formula

$$f_i(p) = \frac{1}{g(p)} f'_i(p) \quad \text{for } p \in X.$$

For $p \in X$, we have

$$\sum_{i \in I} f_i(p) = \frac{1}{g(p)} \sum_{i \in I} f'_i(p) = \frac{1}{g(p)} g(p) = 1.$$

Since $f_i(p) \in [0, \infty)$ for all $i \in I$, this implies that the image of each f_i lies in I and that the f_i form a partition of unity subordinate to $\mathfrak{U} = \{U_i\}_{i \in I}$. $\square$

10.7. Application: extending functions

Here is a typical application of partitions of unity:

LEMMA 10.7.1. *Let X be a paracompact space, let $A \subset X$ be a subspace, and let $f: A \rightarrow \mathbb{R}$ be continuous. For all $a \in A$, assume that there is a neighborhood U_a of a and an extension of $f|_{U_a \cap A}$ to $F_a: U_a \rightarrow \mathbb{R}$. Set $U = \cup_{a \in A} U_a$. Then f can be extended to a continuous function $F: U \rightarrow \mathbb{R}$.*

REMARK 10.7.2. If A is closed, then the Tietze extension theorem (Theorem 6.9.1) says that f can be extended to the whole space X . This can fail for non-closed subspaces. For instance, consider the subspace $\mathbb{Q}$ of $\mathbb{R}$. The continuous function $f: \mathbb{Q} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} -1 & \text{if } x < \sqrt{2}, \\ 1 & \text{if } x > \sqrt{2} \end{cases} \quad \text{for } x \in \mathbb{Q}.$$

can be extended to a continuous function on the open set $\mathbb{R} \setminus \{\sqrt{2}\}$, but cannot be extended to a continuous function on $\mathbb{R}$. $\square$

PROOF. Replacing X by U , we can assume that $\mathfrak{U} = \{U_a \mid a \in A\}$ is an open cover of X . Let $\{\phi_{U_a}: X \rightarrow \mathbb{R} \mid a \in A\}$ be a partition of unity subordinate to $\mathfrak{U}$. Since $\text{supp}(\phi_{U_a}) \subset U_a$, the function $F_a \phi_{U_a}: U_a \rightarrow \mathbb{R}$ can be extended to a continuous function $G_a: X \rightarrow \mathbb{R}$ by letting $G_a(x) = 0$ for $x \in X \setminus U_a$. We have $\text{supp}(G_a) \subset \text{supp}(\phi_a)$ for $a \in A$, so since the set of supports of the ϕ_a are locally finite we can define $F: X \rightarrow \mathbb{R}$ via the formula $F = \sum_{a \in A} G_a$. For $a \in A$, we have

$$F(a) = \sum_{a \in A} F_a(a) \phi_{U_a}(a) = f(a) \sum_{a \in A} \phi_{U_a}(a) = f(a),$$

so F is an extension of f . $\square$

10.8. Exercises

EXERCISE 10.1. Give an example of a space X and a non-locally finite open cover $\mathfrak{U}$ of X such that for all $p \in X$ there are only finitely many $U \in \mathfrak{U}$ with $p \in U$. $\square$

EXERCISE 10.2. Let X be a space and let $\mathfrak{Z}$ be a collection of subsets of X .

(a) If $\mathfrak{Z}$ is locally finite, prove that

$$\overline{\bigcup_{Z \in \mathfrak{Z}} Z} = \bigcup_{Z \in \mathfrak{Z}} \overline{Z}.$$

(b) Give an example to show that local finiteness is needed in the previous part. $\square$

EXERCISE 10.3. Let X be a space, let $\mathfrak{U}$ be an open cover of X , and let $\mathfrak{V}$ be an open cover of X that refines $\mathfrak{U}$. Assume that $\mathfrak{V}$ has a finite subcover. Prove that $\mathfrak{U}$ has a finite subcover. $\square$

EXERCISE 10.4. Let X be paracompact and let $A \subset X$ be closed. Topologize X/A using the quotient topology (see Example 3.2.8). Prove that X/A is paracompact. $\square$

EXERCISE 10.5. Let X and Y be paracompact spaces, let $A \subset X$ be closed, and let $\phi: A \rightarrow Y$ be a closed map. Let Z be the result of gluing X to Y using ϕ (see Example 3.2.1). Prove that Z is paracompact. $\square$

EXERCISE 10.6. Let X be a space and let $\{V_i\}_{i \in I}$ be a locally finite collection of open subsets of X . For each $i \in I$, let $h_i: X \rightarrow \mathbb{R}$ be a continuous function such that $\text{supp}(h_i) \subset V_i$. Define $h: X \rightarrow \mathbb{R}$ via the formula

$$h(p) = \sum_{i \in I} h_i(p) \quad \text{for } p \in X.$$

Prove that $h: X \rightarrow \mathbb{R}$ is continuous. □

EXERCISE 10.7. Let S be the Sorgenfrey line (see Example 2.6.5), so S is $\mathbb{R}$ with the topology given by the basis $\{[a, b) \mid a < b\}$ of all half-open intervals in $\mathbb{R}$. Prove that S is paracompact. Warning: this is harder than most of our other exercises. Exercises 6.12 and 7.11 might be helpful. These exercises show that S is a normal Lindelöf space, and in fact all normal Lindelöf spaces are paracompact.¹ □

Bibliography

- [1] M. Rudin, A new proof that metric spaces are paracompact, Proc. Amer. Math. Soc. 20 (1969), 603.
- [2] A. Stone, Paracompactness and product spaces, Bull. Amer. Math. Soc. 54 (1948), 977–982.

¹In fact, only regularity is needed here since all regular Lindelöf spaces are normal.

Products and Tychonoff's theorem

We now discuss products of spaces, generalizing the topology on $X \times Y$ from Example 2.6.4.

11.1. Finite products

Let $X_1, \dots, X_n$ be spaces. As a set, $X_1 \times \cdots \times X_n$ consists of tuples $(x_1, \dots, x_n)$ with $x_i \in X_i$ for $1 \leq i \leq n$. Give this the topology with the basis consisting of products $U_1 \times \cdots \times U_n$ with $U_i \subset X_i$ open for $1 \leq i \leq n$. We will call these the *basic open sets* of the product. A general open set $V \subset X_1 \times \cdots \times X_n$ can therefore be written a union of basic open sets. Equivalently, $V \subset X_1 \times \cdots \times X_n$ is open if and only if for all $(p_1, \dots, p_n) \in V$, there exist open neighborhoods $U_i \subset X_i$ of each p_i such that $U_1 \times \cdots \times U_n \subset V$.

EXAMPLE 11.1.1. This gives the usual topology on $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ (see Exercise 11.1). $\square$

11.2. Finite universal property

Let $\pi_i: X_1 \times \cdots \times X_n \rightarrow X_i$ be the projection. This is continuous: if $U_i \subset X_i$ is open, then

$$\pi_i^{-1}(U_i) = X_1 \times \cdots \times X_{i-1} \times U_i \times X_{i+1} \times \cdots \times X_n.$$

Now let Y be another space, and for $1 \leq i \leq n$ let $f_i: Y \rightarrow X_i$ be a continuous map. Let $f_1 \times \cdots \times f_n: Y \rightarrow X_1 \times \cdots \times X_n$ be the map where $f_1 \times \cdots \times f_n(y) = (f_1(y), \dots, f_n(y))$ for $y \in Y$. This is continuous; indeed, if $U_i \subset X_i$ is open for $1 \leq i \leq n$ then

$$(f_1 \times \cdots \times f_n)^{-1}(U_1 \times \cdots \times U_n) = U_1 \cap \cdots \cap U_n.$$

Conversely, if $F: Y \rightarrow X_1 \times \cdots \times X_n$ is a continuous map, then letting $f_i = \pi_i \circ F$ we have $F = f_1 \times \cdots \times f_n$. We summarize this informally as:

- A continuous map $F: Y \rightarrow X_1 \times \cdots \times X_n$ is the same thing as a collection of continuous maps $f_i: Y \rightarrow X_i$ for all $1 \leq i \leq n$.

Just like for quotient spaces in §3.4, this is an example of a universal mapping property. We will say more about it in §11.6 below.

11.3. Homotopies, products, and quotient maps

One place where products show up in algebraic topology is in the definition of a homotopy. Roughly speaking, a homotopy is a continuous deformation of a map. The precise definition is as follows. Let $f_0, f_1: Y \rightarrow Z$ be maps. A *homotopy* from f_0 to f_1 is a map $H: Y \times I \rightarrow Z$ such that $H(y, 0) = f_0(y)$ and $H(y, 1) = f_1(y)$ for $y \in Y$. If such a homotopy exists, we say that f_0 and f_1 are *homotopic*.

For $t \in I$, we can let $f_t: Y \rightarrow Z$ be the map defined by $f_t(y) = H(y, t)$ for $y \in Y$. The maps $f_t: Y \rightarrow Z$ can be viewed informally as a continuous family of maps connecting f_0 to f_1 . See §13.7 for how to topologize the space of maps $Y \rightarrow Z$ and make this precise.

EXAMPLE 11.3.1. Any two maps $f_0, f_1: Y \rightarrow \mathbb{R}^n$ are homotopic via the homotopy $H: Y \times I \rightarrow \mathbb{R}^n$ defined by $H(y, t) = (1 - t)f_0(y) + tf_1(y)$. $\square$

EXAMPLE 11.3.2. Let $Y = \{*\}$ be a one-point space. Two maps $f_0, f_1: Y \rightarrow Z$ are homotopic if and only if $f_0(*)$ and $f_1(*)$ lie in the same path component of Z . $\square$

Now assume that $q: X \rightarrow Y$ is a quotient map (see §3.3), so q is surjective and $U \subset Y$ is open if and only if $q^{-1}(U) \subset X$ is open. Given $f_0, f_1: Y \rightarrow Z$, it is natural to try to construct a homotopy from f_0 to f_1 as follows:

- Define $g_0 = f_0 \circ q$ and $g_1 = f_1 \circ q$. Construct a homotopy $\tilde{H}: X \times I \rightarrow Z$ from g_0 to g_1 .
- Next, use the universal property of the quotient map from §3.4 to show that $\tilde{H}$ descends to a homotopy $H: Y \times I \rightarrow Z$.

Here are an example of how this might work:

EXAMPLE 11.3.3. We have $\mathbb{D}^n / \partial\mathbb{D}^n \cong \mathbb{S}^n$ (see Example 3.2.10). A map $f: \mathbb{S}^n \rightarrow Z$ is thus the same as a map $g: \mathbb{D}^n \rightarrow Z$ such that $g|_{\partial\mathbb{D}^n}$ is constant. Given $f_0, f_1: \mathbb{S}^n \rightarrow Z$, let $g_0, g_1: \mathbb{D}^n \rightarrow Z$ be the corresponding maps. To construct a homotopy from f_0 to f_1 , it is natural to instead try to construct a homotopy g_t from g_0 to g_1 such that $g_t|_{\partial\mathbb{D}^n}$ is constant for all t . $\square$

However, there is a flaw in the above reasoning: if $q: X \rightarrow Y$ is a quotient map, it not clear that $q \times \mathbb{1}: X \times I \rightarrow Y \times I$ is a quotient map. Indeed, there are counterexamples if I is replaced by a more complicated space. However, for nice spaces like I this is not a problem. More generally:

LEMMA 11.3.4. *Let $q: X \rightarrow Y$ be a quotient map and let Z be a locally compact space. Then the map $q \times \mathbb{1}: X \times Z \rightarrow Y \times Z$ is a quotient map.*

PROOF.¹ The map $q \times \mathbb{1}: X \times Z \rightarrow Y \times Z$ is continuous, so for every open set $U \subset Y \times Z$ we have $q^{-1}(U)$ open. We must prove the converse. In other words, letting $U \subset Y \times Z$ be a set such that $q^{-1}(U)$ is open, we must prove that U is open. Letting $(y, z) \in U$, it is enough to find an open neighborhood of (y, z) that is contained in U .

Pick $x \in X$ with $q(x) = y$. We have $(x, z) \in q^{-1}(U)$. Since $q^{-1}(U) \subset X \times Z$ is open and Z is locally compact, we can find an open neighborhood $V_1 \subset X$ of x and a compact neighborhood $K \subset Z$ of z such that $V_1 \times K \subset q^{-1}(U)$. We have

$$(y, z) \in q(V_1 \times \text{Int}(K)) = q(V_1) \times \text{Int}(K) \subset U.$$

If $q(V_1) \subset Y$ were open, then $q(V_1) \times \text{Int}(K)$ would be an open neighborhood of (y, z) contained in U and we would be done.

Unfortunately, $q(V_1)$ might not be open since $q^{-1}(q(V_1))$ might be larger than V_1 . We do have $q^{-1}(q(V_1)) \times K \subset q^{-1}(U)$. Since K is compact and $q^{-1}(U)$ is open, we can find an open neighborhood V_2 of $q^{-1}(q(V_1))$ with $V_2 \times K \subset q^{-1}(U)$ (see Exercise 7.8; this is often called the “tube lemma”). Just like for V_1 , there is no reason to expect $q(V_2) \subset Y$ to be open since $q^{-1}(q(V_2))$ might be larger than V_2 . However, we can iterate the procedure we used to find V_2 . The result is an increasing sequence $V_1 \subset V_2 \subset \dots$ of open subsets of Y such that for all $n \geq 1$ we have:

- $V_n \times K \subset q^{-1}(U)$ and $q^{-1}(q(V_n)) \subset V_{n+1}$.

The set $V = \bigcup_{n \geq 1} V_n$ is then an open subset of Y with $V \times K \subset q^{-1}(U)$ and $q^{-1}(q(V)) = V$. It follows that $q(V)$ is an open subset of Y , so $q(V) \times \text{Int}(K)$ is an open neighborhood of (y, z) with $q(V) \times \text{Int}(K) \subset U$, as desired. $\square$

11.4. Tychonoff's theorem, finite case

We have the following basic result:

THEOREM 11.4.1 (Tychonoff's theorem, finite case). *Let $X_1, \dots, X_n$ be compact spaces. Then $X_1 \times \dots \times X_n$ is compact.*

PROOF. By induction, it is enough to prove this for $n = 2$. Let $\mathfrak{U}$ be an open cover of $X_1 \times X_2$. We must prove that $\mathfrak{U}$ has a finite subcover. In fact, it is enough to prove that some refinement of $\mathfrak{U}$ has a finite subcover (see Exercise 10.3). Each element of $\mathfrak{U}$ is a union of basic open sets. Letting $\mathfrak{V}$ be the set of all basic open sets V such that there exists some $U \in \mathfrak{U}$ with $V \subset U$, it is therefore enough to prove that $\mathfrak{V}$ has a finite subcover.

¹We will give an alternate proof in Chapter 13 which is shorter but more abstract. See §13.8.

For $p \in X_1$, let $Z(p) = p \times X_2$. By assumption, $Z(p) \cong X_2$ is compact. We can therefore find a finite subset $\mathfrak{V}(p)$ of $\mathfrak{V}$ that covers $Z(p)$. Since $\mathfrak{V}$ consists of basic open sets, we can write

$$\mathfrak{V}(p) = \{V_1(p) \times V'_1(p), \dots, V_{m_p}(p) \times V'_{m_p}(p)\}$$

with $V_i(p) \subset X_1$ and $V'_i(p) \subset X_2$ for $1 \leq i \leq m_p$. Discarding unneeded terms if necessary, we can assume that $p \in V_i(p)$ for all $1 \leq i \leq m_p$. Letting $V(p) = V_1(p) \cap \dots \cap V_{m_p}(p)$, it follows that $V(p)$ is an open neighborhood of p and $\mathfrak{V}(p)$ covers $V(p) \times X_2$.

The set $\{V(p) \mid p \in X_1\}$ is an open cover of the compact space X_1 , so we can find $p_1, \dots, p_d \in X_1$ such that $X_1 = V(p_1) \cup \dots \cup V(p_d)$. Since $\mathfrak{V}(p_i)$ is a finite cover of $V(p_i) \times X_2$ for $1 \leq i \leq d$, we conclude that $\mathfrak{V}(p_1) \cup \dots \cup \mathfrak{V}(p_d)$ is a finite subset of $\mathfrak{V}$ that covers $X_1 \times X_2$. $\square$

11.5. Infinite products

Now let $\{X_i\}_{i \in I}$ be an arbitrary collection of spaces. As a set, the product $\prod_{i \in I} X_i$ consists of tuples $(x_i)_{i \in I}$ with $x_i \in X_i$ for $i \in I$. The obvious first guess for a topology on $\prod_{i \in I} X_i$ is the one with basis the collection of products $\prod_{i \in I} U_i$ with $U_i \subset X_i$ open for all $i \in I$. However, this topology turns out to be pathological. The issue is that it has too many open sets, and there are maps into it that should be continuous but are not. Here is a key example:

EXAMPLE 11.5.1. Let X be a space and let I be an infinite indexing set. Consider the diagonal map $\Delta: X \rightarrow \prod_{i \in I} X$, so $\Delta(x) = (x)_{i \in I}$ for all $x \in X$. If $U_i \subset X$ is an open set for all $i \in I$, then

$$\Delta^{-1}\left(\prod_{i \in I} U_i\right) = \bigcap_{i \in I} U_i.$$

Since the collection of open sets is *not* closed under infinite intersections, this is not always open. It follows that Δ will generally not be continuous if all such sets of the form $\prod_{i \in I} U_i$ are open. $\square$

To eliminate this pathology, we must avoid infinite intersections of open sets. This can be done as follows. A *basic open set* in $\prod_{i \in I} X_i$ is a product $\prod_{i \in I} U_i$ such that:

- $U_i \subset X_i$ is open for all $i \in I$; and
- $U_i = X_i$ for all but finitely many $i \in I$.

The *product topology* on $\prod_{i \in I} X_i$ is the topology with basis the basic open sets, so a subset of $\prod_{i \in I} X_i$ is open if and only if it is a union of basic open sets. To simplify our notation when talking about these infinite products, we introduce the following convention:

CONVENTION 11.5.2. We regard the indexing set I as being unordered, and thus if $I = J \sqcup K$ we identify

$$\left(\prod_{j \in J} X_j\right) \times \left(\prod_{k \in K} X_k\right) \quad \text{and} \quad \prod_{i \in I} X_i$$

in the obvious way. $\square$

With this notational convention, the basic open sets in $\prod_{i \in I} X_i$ are those that for some distinct $j_1, \dots, j_n \in I$ can be written as

$$U_{j_1} \times \dots \times U_{j_n} \times \prod_{i \in I \setminus \{j_1, \dots, j_n\}} X_i \quad \text{with } U_{j_k} \subset X_{j_k} \text{ open for } 1 \leq k \leq n.$$

REMARK 11.5.3. The topology on $\prod_{i \in I} X_i$ with basis arbitrary products $\prod_{i \in I} U_i$ with $U_i \subset X_i$ open is sometimes called the *box topology*. It is rarely useful. $\square$

11.6. Infinite universal property

Continue to let $\{X_i\}_{i \in I}$ be an arbitrary collection of spaces. For $j \in I$, let $\pi_j: \prod_{i \in I} X_i \rightarrow X_j$ be the projection. The map π_j is continuous; indeed, if $U_j \subset X_j$ is open, then

$$\pi_j^{-1}(U_j) = U_j \times \prod_{i \in I \setminus \{j\}} X_i.$$

Now let Y be another space, and for $i \in I$ let $f_i: Y \rightarrow X_i$ be a continuous map. Let $\prod_{i \in I} f_i: Y \rightarrow \prod_{i \in I} X_i$ be the map

$$\left(\prod_{i \in I} f_i \right) (y) = (f_i(y))_{i \in I} \quad \text{for } y \in Y.$$

This map is continuous; indeed, if $\prod_{i \in I} U_i$ is a basic open set then

$$\left(\prod_{i \in I} f_i \right)^{-1} \left(\prod_{i \in I} U_i \right) = \bigcap_{i \in I} f_i^{-1}(U_i).$$

This is open since $f_i^{-1}(U_i) = f_i^{-1}(X_i) = Y$ for all but finitely many $i \in I$, so this intersection is actually a finite intersection. Conversely, if $F: Y \rightarrow \prod_{i \in I} X_i$ is a continuous map, then letting $f_i = \pi_i \circ F$ we have $F = \prod_{i \in I} f_i$. We summarize this informally as:

- A continuous map $F: Y \rightarrow \prod_{i \in I} X_i$ is the same thing as a collection of continuous maps $f_i: Y \rightarrow X_i$ for all $i \in I$.

Having this universal property is one of the reasons we defined the product topology like we did. It follows from Exercise 11.3 that this universal property categorizes $\prod_{i \in I} X_i$ up to homeomorphism.

11.7. Categorical interpretation

If $\mathbf{C}$ is a category and $\{C_i\}_{i \in I}$ are objects of $\mathbf{C}$, then a *categorical product* of the C_i is an object D of $\mathbf{C}$ together with morphisms $\{f_i: C_i \rightarrow D\}_{i \in I}$ such that the following holds:

- For all objects E of $\mathbf{C}$, there is a bijection between morphisms $\phi: E \rightarrow D$ and collections of morphisms $\{\Phi_i: E \rightarrow C_i\}_{i \in I}$ taking a morphism $\phi: E \rightarrow D$ to the collection of morphisms $\{f_i \circ \phi: E \rightarrow C_i\}_{i \in I}$.

Categorical products might or might not exist, but if they do exist they are unique up to isomorphism (see Exercise 11.3). The universal property from §11.6 shows that if $\{X_i\}_{i \in I}$ is a collection of spaces, the product $P = \prod_{i \in I} X_i$ equipped with the projections $\{\pi_i: P \rightarrow X_i\}_{i \in I}$ is the categorical product of the $\{X_i\}_{i \in I}$ in the category $\mathbf{Top}$ of topological spaces. See Exercise 11.4 for the product in the category of abelian groups.

REMARK 11.7.1. This should be compared to the categorical sum from Remark 3.4.2, which we now recall. If $\mathbf{C}$ is a category and $\{C_i\}_{i \in I}$ are objects of $\mathbf{C}$, then a *categorical sum* of the C_i is an object D of $\mathbf{C}$ together with morphisms $\{f_i: C_i \rightarrow D\}_{i \in I}$ such that the following holds:

- For all objects E of $\mathbf{C}$, there is a bijection between morphisms $\phi: D \rightarrow E$ and collections of morphisms $\{\Phi_i: C_i \rightarrow E\}_{i \in I}$ taking a morphism $\phi: D \rightarrow E$ to the collection of morphisms $\{\phi \circ f_i: C_i \rightarrow E\}_{i \in I}$.

As we discussed in Remark 3.4.2, the categorical sum of a collection of spaces $\{X_i\}_{i \in I}$ is the disjoint union $\sqcup_{i \in I} X_i$. □

11.8. Metrics on countable products

Arbitrary products of metrizable spaces need not be metrizable. However, it turns out that countable products of metrizable spaces are metrizable. This would not be true if we used the box topology. This is easy for finite products (see Exercise 11.6), so we focus on the countably infinite case:

LEMMA 11.8.1. *For each $n \geq 1$, let M_n be a metrizable space. Then $\prod_{n=1}^{\infty} M_n$ is metrizable.*

PROOF. Let $\mathfrak{d}_n$ be a metric on M_n inducing its topology. Let $\mathfrak{d}'_n$ be the metric on M_n defined by $\mathfrak{d}'_n(p, q) = \min\{\mathfrak{d}_n(p, q), 1\}$. This induces the same topology on M_n as $\mathfrak{d}_n$ (see Exercise 2.8). We can then define a two-variable real-valued function on $\prod_{n=1}^{\infty} M_n$ via the formula

$$\mathfrak{d}((p_n)_{n \geq 1}, (q_n)_{n \geq 1}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \mathfrak{d}'_n(p_n, q_n).$$

This is a metric on $\prod_{n=1}^{\infty} M_n$ that induces the product topology (see Exercise 11.7). □

11.9. Tychonoff's theorem, countable case

Tychonoff's theorem generalizes to arbitrary products of compact spaces. We start by proving this for countable products. The proof of the general case is similar, but requires more set theoretic technology.

THEOREM 11.9.1 (Tychonoff's theorem, countable case). *Let $\{X_i\}_{i \geq 1}$ be a countable collection of compact spaces. Then $\prod_{i \geq 1} X_i$ is compact.*

PROOF. Unlike in the finite case, we cannot prove this by induction. However, we will see that the argument we gave in the finite case is almost enough. Only one new idea is needed. Let $\mathfrak{U}$ be an open cover of $\prod_{i \geq 1} X_i$. We must prove that $\mathfrak{U}$ has a finite subcover. In fact, it is enough to prove that some refinement of $\mathfrak{U}$ has a finite subcover (see Exercise 10.3). Each element of $\mathfrak{U}$ is a union of basic open sets. Letting $\mathfrak{V}$ be the set of all basic open sets V such that there exists some $U \in \mathfrak{U}$ with $V \subset U$, it is therefore enough to prove that $\mathfrak{V}$ has a finite subcover.

Assume for the sake of contradiction that $\mathfrak{V}$ has no finite subcover. The proof now has two steps:

STEP 1. *For all $i \geq 1$, there exists some $p_i \in X_i$ such that no finite subset of $\mathfrak{V}$ covers $p_1 \times \cdots \times p_n \times \prod_{i \geq n+1} X_i$ for any $n \geq 1$.*

We construct the p_i inductively. Assume that for some $n \geq 1$ we have found $p_i \in X_i$ for $1 \leq i \leq n-1$ such that no finite subset of $\mathfrak{V}$ covers $p_1 \times \cdots \times p_{n-1} \times \prod_{i \geq n} X_i$. For $n=1$, this is simply our assumption that the open cover $\mathfrak{V}$ of $\prod_{i \geq 1} X_i$ has no finite subcover. We find $p_n \in X_n$ as follows. For $p \in X_n$, let

$$Z(p) = p_1 \times \cdots \times p_{n-1} \times p \times \prod_{i \geq n+1} X_i.$$

Assume for the sake of contradiction that for all $p \in X_n$, there exists a finite subset $\mathfrak{V}(p)$ of $\mathfrak{V}$ that covers $Z(p)$. Since $\mathfrak{V}$ consists of basic open sets, we can write

$$\mathfrak{V}(p) = \left\{ \prod_{i \geq 1} V_{i,j}(p) \mid 1 \leq j \leq m_p \right\}$$

with $V_{i,j}(p) \subset X_i$ for all $i \geq 1$ and $1 \leq j \leq m_p$. Discarding unneeded terms if necessary, we can assume that $p_i \in V_{i,j}(p)$ for all $1 \leq i \leq n-1$ and $1 \leq j \leq m_p$, and also that $p \in V_{n,j}(p)$ for all $1 \leq j \leq m_p$. Define

$$V_i(p) = \bigcap_{j=1}^{m_p} V_{i,j}(p) \quad \text{for } 1 \leq i \leq n,$$

$$V(p) = V_1(p) \times \cdots \times V_n(p).$$

It follows that $V(p)$ is an open neighborhood of $(p_1, \dots, p_{n-1}, p) \in X_1 \times \cdots \times X_n$ and that $\mathfrak{V}(p)$ covers $V(p) \times \prod_{i \geq n+1} X_i$.

The set $\{V(p) \mid p \in X_n\}$ is an open cover of the compact space $p_1 \times \cdots \times p_{n-1} \times X_n$, so we can find $q_1, \dots, q_d \in X_n$ such that

$$p_1 \times \cdots \times p_{n-1} \times X_n \subset V(q_1) \cup \cdots \cup V(q_d).$$

Since $\mathfrak{V}(q_k)$ is a finite cover of $V(q_k) \times \prod_{i \geq n+1} X_i$ for $1 \leq k \leq d$, we conclude that $\mathfrak{V}(q_1) \cup \cdots \mathfrak{V}(q_d)$ is a finite subset of $\mathfrak{V}$ that covers $p_1 \times \cdots \times p_{n-1} \times \prod_{i \geq n} X_i$, contradicting the fact that no such finite cover exists.

