Yet another book on algebraic topology 0: Prerequisites

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Table of contents

Part 1

Point set topology

CHAPTER 1

Introduction

This part of the book is a short course on point-set topology. Our goal is to emphasize definitions and examples that are important for algebraic topology, and also to explain a few things at a slightly more sophisticated level than the usual undergraduate textbooks. I first learned this material from Munkres's undergraduate textbook [**4**]. See [**1, 2, 3**] for more advanced references.

Remark 1.0.1. The only formal prerequisite for this part of the book is a solid undergraduate course on real analysis that covers metric spaces, say at the level of [**5**]. Since we move at a brisk pace in the beginning, it would also be helpful for a reader to have seen the elementary properties and examples of topological spaces before. □

Remark 1.0.2. We focus on examples that are relevant for algebraic topology. There are a vast range of other kinds of examples of topological spaces. See [**6**] for a taste of the possibilities. □

Bibliography

[1] N. Bourbaki, *General topology. Chapters 1–4*, translated from the French Reprint of the 1989 English translation, Elements of Mathematics (Berlin), Springer, Berlin, 1998.

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[3] J. Kelley, *General topology*, D. Van Nostrand Co., Inc., Toronto-New York-London, 1955.

[4] J. Munkres, *Topology*, 2nd edition, Prentice Hall, Upper Saddle River, NJ, 2000.

[5] W. Rudin, *Principles of mathematical analysis*, second edition, McGraw-Hill, New York, 1964.

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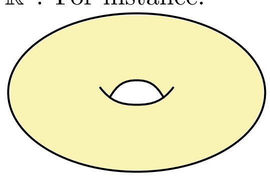
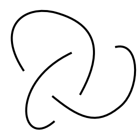
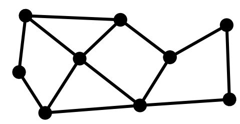
CHAPTER 2

From Euclidean space to metric spaces

We first describe the naive notion of a space. We then make an initial attempt to formalize this via metric spaces and discuss the ways in which this is unsatisfactory.

2.1. Naive spaces

The most familiar spaces are $\mathbb{R}^n$ and its subspaces. Indeed, since we live in $\mathbb{R}^3$ our drawings necessarily lie in $\mathbb{R}^3$. For instance:

We can imagine subspaces of $\mathbb{R}^n$ for $n \geq 4$ by analogy with $\mathbb{R}^3$. These are the geometric objects studied by mathematicians going back to the ancient Greeks.

Modern formalizations of the notion of "space" give a precise language for talking about these spaces and extending our geometric imagination to spaces that are less easily visualized. However, it is important to keep in mind that mathematicians have been studying geometry for thousands of years. The formal language might change and the scope of the field might expand, but it is still the same subject.

2.2. Metric spaces

Perhaps the easiest modern formalization is the notion of a metric space, which was introduced by Hausdorff [1]. A *metric space* is a pair $(M, \mathfrak{d})$ where M is a set and $\mathfrak{d}$ is a distance function $\mathfrak{d}\colon M \times M \to \mathbb{R}$ such that:

- For all $p, q \in M$, we have $\mathfrak{d}(p,q) \geq 0$ with equality if and only if $p = q$.
- For all $p, q \in M$, we have $\mathfrak{d}(p,q) = \mathfrak{d}(q,p)$.
- For all $p, q, r \in M$, we have the triangle inequality $\mathfrak{d}(p,q) \leq \mathfrak{d}(p,r) + \mathfrak{d}(r,q)$.

Sometimes we will not mention $\mathfrak{d}$ and just say that M is a metric space. Subspaces of $\mathbb{R}^n$ fit into this framework as follows:

EXAMPLE 2.2.1. Let $\|\cdot\|$ be the usual norm on $\mathbb{R}^n$:

$$\|(x_1, \ldots, x_n)\| = \sqrt{x_1^2 + \cdots + x_n^2} \quad \text{for all } (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Consider $M \subset \mathbb{R}^n$. For $x, y \in M$, define $[\mathfrak{d}(x,y) = \|x - y\|$. This makes M into a metric space. □

Spaces of functions provide other important examples. For instance:

EXAMPLE 2.2.2. Let $I = [0,1]$ be the closed interval and let $\mathcal{C}(I, \mathbb{R})$ be the set of all continuous functions $f\colon I \to \mathbb{R}$. Define a metric on $\mathcal{C}(I, \mathbb{R})$ as follows:

$$\mathfrak{d}(f,g) = \max\{|f(x) - g(x)| \mid x \in I\} \quad \text{for all continuous } f, g\colon I \to \mathbb{R}.$$

Since I is compact, this maximum makes sense. This makes $\mathcal{C}(I, \mathbb{R})$ into a metric space. □

2.3. Continuity

Once we have defined metric spaces, we can define continuity by imitating the classical definition from real analysis. Let $(M, \mathfrak{d}_M)$ and $(N, \mathfrak{d}_N)$ be metric spaces and let $f\colon M \to N$ be a function. Then:

- f is *continuous at* $p \in M$ if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $q \in M$ with $\mathfrak{d}_M(p, q) < \delta$ we have $\mathfrak{d}_N(f(p), f(q)) < \epsilon$.
- f is *continuous* if it is continuous at all $p \in M$.

The identity function $\mathbb{1}\colon M \to M$ is continuous and composition of two continuous functions is continuous (see Exercise 2.4).

2.4. Categories of metric spaces

The collection of metric spaces can be organized using the notion of a category, which we now discuss. A *category* consists of the following data:

- A collection of objects $\mathfrak{C}$.
- For all $X, Y \in \mathfrak{C}$, a set $\mathrm{Hom}_{\mathfrak{C}}(X, Y)$ of morphisms between X and Y. We will write $f\colon X \to Y$ to indicate that f is a morphism from X to Y.
- For all $X \in \mathfrak{C}$, a distinguished unit morphism $\mathbb{1}_X\colon X \to X$.
- For all $X, Y, Z \in \mathfrak{C}$ a composition map $\mathrm{Hom}_{\mathfrak{C}}(Y, Z) \times \mathrm{Hom}_{\mathfrak{C}}(X, Y) \to \mathrm{Hom}_{\mathfrak{C}}(X, Z)$. For morphisms $f\colon X \to Y$ and $g\colon Y \to Z$, we write their composition as $g \circ f\colon X \to Z$.

