Math 60330: Basic Geometry & Topology, Homework 2

1. Let $f\colon M^m \to N^n$ be a smooth map. Assume that M^m is connected and that $D_p f = 0$ for all $p \in M^m$. Prove that f is a constant map.

2. For $n, m \geqslant 1$, identify $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ with $\mathbb{R}^{nm}$. Let $\mathrm{Sur}(\mathbb{R}^n, \mathbb{R}^m)$ be the subspace of $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ consisting of surjections. Prove that $\mathrm{Sur}(\mathbb{R}^n, \mathbb{R}^m)$ is open in $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$.

3. Let M^n be an n-dimensional manifold in $\mathbb{R}^d$. Define

$$TM = \left\{(p, v) \in \mathbb{R}^d \times \mathbb{R}^d \mid p \in M^n \text{ and } v \in T_p(M)\right\}.$$

   (a) Prove that TM is a $2n$-dimensional manifold in $\mathbb{R}^{2d}$.

   (b) Define the *normal bundle* to M to be

$$T^{\perp}M = \left\{(p, v) \in \mathbb{R}^d \times \mathbb{R}^d \mid p \in M^n \text{ and } v \in (T_p(M))^{\perp}\right\},$$

   where $\perp$ is taken with respect to the ordinary dot product on $\mathbb{R}^d$. Prove that $T^{\perp}M$ is a smooth d-dimensional manifold in $\mathbb{R}^{2d}$.

   (c) Prove that if $f\colon M^m \to N^n$ is a smooth map, then f induces a smooth map $Df\colon TM \to TN$ that for $p \in M$ restricts to the derivative $D_p f\colon T_pM \to T_{f(p)}N$.

4. A polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ is *homogeneous* of degree d if $f(tx_1, \ldots, tx_n) = t^d f(x_1, \ldots, x_n)$.

   (a) Prove *Euler's identity*: for a homogeneous polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d, we have

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = df.$$

   (b) Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree d. For $a \in \mathbb{R}$, let $M(a) = \{x \in \mathbb{R}^n \mid f(x) = a\}$. Prove that $M(a)$ is a smooth $(n-1)$-dimensional manifold for $a \neq 0$.

   (c) Prove that $M(a) \cong M(a')$ for $a, a' > 0$ and that $M(b) \cong M(b')$ for $b, b' < 0$.

5. (a) Identify the space $\mathrm{Mat}_n(\mathbb{R})$ of $n \times n$ real matrices with $\mathbb{R}^{n^2}$. Prove that $\mathrm{SL}_n(\mathbb{R})$ is a smooth (n^2-1)-dimensional submanifold of $\mathrm{Mat}_n(\mathbb{R})$.

   (b) Identify the tangent spaces of $\mathrm{Mat}_n(\mathbb{R})$ with $\mathrm{Mat}_n(\mathbb{R})$. Prove that the tangent space to $\mathrm{SL}_n(\mathbb{R})$ at the identity matrix consists of all matrices $A \in \mathrm{Mat}_n(\mathbb{R})$ with trace 0.