STEP 2. *No finite subset of $\mathfrak{V}$ covers $\prod_{i \geq 1} X_i$.*

Pick $V \in \mathfrak{V}$ such that $(p_i)_{i \geq 1} \in V$. Since $\mathfrak{V}$ consists of basic open sets, we can write $V = \prod_{i \geq 1} V_i$ with $V_i \subset X_i$ open for all $i \geq 1$. Moreover, we have $V_i = X_i$ for all but finitely many $i \geq 1$. This implies that there exists some $n \geq 1$ such that $V_i = X_i$ for $i \geq n+1$. It follows that

$$p_1 \times \cdots \times p_n \times \prod_{i \geq n+1} X_i \subset V \in \mathfrak{V}.$$

This contradicts the fact that no finite subset of $\mathfrak{V}$ covers $p_1 \times \cdots \times p_n \times \prod_{i \geq n+1} X_i$. □

11.10. Well-ordered sets

To generalize the above proof of Tychonoff's theorem to arbitrary products, we need some set-theoretic technology. Recall from §10.3 that a well-ordered set is a set I equipped with a total ordering $\leq$ such that every nonempty subset $S \subset I$ has a minimal element. We call $\leq$ a well-ordering. A remarkable consequence of the axiom of choice is that every set can be equipped with a well-ordering.

EXAMPLE 11.10.1. All total orderings on finite sets are well-orderings. The canonical example of an infinite well-ordered set is $\mathbb{N}$ with its standard ordering. See Exercise 11.8 for more examples. $\square$

If I is a well-ordered set with ordering $\leq$, then an *initial segment* of I is a subset $J \subset I$ such that for all $j \in J$ and $i \in I$ with $i \leq j$ we have $i \in J$. If $J_1, J_2 \subset I$ are initial segments, then either $J_1 \subset J_2$ or $J_2 \subset J_1$. Indeed, assume that J_1 is not a subset of J_2 and pick $j_1 \in J_1 \setminus J_2$. For $j_2 \in J_2$, we cannot have $j_1 \leq j_2$ since $j_1 \notin J_2$. It follows that $j_2 \leq j_1$, so $j_2 \in J_1$ and thus $J_2 \subset J_1$. The initial segments of I are thus totally ordered under inclusion. They fall into three classes:

- The empty set $\emptyset$, which is the unique initial segment that is contained in all initial segments.
- The *successor segments*, which are initial segments $J \subset I$ of the form $J = J' \sqcup \{n\}$ for some initial segment $J' \subsetneq J$ and some $n \in J \setminus J'$.
- The *limit segments*, which are nonempty initial segments $J \subset I$ that are not successor segments. These J are the union of the initial segments $J' \subsetneq J$.

EXAMPLE 11.10.2. For $\mathbb{N}$, the successor segments are of the form $\{1, \dots, n\}$ and the whole set $\mathbb{N}$ is the only limit segment. $\square$

11.11. Transfinite induction

Assume now that I is a well-ordered set and for each $i \in I$ we have a set X_i . Our goal is to construct some $p_i \in X_i$ for all $i \in I$. For each initial segment $J \subset I$, we want some property $\mathcal{P}(J)$ to hold that only refers to the $p_i \in X_i$ for $i \in J$. To simplify our exposition, assume that if $\mathcal{P}(J)$ holds then so does $\mathcal{P}(J')$ for all initial segments $J' \subsetneq J$.

We can construct the $p_i \in X_i$ by *transfinite induction*.² For this, we must prove three things:

- (0) The property $\mathcal{P}(\emptyset)$ holds. This makes sense since $\mathcal{P}(\emptyset)$ makes no reference to any p_i .
- (1) Let J be a successor segment of the form $J = J' \sqcup \{n\}$ for some initial segment $J' \subsetneq J$. Assume that we have already constructed $p_i \in X_i$ for all $i \in J'$ such that $\mathcal{P}(J')$ holds. We must show how to construct $p_n \in X_n$ such that $\mathcal{P}(J)$ holds.
- (2) Let J be a limit segment. Assume that we have constructed $p_i \in X_i$ for all $i \in J$ such that $\mathcal{P}(J')$ holds for all initial segments $J' \subsetneq J$. We must prove that $\mathcal{P}(J)$ holds.

We can then construct $p_i \in X_i$ for all $i \in I$ such that $\mathcal{P}(J)$ holds for all initial segments $J \subset I$. Indeed, let $\mathfrak{J}$ be the set of all initial segments $J \subset I$ for which we can construct $p_i \in X_i$ for each $i \in J$ such that $\mathcal{P}(J)$ holds. The set $\mathfrak{J}$ is linearly ordered by inclusion and nonempty since $\emptyset \in \mathfrak{J}$. Let $J_0 = \cup_{J \in \mathfrak{J}} J$. By (1) and (2), we have $J_0 \in \mathfrak{J}$. We must prove that $J_0 = I$. Indeed, assume that $J_0 \subsetneq I$. Since I is well-ordered, there is a minimal $n \in I \setminus J_0$. It follows that $J_0 \sqcup \{n\}$ is an initial segment, and by (1) we have $J_0 \sqcup \{n\} \in \mathfrak{J}$, contradicting the fact that $J \subset J_0$ for all $J \in \mathfrak{J}$.

REMARK 11.11.1. Isomorphism classes of well-ordered sets are called *ordinals*. Any set of ordinals has a well-ordering where $\mathcal{O}_1 \leq \mathcal{O}_2$ when $\mathcal{O}_1$ is isomorphic to an initial segment of $\mathcal{O}_2$. Transfinite induction is typically discussed using ordinals. $\square$

11.12. Tychonoff's theorem, general case

We now use transfinite induction to prove the general case of Tychonoff's theorem:

THEOREM 11.12.1 (Tychonoff's theorem). *Let $\{X_i\}_{i \in I}$ be a collection of compact spaces. Then $\prod_{i \in I} X_i$ is compact.*

²Since we are constructing things, this is sometimes called *transfinite recursion*.

PROOF. The proof will be almost identical to proof in the countable case, but with some small complications due to the need for transfinite induction. Let $\mathfrak{U}$ be an open cover of $\prod_{i \in I} X_i$. We must prove that $\mathfrak{U}$ has a finite subcover. In fact, it is enough to prove that some refinement of $\mathfrak{U}$ has a finite subcover (see Exercise 10.3). Each element of $\mathfrak{U}$ is a union of basic open sets. Letting $\mathfrak{V}$ be the set of all basic open sets V such that there exists some $U \in \mathfrak{U}$ with $V \subset U$, it is therefore enough to prove that $\mathfrak{V}$ has a finite subcover.

Assume for the sake of contradiction that $\mathfrak{V}$ has no finite subcover. Choose a well-ordering on the indexing set I . By transfinite induction, for each $i \in I$ we will construct some $p_i \in X_i$ such that the following holds for all initial segments $J \subset I$:

(♠_J) No finite subset of $\mathfrak{V}$ covers $Y(J) = \prod_{j \in J} p_j \times \prod_{i \in I \setminus J} X_i$.

The special case (♠_I) says that no finite subset of $\mathfrak{V}$ covers the one-point set $Y(I) = \prod_{i \in I} p_i$, which will be our contradiction. We have (♠_∅) from our assumption that no finite subset of $\mathfrak{V}$ covers $Y(\emptyset) = \prod_{i \in I} X_i$. According to the transfinite induction scheme discussed in §11.11, to prove that (♠_J) holds for all initial segments $J \subset I$ we must prove:

STEP 1. Let $J \subset I$ be a successor segment, so $J = J' \sqcup \{n\}$ for some initial segment $J' \subset J$ and $n \in J \setminus J'$. Assume that we have constructed $p_i \in X_i$ for all $i \in J'$ such that (♠_{J'}) holds. We can then construct $p_n \in X_n$ such that (♠_J) holds.

For $p \in X_n$, let

$$Z(p) = p \times \prod_{j' \in J'} p_{j'} \times \prod_{i \in I \setminus J} X_i.$$

Assume for the sake of contradiction that for all $p \in X_n$, there exists a finite subset $\mathfrak{V}(p)$ of $\mathfrak{V}$ that covers $Z(p)$. Since $\mathfrak{V}$ consists of basic open sets, we can write

$$\mathfrak{V}(p) = \left\{ \prod_{i \in I} V_{i,k}(p) \mid 1 \leq k \leq m_p \right\}$$

with $V_{i,k}(p) \subset X_i$ for all $i \in I$ and $1 \leq k \leq m_p$. Discarding unneeded terms if necessary, we can assume that $p_{j'} \in V_{j',k}(p)$ for all $j' \in J'$ and $1 \leq k \leq m_p$, and also that $p \in V_{n,k}(p)$ for all $1 \leq k \leq m_p$. Keeping in mind that $J = J' \sqcup \{n\}$, define

$$V_j(p) = \bigcap_{k=1}^{m_p} V_{j,k}(p) \quad \text{for } j \in J,$$

$$V(p) = V_n(p) \times \prod_{j' \in J'} V_{j'}(p).$$

It follows that $V(p)$ is an open neighborhood of $p \times \prod_{j' \in J'} p_{j'}$ and that $\mathfrak{V}(p)$ covers $V(p) \times \prod_{i \geq I \setminus J} X_i$.

The set $\{V(p) \mid p \in X_n\}$ is an open cover of the compact space $X_n \times \prod_{j' \in J'} p_{j'}$, so we can find $q_1, \dots, q_d \in X_n$ such that

$$X_n \times \prod_{j' \in J'} p_{j'} \subset V(q_1) \cup \dots \cup V(q_d).$$

Since $\mathfrak{V}(q_\ell)$ is a finite cover of $V(q_\ell) \times \prod_{i \in I \setminus J} X_i$ for $1 \leq \ell \leq d$, we conclude that $\mathfrak{V}(q_1) \cup \dots \cup \mathfrak{V}(q_d)$ is a finite subset of $\mathfrak{V}$ that covers

$$X_n \times \prod_{j' \in J'} p_{j'} \times \prod_{i \in I \setminus J} X_i = \prod_{j' \in J'} p_{j'} \times \prod_{i \in I \setminus J'} X_i = Y(J'),$$

contradicting the fact that no such finite cover exists.

STEP 2. Let $J \subset I$ be a limit segment. Assume that we have constructed p_i for all $i \in J$ such that (♠_{J'}) holds for all initial segments $J' \subsetneq J$. Then (♠_J) holds.

Assume for the sake of contradiction that a finite subset $\{V_1, \dots, V_d\}$ of $\mathfrak{V}$ covers $Y(J)$. Each V_k is a basic open set, so we can write

$$V_k = \prod_{i \in I} V_{k,i} \quad \text{with } V_{k,i} \subset X_i \text{ open for all } i \in I.$$

Moreover, we have $V_{k,i} = X_i$ for all but finitely many $i \in I$. For $1 \leq k \leq d$, let $J(k) = \{j \in J \mid V_{k,j} \neq X_j\}$. Set $\hat{J} = J(1) \cup \dots \cup J(d)$. Let J' be the smallest initial segment containing $\hat{J}$. Since $\hat{J}$ is a finite subset of J , we have $J' \subsetneq J$. Since $V_{k,j} = X_j$ for all $1 \leq k \leq d$ and $j \in J \setminus J'$, the fact that $\{V_1, \dots, V_d\}$ covers $Y(J)$ implies that it also covers $Y(J')$. This contradicts the fact that no finite subset of $\mathfrak{V}$ covers $Y(J')$. $\square$

11.13. Exercises

EXERCISE 11.1. Prove that the product topology on $\mathbb{R}^n = \mathbb{R} \times \dots \times \mathbb{R}$ is the same as the metric space topology. $\square$

EXERCISE 11.2. Let $\{X_i\}_{i \in I}$ be spaces. For each $i \in I$, let $A_i \subset X_i$ be a nonempty subspace. We therefore have a nonempty subspace $\prod_{i \in I} A_i$ of $\prod_{i \in I} X_i$. Prove that $\overline{\prod_{i \in I} A_i} = \prod_{i \in I} \overline{A_i}$. $\square$

EXERCISE 11.3. Let $\mathbf{C}$ be a category and let $\{C_i\}_{i \in I}$ be objects of $\mathbf{C}$. For $k = 1, 2$, let $D(k)$ be a categorical product of the $\{C_i\}_{i \in I}$ with morphisms $\{f_i(k): D(k) \rightarrow C_i\}_{i \in I}$. Prove that there exists an isomorphism $\lambda: D(1) \rightarrow D(2)$ such that $f_i(2) = f_i(1) \circ \lambda$ for all $i \in I$. This can be interpreted as saying that categorical products are unique up to isomorphism. $\square$

EXERCISE 11.4. Let $\{A_i\}_{i \in I}$ be a collection of abelian groups. Prove that $P = \prod_{i \in I} A_i$ together with the projections $\{\pi_i: P \rightarrow A_i\}$ is the categorical product of the A_i in the category $\mathbf{AbGrp}$ of abelian groups. We remark that Exercise 3.10 showed that the direct sum

$$\bigoplus_{i \in I} A_i = \left\{ (a_i)_{i \in I} \in \prod_{i \in I} A_i \mid a_i = 0 \text{ for all but finitely many } i \in I \right\}.$$

is the categorical sum of the $\{A_i\}_{i \in I}$ in $\mathbf{AbGrp}$. $\square$

EXERCISE 11.5. Let $\{X_i\}_{i \in I}$ be a collection of spaces. For each $i \in I$, let $\{p(i)_n\}_{n \geq 1}$ be a sequence of points in X_i that converges to $p(i) \in X_i$. For $n \geq 1$, let $p_n = (p(i)_n)_{i \in I} \in \prod_{i \in I} X_i$. Prove that $\{p_n\}_{n \geq 1}$ converges to $(p(i))_{i \in I} \in \prod_{i \in I} X_i$. $\square$

EXERCISE 11.6. For $1 \leq n \leq N$, let $(M_n, \mathfrak{d}_n)$ be a metric space. Define a two-variable real-valued function on $\prod_{n=1}^N M_n$ via the formula

$$\mathfrak{d}((p_1, \dots, p_N), (q_1, \dots, q_N)) = \mathfrak{d}_1(p_1, q_1) + \dots + \mathfrak{d}_N(p_N, q_N).$$

Prove that this is a metric on $\prod_{n=1}^N M_n$ that induces the product topology. $\square$

EXERCISE 11.7. For each $n \geq 1$, let $(M_n, \mathfrak{d}_n)$ be a metric space. For each $n \geq 1$, assume that $\mathfrak{d}_n(p, q) \leq 1$ for all $p, q \in M_n$. Define a two-variable real-valued function on $\prod_{n=1}^\infty M_n$ via the formula

$$\mathfrak{d}((p_n)_{n \geq 1}, (q_n)_{n \geq 1}) = \sum_{n=1}^\infty \frac{1}{2^n} \mathfrak{d}'_n(p_n, q_n).$$

Prove that this is a metric on $\prod_{n=1}^\infty M_n$ that induces the product topology. $\square$

EXERCISE 11.8. This exercise gives more examples of well-ordered sets.

- Let A and B be two well-ordered sets. Let $A \sqcup B$ be their disjoint union, and let $\leq$ be the total ordering on $A \sqcup B$ that restricts to the given orderings on A and B and has $a < b$ for all $a \in A$ and $b \in B$. Prove that $A \sqcup B$ is a well-ordered set.
- Let A and B be two well-ordered sets. Give $A \times B$ the dictionary ordering, so $(a_1, b_1) < (a_2, b_2)$ if $a_1 < a_2$ or if $a_1 = a_2$ and $b_1 < b_2$. Prove that $A \times B$ is a well-ordered set.
- For some $N \geq 1$, give $\{1, \dots, N\} \times \mathbb{N}$ the dictionary ordering. Identify the initial segments, and say which are successor segments and which are limit segments.
- Give $\mathbb{N} \times \mathbb{N}$ the dictionary ordering. Identify the initial segments, and say which are successor segments and which are limit segments. $\square$

EXERCISE 11.9. Let C be the classical Cantor set, i.e., the set of all $x \in I = [0, 1]$ of the form

$$x = \sum_{n=1}^{\infty} \frac{x_n}{3^n} \quad \text{with } x_n \in \{0, 2\} \text{ for all } n \geq 1.$$

Let X be the discrete 2-point space $X = \{0, 2\}$. Prove the following:

- (a) Define a set map $\Psi: \prod_{n=1}^{\infty} X \rightarrow C$ via the formula

$$\Psi((x_n)_{n \geq 1}) = \sum_{n=1}^{\infty} \frac{x_n}{3^n}.$$

Prove that Ψ is a homeomorphism.

- (b) Prove that for all $p, q \in C$ there exists a homeomorphism $f: C \rightarrow C$ such that $f(p) = q$. In other words, every two points of C “look the same”. The technical term for this is that C is *homogeneous*.
- (c) Define a set map $\Phi: \prod_{n=1}^{\infty} X \rightarrow I$ via the formula

$$\Phi((x_n)_{n \geq 1}) = \sum_{n=1}^{\infty} \frac{x_n}{2^n}.$$

Prove that Φ is a continuous surjection.

- (d) For each $d \geq 1$, construct a homeomorphism $\lambda_d: \prod_{n=1}^{\infty} X \rightarrow (\prod_{n=1}^{\infty} X)^{\times d}$.
- (e) For each $d \geq 1$, let $f_d: C \rightarrow I^d$ be the composition

$$C \xrightarrow{\Psi^{-1}} \prod_{n=1}^{\infty} X \xrightarrow{\lambda_d} (\prod_{n=1}^{\infty} X)^{\times d} \xrightarrow{\prod_{i=1}^d \Phi} I^d.$$

Prove that f_d is a continuous surjection that can be extended to a continuous surjection $g_d: I \rightarrow I^d$ (a “space-filling curve”). $\square$

Metrization theorems

This chapter discusses topological properties that ensure that a space is metrizable.

12.1. Nagata–Smirnov metrization theorem

We have already observed that all metrizable spaces are normal (Lemma 6.4.1) and first countable (Lemma 5.1.1). These conditions are not sufficient. It turns out all metrizable spaces satisfy a condition that is intermediate between first and second countability. A space is first countable if it has a countable neighborhood basis at each point, and is second countable if it has a countable basis. Say that a basis $\mathcal{B}$ is *countably locally finite* if it can be written as $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ with each $\mathcal{B}_n$ locally finite. We are not requiring the $\mathcal{B}_n$ to be countable, so this is weaker than being second countable.

The Nagata–Smirnov metrization theorem says that a space X is metrizable if and only if it is regular and has a countably locally finite basis. We remark that these conditions imply that X is normal; see Exercise 12.2. See [1, Chapter 6] for a proof of the Nagata–Smirnov metrization theorem.

12.2. Urysohn metrization theorem

We prove the following weakening of the Nagata–Smirnov metrization theorem, which is often called the Urysohn metrization theorem:

THEOREM 12.2.1. *Let X be a space that is regular and second countable. Then X is metrizable.*

PROOF. Since second countable regular spaces are normal (Lemma 6.10.1), our space X is normal and we can apply Urysohn’s lemma to it. Let $\mathcal{B}$ be a countable basis for X . Set

$$\mathcal{D} = \{(U, V) \mid U, V \in \mathcal{B} \text{ and } \overline{U} \subset V\}.$$

Since $\mathcal{B}$ is countable, the set $\mathcal{D}$ is also countable. Recalling that $I = [0, 1]$, define

$$Z = \prod_{(U,V) \in \mathcal{D}} I.$$

Since countable products of metrizable spaces are metrizable (Lemma 11.8.1), it follows that Z is metrizable. It is therefore enough to construct an embedding $F: X \rightarrow Z$.

For each $(U, V) \in \mathcal{D}$, apply Urysohn’s Lemma (Theorem 6.5.1) to find a continuous function $f_{UV}: X \rightarrow I$ such that $f_{UV}|_{\overline{U}} = 1$ and such that $\text{supp}(f_{UV}) \subset V$. Define a function

$$F: X \longrightarrow Z = \prod_{(U,V) \in \mathcal{D}} I$$

via the formula

$$F(x) = (f_{UV}(x))_{(U,V) \in \mathcal{D}} \quad \text{for } x \in X.$$

This is continuous by the universal property of the product topology (see §11.6). To prove that F is an embedding, we must prove that F is injective and that F takes open sets of X to open sets of $F(X) \subset Z$. We divide this into two steps:

STEP 1. *The map F is injective.*

Consider distinct points $x, y \in X$. Since X is normal it is in particular Hausdorff. We can therefore find an open neighborhood $V \in \mathcal{B}$ with $x \in V$ and $y \notin V$. Applying normality, we can find an open neighborhood $U \in \mathcal{B}$ of x with $\overline{U} \subset V$. It follows that $f_{UV}(x) = 1$ and $f_{UV}(y) = 0$. The (U, V) -coordinates of $F(x)$ and $F(y)$ therefore differ, so $F(x) \neq F(y)$.

STEP 2. *The map F takes open sets of X to open sets of $F(X) \subset Z$.*

Consider $V_0 \in \mathcal{B}$. It is enough to prove that $F(V_0)$ is an open set of $F(X) \subset Z$. Consider some $x_0 \in V_0$. To prove that $F(V_0)$ is an open set of $F(X) \subset Z$, it is enough to prove that there is an open neighborhood $W_0 \subset Z$ of $F(x_0)$ with $W_0 \cap F(X) \subset F(V_0)$. Since X is normal, we can find an open neighborhood U_0 of x_0 with $\overline{U_0} \subset V_0$. We therefore have $f_{U_0 V_0}(x_0) = 1$. Let

$$W_0 \subset Z = \prod_{(U,V) \in \mathcal{D}} I$$

be the open subset consisting of points whose (U_0, V_0) -coordinate is nonzero. Since $f_{U_0 V_0}(x_0) = 1$, we have $F(x_0) \in W_0$. We claim that $W_0 \cap F(X) \subset F(V_0)$. Indeed, consider a point $x_1 \in X$ with $F(x_1) \in W_0$. We must prove that $x_1 \in V_0$. To see this, note that since $F(x_1) \in W_0$ it must be the case that $f_{U_0 V_0}(x_1) \neq 0$. Since $\text{supp}(f_{U_0 V_0}) \subset V_0$, it follows that $x_1 \in V_0$, as desired. $\square$

12.3. Exercises

EXERCISE 12.1. Give an example of space X that metrizable but not second countable. $\square$

EXERCISE 12.2. Prove the following variant of Lemma 6.10.1: if X is a regular space that has a countably locally finite basis, then X is normal. $\square$

EXERCISE 12.3. Let M be a metrizable space. Prove that X has a countably locally finite basis. Hint: the proof uses the fact that metrizable spaces are paracompact. $\square$

EXERCISE 12.4. Let X be a locally compact Hausdorff space and let X^* be the one-point compactification of X (see §8.3). Prove that X^* is metrizable if and only if X is second countable. $\square$

EXERCISE 12.5. Let X be a paracompact space that is *locally metrizable*, i.e., each $p \in X$ has a metrizable open neighborhood.

- (a) Prove that X has a countably locally finite basis. Hint: use the fact that metrizable spaces are countably locally finite.
- (b) Use the Nagata–Smirnov metrization theorem to deduce that X is metrizable.

This is often called the *Smirnov metrization theorem*. It is enlightening to prove it directly, though the proof is too hard for an exercise. $\square$

Bibliography

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Function spaces and the compact-open topology

Let X and Y be spaces and let¹ $\mathcal{C}(X, Y)$ be the set of all continuous maps $f: X \rightarrow Y$. In this chapter we explain how to turn $\mathcal{C}(X, Y)$ into a space.

13.1. Subbasis

Let X be a set and let $\mathfrak{B}$ be a set of subsets of X . We would like to topologize X with the smallest collection of open sets possible to make each $U \in \mathfrak{B}$ open. If for all $U, V \in \mathfrak{B}$ the intersection $U \cap V$ could be written as a union of sets in $\mathfrak{B}$, then $\mathfrak{B}$ would be a basis for a topology as in §2.6. In that case, we could topologize X by saying that $U \subset X$ is open precisely when U is the union of sets in $\mathfrak{B}$.

However, if $\mathfrak{B}$ does not form a basis then this does not work since in the resulting “topology” the collection of open sets is not closed under finite intersections. To fix this, let $\mathfrak{B}'$ be the set of all finite intersections of elements of $\mathfrak{B}$. Here we interpret the intersection of zero sets as X , so $X \in \mathfrak{B}'$. The set $\mathfrak{B}'$ does form a basis for a topology on X . In this case, we say that $\mathfrak{B}$ is a *subbasis* for this topology.

13.2. Compact-open topology

For sets $A, B \subset X$, define

$$B(A, B) = \{f: X \rightarrow Y \mid f(K) \subset U\} \subset \mathcal{C}(X, Y).$$

The *compact-open topology* on $\mathcal{C}(X, Y)$ is the topology with subbasis the collection of all $B(K, U)$ with $K \subset X$ compact and $U \subset Y$ open. In other words, a set $V \subset \mathcal{C}(X, Y)$ is open if for all $f \in V$ there exist $K_1, \dots, K_n \subset X$ compact and $U_1, \dots, U_n \subset Y$ open such that

$$f \in B(K_1, U_1) \cap \dots \cap B(K_n, U_n) \subset V.$$

13.3. Metrics

If $(Y, \mathfrak{d})$ is a metric space, then it is also natural to try to topologize $\mathcal{C}(X, Y)$ using $\mathfrak{d}$. This is easiest for X compact, in which case we can define a metric $\mathfrak{D}$ on $\mathcal{C}(X, Y)$ by letting

$$(13.3.1) \quad \mathfrak{D}(f, g) = \max \{\mathfrak{d}(f(x_1), g(x_1)) \mid x_1, x_2 \in X\} \quad \text{for } f, g: X \rightarrow Y.$$

This makes sense since X is compact, which implies that $f(X)$ and $g(X)$ are compact subsets of the metric space Y and thus that the above maximum is finite and realized. We have:

LEMMA 13.3.1. *Let X be a compact space and let $(Y, \mathfrak{d})$ be a metric space. The compact-open topology on $\mathcal{C}(X, Y)$ and the metric topology on $\mathcal{C}(X, Y)$ coming from (13.3.1) are the same.*

PROOF. We divide the proof into two steps:

STEP 1. *Every open set in the compact-open topology is open in the metric topology.*

Let $K \subset X$ be compact and $U \subset Y$ be open. We must prove that $B(K, U)$ is open in the metric topology. Indeed, consider $f \in B(K, U)$, so $f(K) \subset U$. Since $f(K)$ is a compact subset of U , we can find some $\epsilon > 0$ such that the ϵ -neighborhood of $f(K)$ is contained in U . For $g \in \mathcal{C}(X, Y)$ with $\mathfrak{D}(f, g) < \epsilon$, since $\mathfrak{d}(g(k), f(k)) < \epsilon$ for all $k \in K$ it follows that $g(K)$ is contained in the ϵ -neighborhood of $f(K)$. We thus have $g(K) \subset U$, so $g \in B(K, U)$. We conclude that the ϵ -ball around f is contained in $B(K, U)$, so $B(K, U)$ is open in the metric topology.

¹It is also common to call this space Y^X , but we think the notation $\mathcal{C}(X, Y)$ is easier to understand.