These must satisfy the following axioms:

- For all $f\colon X \to Y$, we have $\mathbb{1}_Y \circ f = f$ and $f \circ \mathbb{1}_X = f$.
- For all $f\colon X \to Y$ and $g\colon Y \to Z$ and $h\colon Z \to W$, we have the associative law $(h \circ g) \circ f = h \circ (g \circ f)$.

Here are some familiar examples:

Example 2.4.1. There is a category `Set` whose objects are sets X and whose morphisms $f\colon X \to Y$ are functions. Here the composition of two functions $f\colon X \to Y$ and $g\colon Y \to Z$ is the function $g \circ f\colon X \to Z$ given by the usual function composition. □

Example 2.4.2. Let $\mathbf{k}$ be a field. There is a category $\mathtt{Vect}_{\mathbf{k}}$ whose objects are vector spaces V over $\mathbf{k}$ and whose morphisms $f\colon V \to W$ are linear maps. Again, composition is the usual function composition. □

Metric spaces fit into this picture as follows:

Example 2.4.3. There is a category `Metric` whose objects are metric space $(M, \mathfrak{d})$ and whose morphisms $f\colon M \to N$ are continuous maps (see Exercise 2.4). □

Remark 2.4.4. There are other natural classes of maps between metric spaces. Let $(M, \mathfrak{d}_M)$ and $(N, \mathfrak{d}_N)$ be metric spaces and let $f\colon M \to N$ be a function. Then:

- The map f is an *isometric embedding* if $\mathfrak{d}_N(f(p), f(q)) = \mathfrak{d}_M(p, q)$ for all $p, q \in M$.
- The map f is *Lipshitz* if there is some $L \geq 0$ such that $\mathfrak{d}_N(f(p), f(q)) \leq L\, \mathfrak{d}_M(p, q)$ for all $p, q \in M$.
- The map f is *bi-Lipshitz* if there is some $L \geq 1$ such that
$$\frac{1}{L}\, \mathfrak{d}_M(p, q) \leq \mathfrak{d}_N(f(p), f(q)) \leq L\, \mathfrak{d}_M(p, q) \quad \text{for all } p, q \in M.$$
- The map f is a *quasi-isometric embedding* if there is some $L \geq 1$ and $K > 0$ such that
$$\frac{1}{L}\, \mathfrak{d}_M(p, q) - K \leq \mathfrak{d}_N(f(p), f(q)) \leq L\, \mathfrak{d}_M(p, q) + K \quad \text{for all } p, q \in M.$$

Isometric embeddings and Lipshitz maps and bi-Lipshitz maps are all continuous, while quasi-isometric embeddings need not be be continuous (see Exercises 2.2 and 2.3). All of these classes of maps form the morphisms in different categories of metric spaces (see Exercise 2.4). □

2.5. Topology

Ordinary geometry concerns distances, angles, etc. At least for distances, metric spaces are a natural context for this. Topology is a primitive kind of geometry where distances are ignored. Instead, topology focuses on tools for studying continuous function between spaces. Here are two examples of the kinds of questions it might ask:

QUESTION 2.5.1. For metric spaces M and N, we say that M and N are *homeomorphic* if there exists a bijection $f\colon M \to N$ such that f and f^{-1} are continuous. Can we classify metric spaces up to homeomorphism? □

QUESTION 2.5.2. Fix metric spaces M and N. An *embedding* of M into N is a continuous injective function $f\colon M \to N$ that is a homeomorphism onto its image. Can we determine whether M can be embedded into N? □

General metric spaces are far too wild for questions like these to have reasonable answers. Typically topologists restrict to classes of spaces like those drawn at the beginning of this chapter.

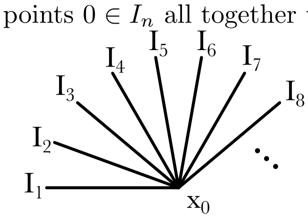
2.6. Downsides of metric spaces

The geometric meaning of the definition of a metric space is easily grasped. However, for topology they have downsides:

- Though continuity is defined in terms of a metric, there are many metrics on a given space that give the same notion of continuity (see Exercise 2.1). In other words, continuity is a more primitive notion than a metric.
- There are many geometric operations one would like to perform on spaces (gluing them together, taking quotients, etc). However, these operations do not always interact well with a metric and often result in "spaces" that are not metric spaces.

Here is an example of this second pathology:

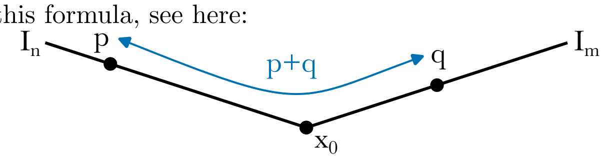
EXAMPLE 2.6.1. For each integer $n \geq 1$, let I_n be a copy of the interval $I = [0, 1]$. Let M be the "space" obtained by identifying the points $0 \in I_n$ all together to a single point x_0:



Each I_n is a subspace of M, so each $p \in M$ lies in some I_n. This I_n is unique unless $p = x_0$. There is a natural choice of a metric on M:

- Consider $p, q \in M$. If there is some $n \geq 1$ such that $p, q \in I_n = [0, 1]$, define $\mathfrak{d}(p, q) = |p - q|$. Otherwise, if $p \in I_n$ and $q \in I_m$ with $n \neq m$, then define $\mathfrak{d}(p, q) = p + q$.

For an explanation of this formula, see here:



Define a function $f\colon M \to \mathbb{R}$ via the formula $f(p) = np$ for $p \in I_n$. This formula makes sense since the map $p \mapsto np$ takes 0 to 0 for all n, so the resulting function f satisfies $f(x_0) = 0$. The restriction of f to each I_n is continuous; however, f itself is not continuous (see Exercise 2.5). □

In this example, it is inconvenient that continuous functions on the I_n do not "glue together" to a continuous function on M. Once we have defined topological spaces, we will be able to turn M into a topological space where this kind of gluing works.

