Math 60330: Basic Geometry & Topology, Homework 1

1. (a) Let $X \subset \mathbb{R}^n$ and $Y \subset X$ be arbitrary sets. Let $f\colon X \to \mathbb{R}^m$ be a smooth map. Prove that $f|_Y\colon Y \to \mathbb{R}^m$ is smooth.

   (b) Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ and $Z \subset \mathbb{R}^\ell$ be arbitrary sets. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be smooth maps. Prove that $g \circ f\colon X \to Z$ is smooth.

2. Let $V \subset \mathbb{R}^n$ be a linear subspace and let $f\colon V \to \mathbb{R}^m$ be a linear map. Prove that f is smooth.

3. Prove the following thing that was asserted in class without proof: for $p \in \mathbb{S}^n$, the tangent space $T_p\mathbb{S}^n$ is the set of all $v \in T_p\mathbb{R}^{n+1}$ that are orthogonal to p.

4. Fix some real numbers $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{n+1}$. Regarding S^n as a subspace of $\mathbb{R}^{n+1}$, define a map $f : S^n \to \mathbb{R}$ via the formula

   $$f(x_1, \ldots, x_{n+1}) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_{n+1} x_{n+1}^2 \qquad \text{for } (x_1, \ldots, x_{n+1}) \in S^n \subset \mathbb{R}^{n+1}.$$

   Recall that a point $p \in S^n$ is a *regular point* of f if the derivative map $D_p f\colon T_p S^n \to T_{f(p)}\mathbb{R}$ is surjective. The *regular values* of f are the set of all $x \in \mathbb{R}$ such that all points of $f^{-1}(x)$ are regular points of f. **Problem**: Prove that the regular values of f are exactly the set $\mathbb{R}\backslash\{\lambda_1, \ldots, \lambda_{n+1}\}$.

5. Let M^n be a smooth manifold and let $p \in M^n$. Let $C^\infty(M^n)$ be the $\mathbb{R}$-algebra of all smooth functions $M^n \to \mathbb{R}$.

   (a) For $\vec{v} \in T_p M^n$, define $\nabla_{\vec{v}}\colon C^\infty(M^n) \to \mathbb{R}$ by letting $\nabla_{\vec{v}}(f)$ equal the image of $\vec{v}$ under the map

   $$T_p M^n \xrightarrow{D_p f} T_{f(p)}\mathbb{R} = \mathbb{R}.$$

   Prove that this satisfies the following properties.

   i. For $\vec{v} \in T_p M^n$, the map $\nabla_{\vec{v}}$ is $\mathbb{R}$-linear.

   ii. For $\vec{v} \in T_p M^n$ and $f, g \in C^\infty(M^n)$, we have

   $$\nabla_{\vec{v}}(f + g) = \nabla_{\vec{v}}(f) + \nabla_{\vec{v}}(g).$$

   iii. For $\vec{v} \in T_p M^n$ and $f, g \in C^\infty(M^n)$, we have the Leibniz rule

   $$\nabla_{\vec{v}}(fg) = g(p)\nabla_{\vec{v}}(f) + f(p)\nabla_{\vec{v}}(g).$$

   (b) Now assume that $\Psi\colon C^\infty(M) \to \mathbb{R}$ is an $\mathbb{R}$-linear map such that

   $$\Psi(f + g) = \Psi(f) + \Psi(g).$$

   and

   $$\Psi(fg) = g(p)\Psi(f) + f(p)\Psi(g)$$

   for all $f, g \in C^\infty(M^n)$. Prove that there exists a unique $\vec{v} \in T_p M^n$ such that $\Psi(f) = \nabla_{\vec{v}}(f)$ for all $f \in C^\infty(M^n)$. Hint: think about what Ψ must do to a function based on the first few terms of that function's Taylor series.

Remark 0.1. Maps $\Psi : C^\infty(M) \to \mathbb{R}$ satisfying the above two properties are called *derivations at p*. The above exercise shows that you can identify the tangent space of M^n at p as the set of derivations at p.

6. This problem generalizes the argument we used to prove the fundamental theorem of algebra. Recall that a map $f : X \to Y$ is a *local homeomorphism* at $p \in X$ if there exists an open neighborhood U of p such that if you let $V = f(U)$, then $f|_U : U \to V$ is a homeomorphism. Also, a map $f : X \to Y$ is *proper* if $f^{-1}(K)$ is compact for all compact $K \subset Y$. Prove:

    - Let X and Y be locally compact Hausdorff spaces and let $f : X \to Y$ be a proper local homeomorphism. Assume that Y is connected. Prove that:
        - For all $y \in Y$, the preimage $f^{-1}(y)$ has finitely many points.
        - For all $y_1, y_2 \in Y$ the preimages $f^{-1}(y_1)$ and $f^{-1}(y_2)$ have the same number of points.

    We remark that once we have defined it, this will give an example of what is called a *covering space*.