STEP 2. *Every open set in the metric topology is open in the compact-open topology.*

Let $f \in \mathcal{C}(X, Y)$ and let $\epsilon > 0$. Let

$$B_\epsilon(f) = \{g \in \mathcal{C}(X, Y) \mid d(g(x), f(x)) < \epsilon \text{ for all } x \in X\}$$

be the open ball around f in the metric topology. It is enough to find compact sets $K_1, \dots, K_n \subset X$ and open sets $U_1, \dots, U_n \subset Y$ such that

$$f \in B(K_1, U_1) \cap \dots \cap B(K_n, U_n) \subset B_\epsilon(f).$$

Since $f(X)$ is a compact subset of Y , we can find $x_1, \dots, x_n \in X$ such that

$$(13.3.2) \quad f(X) \subset B_{\epsilon/3}(f(x_1)) \cup \dots \cup B_{\epsilon/3}(f(x_n)).$$

For $1 \leq i \leq n$, let $K_i = f^{-1}(\overline{B_{\epsilon/3}(f(x_i))})$ and $U_i = B_{\epsilon/2}(f(x_i))$. Since K_i is a closed subset of the compact space X , it follows that K_i is closed. By (13.3.2), the sets K_i cover X . Finally, by construction

$$f \in B(K_1, U_1) \cap \dots \cap B(K_n, U_n).$$

Now consider some $g \in B(K_1, U_1) \cap \dots \cap B(K_n, U_n)$. We must prove that $g \in B_\epsilon(f)$. In other words, letting $x \in X$ we must prove that $d(f(x), g(x)) < \epsilon$. We have $x \in K_i$ for some $1 \leq i \leq n$, so $f(x), g(x) \in U_i$. It follows that $d(f(x), g(x))$ is at most the diameter ϵ of $U_i = B_{\epsilon/2}(f(x_i))$. $\square$

REMARK 13.3.2. If (Y, d) is a metric space but X is not compact, then the metric d induces a topology on $\mathcal{C}(X, Y)$ as follows. For $f \in \mathcal{C}(X, Y)$ and a compact subset $K \subset X$ and $\epsilon > 0$, let

$$B(f, K, \epsilon) = \{g \in \mathcal{C}(X, Y) \mid d(f(x), g(x)) < \epsilon \text{ for all } x \in K\}.$$

These sets form the basis for a topology on $\mathcal{C}(X, Y)$ called the *topology of compact convergence*, and this is the same as the compact-open topology (see Exercise 13.1). $\square$

13.4. Composition

For spaces X and Y and Z , there is a composition map $\mathfrak{c}: \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \rightarrow \mathcal{C}(X, Z)$ defined by $\mathfrak{c}(g, f) = g \circ f$ for $g \in \mathcal{C}(Y, Z)$ and $f \in \mathcal{C}(X, Y)$. It is natural to hope that this is continuous. Unfortunately, this does not hold in general. However, it does hold if Y is locally compact:

LEMMA 13.4.1. *Let X and Y and Z be spaces with Y locally compact. Then the composition map $\mathfrak{c}: \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \rightarrow \mathcal{C}(X, Z)$ is continuous.*

PROOF. Let $K \subset X$ be compact and $U \subset Z$ be open. We must prove that $\mathfrak{c}^{-1}(B(K, U))$ is open. Let $(g, f) \in \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y)$ satisfy $\mathfrak{c}(g, f) \in B(K, U)$. It is enough to find an open neighborhood of (g, f) that is mapped by $\mathfrak{c}$ into $B(K, U)$. Since $g \circ f \in B(K, U)$, we have $f(K) \subset g^{-1}(U)$. Since $f(K)$ is a compact subset of the open subset $g^{-1}(U) \subset Y$ and Y is locally compact, there is a compact neighborhood L of $f(K)$ with $L \subset g^{-1}(U)$ (see Exercise 8.3). It follows that $\mathfrak{c}$ takes the open neighborhood $B(L, U) \times B(K, g^{-1}(U))$ of (g, f) into $B(K, U)$, as desired. $\square$

13.5. Evaluation

For spaces X and Y , there is an evaluation map $\mathfrak{e}: \mathcal{C}(X, Y) \times X \rightarrow Y$ defined by $\mathfrak{e}(f, x) = f(x)$ for $f \in \mathcal{C}(X, Y)$ and $x \in X$. Just like for the composition map, to ensure this is continuous we need to assume that X is locally compact:

LEMMA 13.5.1. *Let X and Y be spaces with X locally compact. Then the evaluation map $\mathfrak{e}: \mathcal{C}(X, Y) \times X \rightarrow Y$ is continuous.*

PROOF. Let p_0 be a one-point space. We have $\mathcal{C}(p_0, X) = X$ and $\mathcal{C}(p_0, Y) = Y$. Applying these identities, the evaluation map becomes the composition map $\mathcal{C}(X, Y) \times \mathcal{C}(p_0, X) \rightarrow \mathcal{C}(p_0, Y)$, which is continuous by Lemma 13.4.1. $\square$

13.6. Parameterized maps

Let X and Y and Z be spaces. It is natural to expect maps $\phi: X \times Z \rightarrow Y$ and $\Phi: Z \rightarrow \mathcal{C}(X, Y)$ to be closely related. Indeed, if we were working with sets rather than spaces then such maps would be in bijection with each other: a map $\Phi: Z \rightarrow \mathcal{C}(X, Y)$ would correspond to the map $\phi: X \times Z \rightarrow Y$ defined by $\phi(x, z) = \Phi(z)(x)$. The following shows that this holds topologically if X is locally compact:

LEMMA 13.6.1. *Let X and Y and Z be spaces. The following holds:*

- (i) *Let $\phi: X \times Z \rightarrow Y$ be continuous. Define $\Phi: Z \rightarrow \mathcal{C}(X, Y)$ to be the map that takes $z \in Z$ to the map $X \rightarrow Y$ taking $x \in X$ to $\phi(x, z) \in Y$. Then Φ is continuous.*
- (ii) *Assume that X is locally compact. Let $\Psi: Z \rightarrow \mathcal{C}(X, Y)$ be continuous. Define $\psi: X \times Z \rightarrow Y$ to be the map taking $(x, z) \in X \times Z$ to $\Psi(z)(x) \in Y$. Then ψ is continuous.*

PROOF. For (i), let $\phi: X \times Z \rightarrow Y$ be continuous and define $\Phi: Z \rightarrow \mathcal{C}(X, Y)$ as in (i). Let $K \subset X$ be compact and $U \subset Y$ be open. We must prove that $\Phi^{-1}(B(K, U)) \subset Z$ is open. Let $z_0 \in \Phi^{-1}(B(K, U))$, so $K \times z_0 \subset \phi^{-1}(U)$. Since $K \subset X$ is compact and $\phi^{-1}(U)$ is an open neighborhood of $K \times z_0$, Exercise 7.8 (the “tube lemma”) gives an open neighborhood $V \subset Z$ of z_0 with $K \times V \subset \phi^{-1}(U)$. It follows that V is an open neighborhood of z_0 with $V \subset \Phi^{-1}(B(K, U))$, as desired.

We now prove (ii). Assume that X is locally compact and that $\Psi: Z \rightarrow \mathcal{C}(X, Y)$ is continuous. The map $\psi: X \times Z \rightarrow Y$ defined in (ii) is the composition

$$X \times Z \xrightarrow{1 \times \Psi} X \times \mathcal{C}(X, Y) \xrightarrow{\epsilon} Y,$$

where $\epsilon: X \times \mathcal{C}(X, Y) \rightarrow Y$ is the evaluation map $\epsilon(x, f) = f(x)$. Lemma 13.5.1 implies that ϵ is continuous, so we conclude that ψ is continuous. $\square$

13.7. Homotopies and the compact-open topology

Let $f_0, f_1: X \rightarrow Y$ be maps. Recall that a homotopy from f_0 to f_1 is a continuous map $H: X \times I \rightarrow Y$ with $H(x, 0) = f_0(x)$ and $H(x, 1) = f_1(x)$ for all $x \in X$. Lemma 13.6.1 implies that such a homotopy gives a map $h: I \rightarrow \mathcal{C}(X, Y)$. This map h can be viewed as a path from $h(0) = f_0$ to $h(1) = f_1$. Conversely, if X is locally compact then Lemma 13.6.1 implies that a path in $\mathcal{C}(X, Y)$ from f_0 to f_1 gives a homotopy from f_0 to f_1 .

13.8. Quotient maps and the compact-open topology

As an application of our results, we give another proof of the following result from §11.3:

LEMMA 11.3.4. *Let $q: X \rightarrow Y$ be a quotient map and let Z be a locally compact space. Then the map $q \times 1: X \times Z \rightarrow Y \times Z$ is a quotient map.*

PROOF. As we discussed in §3.4, the quotient map $q: X \rightarrow Y$ satisfies the following universal property. Let $\sim$ be the equivalence relation on X where $x_1 \sim x_2$ if $q(x_1) = q(x_2)$. A map $F: X \rightarrow W$ is $\sim$ -invariant if $F(x_1) = F(x_2)$ whenever $x_1 \sim x_2$. For all spaces W , the following holds:

- There is a bijection between maps $f: Y \rightarrow W$ and $\sim$ -invariant maps $F: X \rightarrow W$ taking $f: Y \rightarrow W$ to $f \circ q: X \rightarrow W$.

In fact, this universal properties characterizes the quotient topology (see Exercise 3.8). We must therefore verify the analogue of it for the map $q \times 1: X \times Z \rightarrow Y \times Z$.

Consider a space W and a map $G: X \times Z \rightarrow W$ that is $\sim$ -invariant in the sense that $G(x_1, z) = G(x_2, z)$ for all $x_1, x_2 \in X$ and $z \in Z$ with $x_1 \sim x_2$. We must construct a map $g: Y \times Z \rightarrow W$ such that $G = g \circ (q \times 1)$. Let $F: X \rightarrow \mathcal{C}(Z, W)$ be the map defined by

$$F(x)(z) = G(x, z) \quad \text{for all } x \in X \text{ and } z \in Z.$$

By Lemma 13.6.1, the map F is continuous. Since G is $\sim$ -invariant, so is F . It follows that there is a map $f: Y \rightarrow \mathcal{C}(Z, W)$ with $F = f \circ q$. Let $g: Y \times Z \rightarrow W$ be the map defined by

$$g(y, z) = f(y)(z) \quad \text{for all } y \in Y \text{ and } z \in Z.$$

Since Z is locally compact, Lemma 13.6.1 says that g is continuous. By construction we have $G = g \circ (q \times 1)$, as desired. $\square$

13.9. Parameterized maps, II

Let X and Y and Z be spaces with X locally compact. Lemma 13.6.1 gives a bijection between $\mathcal{C}(X \times Z, Y)$ and $\mathcal{C}(Z, \mathcal{C}(X, Y))$. The following lemma says that this bijection is a homeomorphism if X and Z are Hausdorff:

LEMMA 13.9.1. *Let X and Y and Z be spaces with X locally compact Hausdorff and Z Hausdorff. Let $\lambda: \mathcal{C}(X \times Z, Y) \rightarrow \mathcal{C}(Z, \mathcal{C}(X, Y))$ be the map taking $\phi: X \times Z \rightarrow Y$ to the map $\Phi: Z \rightarrow \mathcal{C}(X, Y)$ defined by*

$$\Phi(z)(x) = \phi(x, z) \in Y \quad \text{for all } z \in Z \text{ and } x \in X.$$

Then λ is a homeomorphism.

PROOF. Lemma 13.6.1 says that λ is a bijection. For $K \subset X$ and $L \subset Z$ compact and $U \subset Y$ open the map λ restricts to a bijection between $B(K \times L, U)$ and $B(L, B(K, U))$. To prove the lemma, it is enough to prove that open sets of these forms are subbases for the topologies on $\mathcal{C}(X \times Z, Y)$ and $\mathcal{C}(Z, \mathcal{C}(X, Y))$:

- For $\mathcal{C}(X \times Z, Y)$, we prove this in Lemma 13.9.2 below.
- For $\mathcal{C}(Z, \mathcal{C}(X, Y))$, in Lemma 13.9.3 below we prove more generally that if $\mathcal{B}$ is any subbasis for the topology on a space W , then sets of the form $B(L, V)$ with $L \subset Z$ compact and $V \in \mathcal{B}$ form a subbasis for $\mathcal{C}(Z, W)$. $\square$

The above proof used the following two results:

LEMMA 13.9.2. *Let X and Y and Z be spaces with X and Z Hausdorff. Then the set of all $B(K \times L, U)$ with $K \subset X$ compact and $L \subset Z$ compact and $U \subset Y$ open forms a subbasis for the compact-open topology on $\mathcal{C}(X \times Z, Y)$.*

PROOF. Let $C \subset X \times Z$ be compact and $U \subset Y$ be open. We must prove that $B(C, U)$ is open in the topology with the indicated subbasis. Consider $f \in B(C, U)$. It is enough to find $K_1, \dots, K_n \subset X$ compact and $L_1, \dots, L_n \subset Z$ compact such that

$$f \in B(K_1 \times L_1, U) \cap \dots \cap B(K_n \times L_n, U) \subset B(C, U).$$

Unwrapping this, we need the K_i and L_i to satisfy the following:

- $C \subset \cup_{i=1}^n K_i \times L_i$; and
- $K_i \times L_i \subset f^{-1}(U)$ for all $1 \leq i \leq n$.

Let $C(X) \subset X$ and $C(Z) \subset Z$ be the projections of $C \subset X \times Z$. Both $C(X)$ and $C(Z)$ are compact Hausdorff spaces, and $C \subset C(X) \times C(Z)$. Replacing X with $C(X)$ and Z with $C(Z)$, we can therefore assume without loss of generality that X and Z are compact Hausdorff spaces. The space $X \times Z$ is thus also a compact Hausdorff space, and in particular is normal (see Lemma 7.2.3).

The set $f^{-1}(U)$ is an open neighborhood of C . Since $X \times Z$ is normal, for each $c \in C$ we can find open sets $V_c \subset X$ and $W_c \subset Z$ such that $c \in V_c \times W_c$ and $\overline{V_c} \times \overline{W_c} \subset f^{-1}(U)$. Since C is compact, we can find $c_1, \dots, c_n$ such that $C \subset \cup_{i=1}^n V_{c_i} \times W_{c_i}$. Let $K_i = \overline{V_{c_i}} \subset X$ and $L_i = \overline{W_{c_i}} \subset Z$, so $K_i \times L_i \subset f^{-1}(U)$. Since X and Z are compact, the closed sets K_i and L_i are also compact. By construction we have $C \subset \cup_{i=1}^n K_i \times L_i$, as desired. $\square$

LEMMA 13.9.3. *Let Z and W be spaces with Z Hausdorff and let $\mathcal{B}$ be a subbasis for the topology on W . Then the set of all $B(K, V)$ with $K \subset Z$ compact and $V \in \mathcal{B}$ forms a subbasis for the compact-open topology on $\mathcal{C}(Z, W)$.*

PROOF. See Exercise 13.2. $\square$

REMARK 13.9.4. It is a little annoying that the above results require local compactness. Unfortunately, they are false in general. There is a way around this using the theory of compactly generated spaces, which we describe in Essay H. $\square$

13.10. Exercises

EXERCISE 13.1. Let X be a space and let $(Y, \mathfrak{d})$ be a metric space. For $f \in Y^X$ and a compact subset $K \subset X$ and $\epsilon > 0$, let

$$B(f, K, \epsilon) = \{g \in Y^X \mid \mathfrak{d}(f(x), g(x)) < \epsilon \text{ for all } x, y \in K\}.$$

Prove that these sets form the basis for a topology on Y^X , and this topology is the same as the compact-open topology. $\square$

EXERCISE 13.2. Let Z and W be spaces with Z Hausdorff and let $\mathcal{B}$ be a subbasis for the topology on W . Prove that the set of all $B(K, V)$ with $K \subset Z$ compact and $V \in \mathcal{B}$ forms a subbasis for the compact-open topology on $\mathcal{C}(Z, W)$. $\square$

EXERCISE 13.3. Topologize $\text{GL}_n(\mathbb{R})$ by identifying it as a subspace of $\text{Mat}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. For each $M \in \text{GL}_n(\mathbb{R})$, multiplication by M gives a linear map $\phi_M: \mathbb{R}^n \rightarrow \mathbb{R}^n$. Prove that the map $\iota: \text{GL}_n(\mathbb{R}) \rightarrow \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n)$ defined by $\iota(M) = \phi_M$ is a closed embedding. $\square$

EXERCISE 13.4. Let $(M, \mathfrak{d})$ be a compact metric space. An isometry of M is a bijection $f: M \rightarrow M$ such that $\mathfrak{d}(f(p), f(q)) = \mathfrak{d}(p, q)$ for all $p, q \in M$. Let $\text{Isom}(M)$ be the group of isometries of M . Topologize $\text{Isom}(M)$ using the compact-open topology, i.e., by identifying $\text{Isom}(M)$ with a subspace of $\mathcal{C}(M, M)$. Prove that $\text{Isom}(M)$ is compact. $\square$

EXERCISE 13.5. Let $\text{Homeo}(\mathbb{S}^1)$ be the set of homeomorphisms $f: \mathbb{S}^1 \rightarrow \mathbb{S}^1$. Topologize $\text{Homeo}(\mathbb{S}^1)$ using the compact-open topology, i.e., by identifying $\text{Homeo}(\mathbb{S}^1)$ with a subspace of $\mathcal{C}(\mathbb{S}^1, \mathbb{S}^1)$. Prove that $\text{Homeo}(\mathbb{S}^1)$ is not locally compact. $\square$

Part 2

Essays on manifolds and related topics

Topological manifolds

In this essay, we use the tools we have developed to study manifolds, which are perhaps the most important class of spaces in topology.

A.1. Basic definitions

An n -dimensional manifold (or simply an n -manifold) is a second countable Hausdorff space M^n that is locally homeomorphic to $\mathbb{R}^n$ in the following sense:

- For all $p \in M^n$, there exists an open neighborhood U of p that is homeomorphic to an open subset of $\mathbb{R}^n$.

A *chart* on M^n is a homeomorphism $\phi: U \rightarrow V$ with $U \subset M^n$ and $V \subset \mathbb{R}^n$ open sets. If U is an open neighborhood of $p \in M^n$, we call this chart $\phi: U \rightarrow V$ a chart around p . An *atlas* for M^n is a collection of charts $\{\phi_i: U_i \rightarrow V_i\}_{i \in I}$ such that the U_i cover M^n .

Here are several basic examples:

EXAMPLE A.1.1. The whole space $\mathbb{R}^n$ is an n -manifold with an atlas consisting of a single chart $\mathbb{1}: \mathbb{R}^n \rightarrow \mathbb{R}^n$. More generally, an open set $U \subset \mathbb{R}^n$ is an n -manifold, again with an atlas consisting of a single chart $\mathbb{1}: U \rightarrow U$. $\square$

EXAMPLE A.1.2. More generally, if M^n is an n -manifold and $W \subset M^n$ is open, then W is an n -manifold. Indeed, for $p \in W$ let $\phi: U \rightarrow V$ be a chart around p for M^n . Letting $U' = U \cap W$ and $V' = \phi(U')$, the homeomorphism $\phi|_{U'}: U' \rightarrow V'$ is a chart around p for W . $\square$

EXAMPLE A.1.3. Let $\mathbb{S}^n$ be the n -sphere, so

$$\mathbb{S}^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\}.$$

This is an n -manifold. Indeed, for $1 \leq k \leq n+1$ let

$$U_{x_k > 0} = \{(x_1, \dots, x_{n+1}) \in \mathbb{S}^n \mid x_k > 0\},$$

$$U_{x_k < 0} = \{(x_1, \dots, x_{n+1}) \in \mathbb{S}^n \mid x_k < 0\}.$$

Letting $B = B_1(0) \subset \mathbb{R}^n$ be the open unit ball, we have homeomorphisms $\phi_{x_k > 0}: U_{x_k > 0} \rightarrow B$ and $\phi_{x_k < 0}: U_{x_k < 0} \rightarrow B$ taking a point $(x_1, \dots, x_{n+1})$ to $(x_1, \dots, \widehat{x_k}, \dots, x_{n+1}) \in B$, where the hat in $\widehat{x_j}$ indicates that this coordinate is being omitted. The set

$$\{\phi_{x_k > 0}: U_{x_k > 0} \rightarrow B, \phi_{x_k < 0}: U_{x_k < 0} \rightarrow B \mid 1 \leq k \leq n+1\}$$

is an atlas for $\mathbb{S}^n$. $\square$

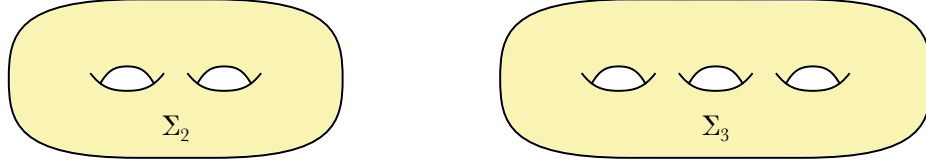
EXAMPLE A.1.4. Let $\mathbb{RP}^n$ be the set of lines through the origin in $\mathbb{R}^{n+1}$. There is a projection map $q: \mathbb{R}^{n+1} \setminus 0 \rightarrow \mathbb{RP}^n$ taking $x \in \mathbb{R}^{n+1} \setminus 0$ to the line through 0 and x . We endow $\mathbb{RP}^n$ with the quotient topology from this projection, so $U \subset \mathbb{RP}^n$ is open if and only if $q^{-1}(U) \subset \mathbb{R}^{n+1} \setminus 0$ is open. The space $\mathbb{RP}^n$ is known as the n -dimensional real projective space. As notation, for $(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \setminus 0$ we write $[x_1, \dots, x_{n+1}]$ for the corresponding point of $\mathbb{RP}^n$, so for $\lambda \in \mathbb{R}$ nonzero we have $[\lambda x_1, \dots, \lambda x_{n+1}] = [x_1, \dots, x_{n+1}]$.

The space $\mathbb{RP}^n$ is an n -manifold. Unlike our previous examples, it is not totally obvious that it is second countable and Hausdorff, so we leave this as an exercise (Exercise A.1). We prove it is locally Euclidean by exhibiting an atlas as follows. For $1 \leq k \leq n+1$, let $U_k = \{[x_1, \dots, x_{n+1}] \in \mathbb{RP}^n \mid x_k \neq 0\}$. This set is well-defined, and the map $\phi_k: U_k \rightarrow \mathbb{R}^n$ defined by

$$\phi_k([x_1, \dots, x_{n+1}]) = (x_1/x_k, \dots, \widehat{x_k/x_k}, \dots, x_{n+1}/x_k) \quad \text{for } [x_1, \dots, x_{n+1}] \in \mathbb{RP}^n$$

is a well-defined homeomorphism (see Exercise A.1). The set $\{\phi_k: U_k \rightarrow \mathbb{R}^n \mid 1 \leq k \leq n+1\}$ is an atlas for $\mathbb{RP}^n$. $\square$

REMARK A.1.5. It is clear that the only connected 0-manifold is a single point. It turns out that $\mathbb{R}$ and $\mathbb{S}^1$ are the only connected 1-manifolds. There is also a very beautiful classification of compact connected 2-manifolds. Here are two examples of such 2-manifolds:



We describe the classification of 2-manifolds in Essay B. The exercises in that essay also outline a proof of the classification of 1-manifolds. In higher dimensions, things are much more complicated. $\square$

REMARK A.1.6. The requirement that manifolds be second countable and Hausdorff is needed to rule out various pathological examples. Without them, even 1-manifolds would not have a simple classification. We describe some of these pathological examples later in this chapter. $\square$

A.2. Basic properties

The following summarizes some of the basic point-set topological properties of manifolds:

LEMMA A.2.1. *Let M^n be an n -manifold. Then:*

- M^n is perfectly normal.
- M^n is locally compact.
- M^n is paracompact.
- M^n is metrizable.
- M^n is locally path connected, so its path components and connected components coincide and are clopen.

PROOF. Since M^n is locally homeomorphic to $\mathbb{R}^n$, the fact that M^n is locally compact and locally path connected follows immediately from the fact that $\mathbb{R}^n$ is locally compact and locally path connected. Since M^n is second countable, Hausdorff, and locally compact, it follows that M^n is paracompact (see Corollary 10.2.2). This implies that M^n is normal (see Lemma 10.4.1). We can therefore use the Urysohn metrization theorem (Theorem 12.2.1) to see that M^n is metrizable and hence perfectly normal (Lemma 6.7.1). $\square$

REMARK A.2.2. One basic property of manifolds we do not list above is that their dimension is well-defined. In fact, it is true that if M is both an n -manifold and an m -manifold then $n = m$, but this is a difficult theorem called the *invariance of domain*. The most natural proof of invariance of domain uses homology. $\square$

A.3. Embedding manifolds into Euclidean space

Many n -manifolds are constructed as subspaces of some $\mathbb{R}^d$, but some manifolds like $\mathbb{RP}^n$ do not have obvious embeddings into any Euclidean space. However, it turns out that all manifolds can be embedded in some $\mathbb{R}^d$:

THEOREM A.3.1. *Let M^n be an n -manifold. Then for some $N \gg 0$ there exists a proper embedding $\iota: M^n \hookrightarrow \mathbb{R}^N$. Since ι is proper it is a closed map, so this implies that M^n is homeomorphic to a closed subspace of $\mathbb{R}^N$.*

In fact, in Essay C we will use dimension theory to embed M^n into $\mathbb{R}^{2n+1}$. We include the proof Theorem A.3.1 here both because it is easier than the dimension theory proof and because its details will be useful in a later volume when we study smooth manifolds.

PROOF OF THEOREM A.3.1. The first step is to show that M^n has a finite atlas

$$\{\phi_k: U_k \rightarrow V_k \mid 1 \leq k \leq m\}$$

such that each $\phi_k: U_k \rightarrow V_k$ extends to a homeomorphism $\bar{\phi}_k: \bar{U}_k \rightarrow \bar{V}_k$. This is obvious if M^n is compact, while for M^n non-compact it follows from Lemma A.4.1 below. This might seem paradoxical if M^n is non-compact, but note that we are not requiring the U_k to be connected. In fact, our construction in Lemma A.4.1 will in general give an atlas in which each U_k has infinitely many path components.

Since M^n is paracompact, there is a strong refinement of the cover $\{U_1, \dots, U_m\}$, i.e., a cover $\{W_1, \dots, W_m\}$ such that $\bar{W}_k \subset U_k$ for each $1 \leq k \leq m$ (see Lemma 10.5.1). Since M^n is perfectly normal, for each $1 \leq k \leq m$ we can find a continuous map $f_k: X \rightarrow I$ such that $f_k^{-1}(1) = \bar{W}_k$ and $\text{supp}(f_k) \subset U_k$; see §6.7. Multiplying ϕ_k by f_k , we get a map $f_k \phi_k: U_k \rightarrow \mathbb{R}^n$. Since $\text{supp}(f_k) \subset U_k$, we can extend $f_k \phi_k: U_k \rightarrow \mathbb{R}^n$ to a continuous map $G_k: M^n \rightarrow \mathbb{R}^n$ with $G_k(p) = 0$ for $p \notin U_k$. Let $\iota: M^n \rightarrow \mathbb{R}^{nm+m}$ be the map defined by

$$\iota(p) = (G_1(p), f_1(p), \dots, G_m(p), f_m(p)) \in (\mathbb{R}^n \times \mathbb{R}^1)^{\times m} = \mathbb{R}^{nm+m} \quad \text{for } p \in M^n.$$

We must prove that ι is injective and proper:

STEP 1. *The map ι is injective.*

Consider $p, q \in M^n$ with $\iota(p) = \iota(q)$. Pick $1 \leq k \leq m$ with $p \in \bar{W}_k$. We thus have $f_k(p) = 1$, so since $\iota(p) = \iota(q)$ we also have $f_k(q) = 1$. Since $f_k^{-1}(1) = \bar{W}_k$, it follows that $q \in \bar{W}_k$. On $\bar{W}_k$ we have $G_k = \phi_k$, so since $p, q \in \bar{W}_k$ and $\iota(p) = \iota(q)$ we have $\phi_k(p) = \phi_k(q)$. Since $\phi_k: U_k \rightarrow V_k$ is a homeomorphism and $\bar{W}_k \subset U_k$, we conclude that $p = q$.

