Math 60330: Basic Geometry & Topology, Homework 3

1. Let $A \subset \mathbb{R}^n$ be arbitrary and let $f\colon A \to \mathbb{R}^m$ be a continuous map. Assume that for all $a \in A$, there exists an open neighborhood $U \subset \mathbb{R}^n$ of a such that $f|_{A\cap U}\colon A \cap U \to \mathbb{R}^m$ extends to a smooth map $U \to \mathbb{R}^m$. Prove that f is smooth, i.e., there exists an open neighborhood $V \subset \mathbb{R}^m$ of A such that f extends to a smooth map $V \to \mathbb{R}^m$. Hint: partitions of unity!

2. Let M be a smooth manifold and let $f, g\colon M \to \mathbb{S}^n$ be two smooth maps such that $\|f(p) - g(p)\| < 2$ for all $p \in M$. Prove that f and g are smoothly homotopic.

3. Let $f\colon M_1^n \to M_2^n$ and $g\colon M_2^n \to M_3^n$ be smooth maps between n-dimensional manifolds with M_1 and M_2 compact. Letting $\deg_2$ denote the mod-2 degree, prove that $\deg_2(g \circ f) = \deg_2(g)\deg_2(f)$. Hint: part of this is showing that you can pick regular values that work for all the maps in sight!

4. Let $f\colon M_1^n \to M_2^n$ be a smooth map between n-manifolds with M_1 compact and M_2 non-compact. Prove that $\deg_2(f) = 0$.

5. Let M be a smooth manifold and let ν be a vector field on M. Recall from Homework 1 that given a function $f\colon M \to \mathbb{R}$ and a tangent vector $\vec{v} \in T_pM$, we can define $\nabla_{\vec{v}}(f) \in \mathbb{R}$. Given a smooth function $f\colon M \to \mathbb{R}$, we define $\nu(f)\colon M \to \mathbb{R}$ via the formula

$$\nu(f)(p) = \nabla_{\nu(p)}(f) \qquad (p \in M).$$

   (a) Let $\phi_t\colon M \to M$ be the flow generated by ν. Prove that

$$\nu(f)(p) = \frac{(f \circ \phi_t)(p)}{\partial t}|_{t=0}$$

   for all $p \in M$.

   (b) Given vector fields ν_1 and ν_2 on M, prove that there exists a unique vector field $[\nu_1, \nu_2]$ on M such that

$$[\nu_1, \nu_2](f) = \nu_1(\nu_2(f)) - \nu_2(\nu_1(f))$$

   for all smooth functions $f\colon M \to \mathbb{R}$.

   (c) Prove that if ν_1 and ν_2 and ν_3 are smooth vector fields on M, then

$$[\nu_1, [\nu_2, \nu_3]] + [\nu_2, [\nu_3, \nu_1]] + [\nu_3, [\nu_1, \nu_2]] = 0.$$

6. Let G be a Lie group, that is, a smooth manifold that is also a group such that the following hold:

   - For all $g \in G$, the map $m_g\colon G \to G$ defined by $m_g(x) = gx$ is smooth.

- The inversion map $\iota\colon G \to G$ is smooth.

For instance, G might be $\mathrm{GL}_n(\mathbb{R})$. A vector field ν on G is said to be *left invariant* if for all $g \in G$, we have

$$D_x m_g(\nu(x)) = \nu(gx) \quad \text{for all } x \in G.$$

The set of all left-invariant vector fields on G is a vector space called the *Lie algebra* of G.

(a) Letting e be the identity element of G, construct a vector space isomorphism between the Lie algebra of G and $T_e G$.

(b) Prove that if ν_1 and ν_2 are left-invariant vector fields on G, then $[\nu_1, \nu_2]$ is also a left invariant vector fields on G.

(c) Look up the definition of a Lie algebra (say on wikipedia) and verify that with the aforementioned bracket operation the Lie algebra of G is indeed a Lie algebra.

(d) Prove that the Lie algebra of $\mathrm{GL}_n(\mathbb{R})$ is precisely the set of $n \times n$ real matrices with the bracket

$$[A, B] = AB - BA.$$