2.7. Open sets and continuity

To give a hint for how to discuss continuity without a metric, we review some other facts about metric spaces. Fix a metric space $(M, \mathfrak{d})$. For $p \in M$ and $r > 0$, let

$$B_r(p) = \{q \in M \mid \mathfrak{d}(p, q) < r\} .$$

This is called the *open ball* of radius r around p. A set $U \subset M$ is *open* if for all $p \in U$, there exists some $r > 0$ such that $B_p(r) \subset U$. We then have:

LEMMA 2.7.1. *Let M_1 and M_2 be metric spaces and let $f : M_1 \to M_2$ be a function. Then f is continuous if and only if for all $U \subset M_2$ open we have $f^{-1}(U) \subset M_1$ open.*

PROOF. See Exercise 2.6. □

2.8. Exercises

EXERCISE 2.1. Prove the following:

(a) Let $(M, \mathfrak{d})$ be a metric space. Define $\mathfrak{d}' : M \times M \to \mathbb{R}$ via the formula $\mathfrak{d}'(p, q) = \min\{\mathfrak{d}(p, q), 1\}$. Prove that $\mathfrak{d}'$ is a metric on M that induces the same topology on M that $\mathfrak{d}$ does.

(b) Let $\| - \|$ be the following standard norm on $\mathbb{R}^n$:

$$\|(x_1, \dots, x_n)\| = \sqrt{x_1^2 + \cdots + x_n^2} \quad \text{for all } (x_1, \dots, x_n) \in \mathbb{R}^n.$$

This induces the metric $\mathfrak{d}(p, q) = \|p - q\|$ on $\mathbb{R}^n$. Now let $\| - \|'$ be an arbitrary norm on the vector space $\mathbb{R}^n$. Define a function $\mathfrak{d}' : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ via the formula

$$\mathfrak{d}'(p, q) = \|p - q\|' \quad \text{for } p, q \in \mathbb{R}^n.$$

Prove that $\mathfrak{d}'$ is a metric on $\mathbb{R}^n$ and that $\mathfrak{d}'$ induces the same topology on $\mathbb{R}^n$ as $\mathfrak{d}$. □

EXERCISE 2.2. Let $(M, \mathfrak{d}_M)$ and $(N, \mathfrak{d}_N)$ be metric spaces and let $f : M \to N$ be a function. Prove that f is continuous if any of the following hold:

(a) f is an isometric embedding.
(b) f is Lipshitz.
(c) f is bi-Lipshitz. □

EXERCISE 2.3. Endow $\mathbb{R}$ and $\mathbb{Z}$ with their standard metrics, so $\mathfrak{d}(x, y) = |x - y|$ for x and y in either $\mathbb{R}$ or $\mathbb{Z}$. Construct a quasi-isometric embedding $f : \mathbb{R} \to \mathbb{Z}$. This shows that quasi-isometric embeddings need not be continuous. □

EXERCISE 2.4. Prove that there are categories whose objects are metric spaces and whose morphisms are:

(a) Continuous maps.
(b) Isometric embeddings.
(c) Lipshitz maps.
(d) Bi-Lipshitz maps.
(e) Quasi-isometric embeddings.

In all five cases, the composition is the usual composition of maps. □

EXERCISE 2.5. Prove that the function $f : M \to \mathbb{R}$ from Example 2.6.1 is not continuous. □

EXERCISE 2.6. Let M_1 and M_2 be metric spaces. Let $f : M_1 \to M_2$ be a function. Prove that f is continuous (defined using the ϵ-δ definition) if and only if for all $U \subset M_2$ open we have $f^{-1}(U) \subset M_1$ open. □

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[1] F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig Viet, 1914.

CHAPTER 3

Topological spaces

Since continuity for metric spaces can be described entirely in terms of open sets, it is natural to abstract the notion of "open sets".

3.1. Definitions and basic examples

A *topological space* is a set X equipped with a collection of subsets of X called the *open sets*. These open sets should satisfy the following three properties:

- The whole space X and the empty set $\emptyset$ are both open.
- The collection of open sets is closed under arbitrary unions: if $\{U_i\}_{i \in I}$ is any collection of open sets, then $\cup_{i \in I} U_i$ is open.
- The collection of open sets is closed under finite intersections: if $U_1, \ldots, U_n$ are open sets, then $U_1 \cap \cdots \cap U_n$ is open.

We call the collection of open sets on X a *topology* on X. A key example is:

Example 3.1.1. If M is a metric space, then the collection of open sets in M makes M into a topological space (see Exercise 3.1). □

Convention 3.1.2. Whenever we draw a figure in $\mathbb{R}^n$, we give it the topology it inherits as a metric space via the Euclidean metric on $\mathbb{R}^n$ discussed in Example 2.2.1. □

Remark 3.1.3. The notion of a topological space has a long pre-history. The definition we gave above first appeared in Bourbaki [**1**], but earlier Hausdorff [**2**] defined something very close to it. We recommend the historical notes in [**1**] for a more thorough discussion of its history. □

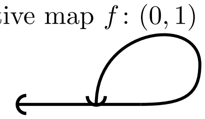
3.2. Continuity

A map $f\colon X \to Y$ between topological spaces is *continuous* if for all $U \subset Y$ open, its preimage $f^{-1}(U) \subset X$ is open. By Lemma 2.7.1, this is equivalent to the usual ϵ-δ definition if X and Y are metric spaces. There is a category Top whose objects are topological spaces and whose morphisms are continuous maps (see Exercise 3.7). The category Top is the natural home for topology.

If $\mathfrak{C}$ is a category, then a morphism $m\colon A \to B$ in $\mathfrak{C}$ is an *isomorphism* if there exists a morphism $m^{-1}\colon B \to A$ such that $m^{-1} \circ m = \mathbb{1}_A$ and $m \circ m^{-1} = \mathbb{1}_B$. If there exists an isomorphism from A to B, then we say that A and B are *isomorphic*. An isomorphism in Top is called a *homeomorphism*. Unwinding the definition, a homeomorphism between topological spaces X and Y is a continuous map $f\colon X \to Y$ that is bijective and whose inverse $f^{-1}\colon Y \to X$ is continuous. If there exists a homeomorphism from X to Y, then we say that X and Y are *homeomorphic*.

Here is an example to show that the continuity of f^{-1} is not immediate:

Example 3.2.1. Consider the injective map $f\colon (0,1) \to \mathbb{R}^2$ whose image X is as follows:



The map $f\colon (0,1) \to X$ is continuous and bijective, but the inverse map $f^{-1}\colon X \to (0,1)$ is not continuous (see Exercise 3.9). □

Remark 3.2.2. For metric spaces, another way to characterize continuity is to use limits:

- If $(M, \mathfrak{d})$ is a metric space, then a sequence of points $\{x_n\}_{n \geq 1}$ of M converges to $y \in M$ if for all $\epsilon > 0$, there exists some $N \geq 1$ such that $\mathfrak{d}(x_n, y) < \epsilon$ for $n \geq N$. We write this as $\lim_{n\mapsto\infty} x_n = y$, and if we do not want to specify y we simply say that $\{x_n\}_{n\geq 1}$ is convergent.