STEP 2. *The map ι is a proper map. We remark that this is trivial if M^n is compact, so this step is only needed in the non-compact case.*

Let $L \subset \mathbb{R}^{nm+m}$ be compact. Applying Lemma 9.2.2, we must prove that $\iota^{-1}(L)$ is compact. Since

$$\iota^{-1}(L) = (\iota^{-1}(L) \cap \bar{W}_1) \cup \dots \cup (\iota^{-1}(L) \cap \bar{W}_m),$$

it is enough to prove that each $\iota^{-1}(L) \cap \bar{W}_k = (\iota|_{\bar{W}_k})^{-1}(L)$ is compact. We will prove this for $k = 1$; the other cases are identical up to changes in notation. Let $\pi: \mathbb{R}^{nm+m} \rightarrow \mathbb{R}^n$ be the projection onto the first n coordinates and let $L' = \pi(L)$, so $L' \subset \mathbb{R}^n$ is compact. We have

$$(\iota|_{\bar{W}_1})^{-1}(L) \subset (\pi \circ \iota|_{\bar{W}_1})^{-1}(L').$$

Since $(\iota|_{\bar{W}_1})^{-1}(L)$ is a closed subspace of $\bar{W}_1$, we see that it is enough to prove that $(\pi \circ \iota|_{\bar{W}_1})^{-1}(L')$ is compact. On $\bar{W}_1$, we have $f_1 = 1$ and $G_1 = \phi_1$. It follows that $\pi \circ \iota|_{\bar{W}_1} = \phi_1|_{\bar{W}_1}$, so what we must prove is that $(\phi_1|_{\bar{W}_1})^{-1}(L')$ is compact. We assumed at the beginning of the proof that the homeomorphism $\phi_1: U_1 \rightarrow V_1$ extends to a homeomorphism $\bar{\phi}_1: \bar{U}_1 \rightarrow \bar{V}_1$. The closed subspace $\bar{V}_1 \cap L'$ of the compact subspace L' is compact, so $\bar{\phi}_1^{-1}(L')$ is compact. Since $(\phi_1|_{\bar{W}_1})^{-1}(L') = \bar{\phi}_1^{-1}(L') \cap \bar{W}_1$, we conclude that $(\phi_1|_{\bar{W}_1})^{-1}(L')$ is compact, as desired. $\square$

A.4. Finite atlases for noncompact manifolds

At least in the non-compact case, the above proof of Theorem A.3.1 depended on the following:

LEMMA A.4.1. *All n -manifolds M^n have finite atlases $\{\phi_k: U_k \rightarrow V_k \mid 1 \leq k \leq m\}$ such that each homeomorphism $\phi_k: U_k \rightarrow V_k$ extends to a homeomorphism $\bar{\phi}_k: \bar{U}_k \rightarrow \bar{V}_k$.*

PROOF. Since M^n is second countable, we can find an atlas $\{\phi_k: U_k \rightarrow V_k \mid k \geq 1\}$ such that each homeomorphism $\phi_k: U_k \rightarrow V_k$ extends to a homeomorphism $\bar{\phi}_k: \bar{U}_k \rightarrow \bar{V}_k$ and each $\bar{V}_k$ is a bounded subset of $\mathbb{R}^n$. We now invoke a result from dimension theory we will prove in Essay C below:

- We can shrink each U_k to a possibly smaller open set and ensure that for each $p \in M^n$ the set $\{k \mid p \in U_k\}$ has cardinality at most¹ $n + 1$. In the terminology we will introduce in Essay C, this means that the cover $\{U_k \mid k \geq 1\}$ has order at most $n + 1$.

¹Of course, this reflects the fact that M^n is n -dimensional. We remark that the precise constant $n + 1$ will not matter for our proof here, and could be replaced by any N that does not depend on the point $p \in M^n$.

Since M^n is paracompact, there is a partition of unity subordinate to the cover $\{U_k \mid k \geq 1\}$, i.e., a collection of continuous functions $\{f_k: M^n \rightarrow I \mid k \geq 1\}$ such that $\text{supp}(f_k) \subset U_k$ for each $k \geq 1$ and such that for all $p \in M^n$ we have

$$\sum_{k=1}^{\infty} f_k(p) = 1.$$

Since each p lies in at most $n+1$ of the U_k , all but finitely many terms in this sum are 0. For each unordered tuple $\{k_1, \dots, k_d\}$ of distinct positive integers, let

$$U_{k_1, \dots, k_d} = \{p \in M^n \mid f_i(p) < f_{k_j}(p) \text{ for all } i \geq 1 \text{ and } 1 \leq j \leq d \text{ with } i \notin \{k_1, \dots, k_d\}\}.$$

These are open sets (see Exercise A.2). Since the inequality in the definition of $U_{k_1, \dots, k_d}$ is strict, it follows that for $p \in U_{k_1, \dots, k_d}$ we have $f_{k_j}(p) > 0$ for all $1 \leq j \leq d$. We thus have

$$U_{k_1, \dots, k_d} \subset \bigcap_{j=1}^d U_{k_j}.$$

Each ϕ_{k_j} therefore restricts to a homeomorphism between $U_{k_1, \dots, k_d}$ and an open set in $\mathbb{R}^n$. Let $V_{k_1, \dots, k_d}$ be the image of one of these restrictions and let $\phi_{k_1, \dots, k_d}: U_{k_1, \dots, k_d} \rightarrow V_{k_1, \dots, k_d}$ be the restricted homeomorphism. From the ϕ_{k_j} these inherit the property that they extend to homeomorphisms $\bar{\phi}_{k_1, \dots, k_d}: \bar{U}_{k_1, \dots, k_d} \rightarrow \bar{V}_{k_1, \dots, k_d}$ and that $\bar{V}_{k_1, \dots, k_d}$ is a bounded subset of $\mathbb{R}^n$. We now prove two key properties of the $U_{k_1, \dots, k_d}$:

CLAIM 1. *For some $d \geq 1$, let $\{k_1, \dots, k_d\}$ and $\{\ell_1, \dots, \ell_d\}$ be distinct unordered d -tuples of positive integers. Then $U_{k_1, \dots, k_d} \cap U_{\ell_1, \dots, \ell_d} = \emptyset$.*

PROOF OF CLAIM. Assume for the sake of contradiction that there is a point p in this intersection. After reordering the k_j and ℓ_j , we can assume that k_1 does not appear in $(\ell_1, \dots, \ell_d)$ and that ℓ_1 does not appear in $(k_1, \dots, k_d)$. We then have that $f_{\ell_1}(p) < f_{k_1}(p)$ and that $f_{k_1}(p) < f_{\ell_1}(p)$, a contradiction. $\square$

CLAIM 2. *The collection of open sets $\{U_{k_1, \dots, k_d} \mid 1 \leq d \leq n+1 \text{ and } k_1, \dots, k_d \geq 1\}$ covers M^n .*

PROOF OF CLAIM. Let $p \in M^n$. By assumption, p lies in at most $n+1$ of the U_k . Enumerating the set $\{k \mid f_k(p) > 0\}$ as $\{k_1, \dots, k_d\}$, we therefore have $1 \leq d \leq n+1$. Since $f_i(p) = 0$ for all $i \notin \{k_1, \dots, k_d\}$, it follows that $p \in U_{k_1, \dots, k_d}$. $\square$

In the rest of the proof, we not need the definition of the $U_{k_1, \dots, k_d}$ but only the above two claims and the fact that we have homeomorphisms $\phi_{k_1, \dots, k_d}: U_{k_1, \dots, k_d} \rightarrow V_{k_1, \dots, k_d}$ that extend to homeomorphisms $\bar{\phi}_{k_1, \dots, k_d}: \bar{U}_{k_1, \dots, k_d} \rightarrow \bar{V}_{k_1, \dots, k_d}$. Since M^n is paracompact, we can shrink the $U_{k_1, \dots, k_d}$ so that they still cover M^n but now satisfy the following strengthening of Claim 1:

- (i) For some $d \geq 1$, let $\{k_1, \dots, k_d\}$ and $\{\ell_1, \dots, \ell_d\}$ be distinct unordered d -tuples of positive integers. Then $\bar{U}_{k_1, \dots, k_d} \cap \bar{U}_{\ell_1, \dots, \ell_d} = \emptyset$.

Moreover, by moving the bounded sets $\bar{V}_{k_1, \dots, k_d} \subset \mathbb{R}^n$ around we can ensure that:

- (ii) For some $d \geq 1$, let $\{k_1, \dots, k_d\}$ and $\{\ell_1, \dots, \ell_d\}$ be distinct unordered d -tuples of positive integers. Then $\bar{V}_{k_1, \dots, k_d} \cap \bar{V}_{\ell_1, \dots, \ell_d} = \emptyset$.

For each $1 \leq d \leq n+1$, let $A_d \subset M^n$ be the union of the disjoint sets $\{U_{k_1, \dots, k_d} \mid k_1, \dots, k_d \geq 1\}$ and let $B_d \subset \mathbb{R}^n$ be the union of the disjoint sets $\{V_{k_1, \dots, k_d} \mid k_1, \dots, k_d \geq 1\}$. The homeomorphisms $\phi_{k_1, \dots, k_d}: U_{k_1, \dots, k_d} \rightarrow V_{k_1, \dots, k_d}$ assemble into a homeomorphism $\psi_d: A_d \rightarrow B_d$. By (i) and (ii) above, the homeomorphism ψ_d extends to a homeomorphism $\bar{\psi}_d: \bar{A}_d \rightarrow \bar{B}_d$. The desired finite atlas is then $\{\psi_d: A_d \rightarrow B_d \mid 1 \leq d \leq n+1\}$. $\square$

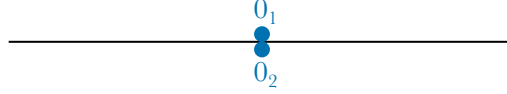
A.5. Non-Hausdorff manifolds

Recall that we require manifolds to be Hausdorff and second countable. Removing these hypotheses gives many exotic generalized manifolds, even in dimension 1. We have already seen one example of a non-Hausdorff 1-manifold, namely the line with two origins from Example 6.1.1. We recall the construction:

EXAMPLE A.5.1. As a set, let $Y = (\mathbb{R} \setminus \{0\}) \sqcup \{0_1, 0_2\}$. For $i = 1, 2$, let $f_i: \mathbb{R} \rightarrow Y$ be the map defined by $f_i(x) = x$ for $x \in \mathbb{R} \setminus \{0\}$ and $f_i(0) = 0_i$. Give Y the identification space topology, so:

- a set $U \subset Y$ is open if and only if $f_1^{-1}(U)$ and $f_2^{-1}(U)$ are open in $\mathbb{R}$.

Here is a picture of this:



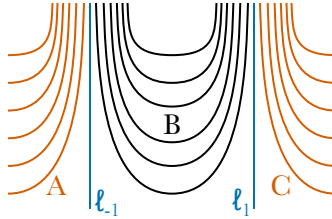
With this topology, the subspaces $Y \setminus \{0_2\} = f_1(\mathbb{R})$ and $Y \setminus \{0_1\} = f_2(\mathbb{R})$ are open subsets of Y that are both homeomorphic to $\mathbb{R}$. It follows that Y is a second-countable non-Hausdorff 1-manifold. $\square$

This example might not seem very geometrically interesting. The theory of foliations of the plane gives non-Hausdorff 1-manifolds with a closer connection to geometry. See [1] for a beautiful discussion of this. We content ourselves here with one example:

EXAMPLE A.5.2. For $c \in \mathbb{R}$, let $X_c = \{(x, y) \mid (x^2 - 1)e^y = c\} \subset \mathbb{R}^2$. Define

$$\mathfrak{F} = \{L \mid L \text{ is a connected component of } X_c \text{ for some } c \in \mathbb{R}\}.$$

The set $\mathfrak{F}$ is what is called a foliation of $\mathbb{R}^2$. Each $L \in \mathfrak{F}$ is called a *leaf* of the foliation. Here is a picture of $\mathfrak{F}$:

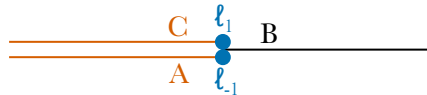


Each leaf L is homeomorphic to $\mathbb{R}$, and $\mathbb{R}^2$ is the disjoint union of the $L \in \mathfrak{F}$. The set X_0 consists of two vertical lines ℓ_{-1} and ℓ_1 where $x = \pm 1$. For $c > 0$, the set X_c consists of two arcs, one lying in the region to the left of ℓ_{-1} labeled A and one lying in the region to the right of ℓ_1 labeled C . For $c < 0$, the set X_c consists of a single arc in the region between ℓ_{-1} and ℓ_1 labeled B .

Let $\mathcal{L}$ be the quotient space of $\mathbb{R}^2$ obtained by collapsing each $L \in \mathfrak{F}$ to a point. This is called the leaf space of the foliation $\mathfrak{F}$. The space $\mathcal{L}$ is a non-Hausdorff 1-manifold. To describe it, let R_1 and R_2 be copies of $\mathbb{R}$. The space $\mathcal{L}$ is obtained by gluing R_1 to R_2 so as to identify each $t \in R_1$ with $t > 0$ with the corresponding $t \in R_2$. The various types of leaves correspond to the following points:

- The points $0 \in R_1$ and $0 \in R_2$ correspond to ℓ_{-1} and ℓ_1 .
- The points $t \in R_1$ with $t < 0$ correspond to the arcs in the region A .
- The points $t \in R_2$ with $t < 0$ correspond to the arcs in region C .
- The points $t \in R_1$ and $t \in R_2$ with $t > 0$ that are glued together correspond to the arcs in the region B .

The picture is as follows:



This space is non-Hausdorff since the points corresponding to ℓ_{-1} and ℓ_1 do not have disjoint neighborhoods. You will verify all of this in Exercise A.3. $\square$

A.6. Non-second countable manifolds

The theory of non-second countable manifolds has a set-theoretic flavor. It turns out that in dimension one there is a single example of a connected non-second countable Hausdorff 1-manifold called the *long line* L . We close this chapter with a brief discussion of it. The space L has the following seemingly paradoxical properties:

- L is a path-connected Hausdorff non-second-countable 1-manifold.
- Like $\mathbb{R}$, the points of L are endowed with a total ordering.

- For $x, y \in L$ with $x < y$, the “interval”

$$[x, y] = \{z \in L \mid x \leq z \leq y\}$$

is homeomorphic to the closed interval $I = [0, 1]$. This accounts for L being path connected.

- On the other hand, since L is not second countable it contains uncountably many subspaces homeomorphic to the open interval $(0, 1)$.

Before we can construct L , we need to discuss some more details about well-ordered sets, which we introduced in §11.10 to set up the process of transfinite induction.

A.6.1. Minimal uncountable well-ordered set. Let S be an uncountable set. Pick a well-ordering on S . Let $\mathfrak{C}$ be the set of all initial segments of S that are either finite or countably infinite. The set $\mathfrak{C}$ is nonempty since $\emptyset \in \mathfrak{C}$. In fact, by starting with $\emptyset$ and repeatedly adding the minimal element we have not yet chosen we see that there exists a countably infinite set in $\mathfrak{C}$. As we discussed in §11.10, the initial segments of S are totally ordered under inclusion. Let

$$S_\Omega = \bigcup_{J \in \mathfrak{C}} J.$$

The set S_Ω is an initial segment of S . By construction, all initial segments J with $J \subsetneq S_\Omega$ are countable. We claim that S_Ω is not countable. Indeed, let s_0 be the minimal element of $S \setminus S_\Omega$. The initial segment $S_\Omega \sqcup \{s_0\}$ cannot lie in $\mathfrak{C}$, so $S_\Omega \sqcup \{s_0\}$ is uncountable. This implies that S_Ω is uncountable. The totally ordered set S_Ω is called the *minimal uncountable well-ordered set*.² It is unique up to isomorphism, but we will not need this. All we need to know about S_Ω is that it is uncountable but all proper initial segments of S_Ω are finite or countably infinite.

A.6.2. Constructing the long line. Let $\widehat{L} = S_\Omega \times [0, 1)$. Both S_Ω and $[0, 1)$ have total orderings. Give $\widehat{L}$ the dictionary ordering, so $(s, x) \leq (s', x')$ if $s < s'$ or if $s = s'$ and $x < x'$. An *open interval* in $\widehat{L}$ is a set of the form $(\theta_1, \theta_2) = \{\nu \mid \theta_1 < \nu < \theta_2\}$ for some $\theta_1, \theta_2 \in \widehat{L}$ with $\theta_1 < \theta_2$. This is a basis for a topology called the *order topology* (see Example 2.6.3). We endow $\widehat{L}$ with the order topology.

To form the long line L , let $s_0 \in S_\omega$ be the minimal element. It follows that $(s_0, 0) \in \widehat{L}$ is the minimal element of $\widehat{L}$. Define $L = \widehat{L} \setminus \{(s_0, 0)\}$. As you will verify in Exercise A.4, this has the properties claimed in §A.6.

A.7. Exercises

EXERCISE A.1. Prove the following:

- The space $\mathbb{R}P^n$ is Hausdorff and second countable.
- Letting $U_k = \{[x_1, \dots, x_{n+1}] \in \mathbb{R}P^n \mid x_k \neq 0\}$, the map $\phi_k: U_k \rightarrow \mathbb{R}^n$ defined by

$$\phi_k([x_1, \dots, x_{n+1}]) = (x_1/x_k, \dots, \widehat{x_k/x_k}, \dots, x_{n+1}/x_k) \quad \text{for } [x_1, \dots, x_{n+1}] \in \mathbb{R}P^n$$

is a well-defined homeomorphism. □

EXERCISE A.2. Let M^n be an n -manifold, let $\{U_k \mid k \geq 1\}$ be an open cover of M^n , and let $\{f_k: M^n \rightarrow I \mid k \geq 1\}$ be a partition of unity subordinate to $\{U_k \mid k \geq 1\}$. For an unordered tuple $\{k_1, \dots, k_d\}$ of distinct positive integers, let

$$U_{k_1, \dots, k_d} = \{p \in M^n \mid f_i(p) < f_{k_j}(p) \text{ for all } i \geq 1 \text{ and } 1 \leq j \leq d \text{ with } i \notin \{k_1, \dots, k_d\}\}.$$

Prove that $U_{k_1, \dots, k_d}$ is open. □

EXERCISE A.3. Verify the description of $\mathcal{L}$ in Example A.5.2. □

EXERCISE A.4. Let L be the long line constructed in §A.6.2. Prove the following:

- For $x, y \in L$ with $x < y$, the closed interval

$$[x, y] = \{z \in L \mid x \leq z \leq y\}$$

is homeomorphic to the closed interval $I = [0, 1]$.

²Or the minimal uncountable ordinal, but we have chosen not to use that terminology.

- (b) The space L is path-connected.
- (c) The space L contains uncountably many subspaces homeomorphic to the open interval $(0, 1)$.
- (d) The space L is a Hausdorff non-second-countable 1-manifold. □

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- [1] A. Haefliger & G. Reeb, One dimensional non-Hausdorff manifolds and foliations of the plane, in *Geometric methods in group theory—papers dedicated to Ruth Charney*, 225–241, Sémin. Congr., 34, Soc. Math. France, Paris.

Classification of surfaces

B.1. Introduction

An enormous amount of algebraic topology was developed to help classify manifolds up to homeomorphism. This classification is easy in dimensions 0 and 1, where the only connected examples are a point, a circle $\mathbb{S}^1$, and the real line $\mathbb{R}$ (see Exercise B.9). The first interesting dimension is 2, i.e., surfaces. Here there are infinitely many examples, but there is an elegant and easy-to-state classification (at least in the compact case) whose origins go back to 19th century work of Möbius.

B.1.1. Our goal. In this essay, we prove the classification of surfaces. Our goal is to emphasize geometric reasoning. There is a large expository gulf between the geometric topology literature and accounts of the classification of surfaces, which are typically aimed at beginning students and involve elaborate manipulations of triangulations. We include many examples and pictures, but some of our proofs and definitions are a little informal. Making them rigorous will (hopefully) be routine to readers who are experienced with smooth manifolds.

B.1.2. History and sources. The idea of our proof goes back to Zeeman [13]. Here are other accounts geared to students earlier in their education:

- See [9] or [12] for the classical combinatorial proof. I first learned this material from [9] when I was an undergraduate.
- See [2] for a proof similar to the one we give.

There are other possible proofs of this result. One that is particularly charming is Conway’s “ZIP” proof, which can be found in [4]. For a history of the classification, see [5].

B.1.3. Assumed results. To avoid getting bogged down with foundational results, we will carefully state but not prove two important results:

- The existence of triangulations of surfaces. Actually, we will use the more flexible notion of “polygonal decompositions”.
- The fact that the Euler characteristic of a surface is a topological invariant. This result is very easy once the theory of homology is introduced, so we see little point in giving a combinatorial proof that uses special features of surfaces.

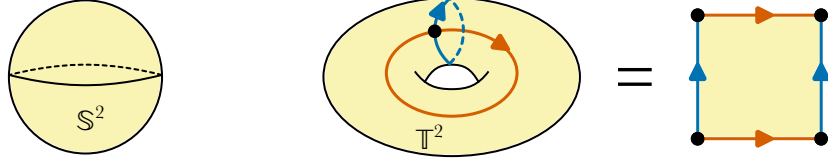
We will also freely use standard results about smooth manifolds, often without mentioning them explicitly.

B.1.4. Outline. In §B.2 we give examples of surfaces and state a first version of the classification theorem. Next, as a warm-up to the proof in §B.3 we discuss graphs and their Euler characteristics. We then introduce polygonal decompositions and prove some basic results about the Euler characteristic in §B.4, which ends with a refined version of the classification. We prove the classification in the next two sections: §B.5 proves the “Poincaré conjecture” characterizing the sphere, and §B.6 proves the rest of the classification. Finally, §B.7 gives some extensions and generalizations of the classification.

B.2. Examples of surfaces

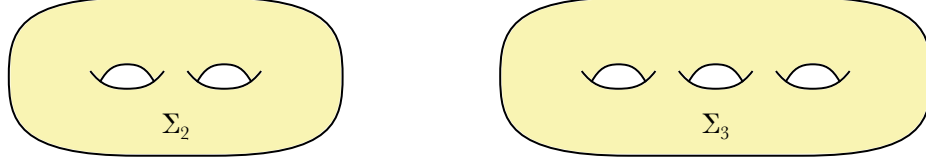
A *surface* is a 2-dimensional manifold, possibly with boundary. Our focus will be on surfaces that are connected and *closed*, that is, compact and without boundary. This section focuses on examples.

B.2.1. Basic examples. The most familiar surfaces are the 2-sphere $\mathbb{S}^2$ and the 2-torus $\mathbb{T}^2 = (\mathbb{S}^1)^{\times 2}$:



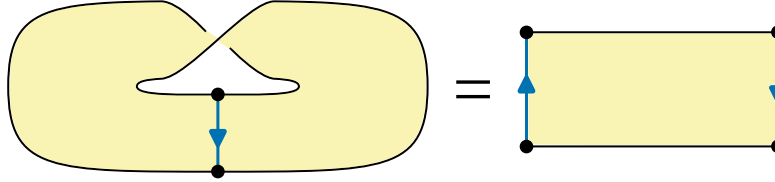
As is shown here, $\mathbb{T}^2$ can be obtained from $\mathbb{D}^2$ by identifying $\mathbb{D}^2$ with a square and identifying parallel sides. The four vertices of the square are all identified to a single point. The sphere $\mathbb{S}^2$ can also be obtained from $\mathbb{D}^2$ by identifying the entire boundary $\partial\mathbb{D}^2 = \mathbb{S}^1$ to a single point.

The torus is the surface of an ordinary donut. More generally, a *genus- g surface*, denoted Σ_g is the surface of a donut with g holes:



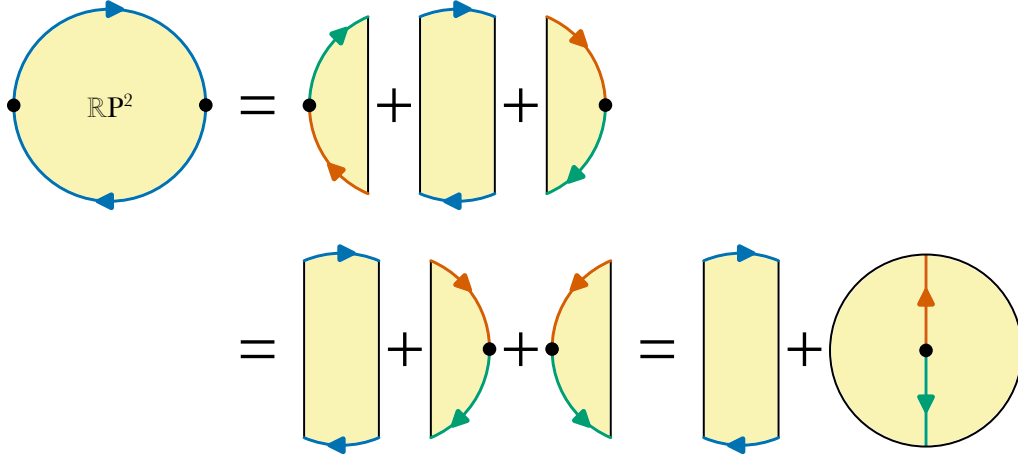
We therefore have $\Sigma_0 \cong \mathbb{S}^2$ and $\Sigma_1 \cong \mathbb{T}^2$. As we will discuss in §B.4 below, for $g \geq 1$ the surface Σ_g can be obtained from a $4g$ -gon by identifying sides in an appropriate way.

B.2.2. Möbius band and real projective plane. The surfaces Σ_g are all orientable. We will assume that this notion is familiar from theory of manifolds. The most basic example of a non-orientable surface is a Möbius band:



The Möbius band has one boundary component. To obtain a closed surface, we glue a disk $\mathbb{D}^2$ to this boundary component to form the *real projective plane* $\mathbb{RP}^2$. You might worry that the result depends on the choice of a homeomorphism between $\partial\mathbb{D}^2 = \mathbb{S}^1$ and the boundary component of the Möbius band, but it turns out that the result is independent of the gluing. This holds in great generality; see Exercise B.11. We will use this fact silently throughout this essay.

Pictures of $\mathbb{RP}^2$ are not particularly enlightening,¹ but as the following shows it can be obtained from $\mathbb{D}^2$ by identifying antipodal points on the boundary $\partial\mathbb{D}^2$:



Another way of viewing $\mathbb{RP}^2$ is as the space of lines through the origin in $\mathbb{R}^3$. To connect this with

¹It cannot be embedded in $\mathbb{R}^3$, but only in $\mathbb{R}^4$. There is a way of drawing it in $\mathbb{R}^3$ with self-intersections called the “Boy’s Surface”, but this picture does not shed much light on its nature.

the above picture, note that every such line intersects the upper hemisphere

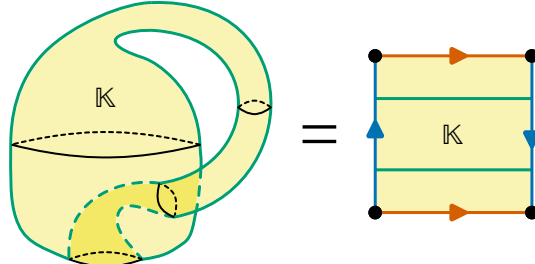
$$U = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1, z \geq 0\} \cong \mathbb{D}^2.$$

This intersection is unique except for lines lying in the xy -plane, which intersect

$$\partial U = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\} \cong \partial \mathbb{D}^2$$

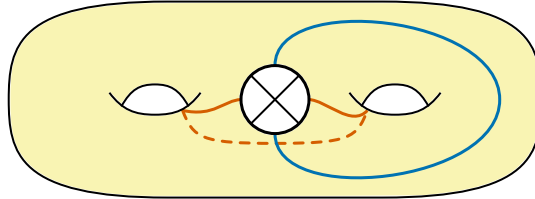
in two antipodal points. The space of lines through the origin can thus be identified with $U \cong \mathbb{D}^2$, but with antipodal points on $\partial U \cong \partial \mathbb{D}^2$ identified.

B.2.3. Klein bottle. Another important example of a non-orientable surface is the Klein bottle $\mathbb{K}$, which is obtained by gluing two Möbius bands together along their boundary. Unlike $\mathbb{RP}^2$, there is a somewhat enlightening way of drawing the $\mathbb{K}$, though necessarily this picture has self-intersections. See here for this and also how to get $\mathbb{K}$ by identifying the sides of a rectangle:

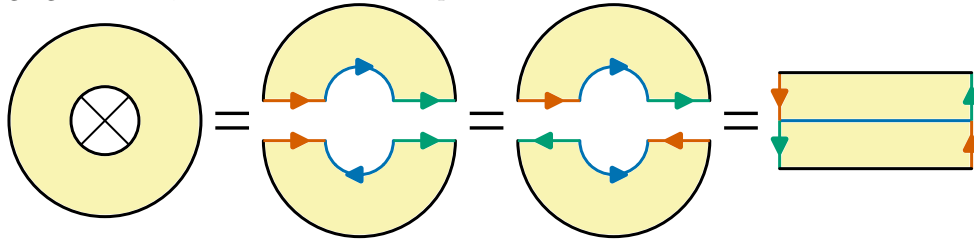


The green curve in both figures is a circle, and when you cut either open along it you get two Möbius bands. This shows that these two surfaces are indeed homeomorphic.

