Math 60330: Basic Geometry & Topology, Homework 4

1. Recall that we oriented $\mathbb{S}^n$ by saying that an ordered basis $(v_1, \ldots, v_n)$ for $T_p\mathbb{S}^n$ is positively oriented if $(p, v_1, \ldots, v_n)$ is a positively oriented basis for $\mathbb{R}^{n+1}$. Prove that this orientation depends smoothly on p (as defined in class).

2. Let $f\colon X \to Y$ be a diffeomorphism between connected oriented manifolds. Assume that there exists some $p_0 \in X$ such that $D_{p_0}f\colon T_{p_0}X \to T_{f(p_0)}Y$ is orientation-preserving. Prove that for all $p \in X$ the map $D_{p_0}f\colon T_{p_0}X \to T_{f(p_0)}Y$ is orientation-preserving.

3. Let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ be a smooth map with $\deg(f) \neq (-1)^{n+1}$. Prove that there is some $x \in \mathbb{S}^n$ with $f(x) = x$.

4. Let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ be a map with $\deg(f)$ odd. Prove that there exists some $x \in \mathbb{S}^n$ such that $f(-x) = -f(x)$.

5. Let M^m and N^n be compact oriented submanifolds of $\mathbb{R}^{k+1}$ such that $M^m \cap N^n = \varnothing$. Assume that $m + n = k$. Define the map $\lambda\colon M^m \times N^n \to \mathbb{S}^k$ via
$$\lambda(x, y) = \frac{x - y}{\|x - y\|}.$$
The *linking number* of M^m and N^n is then the degree $\mathrm{lk}(M^m, N^n)$ of the linking map λ. Prove the following:
   (a) $\mathrm{lk}(N^n, M^m) = (-1)^{(m+1)(n+1)}\,\mathrm{lk}(M^m, N^n)$.
   (b) If M^m is the boundary of a compact oriented manifold X^{m+1} such that $X^{m+1} \cap N^n = \varnothing$, then $\mathrm{lk}(N^n, M^m) = 0$.
   (c) Now assume that M^m and N^n lie in $\mathbb{S}^{k+1}$ rather than $\mathbb{R}^{k+1}$. For each $p \in \mathbb{S}^{k+1}$, construct an orientation-preserving diffeomorphism $f_p\colon \mathbb{S}^{k+1}\backslash p \to \mathbb{R}^{k+1}$. If p is disjoint from M^m and N^n, we can then define $\mathrm{lk}(f_p(M^m), f_p(N^n))$. Prove that this does not depend on p. We call this common value $\mathrm{lk}(M^m, N^n)$.

6. Let $f\colon \mathbb{S}^{2p-1} \to \mathbb{S}^p$ be a smooth map. For distinct regular values p and q of f, we have that $f^{-1}(p)$ and $f^{-1}(q)$ are disjoint $(p-1)$-manifolds in $\mathbb{S}^{2p-1}$. Orienting them as described in Chapter 5 of Milnor, we can talk about $\mathrm{lk}(f^{-1}(p), f^{-1}(q))$. Prove the following:
   (a) For regular values p and q of f, the integer $\mathrm{lk}(f^{-1}(p), f^{-1}(q))$ is locally constant as a function of q.
   (b) Let $g\colon \mathbb{S}^{2p-1} \to \mathbb{S}^p$ be another smooth map such that p and q are regular values of both f and g. Assume that $\|f(x) - g(x)\| < \|p - q\|$ for all $x \in \mathbb{S}^{2p-1}$. Prove that
$$\mathrm{lk}(f^{-1}(p), f^{-1}(q)) = \mathrm{lk}(g^{-1}(p), f^{-1}(q)) = \mathrm{lk}(g^{-1}(p), g^{-1}(q)).$$

(c) Prove that $\mathrm{lk}(f^{-1}(p), f^{-1}(q))$ depends only on the smooth homotopy class of f and is independent of the choice of regular values p and q.

The common value $\mathrm{lk}(f^{-1}(p), f^{-1}(q))$ is called the *Hopf invariant* of f, and is written $H(f)$.

7. Prove the following results about the Hopf invariant:

   (a) Let p be odd and let $f\colon \mathbb{S}^{2p-1} \to \mathbb{S}^p$ be a smooth map. Then $H(p) = 0$.

   (b) Let $f\colon \mathbb{S}^{2p-1} \to \mathbb{S}^p$ be a smooth map and let $g\colon \mathbb{S}^p \to \mathbb{S}^p$ be another smooth map. Prove that $H(g \circ f) = \deg(g)^2 H(f)$.

   (c) Go and read the wikipedia article on the Hopf fibration $\pi\colon \mathbb{S}^3 \to \mathbb{S}^2$. Prove that $H(\pi) = 1$.