- A map $f\colon M \to N$ between metric spaces is continuous if and only if for all sequences $\{x_n\}_{n\geq 1}$ of points in M converging to $y \in M$, the sequence $\{f(x_n)\}_{n\geq 1}$ converges to $f(y)$. In other words, for a convergent sequence $\{x_n\}_{n\geq 1}$ in M we require $f(\lim_{n\mapsto\infty} x_n) = \lim_{n\mapsto\infty} f(x_n)$.

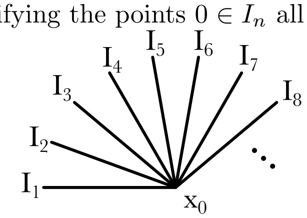
This definition could also be generalized to topological spaces, though with some subtleties (for instance, limits of sequences need not be unique). However, without some additional assumptions it would give a different notion of continuity. See §6.2 below for more about this. □

Convention 3.2.3. Henceforth, we will use the word "space" as a synonym for "topological space". Also, unless otherwise specified all maps between spaces are assumed to be continuous. □

3.3. Gluing intervals

We now return to Example 2.6.1 and explain how the notion of a topological space fixes its pathological behavior.

Example 3.3.1. For each integer $n \geq 1$, let I_n be a copy of the interval $I = [0, 1]$. Let M be the topological space obtained by identifying the points $0 \in I_n$ all together to a single point x_0:



Each I_n is a subspace of M, so each $p \in M$ lies in some I_n. This I_n is unique unless $p = x_0$. Endow M with the following topology:

- A set $U \subset M$ is open if and only if $U \cap I_n$ is open for all $n \geq 1$.

It is immediate from this definition that a function $f\colon M \to \mathbb{R}$ is continuous if and only if $f|_{I_n}\colon I_n \to \mathbb{R}$ is continuous for all $n \geq 1$. In particular, the function $f\colon M \to \mathbb{R}$ from Example 2.6.1 defined via the formula $f(p) = np$ for $p \in I_n$ is continuous. □

Remark 3.3.2. The topology we imposed on the space M in Example 3.3.1 is an example of an identification space topology. See Chapter 4 below for more details about this. □

3.4. Basic terminology for topological spaces

Before giving more examples, we introduce some terminology.

3.4.1. Closed sets. If X is a space, then a set $C \subset X$ is *closed* if $X \setminus C$ is open. The collection of closed sets is closed under finite unions and arbitrary intersections. The whole subject could be developed using closed sets instead of open ones (see Exercise 3.10).

3.4.2. Interior, closure, and neighborhoods. If X is a space and $A \subset X$ is a subset, we define the interior $\text{Int}(A)$ and the closure $\overline{A}$ as follows:

- The interior $\text{Int}(A)$ is the union of all open sets U with $U \subset A$. In other words, $\text{Int}(A)$ is the largest open set contained in A.
- The closure $\overline{A}$ is the intersection of all closed sets C with $A \subset C$. In other words, $\overline{A}$ is the smallest closed set containing A.

For $p \in X$, a *neighborhood* of p is a set A with $p \in \text{Int}(A)$. More generally, for a set $B \subset X$, a *neighborhood* of B is a set A with $B \subset \text{Int}(A)$. The most important special case of this terminology is an open neighborhood of $B \subset X$, which is an open set U with $B \subset U = \text{Int}(U)$.

3.4.3. Basis for a topology. A *basis* for a topology on a set X consists of a set $\mathfrak{B}$ of subsets of X such that:

- all points of X lie in some $U \in \mathfrak{B}$; and
- for all $U, V \in \mathfrak{B}$, the intersection $U \cap V$ can be written as a union of sets in $\mathfrak{B}$.

Given such a basis, the corresponding topology is the one where a set $U \subset X$ is open if and only if U is a union of sets in $\mathfrak{B}$. For instance, the topology on a metric space M has for a basis the set of open balls in M.

REMARK 3.4.1. There is a also the weaker notion of a subbasis; see §12.1 below. □

3.4.4. Subspaces. Let X be a space and let $Y \subset X$ be a subset. We would like to make Y into a space. Letting $\iota\colon Y \to X$ be the inclusion, the topology we impose on Y should make ι into a continuous function. For an open set $U \subset X$, we therefore need $\iota^{-1}(U) = U \cap Y$ to be open in Y. This suggests the following: the *subspace topology* on Y is the topology whose open sets $V \subset Y$ are the sets of the form $V = U \cap Y$ for an open set $U \subset X$. Unless we say otherwise, all subspaces are given the subspace topology.

3.4.5. Open and closed maps. Let $f\colon X \to Y$ be a continuous map. By definition, $f^{-1}(U) \subset X$ is open if $U \subset Y$ is open, and also $f^{-1}(C) \subset X$ is closed if $C \subset Y$ is closed. We say that the map f is *open* if $f(V) \subset Y$ is open for all $V \subset X$ open. Similarly, the map f is *closed* if $f(D) \subset Y$ is closed for all $D \subset X$ closed. Homeomorphisms are both open and closed.

3.4.6. Embeddings. A map $f\colon X \to Y$ is an *embedding* if it is a homeomorphism onto its image. In other words, an embedding f is a continuous injection onto a subspace $f(X)$ of Y such that the inverse map $f^{-1}\colon f(X) \to X$ is continuous. An injection which is either open or closed is automatically an embedding. For a subspace A of X, the inclusion $\iota\colon A \to X$ is an embedding. If $A \subset X$ is open (resp. closed), then ι is an open embedding (resp. a closed embedding).

3.5. Other examples

The notion of a topological space is extremely general. Here are a few more examples.