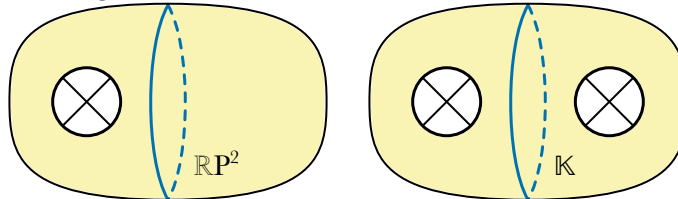
B.2.4. Cross caps. The surfaces $\mathbb{RP}^2$ and $\mathbb{K}$ are the first two elements of an infinite family of non-orientable surfaces. To explain this, we must introduce the notion of a *cross-cap*. A cross-cap on a surface is obtained by removing the interior of a disk and then identifying antipodal points. We denote this by drawing a disk with a cross in it like this:



In this figure, the blue and orange arcs are actually disjoint circles embedded in the surface. As the following figure shows, a disk with a cross-cap in it is a Möbius band:

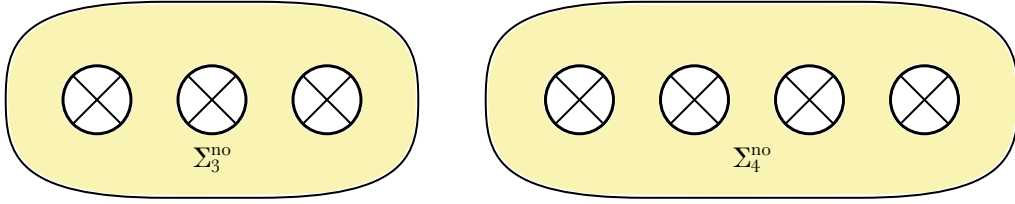


Since $\mathbb{RP}^2$ is a Möbius band with a disk glued to it and $\mathbb{K}$ is two Möbius bands glued together along their boundary, the following are $\mathbb{RP}^2$ and $\mathbb{K}$:



On the left the blue loop divides $\mathbb{RP}^2$ into a Möbius band and a disk, and on the right the blue loop divides $\mathbb{K}$ into two Möbius bands. These pictures suggest the general pattern: the *genus- n*

nonorientable surface, denoted Σ_n^{no} , is a sphere with n cross-caps on it:



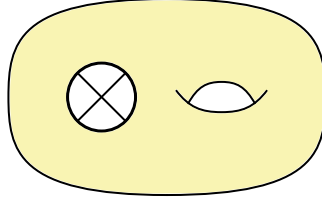
Thus $\Sigma_1^{\text{no}} \cong \mathbb{RP}^2$ and $\Sigma_2^{\text{no}} \cong \mathbb{K}$.

B.2.5. Classification theorem, first version. We can now state a first version of the classification theorem:

THEOREM B.2.1 (Classification of surfaces, weak). *Let Σ be a closed connected surface. Then:*

- If Σ is orientable, then $\Sigma \cong \Sigma_g$ for a unique $g \geq 0$.
- If Σ is non-orientable, then $\Sigma \cong \Sigma_n^{\text{no}}$ for a unique $n \geq 1$.

In some ways this is a very satisfying result, but one weakness is that it does not give an effective way to recognize a given surface. Since it is easy to write down surfaces that do not fit into the above classification in an obvious way, this is a real problem. For instance, consider the following non-orientable surface:

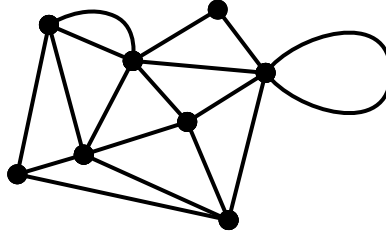


By Theorem B.2.1, this must be homeomorphic to Σ_n^{no} for some $n \geq 1$. However, it is not at all obvious which Σ_n^{no} it is. We will later give a refined classification theorem that will make it clear that the above surface is Σ_3^{no} ; see Theorem B.4.12. Before reading this, it is worth trying to prove it for yourself.

B.3. Graphs and their Euler characteristics

As a warm-up before proving the classification of surfaces, this section discusses aspects of graph theory that can be viewed as a one-dimensional analogue of this classification.

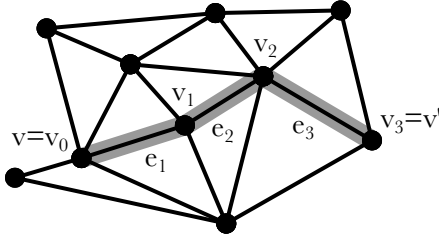
B.3.1. Basic definitions. Recall that a *graph* X is a collection of vertices $V(X)$ and a collection of edges $E(X)$. Each $e \in E(X)$ connects two vertices in $V(X)$. These vertices need not be distinct:



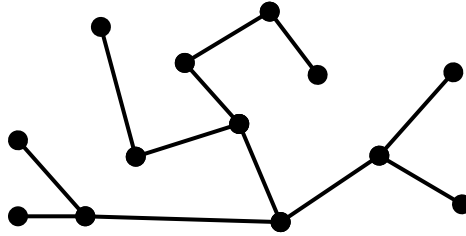
In this essay, we will only consider graphs with finitely many vertices and edges, which we call *finite graphs*. A finite graph is a topological space in a straightforward way. An *edge-path* in a graph from a vertex $v \in V(X)$ to a vertex $v' \in V(X)$ is a sequence of edges $e_1, \dots, e_n$ such that there exist vertices

$$v = v_0, v_1, \dots, v_n = v'$$

such that e_i connects v_{i-1} and v_i for all $1 \leq i \leq n$:

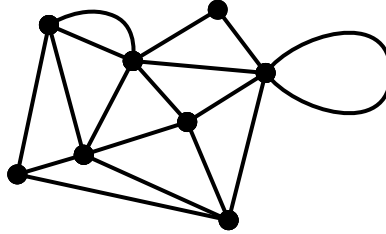


Associated to an edge-path is a continuous map $\gamma: I \rightarrow X$, and we will often confuse an edge-path with the associated map. The edge-path is *closed* if $v = v'$, in which case the associated path is a loop that we can regard as a continuous map $\gamma: \mathbb{S}^1 \rightarrow X$. The edge-path is a *cycle* if it is closed, $n \geq 1$, and all the e_i are distinct. A graph is *connected* if all distinct $v, v' \in V(X)$ are connected by an edge-path. This is equivalent to the graph being path-connected as a topological space. A *tree* is a nonempty connected graph with no cycles:



B.3.2. Euler characteristic of graphs. If X is a finite graph, then the *Euler characteristic* of X is $\chi(X) = |V(X)| - |E(X)|$.

EXAMPLE B.3.1. If X is the graph

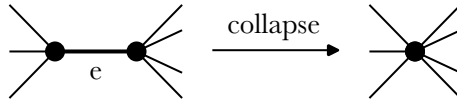


then the Euler characteristic of X is $\chi(X) = |V(X)| - |E(X)| = 8 - 17 = -9$. $\square$

B.3.3. Poincaré conjecture for graphs. The importance of the Euler characteristic is illustrated by the following result, which we think of as the “Poincaré conjecture”² for graphs:

LEMMA B.3.2 (Poincaré conjecture for graphs). *Let X be a finite nonempty connected graph. Then $\chi(X) \leq 1$, with equality if and only if X is a tree.*

PROOF. If e is an edge of X connecting two distinct vertices, then we can collapse e without changing whether or not X is a tree:



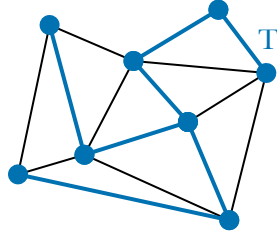
Such a collapse decreases the number of edges and vertices by 1, and thus does not change the Euler characteristic. Collapsing such edges repeatedly, we can therefore assume without loss of generality that all edges of X are loops. Since X is nonempty and connected, this implies that X has one vertex. We therefore have

$$\chi(X) = |V(X)| - |E(X)| = 1 - |E(X)| \leq 1$$

²For manifolds, the Poincaré conjecture is a topological characterization of a sphere. Once we have defined the Euler characteristic for surfaces, the two-dimensional Poincaré conjecture will say that a compact connected surface Σ is homeomorphic to $\mathbb{S}^2$ if and only if its Euler characteristic is 2.

with equality if and only if $|E(X)| = 0$, i.e., if and only if X is a tree.³ $\square$

B.3.4. Maximal trees. If X is a connected nonempty graph, then a maximal tree in X is a subtree T of X containing all the vertices. See here:



These always exist:

LEMMA B.3.3. *Let X be a connected nonempty graph. Then X has a maximal tree.*

The proof for finite graphs X is a little easier, and this is the only case we need. We therefore restrict to this case:

PROOF OF LEMMA B.3.3 FOR FINITE GRAPHS. Assume that X has n vertices. Inductively define subtrees

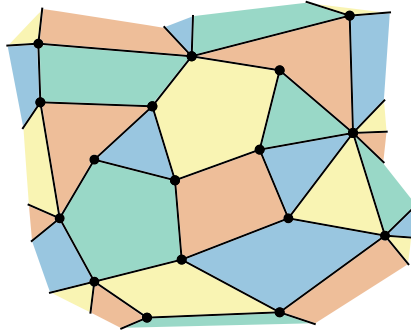
$$T_1 \subset T_2 \subset \cdots \subset T_n$$

of X in the following way. Start by choosing a vertex v_0 of X and letting $T_1 = v_0$. Now assume that T_k has been constructed for some $k < n$. Since X has n vertices, it must be the case that T_k does not contain all the vertices of X . Since X is connected, this implies that there must be an edge e of X that connects a vertex of T_k to a vertex that does not lie in T_k . Let T_{k+1} be the result of adding e to T_k . This process stops at T_n , which contains n vertices and hence is a maximal tree in X . $\square$

B.4. Polygonal decompositions and the Euler characteristic

Our proof of the classification of surfaces will depend on a decomposition of the surface that is a sort of two-dimensional analogue of the decomposition of a graph into vertices and edges.

B.4.1. Basic definitions. A surface equipped with a *polygonal decomposition* is a compact surface Σ (possibly with boundary) together with a finite graph X embedded in Σ such that each path component F of $\Sigma \setminus X$ is homeomorphic to an open disk $\text{Int}(\mathbb{D}^2)$. We will call such an F a *face* of the polygonal decomposition.⁴ Here is a picture of part of a polygonal decomposition, with the faces in different colors to help the reader distinguish them:



Here is some terminology for polygonal decompositions:

- The graph X will be called the *1-skeleton*.
- The vertices and edges of X will be called the vertices and edges of the polygonal decomposition, and the sets of vertices and edges will be written $V(\Sigma)$ and $E(\Sigma)$, respectively.
- The set of faces of the polygonal decomposition will be written $F(\Sigma)$.
- The *Euler characteristic* of the polygonal decomposition is $\chi(\Sigma) = |V(\Sigma)| - |E(\Sigma)| + |F(\Sigma)|$.

³Since X has only one vertex, the only way it can be a tree is if it has no edges.

⁴For non-compact surfaces, one would also need to require that the closure of each face is compact.

B.4.2. Existence. The following theorem will play a basic role in our proof:

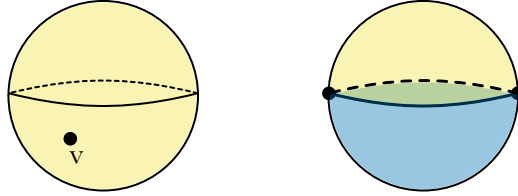
THEOREM B.4.1. *Let Σ be a compact surface, possibly with boundary. Then Σ has a polygonal decomposition.*

See Exercise B.7 for the analogous fact in dimension 1. We will not prove Theorem B.4.1, which requires a long detour into point-set topology. It was originally proved by Radó [11]. See [1] and [10] for modern versions of Radó's proof. I remark that I first learned this proof from [1]. A recent and elegant proof along very different lines can be found in [6]. Amazingly, the proof in [6] uses smooth manifold techniques (even though the surface is not assumed to be smooth), and avoids doing any serious point-set work.

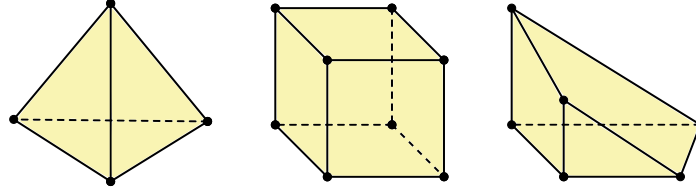
REMARK B.4.2. All of the above sources actually prove the slightly stronger fact that Σ has a triangulation, that is, a polygonal decomposition where the boundaries of each face are length-3 edge-paths in the 1-skeleton. It is easy to subdivide a general polygonal decomposition and turn it into a triangulation. $\square$

B.4.3. Examples of polygonal decompositions. Here are a number of examples:

EXAMPLE B.4.3. Here are two easy polygonal decompositions of $\mathbb{S}^2$:

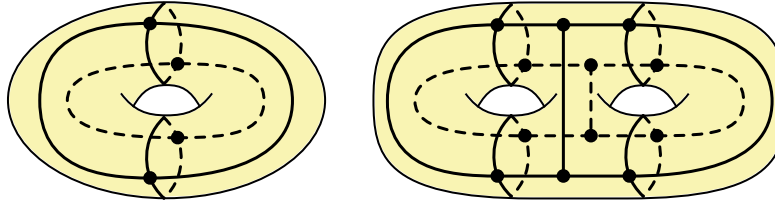


The first has a single vertex v , no edges, and one face $\mathbb{S}^2 \setminus v \cong \mathbb{R}^2 \cong \text{Int}(\mathbb{D}^2)$. Its Euler characteristic is $1 - 0 + 1 = 2$. The second has two vertices, two edges, and two faces. Its Euler characteristic is $2 - 2 + 2 = 2$. Other polygonal decompositions of $\mathbb{S}^2$ can be obtained by identifying the boundaries of polyhedra in $\mathbb{R}^3$ with $\mathbb{S}^2$. For instance:



Here we have stopped trying to draw the faces in different colors. As the reader will check, in each of these cases the Euler characteristic is 2. For instance, the left-most polygonal decomposition has 4 vertices, 6 edges, and 4 faces, so its Euler characteristic is $4 - 6 + 4 = 2$. All these examples reflect a theorem we will discuss below that says that all polygonal decompositions of the same surface have the same Euler characteristic. $\square$

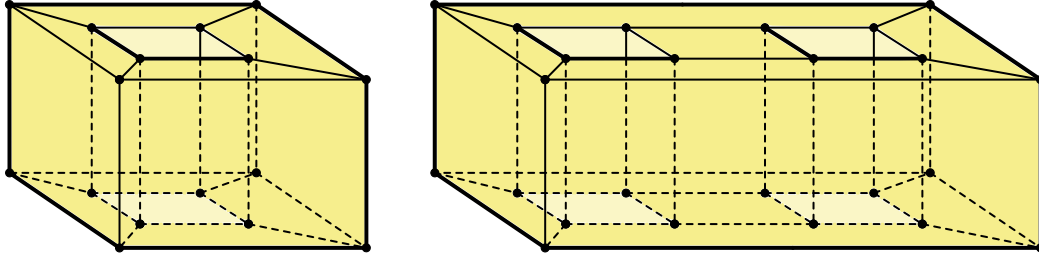
EXAMPLE B.4.4. Here are polygonal decompositions of Σ_1 and Σ_2 :



The Euler characteristics of these polygonal decompositions are

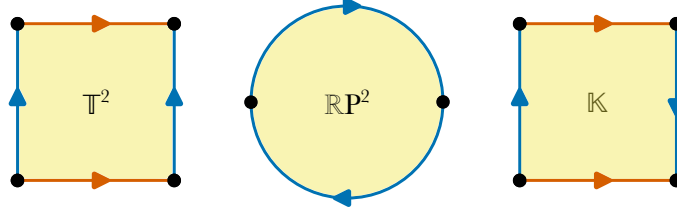
$$\begin{aligned}\chi(\Sigma_1) &= 4 - 8 + 4 = 0, \\ \chi(\Sigma_2) &= 12 - 22 + 8 = -2.\end{aligned}$$

By regarding Σ_1 and Σ_2 as cubes with holes drilled through their centers, we can obtain two other polygonal decompositions for which it is a little easier to see that the faces are open disks:



Again, the reader can verify that these have Euler characteristic 0 and -2 . $\square$

EXAMPLE B.4.5. When we gave examples of surfaces in §B.2, many of our surface came presented as polygons or disks with sides identified. This gives a polygonal decomposition of the surface. For instance, here are pictures of the torus $\mathbb{T}^2$, the real projective plane $\mathbb{RP}^2$, and the Klein bottle $\mathbb{K}$:



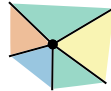
When we identify the boundary points as shown, the boundary becomes a graph and the interior of the disk/polygon gives a single face. In the above examples:

- $\mathbb{T}^2$ has one vertex and two edges and one face, so $\chi(\mathbb{T}^2) = 1 - 2 + 1 = 0$; and
- $\mathbb{RP}^2$ has one vertex and one edge and one face, so $\chi(\mathbb{RP}^2) = 1 - 1 + 1 = 1$; and
- $\mathbb{K}$ has one vertex and two edges and one face, so $\chi(\mathbb{K}) = 1 - 2 + 1 = 0$.

We will give more examples of this later in §B.4.10. $\square$

B.4.4. Local structure of polygonal decompositions. Let Σ be a closed surface equipped with a polygonal decomposition. We now discuss the local structure of this polygonal decomposition. This local structure follows from the definition of a polygonal decomposition. However, the proof is a little technical and we will omit it.⁵ For a reader who cannot prove it on their own, we suggest adding these local results to the definition. The various proofs that polygonal decompositions exist (Theorem B.4.1) give polygonal decompositions where this local structure definitely holds.

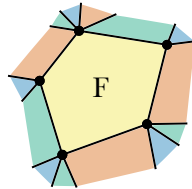
Our statements will be informal, but will be sufficient to understand the proof of the classification. First, a small neighborhood of a vertex looks like this:



Each of the shaded regions is part of a face. These faces need not all be distinct. Next, consider an edge e . We have $e \cong [0, 1]$ and $\text{Int}(e) \cong (0, 1)$. A small neighborhood of $\text{Int}(e)$ looks like this:



Again, the two shaded regions are parts of two faces, though these faces might be the same. We now come to a face F . Recall that $F \cong \text{Int}(\mathbb{D}^2)$. There exists a homeomorphism $\phi: \text{Int}(\mathbb{D}^2) \rightarrow F$ that extends to a continuous map $\phi: \mathbb{D}^2 \rightarrow F$. The restriction of ϕ to $\partial\mathbb{D}^2 = \mathbb{S}^1$ is usually a closed edge-path in the 1-skeleton:



⁵See Exercise B.8 for analogous results in dimension 1.

The edges in the 1-skeleton traversed by this closed edge path need not be distinct. There is one case where this does not hold: for the polygonal decomposition of $\mathbb{S}^2$ with a single vertex and face and no edges (cf. Example B.4.3), the restriction of ϕ to $\partial\mathbb{D}^2 = \mathbb{S}^1$ is the constant map to this single vertex.

REMARK B.4.6. Polygonal decompositions also are useful for surfaces with boundary, but the local structure described above needs to be modified for vertices and edges that are contained in the boundary. $\square$

B.4.5. Well-definedness of Euler characteristic. In our examples above, the Euler characteristics of different polygonal decompositions of the same surface were always the same. This always holds:

THEOREM B.4.7. *Let Σ be a compact surface, possibly with boundary. Then the Euler characteristics of any two polygonal decompositions of Σ are the same.*

The most conceptual proof of this theorem uses homology. For any reasonable compact space X , that theory produces integers $b_i(X) \geq 0$ for each $i \geq 0$ called the *Betti numbers* of X . The Betti number of X are manifestly invariants of X , and for any polygonal decomposition of a compact surface Σ we have

$$\chi(\Sigma) = b_0(\Sigma) - b_1(\Sigma) + b_2(\Sigma).$$

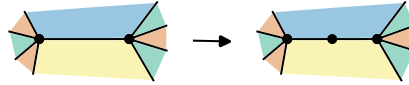
In fact, there are higher-dimensional versions of polygonal decompositions called *CW complex structures*. For a compact space X equipped with a CW complex structure, we have $b_i(X) = 0$ for $i \gg 0$, so the a priori infinite alternating sum

$$\chi(X) = b_0(X) - b_1(X) + b_2(X) - \cdots + (-1)^i b_i(X) + \cdots$$

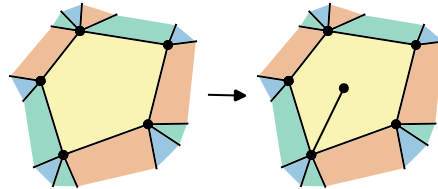
is a finite sum, called the Euler characteristic of X . One of the basic structural theorems about Betti numbers is that $\chi(X)$ also equals an alternating sum of the i -dimensional cells of X . We will prove this more general version in Volume 2 when we discuss homology.

For surfaces, there is also an alternate approach as follows. Along with the existence of polygonal decompositions (Theorem B.4.1), there is also a uniqueness statement saying that any two polygonal decompositions of a compact surface Σ have what is called a *common subdivision*. What this means is that after applying a sequence of the following three moves, any two polygonal decompositions differ by a homeomorphism of Σ :

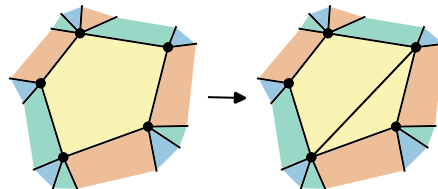
- Subdivide an edge as follows:



- Add a vertex and edge in the interior of a face as follows:



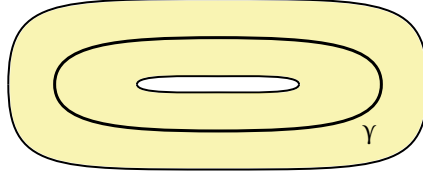
- Add an edge connecting two vertices of a face as follows:



To prove that the Euler characteristic does not depend on the polygonal decomposition, it is thus enough to observe that the above three moves do not change it. For instance, subdividing an edge adds one vertex and one edge, and these cancel out when calculating the Euler characteristic.

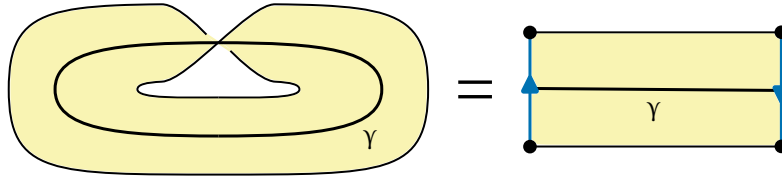
B.4.6. Cutting. We now explore the effect on the Euler characteristic of cutting along simple closed curves. Let Σ be a compact surface, possibly with boundary. Let γ be a simple closed curve lying in the interior of Σ . From the theory of manifolds, γ has what is called a *tubular neighborhood*. There are two cases:

- This tubular neighborhood is an annulus with γ in its interior like this:



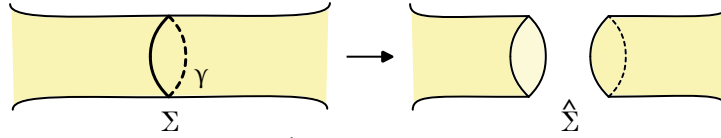
In this case, we say that γ is a *two-sided curve*. If Σ is orientable, all γ are two-sided.

- This tubular neighborhood is a Möbius band with γ in its interior like this:



In this case, we say that γ is a *one-sided curve*.

Cutting Σ open along γ turns Σ into a surface $\hat{\Sigma}$. If γ is two-sided, the surface $\hat{\Sigma}$ has two more boundary components than Σ , see here:

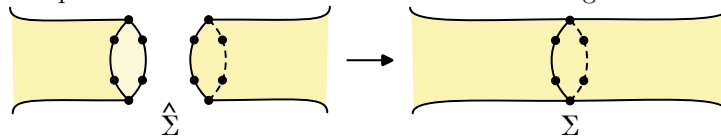


On the other hand, if γ is one-sided then $\hat{\Sigma}$ has only more more boundary component. We remark that $\hat{\Sigma}$ might be disconnected.

We will prove:

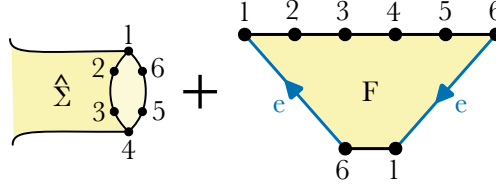
LEMMA B.4.8. *Let Σ be a compact surface, possibly with boundary. Let γ be a simple closed curve in the interior of Σ and let $\hat{\Sigma}$ be the result of cutting Σ open along γ . Then $\chi(\hat{\Sigma}) = \chi(\Sigma)$.*

PROOF. Fix a polygonal decomposition of $\hat{\Sigma}$. Assume first that γ is two-sided. In this case, there are boundary components ∂_1 and ∂_2 of $\hat{\Sigma}$ such that Σ can be obtained by gluing ∂_1 to ∂_2 . Necessarily ∂_1 and ∂_2 are cycles in the 1-skeleton of $\hat{\Sigma}$. Subdividing edges in the ∂_i if necessary, we can assume that ∂_1 and ∂_2 contain the same number of n of vertices. As the following shows, Σ then has a polygonal decomposition with n fewer vertices and n fewer edges than $\hat{\Sigma}$:



This figure does not include the part of the polygonal decomposition lying in the interior of $\hat{\Sigma}$, which is irrelevant for this calculation. These changes in the numbers of vertices and edges cancel out when we calculate the Euler characteristic, giving that $\chi(\hat{\Sigma}) = \chi(\Sigma)$.

Now assume that γ is one-sided. In this case, there is a boundary component ∂ of $\hat{\Sigma}$ such that Σ can be obtained by gluing a Möbius band to ∂ . Necessarily ∂ lies in the 1-skeleton of $\hat{\Sigma}$. Assume that ∂ has n vertices. As the following shows, with respect to an appropriate polygonal decomposition Σ has 1 more edge (labeled e) and 1 more face (labeled F) than $\hat{\Sigma}$:



Here we have drawn the Möbius band in a skewed way to emphasize that like on ∂ its vertices are equally spaced around its single boundary component, and again we did not include the part of the polygonal decomposition lying in the interior of $\hat{\Sigma}$. These changes in the numbers of vertices and edges again cancel out when we calculate the Euler characteristic, giving that $\chi(\hat{\Sigma}) = \chi(\Sigma)$. $\square$

B.4.7. Capping. We now study the effect on the Euler characteristic of gluing a disk to a boundary component. Recall that the result does not depend on the gluing map (Exercise B.11).

LEMMA B.4.9. *Let Σ be a compact surface with boundary and let ∂ be a boundary component of Σ . Let $\bar{\Sigma}$ be the result of gluing a disk to ∂ . Then $\chi(\bar{\Sigma}) = \chi(\Sigma) + 1$.*

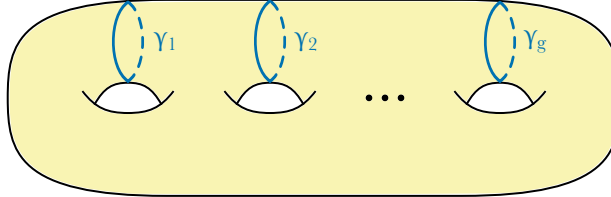
PROOF. Choose a polygonal decomposition of Σ . The disk we glued to ∂ can serve as a face of a polygonal decomposition of $\bar{\Sigma}$, giving a polygonal decomposition of $\bar{\Sigma}$ with the same number of vertices and edges as Σ and one more face. It follows that

$$\chi(\bar{\Sigma}) = |V(\bar{\Sigma})| - |E(\bar{\Sigma})| + |F(\bar{\Sigma})| = |V(\Sigma)| - |E(\Sigma)| + |F(\Sigma)| + 1 = \chi(\Sigma) + 1. \quad \square$$

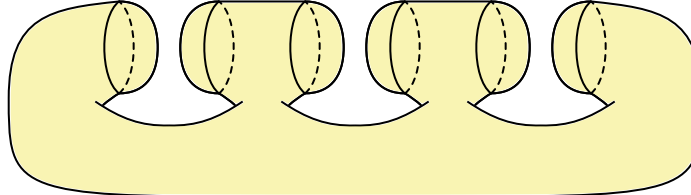
B.4.8. Euler characteristic calculations. Our results about cutting and capping make it easy to calculate the Euler characteristics of surfaces without needing to find explicit polygonal decompositions of them. As examples of this, we calculate the Euler characteristics of the genus- g surface Σ_g and the nonorientable genus- n surface Σ_n^{no} :

LEMMA B.4.10. *For $g \geq 0$, we have $\chi(\Sigma_g) = 2 - 2g$.*

PROOF. Let $\gamma_1, \dots, \gamma_g$ be the following two-sided curves on Σ_g :



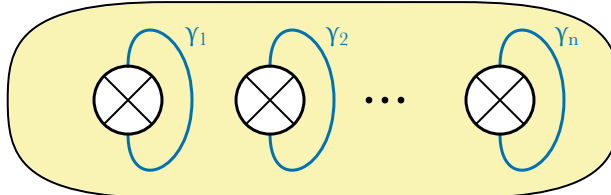
Cutting Σ_g open along the γ_i yields a connected surface $\hat{\Sigma}$ with $2g$ boundary components. Gluing disks to each of these $2g$ boundary components yields $\mathbb{S}^2$; for instance, see here for the case $g = 3$:



Applying Lemmas B.4.8 and B.4.9, we deduce that $\chi(\Sigma_g) = \chi(\hat{\Sigma}) = \chi(\mathbb{S}^2) - 2g = 2 - 2g$. $\square$

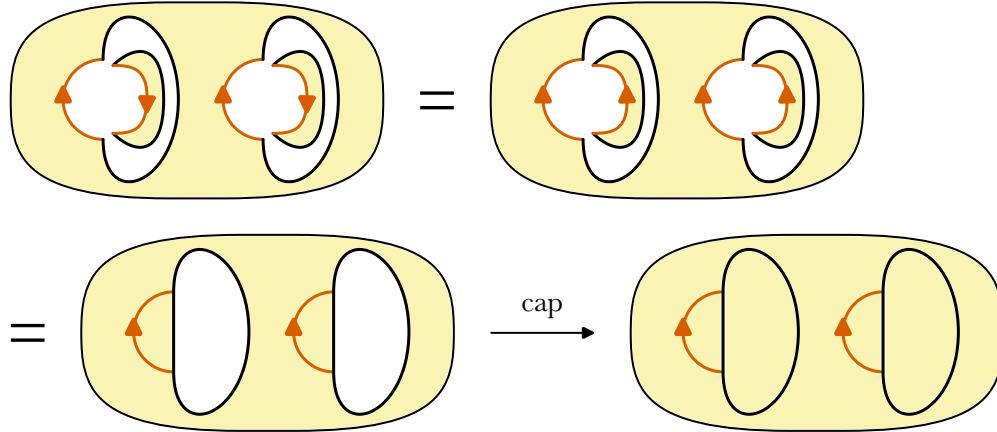
LEMMA B.4.11. *For $n \geq 1$, we have $\chi(\Sigma_n^{\text{no}}) = 2 - n$.*

PROOF. Let $\gamma_1, \dots, \gamma_n$ be the following one-sided curves on Σ_n^{no} :



Cutting Σ_n^{no} open along the γ_i yields a connected surface $\hat{\Sigma}$ with n boundary components. Gluing

disks to each of these n boundary components yields $\mathbb{S}^2$; for instance, see here for the case $n = 2$:



Applying Lemmas B.4.8 and B.4.9, we deduce that $\chi(\Sigma_n^{\text{no}}) = \chi(\widehat{\Sigma}) = \chi(\mathbb{S}^2) - n = 2 - n$. $\square$

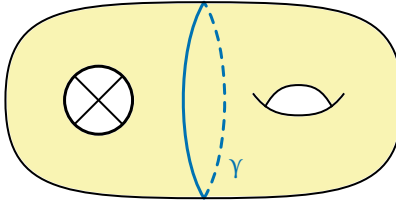
B.4.9. Refined classification. Recall that the classification theorem for surfaces says that every closed connected surface is homeomorphic to some Σ_g or Σ_n^{no} . Since $\chi(\Sigma_g) = 2 - 2g$ and $\chi(\Sigma_n^{\text{no}}) = 2 - n$, this will imply that the homeomorphism type of a closed connected surface is completely determined by its Euler characteristic and whether or not it is orientable. We state this refined version of the classification as follows:

THEOREM B.4.12 (Classification of surfaces). *Let Σ be a closed connected surface. Then:*

- *If Σ is orientable then $\Sigma \cong \Sigma_g$, where $g \geq 0$ satisfies $\chi(\Sigma) = 2 - 2g$. In particular, $\chi(\Sigma)$ is even.*
- *If Σ is non-orientable then $\Sigma \cong \Sigma_n^{\text{no}}$, where $n \geq 1$ satisfies $\chi(\Sigma) = 2 - n$.*

Using this, we can answer the question we posed after stating the first version of the classification (Theorem B.2.1):

EXAMPLE B.4.13. Consider the following surface Σ :



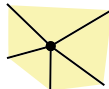
This surface is non-orientable since it contains a cross-cut. To calculate its Euler characteristic, let γ be the curve drawn above. The curve γ is two-sided, and when we cut Σ open along it and cap off the resulting two boundary components we get Σ_1 and Σ_1^{no} . It follows that

$$\chi(\Sigma) = \chi(\Sigma_1) + \chi(\Sigma_1^{\text{no}}) - 2 = 0 + 1 - 2 = -1.$$

By Theorem B.4.12, we deduce that $\Sigma \cong \Sigma_3^{\text{no}}$. $\square$

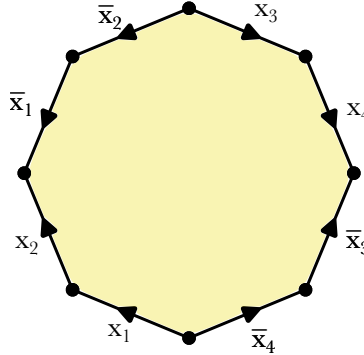
REMARK B.4.14. Many proofs of the classification of surfaces require the homeomorphism $\Sigma \cong \Sigma_3^{\text{no}}$ from Example B.4.13, which must be proved by hand. As we will see, our proof does not need this fact, so the above argument is not circular. $\square$

B.4.10. Polygons with sides identified. We will start the proof of Theorem B.4.12 soon, but first we give a few more examples of how it can be used. One natural way to build a surface is to take a polygon (or several polygons) in $\mathbb{R}^2$ and glue its sides together. We saw several examples of this already in Example B.4.5. If each side is glued to exactly one other side, then the result will always be a surface. Indeed, the only place where it is not obviously a surface is at the vertices, and a neighborhood of a vertex looks like this:

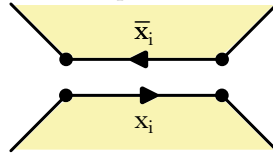


This surface has a natural polygonal decomposition where the glued-together boundary forms the 1-skeleton and the polygon is a single face (or multiple faces if there are multiple polygons). Here are some example of how to use Theorem B.4.12 to identify the resulting surface:

EXAMPLE B.4.15. Let Σ be an octagon with sides identified as follows:



Here we have labeled the sides with letters and oriented them to show how they should be glued. The bars over the letters indicate a reversed orientation on the edge. The surface Σ is orientable since the gluing respects the orientation of the plane:



All the vertices are identified to a single vertex, and the boundary edges glue to 4 edges. Since there is a single face, we see that $\chi(\Sigma) = 1 - 4 + 1 = -2$. It follows that $\Sigma \cong \Sigma_2$. $\square$

REMARK B.4.16. A nice way to give a gluing pattern on the boundary of a $2n$ -gon is to label the paired edges by letters $x_1, \dots, x_n$. You then give a word in letters $\{x_1, \bar{x}_1, \dots, x_n, \bar{x}_n\}$ obtained by going around the polygon clockwise starting from some vertex and recording which edge-labels appear, with a bar indicating that the edge is traversed in the opposite orientation. For instance, in Example B.4.15 the word would be $x_1 x_2 \bar{x}_1 \bar{x}_2 x_3 x_4 \bar{x}_3 \bar{x}_4$. For each $1 \leq i \leq n$, the letter x_i should appear twice (each time possibly with a bar over it). $\square$

EXAMPLE B.4.17. Generalizing Example B.4.15, let Σ be a $4g$ -gon with sides identified according to the pattern

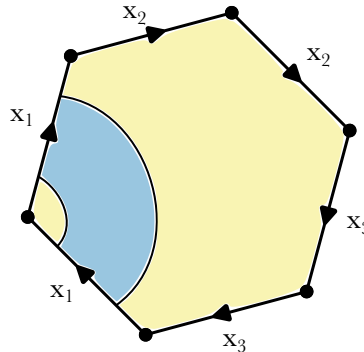
$$x_1 x_2 \bar{x}_1 \bar{x}_2 \cdots x_{2g-1} x_{2g} \bar{x}_{2g-1} \bar{x}_{2g}.$$

Again, all the vertices get identified to a single vertex and there are $2g$ edges and one face, so $\chi(\Sigma) = 1 - 2g + 1 = 2 - 2g$. We thus have $\Sigma \cong \Sigma_g$. $\square$

EXAMPLE B.4.18. Let Σ be a $2n$ -gon with sides identified according to the pattern

$$x_1 x_1 x_2 x_2 \cdots x_n x_n.$$

For instance, for $n = 3$ this is



The blue strip here glues up to a Möbius band, so Σ is non-orientable. All the vertices get identified

to a single vertex and there are n edges and one face, so $\chi(\Sigma) = 1 - n + 1 = 2 - n$. We thus have $\Sigma \cong \Sigma_n^{\text{no}}$. $\square$

B.5. The two-dimensional Poincaré conjecture

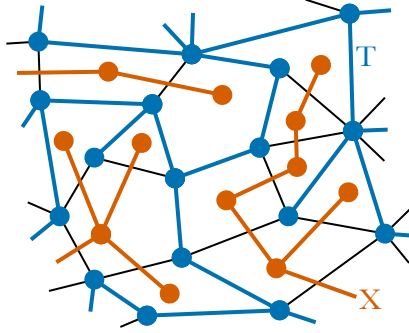
We now start the proof of the classification by proving the two-dimensional Poincaré conjecture, which says that $\mathbb{S}^2$ is the unique connected closed surface with Euler characteristic 2:

THEOREM B.5.1 (Two-dimensional Poincaré conjecture). *Let Σ be a closed connected surface. Then $\chi(\Sigma) \leq 2$, with equality if and only if $\Sigma \cong \mathbb{S}^2$.*

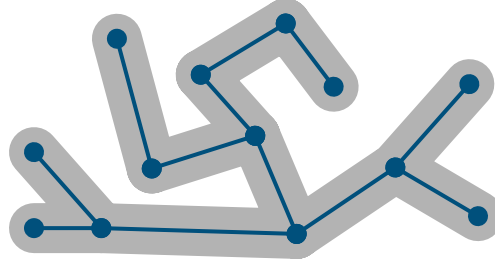
PROOF. Fix a polygonal decomposition of Σ . Its 1-skeleton is a finite graph embedded in Σ . Let T be a maximal tree in the 1-skeleton. Next, let X the following graph, which one can view as a sort of “dual graph” to T :

- Put a vertex of X in the center of each polygon of our polygonal decomposition.
- Connect two vertices of X if their associated polygons share an edge that does not lie in T .

See here:



We claim that the graph X is connected. Equivalently, removing T does not disconnect Σ . The key to this is the fact that a small neighborhood U of T is homeomorphic to $\mathbb{D}^2$:

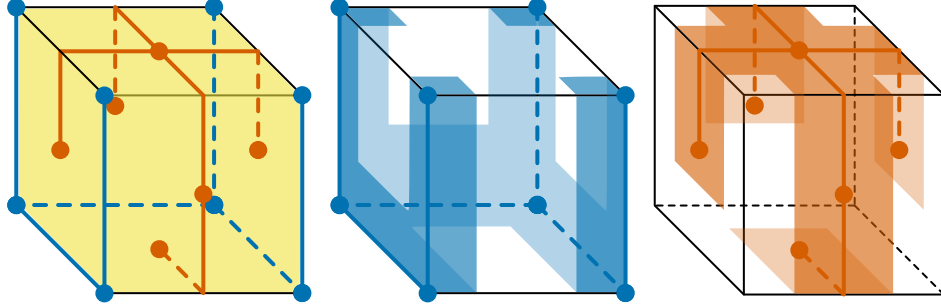


From this, we see that a path in Σ that crosses T can be re-routed to turn and follow the boundary of U until it gets to the other side of T , proving that T does not disconnect the surface.

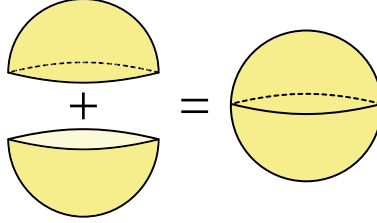
Lemma B.3.2 implies that $\chi(T) = 1$ and that $\chi(X) \leq 1$ with equality if and only if X is a tree. Since each vertex of Σ is a vertex of T , each polygon of Σ contains a unique vertex of X , and each edge of Σ is either an edge of T or is crossed by a unique edge of X , we have

$$\begin{aligned} \chi(\Sigma) &= |V(\Sigma)| - |E(\Sigma)| + |F(\Sigma)| = |V(T)| - (|E(T)| + |E(X)|) + |V(X)| \\ &= \chi(T) + \chi(X) \leq 1 + 1 = 2. \end{aligned}$$

This proves half of the theorem. Equality holds if and only if $\chi(X) = 1$, i.e, if X is a tree. Assume, therefore, that X is a tree. We must prove that $\Sigma \cong \mathbb{S}^2$. To see this, note that just like above if we slightly thicken T and X we get disks D_1 and D_2 . Choosing these thickenings carefully, we can ensure that D_1 and D_2 intersect in their boundaries. To help the reader understand this, here is an example of a polygonal decomposition of a surface where T and X are trees, along with D_1 and D_2 :



We deduce that Σ can be decomposed into two disks meeting along their boundaries:



It follows that $\Sigma \cong \mathbb{S}^2$. □

Before continuing with the classification, we pause to extract the following from the above proof:

COROLLARY B.5.2. *Let Σ be a closed connected surface such that $\chi(\Sigma) < 2$. Then there exists a simple closed curve γ on Σ that is nonseparating, i.e., such that $\Sigma \setminus \gamma$ is connected.*

PROOF. Fix a polygonal decomposition of Σ , and let T and X be as in the proof of Theorem B.5.1. Since $\chi(\Sigma) < 2$, it follows from the proof of Theorem B.5.1 that X is a connected graph that is *not* a tree. It therefore contains a cycle γ . We claim that γ is nonseparating.

In fact, even more is true: $\Sigma \setminus X$ is connected. To see this, note that any point of $\Sigma \setminus X$ can be connected by a path in $\Sigma \setminus X$ to a vertex of the polygonal decomposition. This vertex lies in the maximal tree T , and since T is connected we can follow a path in T to any other vertex of the polygonal decomposition. The claim follows. □

B.6. The classification of surfaces in general

We now come to the proof of the classification of surfaces, whose statement we recall:

THEOREM B.4.12 (Classification of surfaces). *Let Σ be a closed connected surface. Then:*

- *If Σ is orientable then $\Sigma \cong \Sigma_g$, where $g \geq 0$ satisfies $\chi(\Sigma) = 2 - 2g$. In particular, $\chi(\Sigma)$ is even.*
- *If Σ is non-orientable then $\Sigma \cong \Sigma_n^{no}$, where $n \geq 1$ satisfies $\chi(\Sigma) = 2 - n$.*

PROOF. Theorem B.5.1 says that $\chi(\Sigma) \leq 2$. The proof will be by reverse induction on $\chi(\Sigma)$. The base case is when $\chi(\Sigma) = 2$, in which case Theorem B.5.1 says that $\Sigma \cong \mathbb{S}^2 \cong \Sigma_0$. In particular, Σ must be orientable in this case, as claimed in the theorem.

Assume now that $\chi(\Sigma) < 2$ and that the theorem is true for closed connected surfaces with larger Euler characteristics. Corollary B.5.2 implies that there is a nonseparating simple closed curve γ on Σ . As we discussed in §B.4.6, the curve γ is either two-sided or one-sided. Let $\widehat{\Sigma}$ be the connected surface with boundary obtained by cutting Σ open along γ . There are four cases, with the first case being the only one that occurs for Σ orientable:

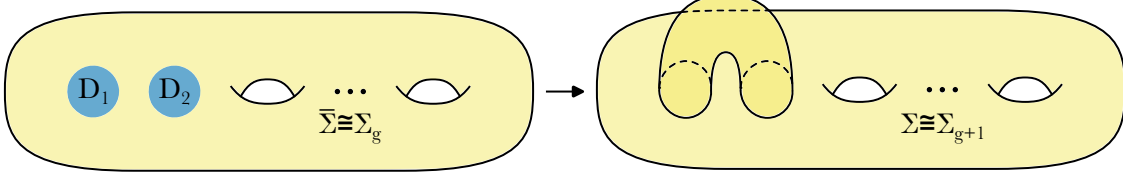
CASE 1. *γ is two-sided and $\widehat{\Sigma}$ is orientable.*

Since γ is two-sided, $\widehat{\Sigma}$ has two boundary components. Let $\overline{\Sigma}$ be the closed connected surface obtained from $\widehat{\Sigma}$ by gluing disks to both of its boundary components. Using Lemmas B.4.8 and B.4.9, we have

$$\chi(\overline{\Sigma}) = \chi(\widehat{\Sigma}) + 2 = \chi(\Sigma) + 2.$$

We can therefore apply our inductive hypothesis to $\bar{\Sigma}$. Since $\hat{\Sigma}$ is orientable, so is $\bar{\Sigma}$. It follows that $\bar{\Sigma} \cong \Sigma_g$, where $g \geq 0$ satisfies $\chi(\bar{\Sigma}) = 2 - 2g$. Since $\chi(\Sigma) = \chi(\bar{\Sigma}) - 2$, we have $\chi(\Sigma) = 2 - 2(g + 1)$. Our goal, therefore, is to prove that $\Sigma \cong \Sigma_{g+1}$.

To see this, note that Σ is obtained from $\bar{\Sigma} \cong \Sigma_g$ by removing two open disks D_1 and D_2 whose closures are disjoint and gluing together the resulting boundary components. In other words, Σ is obtained by attached a handle as follows:



It follows that $\Sigma \cong \Sigma_{g+1}$, as desired.

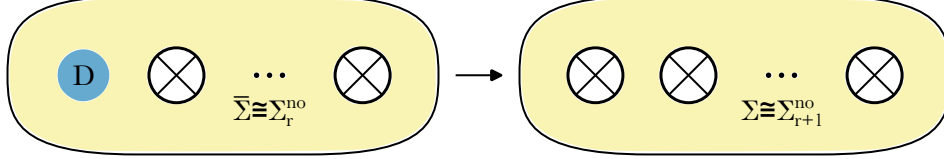
CASE 2. γ is one-sided and $\hat{\Sigma}$ is non-orientable.

Since γ is one-sided, $\hat{\Sigma}$ has one boundary component. Let $\bar{\Sigma}$ be the closed connected surface obtained from $\hat{\Sigma}$ by gluing disks to its boundary component. Using Lemmas B.4.8 and B.4.9, we have

$$\chi(\bar{\Sigma}) = \chi(\hat{\Sigma}) + 1 = \chi(\Sigma) + 1.$$

We can therefore apply our inductive hypothesis to $\bar{\Sigma}$. Since $\hat{\Sigma}$ is non-orientable, so is $\bar{\Sigma}$. It follows that $\bar{\Sigma} \cong \Sigma_n^{\text{no}}$, where $n \geq 1$ satisfies $\chi(\bar{\Sigma}) = 2 - n$. Since $\chi(\Sigma) = \chi(\bar{\Sigma}) - 1$, we have $\chi(\Sigma) = 2 - (n + 1)$. Our goal, therefore, is to prove that $\Sigma \cong \Sigma_{n+1}^{\text{no}}$.

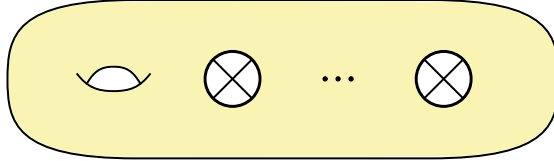
To see this, note that Σ is obtained from $\bar{\Sigma} \cong \Sigma_n^{\text{no}}$ by removing an open disk D and gluing in a Möbius band. In other words, Σ is obtained by adding a cross-cap as follows:



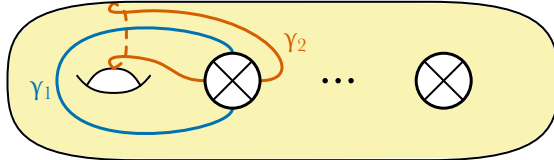
It follows that $\Sigma \cong \Sigma_{n+1}^{\text{no}}$, as desired.

CASE 3. γ is two-sided and $\hat{\Sigma}$ is non-orientable.

Following the argument in the previous two cases, the surface Σ has one genus and $n \geq 1$ cross-caps as follows:



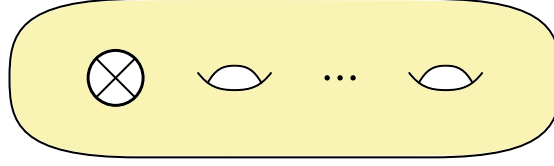
What we did wrong in this case was choose the wrong curve γ to cut along. Let γ_1 and γ_2 be as follows:



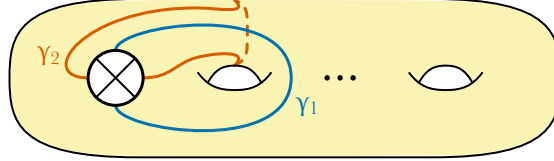
The curves γ_1 and γ_2 are both one-sided. Since γ_2 is one-sided and disjoint from γ_1 , it follows that the surface $\bar{\Sigma}'$ obtained by cutting along γ_1 is non-orientable. This reduces us to Case 2.

CASE 4. γ is one-sided and $\hat{\Sigma}$ is orientable.

This time, if we follow the argument from Cases 1 and 2 we get that Σ has 1 cross-cap and $g \geq 0$ genus as follows:



If $g = 0$ then $\Sigma \cong \Sigma_1^{\text{no}}$ and we are done. Otherwise, we can use the same trick we used in Case 3. Namely, let γ_1 and γ_2 be as follows:

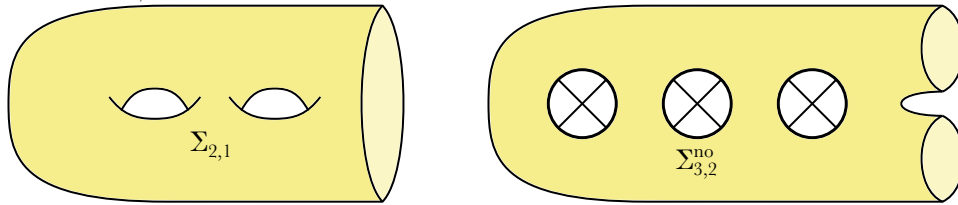


The curves γ_1 and γ_2 are both one-sided. Since γ_2 is one-sided and disjoint from γ_1 , it follows that the surface $\hat{\Sigma}'$ obtained by cutting along γ_1 is non-orientable. This reduces us to Case 2. $\square$

B.7. Extensions of the classification of surfaces

We close this essay by describing two extensions of the classification of surfaces: to compact surfaces with boundary, and to non-compact surfaces.

B.7.1. Compact surfaces with boundary. Let $\Sigma_{g,b}$ be genus- g surface Σ_g with b open disks removed and let $\Sigma_{n,b}^{\text{no}}$ be a non-orientable genus- n surface with b open disks removed. For instance,



Both $\Sigma_{g,b}$ and $\Sigma_{n,b}$ are compact surfaces with b boundary components. These are the only surfaces with boundary:

THEOREM B.7.1 (Classification of surfaces with boundary). *Let Σ be a compact connected surface with $b \geq 0$ boundary components. Then:*

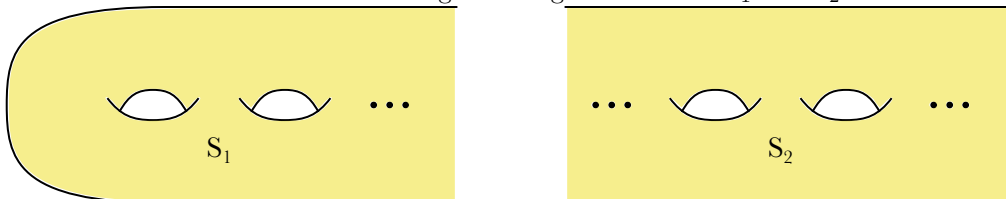
- If Σ is orientable then $\Sigma \cong \Sigma_{g,b}$, where $g \geq 0$ satisfies $\chi(\Sigma) = 2 - 2g - b$.
- If Σ is non-orientable then $\Sigma \cong \Sigma_{n,b}^{\text{no}}$, where $n \geq 1$ satisfies $\chi(\Sigma) = 2 - n - b$.

PROOF. Gluing disks to all the boundary components of Σ gives a closed connected surface $\bar{\Sigma}$ with $\chi(\bar{\Sigma}) = \chi(\Sigma) + b$. By the classification of surfaces (Theorem B.4.12), we either have $\bar{\Sigma} \cong \Sigma_g$ with $\chi(\bar{\Sigma}) = 2 - 2g$ or $\bar{\Sigma} \cong \Sigma_n^{\text{no}}$ with $\chi(\bar{\Sigma}) = 2 - n$ depending on whether or not $\bar{\Sigma}$ (and hence Σ) is orientable. The theorem follows. $\square$

B.7.2. Noncompact surfaces. One way to get a noncompact surface is to remove a finite number of points from the interior of a compact surface with boundary. This gives what is called a surface of *finite type*. However, non-compact surfaces can be much more complicated than this. Here are some examples.

EXAMPLE B.7.2. Let C be a Cantor set embedded in $\mathbb{S}^2$. Then $\mathbb{S}^2 \setminus C$ is a very complicated non-compact surface. $\square$

EXAMPLE B.7.3. Consider the following infinite-genus surfaces S_1 and S_2 :



The difference between them is that S_1 has genus going off to infinity only to the right, while S_2 has genus going off to infinity in both directions. These surfaces are not homeomorphic. One way to distinguish them is to note that for every compact subset $K_1 \subset S_1$, there is only one component C of $S_1 \setminus K_1$ such that $\overline{C}$ is noncompact. However, there exist compact subsets $K_2 \subset S_2$ such that $S_2 \setminus K_2$ has two such components. This can be formalized using the theory of what are called “ends” of a space. $\square$

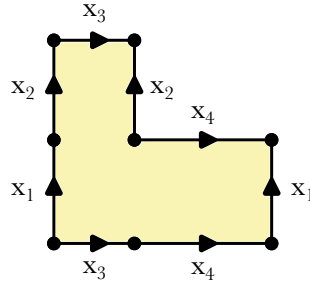
This might lead the reader to think that there is no hope of classifying noncompact surfaces. However, there is such a classification making use of “end data”. It was first stated by Kerékjártó [7, Chapter 5]. His proof had gaps, and the first correct proof was found by Richards [8].