EXAMPLE 3.5.1. Let X be a set. The *discrete topology* on X is the one where all sets are open. The *trivial topology* on X is the one where the only open sets are $\emptyset$ and X. Another topology that can be put on an arbitrary set X is the *cofinite topology* whose open sets are those of the form $X \setminus F$ with F finite. The fact that this is a topology follows from the fact that finite sets are closed under finite unions and arbitrary intersections. □

EXAMPLE 3.5.2. Let $\mathbf{k}$ be a field; for instance, $\mathbf{k}$ might be $\mathbb{C}$ or $\mathbb{R}$. For a polynomial $f \in \mathbf{k}[z_1, \ldots, z_n]$, define the *vanishing* and *non-vanishing* loci of f to be

$$V(f) = \{(x_1, \ldots, x_n) \in \mathbf{k}^n \mid f(x_1, \ldots, x_n) = 0\} \subset \mathbf{k}^n \quad \text{and} \quad NV(f) = \mathbf{k}^n \setminus V(f).$$

The *Zariski topology* on $\mathbf{k}^n$ is the topology whose open sets are the nonvanishing loci $NV(f)$ as f ranges over elements of $\mathbf{k}[z_1, \ldots, z_n]$ (see Exercise 3.2). The closed sets are thus the vanishing loci $Z(f)$. For $n = 1$, the vanishing locus of a polynomial in $\mathbf{k}[z_1]$ can be any finite subset of $\mathbf{k}^1$, so the Zariski topology on $\mathbf{k}^1$ is the cofinite topology. □

REMARK 3.5.3. For $\mathbf{k}$ equal to $\mathbb{C}$ or $\mathbb{R}$, we have now seen two topologies on $\mathbf{k}^n$:

- the classical topology obtained by regarding $\mathbf{k}^n$ as a metric space; and
- the Zariski topology.

Every open set in the Zariski topology is open in the classical topology. We say that the classical topology is *finer* or *stronger* than the Zariski topology, and that the Zariski topology is *coarser* or *weaker* than the classical topology. □

EXAMPLE 3.5.4. A finite set X can be endowed with many topologies. For instance, if $X = \{a, b\}$ then the following are all topologies on X:

- the discrete topology: $\emptyset, \{a\}, \{b\}, \{a, b\}$
- $\emptyset, \{a\}, \{a, b\}$

- $\emptyset, \{b\}, \{a, b\}$
- the topology: $\emptyset, \{a, b\}$

The number of topologies on a finite set X grows very quickly as the size of X grows. Some of these spaces can be given geometric interpretations, but most of them are purely combinatorial objects. □

3.6. Rest of short course

Because the notion of a topological space is so general, there is almost nothing nontrivial that can be said about an arbitrary topological space. They are thus almost never studied for their own sake. Rather, they provide a minimal framework and language for studying continuity as it appears throughout mathematics.

The tools of algebraic topology are most useful for spaces that have some kind of geometric origin. In the rest of this short course, we introduce language to allow us to work with the kinds of spaces that appear in the rest of this book. We try to include enough examples and sample results to make reading this book more interesting than reading a dictionary, but we apologize if at some points it does seem merely like a compendium of definitions. We close with a discussion of topological manifolds, which illustrate most of our tools and play a basic role in the subject.

REMARK 3.6.1. Ultimately, the most natural class of spaces for algebraic topology are CW complexes. We will introduce these in Volume 1. □

3.7. Exercises

EXERCISE 3.1. Let M be a metric space. Prove that the collection of open sets in M makes M into a topological space. □

EXERCISE 3.2. Let $\mathbf{k}$ be a field. Prove that the Zariski topology on $\mathbf{k}^n$ described in Example 3.5.2 is a topology. □

EXERCISE 3.3. Let S be a set with a total ordering $\leq$. For $s_1, s_2 \in S$ with $s_1 < s_2$, let $(s_1, s_2) = \{s \in S \mid s_1 < s < s_2\}$. Prove that the collection of all sets of the form (s_1, s_2) forms a basis for a topology on S. For instance, if $S = \mathbb{R}$ with its usual ordering this is the usual basis for the topology on $\mathbb{R}$. □

EXERCISE 3.4. Let X and Y be topological spaces. Let $\mathcal{B}$ be the set of subsets of $X \times Y$ of the form $U \times V$ with $U \subset X$ and $V \subset Y$ open. Prove that $\mathcal{B}$ is the basis for a topology on $X \times Y$. We call this the *product topology*, and will discuss it extensively in Chapter 11. □

EXERCISE 3.5. Let X be a topological space and let $A \subset X$. Assume that for each $a \in A$ there exists a neighborhood N of a with $N \subset A$. Note that we are not assuming that N is open. Prove that A is open. □

EXERCISE 3.6. Consider the real line $\mathbb{R}$.

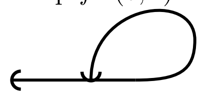
(a) Let $\mathcal{B}$ be the set $\{[a, b) \mid a < b\}$ of all half-open intervals in $\mathbb{R}$. Prove that $\mathcal{B}$ is the basis for a topology on $\mathbb{R}$ called the *lower limit topology*.
(b) Prove that the lower limit topology on $\mathbb{R}$ is finer than the standard topology. □

EXERCISE 3.7. Verify the axioms for a category for the category Top whose objects are topological spaces and whose morphisms are continuous maps. □

EXERCISE 3.8. Let X be a space and $\mathfrak{A}$ be a collection of subsets of X such that $X = \cup_{A \in \mathfrak{A}} A$. Let Y be another space and let $f\colon X \to Y$ be a map of sets (not necessarily continuous) such that $f|_A$ is continuous for all $A \in \mathfrak{A}$.

(a) Assume that each $A \in \mathfrak{A}$ is open. Prove that f is continuous.
(b) Assume that each $A \in \mathfrak{A}$ is closed and that $\mathfrak{A}$ is finite. Prove that f is continuous.
(c) Construct an example where each $A \in \mathfrak{A}$ is closed but f is *not* continuous. □

EXERCISE 3.9. Consider the injective map $f\colon (0,1) \to \mathbb{R}^2$ whose image X is as follows:



Regard f as a bijective map $f\colon (0,1) \to X$. Prove that $f^{-1}\colon X \to (0,1)$ is not continuous. □

EXERCISE 3.10. Formulate a set of axioms for a collection of closed sets to be the closed sets of a topology. Prove that this gives the same notion of a topological space as the axiomization we gave using open sets. □

Bibliography

[1] N. Bourbaki, *General topology. Chapters 1–4*, translated from the French Reprint of the 1989 English translation, Elements of Mathematics (Berlin), Springer, Berlin, 1998.
[2] F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig Viet, 1914.

CHAPTER 4

Identification spaces and the quotient topology

We now explain how to construct a new space from a collection of existing ones by identifying certain points together. This generalizes the construction we gave in Example 3.3.1.

4.1. Identification spaces

Let $\{X_i\}_{i\in I}$ be a collection of spaces. An *identification space* is a topological space Y equipped with maps $f_i\colon X_i \to Y$ for each $i \in I$ such that:

- each $y \in Y$ is in the image of some f_i; and
- a set $U \subset Y$ is open if and only if $f_i^{-1}(U) \subset X_i$ is open for all $i \in I$.