REMARK B.7.4. Noncompact surfaces with boundary are even more complicated, especially if they have noncompact boundary components. However, a classification of them was found by Brown–Messer [3]. $\square$

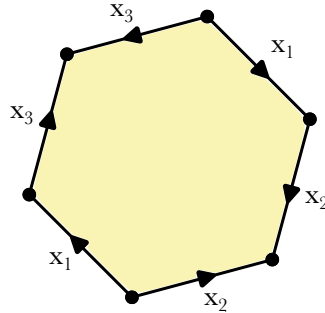
B.8. Exercises

EXERCISE B.1. Determine the surfaces by identifying sides of polygons as follows:

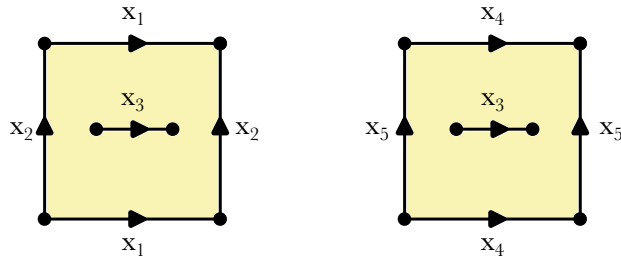
- (a) The L-shaped polygon with opposite sides identified here:



- (b) The hexagon with sides identified here:

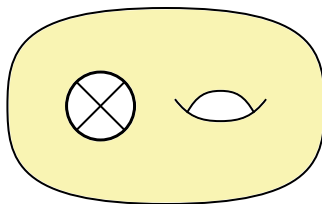


- (c) The two squares with slits here:



For this, make sure you are working with a polygonal decomposition. $\square$

EXERCISE B.2. As we discussed after stating Theorem B.4.12, the following surface is homeomorphic to Σ_3^{no} :

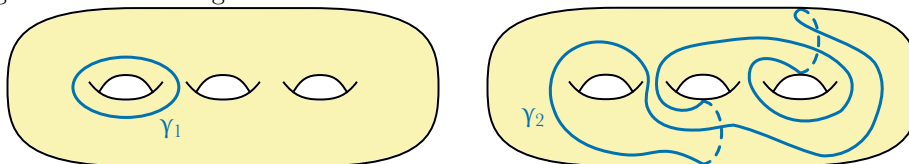


Prove this directly by decomposing both Σ_3^{no} and the above surface into the union of three Möbius bands and $\Sigma_{0,3}$ glued along their boundaries. $\square$

EXERCISE B.3. Using the fact that the Euler characteristic of $\mathbb{S}^2$ is 2, prove that any regular polyhedron in $\mathbb{R}^3$ is either a tetrahedron, a cube, an octahedron, a dodecahedron, or an icosahedron. This goes back to the ancient greeks, and appears in Euclid's Elements. $\square$

EXERCISE B.4. Using the fact that the Euler characteristic of $\mathbb{S}^2$ is 2, determine the number of components of the complement of n great circles on $\mathbb{S}^2$ no three of which pass through a common point. $\square$

EXERCISE B.5. Let γ_1 and γ_2 be two nonseparating simple closed curves on Σ_g . For instance, the γ_i might be the following:



Prove that there is a homeomorphism $f: \Sigma_g \rightarrow \Sigma_g$ such that $f(\gamma_1) = \gamma_2$. Hint: use the classification of surfaces with boundary to prove that the surfaces you get by cutting Σ_g open along the γ_i are homeomorphic. $\square$

EXERCISE B.6. Let $f: \tilde{\Sigma} \rightarrow \Sigma$ be a degree- d cover between closed connected surfaces. Prove that $\chi(\tilde{\Sigma}) = d\chi(\Sigma)$. Hint: Figure out how to lift a polygonal decomposition of Σ to one of $\tilde{\Sigma}$. $\square$

EXERCISE B.7. Let M be a 1-manifold. A *triangulation* of M is a closed discrete subset $V \subset M$ call the *vertices* such that each path-component E of $M \setminus V$ is homeomorphic to $(0, 1)$ and has compact closure. We will call these E the *edges* of the triangulation. Prove that M has a triangulation by following these steps:

- (a) First prove that there is a countable open cover $\{U_i \mid i \in I\}$ of M with the following properties:
 - For each $i \in I$, the closure $\overline{U}_i$ is a closed subset of M homeomorphic to $[0, 1]$ via a homeomorphism taking U_i to $(0, 1)$.
 - The set $\{\overline{U}_i \mid i \in I\}$ is locally finite, i.e., for each compact subset $K \subset M$ the set $\{i \in I \mid \overline{U}_i \cap K \neq \emptyset\}$ is finite.

We remark that this will use the fact that M is second countable and Hausdorff.

- (b) Letting $\{U_i \mid i \in I\}$ be as in (a), set

$$V = \bigcup_{i \in I} \overline{U}_i \setminus U_i.$$

Prove that V is the set of vertices of a triangulation of M . We remark that this will also use the fact that M is Hausdorff. $\square$

EXERCISE B.8. Let M be a 1-manifold. Fix a triangulation of M as in Exercise B.7. We will verify that M has local properties similar to those of polygonal decompositions of surfaces; cf. §B.4.4.

- (a) Prove that for each vertex v of the triangulation, there exists an open neighborhood U of v along with a homeomorphism $f: (-1, 1) \rightarrow U$ such that $f(0) = v$ and such that f takes both $(-1, 0)$ and $(0, 1)$ into open subsets of edges of the triangulation. These edges need not be distinct.

- (b) Let E be an edge of the triangulation. Prove that there is a homeomorphism $g: (0, 1) \rightarrow E$ that extends to a continuous map $G: [0, 1] \rightarrow M$ with $G(0)$ and $G(1)$ vertices of the triangulation. Hint: This exercise will require you to use the fact that manifolds are Hausdorff. $\square$

EXERCISE B.9. Let M be a connected 1-manifold. Prove that M is homeomorphic to either $\mathbb{S}^1$ or $\mathbb{R}$. Hint: use a triangulation as in Exercises B.7 and B.8. We remark that this relies on the fact that manifolds are second countable and Hausdorff. Without these conditions there are many more examples of connected 1-manifolds. Even assuming second countability, as far as I am aware there is no reasonable classification of non-Hausdorff 1-manifolds. $\square$

EXERCISE B.10. Let Σ be either $\Sigma_{g,b}$ or $\Sigma_{n,b}^{\text{no}}$ with $b \geq 1$, let ∂ be a boundary component of Σ , and let $f: \partial \rightarrow \partial$ be a homeomorphism. In this exercise, you will prove that f extends to a homeomorphism $F: \Sigma \rightarrow \Sigma$.

- (a) Prove that if $\phi: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is an orientation-preserving homeomorphism, then there is a homotopy $\phi_t: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ such that $\phi_0 = \phi$ and $\phi_1 = \text{id}_{\mathbb{S}^1}$ and such that each ϕ_t is a homeomorphism. Hint: First reduce to the case where $\phi(1) = 1$. Identify ϕ with a path $\psi: I \rightarrow \mathbb{S}^1$ with $\psi(0) = \psi(1) = 1$, and then try lifting ϕ to the universal cover $p: \mathbb{R} \rightarrow \mathbb{S}^1$ of $\mathbb{S}^1$.
- (b) Using (a), prove the exercise for f orientation-preserving. Hint: Use the fact⁶ that ∂ has a *collar neighborhood*, i.e., an embedding $\iota: \partial \times [0, 1] \rightarrow \Sigma$ such that $\iota(x, 0) = x$ for all $x \in \partial$.
- (c) Conclude by proving the exercise for f orientation-reversing. Hint: using (b), show that it is enough to exhibit a single orientation-reversing homeomorphism $\phi: \Sigma \rightarrow \Sigma$ with $\phi(\partial) = \partial$. $\square$

EXERCISE B.11. Let Σ and Σ' be compact connected surfaces with boundary. Assume that Σ is either $\Sigma_{g,b}$ or $\Sigma_{n,b}^{\text{no}}$ with $b \geq 1$. Let ∂ be a boundary component of Σ and let ∂' be a boundary component of Σ' . Let $f_1, f_2: \partial \rightarrow \partial'$ be two homeomorphisms and let S_i be the result of gluing $\partial \subset \Sigma$ to $\partial' \subset \Sigma'$ using f_i . Prove that S_1 is homeomorphic to S_2 . Hint: Exercise B.10 will be helpful. $\square$

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<https://www.maths.ed.ac.uk/~v1ranick/surgery/zeeman.pdf>

⁶This fact clearly holds for the Σ in the exercise. In fact, a standard theorem in manifold topology shows that collar neighborhoods exist for boundary components of arbitrary compact manifolds.

ESSAY C

Dimension theory (to be written)

Part 3

Essays on classical geometric results

The Brouwer fixed point theorem

This chapter gives an elementary proof of the Brouwer fixed point theorem based on Sperner's Lemma, which is a classic result about the combinatorics of triangulations.

D.1. Simplices and triangulations

Let $P \subset \mathbb{R}^d$ be a nonempty finite subset. Enumerate P as $P = \{\mathbf{p}_1, \dots, \mathbf{p}_k\}$. An *affine linear combination* of the points of P is a point in $\mathbb{R}^d$ of the form

$$\lambda_1 \mathbf{p}_1 + \dots + \lambda_k \mathbf{p}_k \in \mathbb{R}^d \quad \text{with } \lambda_1, \dots, \lambda_k \in \mathbb{R} \text{ such that } \lambda_1 + \dots + \lambda_k = 1.$$

The *affine hull* of P , denoted $\text{Aff}(P)$, is the collection of all points that are affine linear combinations of the points of P . We say that P is *affinely dependent* if a point can be expressed as an affine linear combination of points of P in two different ways. Here are two equivalent ways of expressing this (see Exercise D.1):

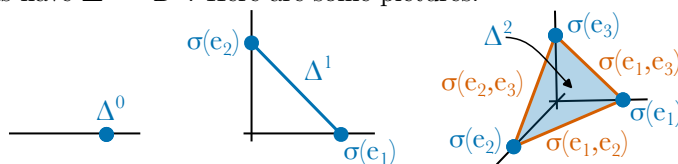
- There exist $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ with $\lambda_1 + \dots + \lambda_k = 0$ and $(\lambda_1, \dots, \lambda_k) \neq (0, \dots, 0)$ such that $\lambda_1 \mathbf{p}_1 + \dots + \lambda_k \mathbf{p}_k = 0$.
- The vectors $\{\mathbf{p}_2 - \mathbf{p}_1, \dots, \mathbf{p}_k - \mathbf{p}_1\}$ in $\mathbb{R}^d$ are linearly dependent.

If P is affinely dependent, then there exists some $1 \leq i \leq k$ such that $\text{Aff}(P \setminus \mathbf{p}_i) = \text{Aff}(P)$. If P is not affinely dependent, then we say that P is *affinely independent*. The *convex hull* of P is the subspace

$$\sigma(P) = \{\lambda_1 \mathbf{p}_1 + \dots + \lambda_k \mathbf{p}_k \mid \lambda_1, \dots, \lambda_k \geq 0 \text{ and } \lambda_1 + \dots + \lambda_k = 1\}$$

of $\text{Aff}(P)$. A k -*simplex* is a subspace of $\mathbb{R}^d$ of the form $\sigma(P)$ with P an affinely independent set with $|P| = k + 1$. We will just call this a *simplex* if we do not want to specify k . A k -simplex is homeomorphic to $\mathbb{D}^k$ (see Exercise D.2). The *faces* of $\sigma(P)$ are the subspaces of the form $\sigma(P')$ with $P' \subset P$ nonempty. The *dimension* of a face $\sigma(P')$ is its dimension as a simplex, namely $|P'| - 1$.

Letting $B = \{\mathbf{e}_1, \dots, \mathbf{e}_{n+1}\}$ be the standard basis for $\mathbb{R}^{n+1}$, the *standard n -simplex* is the simplex $\Delta^n = \sigma(B)$. We thus have $\Delta^n \cong \mathbb{D}^n$. Here are some pictures:

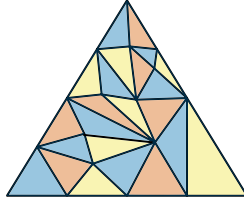


The different faces are indicated.

A *triangulation* of a subspace X of $\mathbb{R}^d$ is a decomposition X into a union of simplices with the following three properties:

- If σ is a simplex of the triangulation, then so are all the faces of σ .
- If σ and σ' are two simplices of the triangulation, then $\sigma \cap \sigma'$ is either empty or is a face of both σ and σ' .
- For each point $p \in X$, there is an open neighborhood U of p that only intersects finitely many simplices in the triangulation. This is automatic if the triangulation has only finitely many simplices, which will be the case in all our examples in this chapter.

Here is an example of a triangulation of a 2-simplex:



Let $\mathfrak{d}$ be the standard distance function on $\mathbb{R}^d$. The *diameter* of a simplex σ is

$$\text{diam}(\sigma) = \max \{ \mathfrak{d}(x, y) \mid x, y \in \sigma \}.$$

The *mesh size* of the triangulation is the maximal diameter of any of its simplices. We have:

LEMMA D.1.1. *Fix $n \geq 1$. For all $\epsilon > 0$, there exists a triangulation of Δ^n whose mesh size is at most ϵ .*

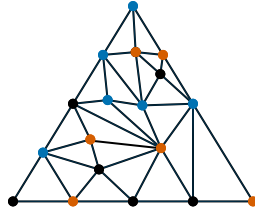
PROOF. See Exercise D.3. □

D.2. Sperner's Lemma

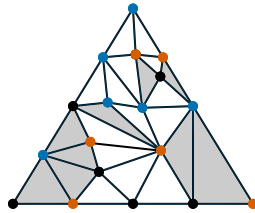
The key to our proofs is a simple combinatorial property of triangulations of simplices. Let $B = \{\mathbf{e}_1, \dots, \mathbf{e}_{n+1}\}$ be the standard basis for $\mathbb{R}^{n+1}$, so $\Delta^n = \sigma(B)$. Fix a triangulation of Δ^n , and let V be the set of 0-simplices of this triangulation (the *vertices*). A *Sperner coloring* of this triangulation is a set map $s: V \rightarrow \{1, \dots, n+1\}$ such that the following holds for each $v \in V$:

- Assume that v lies in a face $\sigma(B')$ for some $B' \subset B$. Write $B' = \{\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_k}\}$. Then $s(v) \in \{i_1, \dots, i_k\}$.

We have $\mathbf{e}_i \in V$, and this rule implies in particular that $s(\mathbf{e}_i) = i$. We view the labels $\{1, \dots, n+1\}$ as “colors” for the vertices. Here is an example of a Sperner coloring of a triangulation of Δ^2 , with the vertices colored with the colors black and blue and orange to mean that they are labeled with 1 and 2 and 3:



A *rainbow n -simplex* of s is an n -simplex $\sigma(v_1, \dots, v_{n+1})$ such that $\{s(v_1), \dots, s(v_{n+1})\} = \{1, \dots, n+1\}$. In the following the rainbow n -simplices are shaded:



Sperner's lemma says that rainbow n -simplices always exist. In fact, even more is true:

LEMMA D.2.1 (Sperner's Lemma). *Fix a triangulation of Δ^n with vertex set V . Let $s: V \rightarrow \{1, \dots, n+1\}$ be a Sperner coloring. Then there are an odd number of rainbow n -simplices of s .*

PROOF. The proof will be by induction on n . The base case $n = 0$ is trivial since Δ^0 consists of a single point. Assume now that $n \geq 1$ and that the lemma is true for Δ^{n-1} .

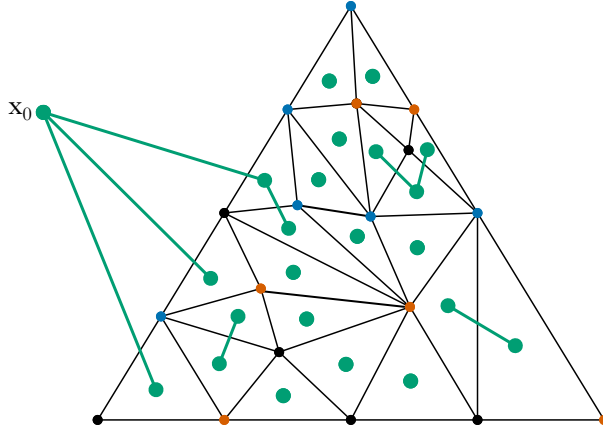
Call an $(n-1)$ -simplex $\sigma(v_1, \dots, v_n)$ of the triangulation a rainbow $(n-1)$ -simplex if we have $\{s(v_1), \dots, s(v_n)\} = \{1, \dots, n\}$. In particular, none of the $s(v_i)$ equal $n+1$. Identify $\Delta^{n-1} \subset \mathbb{R}^n$ with the face of $\Delta^n \subset \mathbb{R}^{n+1}$ that is the convex hull of the first n standard basis vectors of $\mathbb{R}^{n+1}$. Our triangulation restricts to a triangulation of Δ^{n-1} , and the Sperner coloring of our triangulation

restricts to a Sperner coloring of Δ^{n-1} . Our inductive hypothesis thus implies that there are an odd number of rainbow $(n-1)$ -simplices in Δ^{n-1} .

Recall that a graph G is a collection of vertices $\mathcal{V}(G)$ connected by edges $\mathcal{E}(G)$. Let G be the following graph:

- The vertices $\mathcal{V}(G)$ consist of the n -simplices σ of the triangulation plus a single additional vertex x_0 .
- The edges $\mathcal{E}(G)$ consist of the rainbow $(n-1)$ -simplices of the triangulation. The edge corresponding to a rainbow $(n-1)$ -simplex f connects the following two vertices:
 - If f is a face of two n -simplices σ and σ' of the triangulation (so f lies in the “interior” of Δ^n), then f connects the vertices corresponding to σ and σ' .
 - If f is the face of only one n -simplex σ of the triangulation (so f lies in the “boundary” of Δ^n), then f connects the vertex corresponding to σ to x_0 .

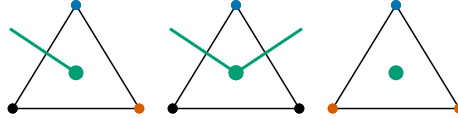
This graph can be visualized by placing a vertex of G in the center of each n -simplex of the triangulation and the vertex x_0 outside of Δ^n and then connecting the vertices according to the above rule. Here is an example, with G drawn in green:



The reader will notice that many vertices of G have no edges coming out of them. As we will soon see, this is not atypical.

Recall that the *degree* of a vertex x of a graph, denoted $\deg(x)$, is the number of edges coming out of x . By definition, $\deg(x_0)$ is the number of rainbow $(n-1)$ -simplices f that are the face of exactly one simplex of the triangulation. Since our coloring is a Sperner coloring, a rainbow $(n-1)$ -simplex f must lie on the face Δ^{n-1} of Δ^n . As we discussed above, our inductive hypothesis therefore implies that $\deg(x_0)$ is odd.

Let $\mathcal{V}_n(G)$ be the set of vertices corresponding to n -simplices, so $\mathcal{V}(G) = \mathcal{V}_n(G) \cup \{x_0\}$. For $\sigma \in \mathcal{V}_n(G)$, we have $\deg(\sigma) \in \{0, 1, 2\}$ with $\deg(\sigma) = 1$ exactly when σ is a rainbow n -simplex (see Exercise D.4). Here are some possibilities when $n = 2$:



From this, we see that to show there are an odd number of rainbow n -simplices, we must prove that

$$\sum_{\sigma \in \mathcal{V}_n(G)} \deg(\sigma) \equiv 1 \pmod{2}.$$

Since each edge connects two vertices, we have

$$\deg(x_0) + \sum_{\sigma \in \mathcal{V}_n(G)} \deg(\sigma) = \sum_{x \in \mathcal{V}(G)} \deg(x) = 2|\mathcal{E}(G)| \equiv 0 \pmod{2}.$$

It follows that

$$\sum_{\sigma \in \mathcal{V}_n(G)} \deg(\sigma) \equiv \deg(x_0) \equiv 1 \pmod{2},$$

as desired. $\square$

D.3. Brouwer fixed point theorem

We now use Sperner's Lemma to prove the following classic result:

THEOREM D.3.1 (Brouwer fixed point theorem). *Let $f: \mathbb{D}^n \rightarrow \mathbb{D}^n$ be a map. Then there exists some $x \in \mathbb{D}^n$ such that $f(x) = x$.*

PROOF. Identify $\mathbb{D}^n$ with the standard n -simplex

$$\Delta^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^n \mid x_1, \dots, x_{n+1} \geq 0 \text{ and } x_1 + \dots + x_{n+1} = 1\};$$

see Exercise D.2. Under this identification, f is a map $f: \Delta^n \rightarrow \Delta^n$. Assume for the sake of contradiction that $f(\mathbf{x}) \neq \mathbf{x}$ for all $\mathbf{x} \in \Delta^n$. We will use this assumption to find $\mathbf{z} \in \Delta^n$ with $f(\mathbf{z}) = \mathbf{z}$ (which is a contradiction, but also what we were trying to find in the first place!).

Define a non-continuous set map

$$s: \Delta^n \rightarrow \{1, \dots, n+1\}$$

as follows. Consider $\mathbf{x} = (x_1, \dots, x_{n+1}) \in \Delta^n$, and let $f(\mathbf{x}) = (y_1, \dots, y_{n+1})$. Since $f(\mathbf{x}) \neq \mathbf{x}$ and the x_i and y_j are nonnegative real numbers satisfying

$$x_1 + \dots + x_{n+1} = y_1 + \dots + y_{n+1} = 1,$$

there must be some $1 \leq i_0 \leq n+1$ such that $y_{i_0} < x_{i_0}$. Choose some such i_0 , and define $s(\mathbf{x}) = i_0$.

The map s is sort of like a Sperner coloring in the following sense. Let $B = \{\mathbf{e}_1, \dots, \mathbf{e}_{n+1}\}$ be the standard basis for $\mathbb{R}^{n+1}$. Then:

- (♠) Consider $\mathbf{x} \in \Delta^n$ such that $\mathbf{x}$ lies in a face $\sigma(B')$ for some $B' \subset B$. Write $B' = \{\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_k}\}$. Then $s(\mathbf{x}) \in \{i_1, \dots, i_k\}$.

Indeed, for $j \notin \{i_1, \dots, i_k\}$ the $\mathbf{e}_j$ -coordinate of $\mathbf{x}$ is 0, so f cannot decrease the $\mathbf{e}_j$ -coordinate of $\mathbf{x}$.

For each triangulation of Δ^n , it follows from (♠) that the restriction of s to the vertices of the triangulation is a Sperner coloring. Sperner's Lemma (Lemma D.2.1) implies that our triangulation has a rainbow n -simplex. For each $\ell \geq 1$, apply this to a triangulation with mesh size less than $1/\ell$ (see Lemma D.1.1) and let $\mathbf{x}(\ell)$ be a point in this rainbow n -simplex. Letting $\mathfrak{d}$ be the standard distance function on $\mathbb{R}^{n+1}$, for all $\ell \geq 1$ we then have:

- (♣) For each $1 \leq i \leq n+1$, there is a point $\mathbf{x}' \in \Delta^n$ with $\mathfrak{d}(\mathbf{x}(\ell), \mathbf{x}') < 1/\ell$ such that $s(\mathbf{x}') = i$. By the Heine–Borel theorem (Theorem 7.5.1), the n -simplex Δ^n is compact and thus sequentially compact. Some subsequence of the sequence $\{\mathbf{x}(\ell)\}_{\ell \geq 1}$ therefore converges to a point $\mathbf{z}$. For all $\epsilon > 0$ and $1 \leq i \leq n+1$, it follows from (♣) that the following holds:

- (♦) There is a point $\mathbf{z}' \in \Delta^n$ with $\mathfrak{d}(\mathbf{z}, \mathbf{z}') < \epsilon$ such that $s(\mathbf{z}') = i$, i.e., such that the $\mathbf{e}_i$ -coordinate of $f(\mathbf{z}')$ is less than the $\mathbf{e}_i$ -coordinate of $\mathbf{z}'$.

Let $\mathbf{z} = (z_1, \dots, z_{n+1})$ and $f(\mathbf{z}) = (w_1, \dots, w_{n+1})$. From (♦), we deduce that $w_i \leq z_i$ for all $1 \leq i \leq n+1$. Since the z_i and w_i are nonnegative numbers satisfying

$$z_1 + \dots + z_{n+1} = w_1 + \dots + w_{n+1} = 1,$$

this implies that $w_i = z_i$ for all $1 \leq i \leq n+1$, i.e., that $f(\mathbf{z}) = \mathbf{z}$, as desired. $\square$

If X is a space and $Y \subset X$ is a subspace, then a *retraction* of X to Y is a map $r: X \rightarrow Y$ such that $r(y) = y$ for all $y \in Y$, i.e., such that $r|_Y = \mathbb{1}_Y$. Note that $\mathbb{S}^{n-1} \subset \mathbb{D}^n$. The Brouwer fixed point theorem implies the following:

COROLLARY D.3.2. *For $n \geq 1$, there does not exist a retraction $r: \mathbb{D}^n \rightarrow \mathbb{S}^{n-1}$.*

PROOF. Assume a retraction $r: \mathbb{D}^n \rightarrow \mathbb{S}^{n-1}$ exists. Let $\alpha: \mathbb{S}^{n-1} \rightarrow \mathbb{S}^{n-1}$ be the antipodal map:

$$\alpha(\mathbf{x}) = -\mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{S}^{n-1} \subset \mathbb{R}^n.$$

Since α has no fixed points, it follows that the map $\alpha \circ r: \mathbb{D}^n \rightarrow \mathbb{D}^n$ has no fixed points, contradicting the Brouwer fixed point theorem. $\square$

REMARK D.3.3. Almost every tool in algebraic topology can be used to prove the Brouwer fixed point theorem. Most of those proofs actually first establish Corollary D.3.2. See Exercise D.5 for how to derive the Brouwer fixed point theorem from Corollary D.3.2. $\square$

D.4. Exercises

EXERCISE D.1. Let $P = \{p_1, \dots, p_k\}$ be a finite set of points in $\mathbb{R}^n$. Prove that the following are equivalent:

- The set P is affinely dependent.
- There exist $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ with $\lambda_1 + \dots + \lambda_k = 0$ and $(\lambda_1, \dots, \lambda_k) \neq (0, \dots, 0)$ such that $\lambda_1 \mathbf{p}_1 + \dots + \lambda_k \mathbf{p}_k = 0$.
- The vectors $\{\mathbf{p}_2 - \mathbf{p}_1, \dots, \mathbf{p}_k - \mathbf{p}_1\}$ in $\mathbb{R}^n$ are linearly dependent. □

EXERCISE D.2. Prove that a k -simplex is homeomorphic to $\mathbb{D}^k$. □

EXERCISE D.3. Fix $n \geq 1$ and $\epsilon > 0$. Prove that there exists a triangulation of Δ^n whose mesh size is at most ϵ . □

EXERCISE D.4. Fix a Sperner coloring of a triangulation of Δ^n . Let σ be an n -simplex of the triangulation, and let d be the number of $(n-1)$ -dimensional faces of σ that are rainbow $(n-1)$ -simplices. Prove that $d \in \{0, 1, 2\}$ with $d = 1$ exactly when σ is a rainbow n -simplex. □

EXERCISE D.5. Prove that Corollary [D.3.2](#) implies the Brouwer fixed point theorem by constructing from a map $f: \mathbb{D}^n \rightarrow \mathbb{D}^n$ with $f(\mathbf{x}) \neq \mathbf{x}$ for all $\mathbf{x} \in \mathbb{D}^n$ a retraction $r: \mathbb{D}^n \rightarrow \mathbb{S}^{n-1}$. Hint: for $\mathbf{x} \in \mathbb{D}^n$, define $r(\mathbf{x})$ to be the intersection point with $\mathbb{S}^{n-1}$ of a ray starting at $f(\mathbf{x})$ and passing through $\mathbf{x}$. □

The Jordan separation theorem

This essay proves the Jordan separation theorem. As an application, we prove that disks and spheres are never homeomorphic.

E.1. Jordan non-separation theorem

The easiest special case of the Jordan non-separation theorem is that no subspace $X \subset \mathbb{S}^n$ with $X \cong \mathbb{D}^0$ can separate $\mathbb{S}^n$ into multiple path components. Since $\mathbb{D}^0$ is a single point, another way to state this is that for all $p_0 \in \mathbb{S}^n$, the space $\mathbb{S}^n \setminus p_0$ is path connected. In fact, even more is true: $\mathbb{S}^n \setminus p_0 \cong \mathbb{R}^n$ (see Exercise E.2).

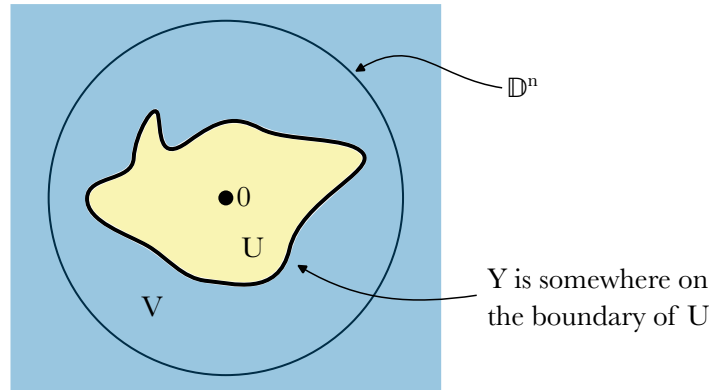
The general case allows X to be homeomorphic to an arbitrary disk. Since a disk can be embedded into $\mathbb{S}^n$ in many ways, this is a far more non-trivial fact. The proof is as follows:

THEOREM E.1.1 (Jordan non-separation theorem). *For some $n \geq 1$ and $m \geq 0$, let $X \subset \mathbb{S}^n$ be a closed subspace with $X \cong \mathbb{D}^m$. Then $\mathbb{S}^n \setminus X$ is path-connected.*

PROOF. Assume that $\mathbb{S}^n \setminus X$ is not path-connected and let U' be a path component of $\mathbb{S}^n \setminus X$. Set $V' = (\mathbb{S}^n \setminus X) \setminus U'$. Both U' and V' are open subsets of $\mathbb{S}^n$. Pick some $q_0 \in V'$. We have $\mathbb{S}^n \setminus q_0 \cong \mathbb{R}^n$ (see Exercise E.2). Pick a homeomorphism $h: \mathbb{S}^n \setminus q_0 \rightarrow \mathbb{R}^n$ and set $Y = h(X)$ and $U = h(U')$ and $V = h(V' \setminus q_0)$. The following hold (see Exercise E.2):

- Y is a closed subset of $\mathbb{R}^n$ satisfying $Y \cong \mathbb{D}^m$; and
- U is a path component of $\mathbb{R}^n \setminus Y$ that is bounded¹ and $V = (\mathbb{R}^n \setminus Y) \setminus U$.

Translating everything and rescaling, we can assume that $0 \in U$ and that U is contained in the unit disk $\mathbb{D}^n$. The picture is as follows:



Since $\mathbb{R}^n$ is locally path connected and $\mathbb{R}^n \setminus Y$ is an open subspace of $\mathbb{R}^n$, it follows that $\mathbb{R}^n \setminus Y$ is locally path-connected. This implies that the path components of $\mathbb{R}^n \setminus Y$ are the same as the connected components and that the connected components are both open and closed. It follows that U is both open and closed in $\mathbb{R}^n \setminus Y$. In particular, there is a closed set $C \subset \mathbb{R}^n$ such that

$$U = C \cap (\mathbb{R}^n \setminus Y) = C \setminus Y.$$

From this we see that

$$A = U \cup Y = (C \setminus Y) \cup Y = C \cup Y$$

¹That is, U is contained in some large ball around the origin.

is closed in $\mathbb{R}^n$. Similarly, the set $B = V \cup Y$ is closed. By construction, A and B are nonempty closed sets with $\mathbb{R}^n = A \cup B$ and $A \cap B = Y$.

Recall that $I = [0, 1]$. Since $Y \cong \mathbb{D}^m$ and $\mathbb{D}^m \cong I^m$ (see Exercise E.1), a map $f: \mathbb{R}^n \rightarrow Y$ can be viewed as an m -tuple of maps $\mathbb{R}^n \rightarrow I$. The space $\mathbb{R}^n$ is normal, so the Tietze extension applies to it. Applying the Tietze extension theorem m times, we can therefore extend the identity map $\mathbb{1}_Y: Y \rightarrow Y$ to a continuous map $f: \mathbb{R}^n \rightarrow Y$. Define $r: \mathbb{R}^n \rightarrow \mathbb{R}^n$ to be the map

$$r(p) = \begin{cases} f(p) & \text{if } p \in A, \\ p & \text{if } p \in B. \end{cases}$$

For $p \in A \cap B = Y$, we have $f(p) = p$, so this definition makes sense and defines a continuous function (see Exercise 2.5). By construction, r is a retraction from $\mathbb{R}^n$ to B . There also exists a retraction $r': \mathbb{D}^n \setminus 0 \rightarrow \mathbb{S}^{n-1}$ (see Exercise E.3). The composition

$$\mathbb{D}^n \xrightarrow{r|_{\mathbb{D}^n}} B \cap \mathbb{D}^n \xrightarrow{r'} \mathbb{S}^{n-1}$$

is then a retraction. Using the Brouwer fixed point theorem, we proved in Essay D that no such retraction exists (see Corollary D.3.2), so this is a contradiction. $\square$

REMARK E.1.2. Deeper than Theorem E.1.1 is the Jordan separation theorem, which says that for any closed subspace $X \subset \mathbb{S}^n$ with $X \cong \mathbb{S}^{n-1}$ the space $\mathbb{S}^n \setminus X$ has two path components. It is not hard to see that this implies a similar result with $\mathbb{S}^n$ replaced by $\mathbb{R}^n$. A special case of this is the Jordan curve theorem, which says that any embedded circle in $\mathbb{R}^2$ separates $\mathbb{R}^2$ into two path components.

I am not aware of any proof of the general high-dimensional Jordan separation theorem that does not use tools from algebraic topology. However, there are elementary proofs of the Jordan curve theorem. We will give one in Essay F. $\square$

E.2. Distinguishing spheres and disks

E.3. Exercises

EXERCISE E.1. Prove that $\mathbb{D}^k$ is homeomorphic to I^k . $\square$

EXERCISE E.2. WRITE IT!!! $\square$

EXERCISE E.3. WRITE IT!!! $\square$

ESSAY F

The Jordan curve theorem (to be written)

Part 4

Essays on classes of spaces

ESSAY G

CW complexes (to be written)

ESSAY H

Compactly generated spaces (to be written)

Part 5

Essays on groups and group actions

Quotients by group actions

This essay discusses the point-set topology of group actions. Our exposition is influenced by the paper [1], to which we refer the reader for many more results along these lines.

I.1. Group actions

Let G be a group and X be a space. In this chapter, actions of G on X are always assumed to be continuous. This means that a (left) action of G on X consists of a left action of G on the points of X such that for all $g \in G$ the left-multiplication map $m_g: X \rightarrow X$ defined by

$$m_g(x) = g \cdot x \quad \text{for } g \in G \text{ and } x \in X$$

is continuous. Since $m_1 = m_{g^{-1}g} = m_{g^{-1}}m_g$, the map m_g is necessarily a homeomorphism.

For a subset $Y \subset X$, let

$$G \cdot Y = \{g \cdot y \mid g \in G \text{ and } y \in Y\}.$$

A G -orbit of X is a set of the form $G \cdot x_0$ for some $x_0 \in X$. For $Y \subset X$, the set $G \cdot Y$ is a union of G -orbits. We write X/G for the space of orbits of G acting on X , equipped with the quotient topology. Letting $\pi: X \rightarrow X/G$ be the projection, a set $U \subset X/G$ is thus open if and only if $\pi^{-1}(U) \subset X$ is open. Here are some examples:

EXAMPLE I.1.1. The group $\mathbb{Z}$ acts on $\mathbb{R}$ by integer translations, and $\mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$. □

EXAMPLE I.1.2. The cyclic group C_2 of order 2 acts on $\mathbb{S}^n$ via the antipodal map. In other words, if $t \in C_2$ is the generator and $p \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$, then $t \cdot p = -p$. The quotient $\mathbb{S}^n/C_2$ is the space of pairs of antipodal points on $\mathbb{S}^n$. A line in $\mathbb{R}^{n+1}$ through the origin intersects $\mathbb{S}^n$ in a pair of antipodal points, so we can identify $\mathbb{S}^n/C_2$ with the space of lines through the origin in $\mathbb{R}^{n+1}$. This space of lines is called *real projective space*, and is denoted $\mathbb{RP}^n = \mathbb{S}^n/C_2$. □

I.2. Easy facts about quotients

Many of our results will depend on the following observation:

LEMMA I.2.1. *Let G be a group acting on a space X . Then the projection $\pi: X \rightarrow X/G$ is an open map.*

PROOF. See Exercise I.2. □

Using this, we see that many properties of spaces pass to quotients by group actions:

LEMMA I.2.2. *Let G be a group acting on a space X . The following hold:*

- (a) *If X is path connected (resp. connected), then X/G is path-connected (resp. connected).*
- (b) *If X is first countable (resp. second countable, resp. separable), then X/G is first countable (resp. second countable, resp. separable).*
- (c) *If X is compact, then X/G is compact.*
- (d) *If X is locally compact, then X/G is locally compact.*

PROOF. Let $\pi: X \rightarrow X/G$ be the projection. These are all consequences of the following facts:

- The map π is a continuous open map (Lemma I.2.1).
- Like all continuous maps, π takes compact sets to compact sets.

See Exercise I.3. □

Lemma I.2.2 does *not* say anything about Hausdorff or normal spaces, and as we will see even fairly tame group actions do not always preserve these properties.

I.3. Finite group actions

We start by observing that these separation properties are always preserved if G is finite:

LEMMA I.3.1. *Let X be a space and let G be a finite group acting on X . Then:*

- (i) *If X is Hausdorff, then X/G is Hausdorff.*
- (ii) *If X is normal, then X/G is normal.*

PROOF. We prove (i) and leave (ii) as Exercise I.4. Assume that X is Hausdorff. Let $\pi: X \rightarrow X/G$ be the quotient map. Consider $x, y \in X$ such that $\pi(x) \neq \pi(y)$. We must find disjoint open neighborhoods of $\pi(x)$ and $\pi(y)$. Let

$$S = G \cdot x = \{g \cdot x \mid g \in G\} \quad \text{and} \quad T = G \cdot y = \{g \cdot y \mid g \in G\}.$$

By assumption, S and T are disjoint finite subsets of X . Since X is Hausdorff, we can find open neighborhoods U of S and V of T with $U \cap V = \emptyset$ (see Exercise 6.1). For all $g \in G$, the open set $g \cdot U$ is an open neighborhood of S and the open set $g \cdot V$ is an open neighborhood of T . Letting

$$U' = \bigcap_{g \in G} g \cdot U \quad \text{and} \quad V' = \bigcap_{g \in G} g \cdot V,$$

it follows that U' is an open neighborhood of S and V' is an open neighborhood of T . By construction, we have $U' \cap V' = \emptyset$ and also $g \cdot U' = U'$ and $g \cdot V' = V'$ for all $g \in G$. Since the quotient map $\pi: X \rightarrow X/G$ is an open map (Lemma I.2.1), we conclude that $\pi(U')$ and $\pi(V')$ are disjoint open neighborhoods of $\pi(x)$ and $\pi(y)$, as desired. $\square$

I.4. Free actions, and why they can be pathological

Nothing like Lemma I.3.1 can be true for actions of infinite groups. Let G be a group acting on a space X . For $x \in X$, let G_x be the stabilizer of x , i.e.,

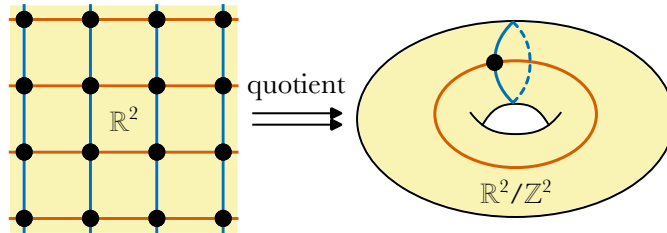
$$G_x = \{g \in G \mid g \cdot x = x\}.$$

One natural condition to impose on an action of a group G on a space X is that it is *free*, that is, that $G_x = 1$ for all $x \in X$. Here is an easy example showing that this can still be pathological:

EXAMPLE I.4.1. Let $\mathbb{Q}$ act by $\mathbb{R}$ by translations. Then $\mathbb{R}/\mathbb{Q}$ is not Hausdorff (see Exercise I.5). $\square$

A reader might think that the source of this pathology is that $\mathbb{Q}$ is “too large”. For instance, though $\mathbb{Q}$ is countable it is not finitely generated. Here is an example showing that even for the simplest infinite group $\mathbb{Z}$ we can still have pathological free quotients:

EXAMPLE I.4.2. Consider the 2-torus $\mathbb{T}^2 = (\mathbb{S}^1)^{\times 2}$. As the following figure shows, we can identify $\mathbb{T}^2$ with $\mathbb{R}^2/\mathbb{Z}^2$ with $\mathbb{Z}^2$ acting on $\mathbb{R}^2$ by integer translations:



Of course, this action is not pathological. Let $G = \mathbb{Z}$ with generator $t = 1$. Fix an irrational number $\alpha \in \mathbb{R}$, and define an action of G on $\mathbb{R}^2$ via the formula

$$t^n \cdot (x, y) = (x + n, y + n\alpha) \quad \text{for all } (x, y) \in \mathbb{R}^2 \text{ and } n \in \mathbb{Z}.$$

This action commutes with the action of $\mathbb{Z}^2$, and thus descends to an action of G on $\mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$. This action is free (see Exercise I.6). It turns out that all the G -orbits in $\mathbb{T}^2$ are dense and that $\mathbb{T}^2/G$ is not Hausdorff (see Exercise I.6). $\square$

I.5. Covering space actions, and why they can be pathological

From the discussion above, we see that we have to avoid dense orbits. One way to exclude them is as follows. An action of a group G on a space X is a *covering space action*¹ if for all $x \in X$ the following holds:

- There exists an open neighborhood U of x such that $U \cap g \cdot U = \emptyset$ for all $g \in G$ with $g \neq 1$. Equivalently, $g_1 \cdot U \cap g_2 \cdot U = \emptyset$ for all distinct $g_1, g_2 \in G$.

Covering space actions are clearly free. We have already seen many covering space actions:

- the action of $\mathbb{Z}$ on $\mathbb{R}$ by translations with $\mathbb{R}/\mathbb{Z} = \mathbb{S}^1$ (Example I.1.1); and
- the action of C_2 on $\mathbb{S}^n$ with $\mathbb{S}^n/C_2 = \mathbb{RP}^n$ (Example I.1.2); and
- the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ by translations with $\mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$ (Example I.4.2).

These examples might suggest that these actions are not pathological. Unfortunately, they can still have pathological quotients. Here is an example:

EXAMPLE I.5.1. Let $X = \mathbb{R}^2 \setminus 0$ and let $G = \mathbb{Z}$ with generator $t = 1$. Define an action of G on X as follows:

$$t^n \cdot (x, y) = (2^n x, 2^{-n} y) \quad \text{for all } (x, y) \in X \text{ and } n \in \mathbb{Z}.$$

This is a covering space action (see Exercise I.8). Let $\pi: X \rightarrow X/G$ be the quotient. The points $\pi(1, 0)$ and $\pi(0, 1)$ then do *not* have disjoint open neighborhoods (see Exercise I.8), so X/G is not Hausdorff. $\square$

REMARK I.5.2. Let $(M, \mathfrak{d})$ be a metric space and let G be a group acting on M . We say that G acts by *isometries* if

$$\mathfrak{d}(g \cdot x, g \cdot y) = \mathfrak{d}(x, y) \quad \text{for all } x, y \in M \text{ and } g \in G.$$

Below in Lemma I.7.2 we will prove that if G acts by isometries via a covering space action, then M/G is Hausdorff. In other words, we can avoid pathological examples like Example I.5.1 if we insist that our groups preserve a metric. $\square$

I.6. Proper actions

Consider a group G acting on a space X . Regard G as a discrete topological space, and consider the map $\sigma: G \times X \rightarrow X \times X$ defined by

$$\sigma(g, x) = (x, g \cdot x) \quad \text{for all } g \in G \text{ and } x \in X.$$

We call σ the *action map* of the group action. In terms of the action map, we have the following criterion for X/G to be Hausdorff:

LEMMA I.6.1. *Let G be a group acting on a space X . Assume that the image of the action map $\sigma: G \times X \rightarrow X \times X$ is closed. Then X/G is Hausdorff.*²

PROOF. Let $\pi: X \rightarrow X/G$ be the projection. Consider $x, y \in X$ with $\pi(x) \neq \pi(y)$. We must find disjoint open neighborhoods of $\pi(x)$ and $\pi(y)$. Let $E \subset X \times X$ be the image of σ , so E is closed. Since $\pi(x) \neq \pi(y)$, we have $(x, y) \notin E$. Since E is closed, we can find open neighborhoods U of x and V of y such that the open neighborhood $U \times V$ of (x, y) is disjoint from E . Since $U \times V$ is disjoint from E , the sets $\pi(U)$ and $\pi(V)$ are disjoint. Since π is an open map (Lemma I.2.1), the sets $\pi(U)$ and $\pi(V)$ are disjoint open neighborhoods of $\pi(x)$ and $\pi(y)$. $\square$

The action of G on X is a *proper action* if its action map is a proper map. We also say that G acts *properly* on X . As we will see below, there are simple criteria that imply this. Since proper maps are closed, Lemma I.6.1 implies:

LEMMA I.6.2. *Let G be a group acting properly on a space X . Then X/G is Hausdorff.*

The author of this book does not know if the following holds:

QUESTION I.6.3. Let G be a group acting properly on a normal space X . Is it necessarily the case that X/G is normal? $\square$

¹The terminology will be explained in a later volume when we discuss covering spaces.

²It is a little strange that we do not need to assume that X is Hausdorff.

However, the following result gives something stronger in many cases.

COROLLARY I.6.4. *Let G be a group acting properly on a second countable locally compact metrizable space X . Then X/G is a second countable locally compact metrizable space.*

PROOF. Lemma I.2.2 implies that X/G is second countable and locally compact. To see that it is metrizable, note that Lemma I.6.2 implies that X/G is Hausdorff. Since second countable locally compact Hausdorff spaces are regular (Lemma 8.2.1), the space X is regular and the Urysohn metrization theorem (Theorem 12.2.1) applies and shows it is metrizable. $\square$

REMARK I.6.5. This applies in particular if X is either an open or a closed subspace of $\mathbb{R}^n$. $\square$

REMARK I.6.6. The local compactness assumption can be removed if for some metric on X the group G acts by isometries. See Exercise I.9. $\square$

I.7. Criteria for proper actions

We now give two criteria that ensure that a group action is proper. Confusingly, these different criteria are both called “proper discontinuity” in the literature. The first criterion is:

LEMMA I.7.1. *Let X be a Hausdorff space that is either first countable or locally compact. Let G be a group acting on X such that the following holds:*

- *For all compact subsets $K \subset X$, the set $\{g \in G \mid K \cap g \cdot K \neq \emptyset\}$ is finite.*

Then G acts properly on X . Consequently, X/G is Hausdorff.

PROOF. Let $\sigma: G \times X \rightarrow X \times X$ be the action map and let $L \subset X \times X$ be compact. By Lemma 9.2.2, it is enough to show that $\sigma^{-1}(L)$ is compact. Let $\pi_1: X \times X \rightarrow X$ and $\pi_2: X \times X \rightarrow X$ be the two projections. Set $K = \pi_1(L) \cup \pi_2(L)$. The set K is compact and $L \subset K \times K$. Since X is Hausdorff, the set L is closed and thus $\sigma^{-1}(L)$ is a closed subset of $\sigma^{-1}(K \times K)$. It is therefore enough to prove that $\sigma^{-1}(K \times K)$ is compact. For this, it follows from the definition of σ that $\sigma^{-1}(K \times K) = \Lambda \times K$ with

$$\Lambda = \{g \in G \mid K \cap g \cdot K \neq \emptyset\}.$$

By assumption Λ is finite, so since K is compact it follows that $\Lambda \times K$ is compact, as desired. $\square$

For our second result, recall that a group G acts on a metric space $(M, \mathfrak{d})$ by isometries if

$$\mathfrak{d}(g \cdot x, g \cdot y) = \mathfrak{d}(x, y) \quad \text{for all } x, y \in M \text{ and } g \in G.$$

Also, for $x \in M$ and $r > 0$ recall that $B_r(x)$ denotes the open r -ball around x . We then have:

LEMMA I.7.2. *Let $(M, \mathfrak{d})$ be a metric space. Let G be a group acting on M by isometries such that the following holds:*

- *For all $x \in M$, there exists some $\epsilon > 0$ such that $\{g \in G \mid B_\epsilon(x) \cap g \cdot B_\epsilon(x) \neq \emptyset\}$ is finite.*

Then G acts properly on M . Consequently, M/G is Hausdorff.

REMARK I.7.3. Lemma I.7.2 applies in particular to actions by isometries on metric spaces that are covering space actions. $\square$

PROOF OF LEMMA I.7.2. Let $K \subset M$ be compact. Set $\Lambda = \{g \in G \mid K \cap g \cdot K \neq \emptyset\}$. By Lemma I.7.1, it is enough to prove that Λ is finite. For the sake of contradiction, assume that Λ is infinite. We can thus find a sequence $\{x_n\}_{n \geq 1}$ of points of K and a sequence $\{g_n\}_{n \geq 1}$ of distinct elements of Λ such that $g_n \cdot x_n \in K$ for all $n \geq 1$. Since K is a compact subset of the metric space M , it follows that K is sequentially compact. Passing to subsequences, we can therefore assume that there are $y, z \in K$ such that

$$\lim_{n \rightarrow \infty} x_n = y \quad \text{and} \quad \lim_{n \rightarrow \infty} g_n \cdot x_n = z.$$

Using our assumption, we can find some $\epsilon > 0$ such that

$$(I.7.1) \quad \{g \in G \mid B_\epsilon(y) \cap g \cdot B_\epsilon(y) \neq \emptyset\}$$

is finite. Choose $N > 0$ such that for $n \geq N$ we have $\mathfrak{d}(x_n, y) < \epsilon/2$ and $\mathfrak{d}(g_n \cdot x_n, z) < \epsilon/4$. For $n \geq N$, we thus have $x_n \in B_\epsilon(y)$. Also, using the fact that G acts by isometries we have

$$\begin{aligned} \mathfrak{d}(g_N^{-1} g_n \cdot x_n, y) &\leq \mathfrak{d}(g_N^{-1} g_n \cdot x_n, x_N) + \mathfrak{d}(x_N, y) \\ &= \mathfrak{d}(g_n \cdot x_n, g_N \cdot x_N) + \mathfrak{d}(x_N, y) \\ &\leq \mathfrak{d}(g_n \cdot x_n, z) + \mathfrak{d}(z, g_N \cdot x_N) + \mathfrak{d}(x_N, y) \\ &< \epsilon/4 + \epsilon/4 + \epsilon/2 = \epsilon, \end{aligned}$$

so $g_N^{-1} g_n \cdot x_n \in B_\epsilon(y)$. In other words, for $n \geq N$ we have $B_\epsilon(y) \cap g_N^{-1} g_n \cdot B_\epsilon(y) \neq \emptyset$. By assumption the $g_N^{-1} g_n$ are all distinct, so this contradicts the fact that (1.7.1) is finite. $\square$

I.8. Exercises

EXERCISE I.1. For the following groups G acting on spaces X , identify X/G :

- (a) The cyclic group $G = C_2$ of order 2 generated by $t \in C_2$ acting on $\mathbb{R}$ via

$$t \cdot x = -x \quad \text{for all } x \in \mathbb{R}.$$

- (b) The group $G = \mathbb{Z}$ generated by $t = 1$ acting on $X = \mathbb{R}^n \setminus 0$ by

$$t^n \cdot x = 2^n x \quad \text{for all } n \in \mathbb{Z} \text{ and } x \in X.$$

- (c) The cyclic group $G = C_2$ of order 2 generated by $t \in C_2$ acting on $X = \mathbb{S}^n$ via

$$t \cdot (x_1, \dots, x_{n+1}) = (-x_1, x_2, \dots, x_{n+1}) \quad \text{for all } (x_1, \dots, x_{n+1}) \in \mathbb{S}^n \subset \mathbb{R}^{n+1}. \quad \square$$

EXERCISE I.2. Let X be a space and let G be a group acting on X . Prove that the projection $\pi: X \rightarrow X/G$ is an open map. $\square$

EXERCISE I.3. Let G be a group acting on a space X . Prove the following:

- (a) If X is path connected, then X/G is path-connected.
- (b) If X is connected, then X/G is connected.
- (c) If X is first countable, then X/G is first countable.
- (d) If X is second countable, then X/G is second countable.
- (e) If X is separable, then X/G is separable.
- (f) If X is compact, then X/G is compact.
- (g) If X is locally compact, then X/G is locally compact. $\square$

EXERCISE I.4. Let X be a normal space and let G be a finite group acting on X . Prove that X/G is normal. $\square$

EXERCISE I.5. Let $\mathbb{Q}$ act by $\mathbb{R}$ by translations. Prove that $\mathbb{R}/\mathbb{Q}$ is not Hausdorff. $\square$

EXERCISE I.6. Let $G = \mathbb{Z}$ with generator $t = 1$. Fix an irrational number $\alpha \in \mathbb{R}$, and define an action of G on $\mathbb{R}^2$ via the formula

$$t^n \cdot (x, y) = (x + n, y + n\alpha) \quad \text{for all } (x, y) \in \mathbb{R}^2 \text{ and } n \in \mathbb{Z}.$$

This action commutes with the action of $\mathbb{Z}^2$, and thus descends to an action of G on $\mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$. Prove the following:

- (a) The action of G on $\mathbb{T}^2$ is free.
- (b) For all $p \in \mathbb{T}^2$, the G -orbit $G \cdot p = \{g \cdot p \mid g \in G\}$ is dense in $\mathbb{T}^2$.
- (c) The quotient $\mathbb{T}^2/G$ is not Hausdorff. $\square$

EXERCISE I.7. Let $X = \mathbb{R}^2 \setminus 0$ and let $G = \mathbb{Z}$ with generator $t = 1$. Define an action of G on X as follows:

$$t^n \cdot (x, y) = (2^n x, 2^{-n} y) \quad \text{for all } (x, y) \in X \text{ and } n \in \mathbb{Z}.$$

Prove the following:

- (a) This is a covering space action.
- (b) Letting $q: X \rightarrow X/G$ be the quotient, the points $q(1, 0)$ and $q(0, 1)$ do not have disjoint open neighborhoods. $\square$

EXERCISE I.8. Let $X = \mathbb{R}^2 \setminus 0$ and let $G = \mathbb{Z}$ with generator $t = 1$. Define an action of G on X as follows:

$$t^n \cdot (x, y) = (2^n x, 2^{-n} y) \quad \text{for all } (x, y) \in X \text{ and } n \in \mathbb{Z}.$$

Let $q: X \rightarrow X/G$ be the quotient. Prove the following:

- (a) The action of G on X is a covering space action.
- (b) The points $q(1, 0)$ and $q(0, 1)$ then do *not* have disjoint open neighborhoods, so X/G is not Hausdorff. □

EXERCISE I.9. Let $(M, \mathfrak{d})$ be a second countable metric space and let G be a group acting on M by isometries. Assume that the action of G on M is proper, so M/G is Hausdorff (Lemma 1.6.2).

- (a) Prove that M/G is regular.
- (b) Prove that M/G is second countable and metrizable. □

Bibliography

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