The second condition ensures that each $f_i\colon X_i \to Y$ is continuous. It also ensures that for a space Z a map of sets $\phi\colon Y \to Z$ if continuous if and only if $\phi \circ f_i\colon X_i \to Z$ is continuous for all $i \in I$ (we will say more about this in §4.4 below).

Example 4.1.1. Let $\{X_i\}_{i\in I}$ be a collection of spaces. As a set, let Y be the disjoint union $\sqcup_{i\in I} X_i$ of the X_i. Topologize Y by letting $U \subset Y$ be open if and only if $U \cap X_i$ is open for all $i \in I$ (see Exercise 4.1). The space Y is an identification space under the inclusion maps $f_i\colon X_i \to Y$. □

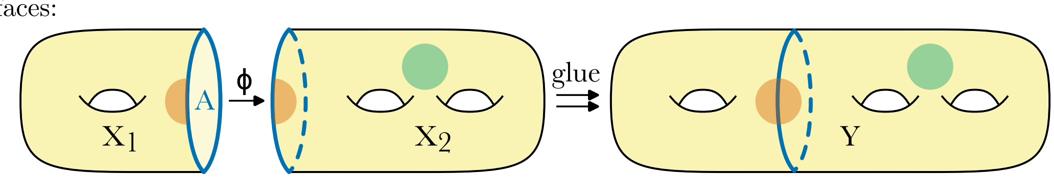
In general, if Y is a set obtained by taking the X_i and identifying some points together, then letting $f_i\colon X_i \to Y$ be the projections we can turn Y into an identification space by imposing the topology discussed above. We will call this the *identification space topology* on Y. If we have a construction of a purported "space" from the points of the X_i, then this gives a canonical way of turning our purported "space" into a topological space.

4.2. Examples

The above discussion is a little abstract. Here are some examples.

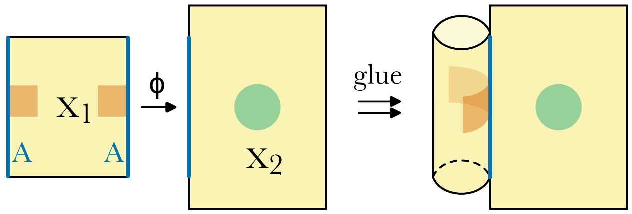
Example 4.2.1 (Gluing). Let X_1 and X_2 be spaces, $A \subset X_1$ be a subspace, and $\phi\colon A \to X_2$ be a map. As a set, let Y be the disjoint union of X_1 and X_2 modulo the equivalence relation that identifies each $a \in A \subset X_1$ with $\phi(a) \in X_2$. There are natural maps $f_1\colon X_1 \to Y$ and $f_2\colon X_2 \to Y$, and we give Y the identification space topology. We call Y the space obtained by *gluing* X_1 to X_2 via the gluing map ϕ. If $A \subset X_1$ is closed, then $f_2\colon X_2 \to Y$ is a closed embedding and $f_1|_{X_1\setminus A}\colon X_1 \setminus A \to Y$ is an open embedding (see Exercise 4.2). If A is not closed, then things can be much more complicated. □

Example 4.2.2. Here is one easy example of gluing. Let X_1 and X_2 and Y be the following surfaces:



As is shown in this figure, Y is obtained by gluing the boundary component $A \cong \mathbb{S}^1$ of X_1 to the boundary component of X_2 via a homeomorphism. Two open sets on Y are drawn together with their preimages in X_1 and X_2. In this example, ϕ is a homeomorphism onto its image and both X_1 and X_2 are subspaces of Y. □

Example 4.2.3. Here is another example of gluing where the gluing map is not injective:

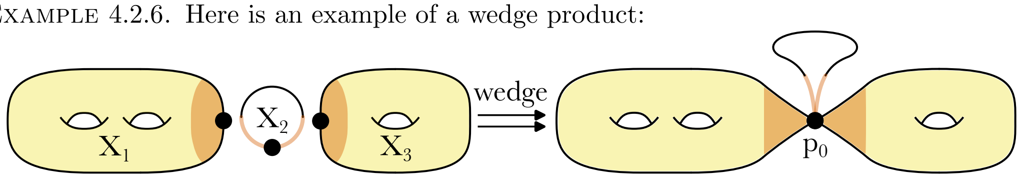


In this figure, ϕ identifies the two blue vertical edges of the rectangle X_1 with a single segment in the left-hand vertical edge of X_2. □

Remark 4.2.4. In general, we will use informal language to describe how we are gluing spaces together, but we always mean this topology. □

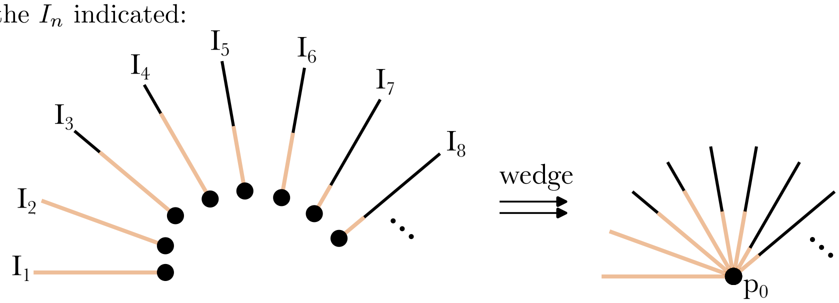
Example 4.2.5 (Wedge product). Let $\{X_i\}_{i \in I}$ be a collection of topological spaces. Assume that each X_i has a distinguished basepoint $x_i \in X_i$. The *wedge product* of the X_i, denoted $\vee_{i \in I} X_i$, is the space obtained by identifying all the x_i together to a single point p_0. There are inclusions $f_i \colon X_i \to \vee_{i \in I} X_i$, and we give $\vee_{i \in I} X_i$ the identification space topology. The maps f_i are all embeddings (see Exercise 4.3). □

Example 4.2.6. Here is an example of a wedge product:



This figure shows an open neighborhood of p_0 together with its preimage in the X_i. □

Example 4.2.7. Example 3.3.1 is the wedge product of countably many intervals $I_n = I$ equipped with the basepoints $0 \in I_n$. Here is a picture of this, with an open neighborhood p_0 together with its preimage in the I_n indicated:

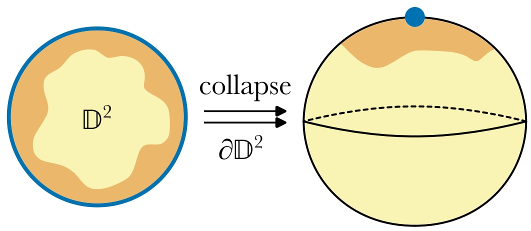


Note that the length of the portion of this open set in I_n is shrinking to 0 at n increases, which would not be possible if we were using the topology coming from a metric. □

Example 4.2.8 (Collapsing subspace). In an identification space, we allow there to only be a single space X. As an example of this, let X be a space and let $A \subset X$ be a subspace. Denote by X/A the result of collapsing A to a single point. The points of X/A are thus the points of $X \setminus A$ together with a single point $[A]$ corresponding to A. Letting $f \colon X \to X/A$ be the projection, we can endow X/A with the identification space topology. If A is a closed (resp. open) set, then the restriction of f to $X \setminus A$ is an open (resp. closed) embedding (see Exercise 4.4). □

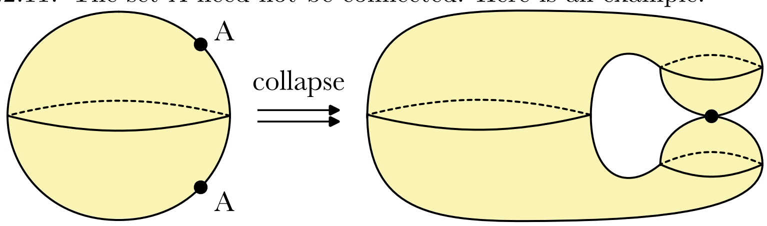
Remark 4.2.9. Collapsing a subspace $A \subset X$ will typically yield a pathological space X/A if A is not closed. For instance, in most natural spaces single points are closed, and this will only hold for the point $[A]$ of X/A if $A \subset X$ is closed. As an example of how pathological this can be, collapsing the subspace $\mathbb{Q}$ of $\mathbb{R}$ gives a terrible space $\mathbb{R}/\mathbb{Q}$. □

Example 4.2.10. Consider the boundary $\partial \mathbb{D}^n = \mathbb{S}^{n-1}$. As the following shows, $\mathbb{D}^n / \partial \mathbb{D}^n \cong \mathbb{S}^n$, with the blue $\partial \mathbb{D}^n$ mapping to the north pole of $\mathbb{S}^n$:



As this figure shows, a neighborhood of the north pole in $\mathbb{S}^n$ lifts to a neighborhood of $\partial\mathbb{D}^n$. □

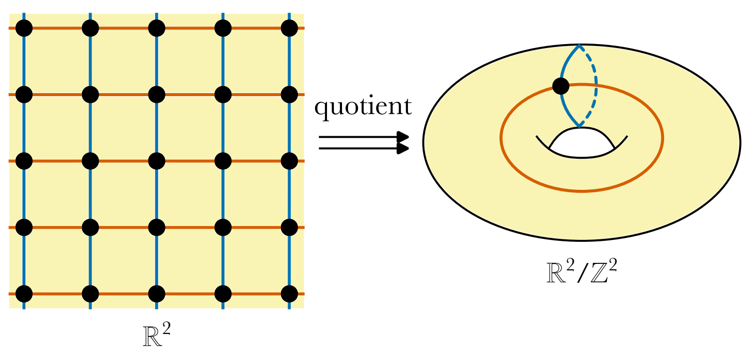
EXAMPLE 4.2.11. The set A need not be connected. Here is an example:



In this example, $X = \mathbb{S}^2$ and A is two points on X. □

EXAMPLE 4.2.12 (Quotienting by group action). Let X be a space and let G be a group acting on X. As a set, X/G consists of the orbits of X under the action of G. Letting $f\colon X \to X/G$ be the quotient map, we endow X/G with the identification space topology. The point-set topological properties of X/G depend both on the properties of X and the qualities of the group action G. We explore this in Chapter 13. □

EXAMPLE 4.2.13. Let the group $\mathbb{Z}^2$ act on $\mathbb{R}^2$ by translations. The quotient $\mathbb{R}^2/\mathbb{Z}^2$ is homeomorphic to the 2-torus:



The orange and blue loops on $\mathbb{R}^2/\mathbb{Z}^2$ lift to the orange and blue parallel lines on $\mathbb{R}^2$. □

4.3. Quotient topology

A map $f\colon X \to Y$ is a *quotient map* if f is surjective and $U \subset Y$ is open if and only if $f^{-1}(U) \subset X$ is open. Given a space X and a surjection of sets $f\colon X \to Y$, the *quotient topology* on Y is the topology making $f\colon X \to Y$ a quotient map. We call Y a *quotient space* of X.

Of course, this is a special case of an identification space. Moreover, given a collection of spaces $\{X_i\}_{i\in}$ and an identification space Y of the X_i with maps $f_i\colon X_i \to Y$, we can realize Y as a quotient space in the following way. Let $\sqcup_{i\in I} X_i$ be the disjoint union of the X_i (see Example 4.1.1). The maps $f_i\colon X_i \to Y$ then assemble to a quotient map $F\colon \sqcup_{i\in I} X_i \to Y$.

REMARK 4.3.1. Many treatments of point-set topology only talk about quotient spaces, but we find it convenient to use the slightly more general notion of identification spaces since like in the examples from earlier in this chapter, we often use them to build a space out of several spaces, not just one. □

4.4. Universal mapping property

Let $f\colon X \to Y$ be a quotient map. Let $\sim$ be the equivalence relation on the set X where $p \sim q$ if and only if $f(p) = f(q)$. The equivalence classes of $\sim$ are the fibers $f^{-1}(y)$ for $y \in Y$. Letting Z be another space and $\phi\colon Y \to Z$ be a continuous map, the composition $\Phi = \phi \circ f$ is a continuous map $\Phi\colon X \to Z$ is $\sim$-invariant, i.e., $\Phi(p) = \Phi(q)$ whenever $p \sim q$. Conversely, if $\Phi\colon X \to Z$ is a continuous $\sim$-invariant map, then there is a set map $\phi\colon Y \to Z$ such that $\Phi = \phi \circ f$ and the quotient topology on Y is set up to ensure that ϕ is continuous.

The above discussion shows that composition with f gives a bijection between continuous maps $\phi\colon Y \to Z$ and $\sim$-invariant continuous maps $\Phi\colon X \to Z$. This is an example of a *universal mapping property*, and we will describe it informally by saying that a map $\phi\colon Y \to Z$ is the same as a $\sim$-invariant map $\Phi\colon X \to Z$. This universal mapping property characterizes quotient spaces (see Exercise 4.6). Here are several examples of it:

Example 4.4.1 (Disjoint union). Let $\{X_i\}_{i \in I}$ be a collection of spaces. Let $Y = \sqcup_{i \in I} X_i$ with the disjoint union topology discussed in Example 4.1.1. For a space Z, maps $\phi\colon \vee_{i \in I} X_i \to Z$ are the same as collections of maps $\Phi_i\colon X_i \to Z$ for each $i \in I$. In category theory, a sum in a category is something satisfying a universal property of the this form. A category theorist would therefore say that $\sqcup_{i \in I} X_i$ is the sum of the X_i in the category of topological spaces. □

Remark 4.4.2. We will discuss the categorical product of spaces in Chapter 11 □

Example 4.4.3 (Wedge product). Let $\{X_i\}_{i \in I}$ be a collection of topological spaces. Assume that each X_i has a distinguished basepoint $x_i \in X_i$. For a space Z, maps $\phi\colon \vee_{i \in I} X_i \to Z$ are the same as collections of maps $\Phi_i\colon X_i \to Z$ such that $\Phi_i(x_i) = \Phi_j(x_j)$ for all $i, j \in I$. In particular, this explains why the quotient topology is the right one to ensure the real-valued function in Example 3.3.1 is continuous. □

Example 4.4.4 (Collapsing subspace). Let X be a space and let $A \subset X$ be a subspace. For a space Z, maps $\phi\colon X/A \to Z$ are the same as maps $\Phi\colon X \to Z$ such that $\Phi(A)$ is a single point. □

Example 4.4.5 (Quotienting by group action). Let X be a space and let G be a group acting on X. For a space Z, maps $\phi\colon X/G \to Z$ are the same as maps $\Phi\colon X \to Z$ that are G-invariant in the sense that $\Phi(g \cdot x) = \Phi(x)$ for all $x \in X$ and $g \in G$. □

4.5. Exercises

Exercise 4.1. Let $\{X_i\}_{i \in I}$ be a collection of spaces. As a set, let Y be the disjoint union $\sqcup_{i \in I} X_i$ of the X_i. Say that $U \subset Y$ is open if and only if $U \cap X_i$ is open for all $i \in I$. Prove that this makes Y into a topological space. □

Exercise 4.2. Let X_1 and X_2 be spaces, let $A \subset X_1$ be a closed subspace, and let $\phi\colon A \to X_2$ is a map. Let Y be the space obtained by gluing X_1 to X_2 via the gluing map ϕ, and let $f_1\colon X_1 \to Y$ and $f_2\colon X_2 \to Y$ be the natural maps. Prove that $f_2\colon X_2 \to Y$ is a closed embedding and $f_1|_{X_1 \setminus A}\colon X_1 \setminus A \to Y$ is an open embedding. □

Exercise 4.3. Let $\{X_i\}_{i \in I}$ be a collection of topological spaces. Assume that each X_i has a distinguished basepoint $x_i \in X_i$, and consider the edge product $\vee_{i \in I} X_i$. For each $i \in I$, let $f_i\colon X_i \to \vee_{i \in I} X_i$ be the inclusion. Prove that f_i is an embedding. Moreover, if each basepoint $x_i \in X_i$ is closed then prove that each f_i is a closed embedding. □

Exercise 4.4. Let X be a space, let $A \subset X$ be a subspace, and let $f\colon X \to X/A$ be the projection.

(a) If A is closed, then prove that the restriction of f to $X \setminus A$ is an open embedding.
(b) If A is open, then prove that the restriction of f to $X \setminus A$ is a closed embedding. □

Exercise 4.5. Let $\sim_1$ and $\sim_2$ be the following equivalence relations on $\mathbb{R}^2$:

$$(x, y) \sim_1 (z, w) \quad \text{if } x + y^2 = z + w^2,$$
$$(x, y) \sim_2 (z, w) \quad \text{if } x^2 + y^2 = z^2 + w^2.$$

For $i = 1, 2$, let X_i be the quotient of $\mathbb{R}^2$ by $\sim_i$. Identify the spaces X_1 and X_2. □

Exercise 4.6. Let X be a space and let $\sim$ be an equivalence relation on X. As a set, let $Y = X/\sim$ and let $f\colon X \to Y$ be the projection. Endow Y with the quotient topology, so $f\colon X \to Y$ is a quotient map. Let Y' be another space and let $f'\colon X \to Y'$ be a map such that the following holds:

- For all spaces Z, composition with f' gives a bijection between continuous maps $\phi\colon Y' \to Z$ and $\sim$-invariant continuous maps $\Phi\colon X \to Z$.

Prove that there is a homeomorphism $g\colon Y \to Y'$ such that $f' = g \circ f$. In other words, the above universal mapping property characterizes the quotient space Y. $\square$

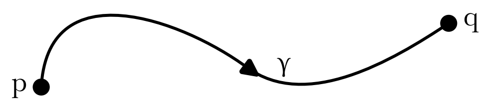
CHAPTER 5

Connectivity properties

Our next topic is connectivity and path connectivity.

5.1. Path connectivity

Recall that $I = [0,1]$. A *path* in a space X from $p \in X$ to $q \in X$ is a map $\gamma\colon I \to X$ with $\gamma(0) = p$ and $\gamma(1) = q$:



The space X is *path connected* if for all $p, q \in X$ there exists a path in X from p to q. The geometric meaning of this is hopefully clear.

EXAMPLE 5.1.1. The space $\mathbb{R}^n$ is path connected. Indeed, for $p, q \in \mathbb{R}^n$ the map $\gamma\colon I \to \mathbb{R}^n$ defined by

$$\gamma(t) = (1-t)p + tq \quad \text{for } t \in I$$

is a path from p to q. □

5.2. Connectivity

We now turn to connectivity. It is easier to say what it means for a space to be disconnected. A space X is *disconnected* if we can write $X = U \cup V$ with $U, V \subset X$ disjoint nonempty open subsets of X. Since $X \setminus U = V$ and $X \setminus V = U$, the sets U and V are necessarily closed as well as open. Sets that are both open and closed are called *clopen sets*.[1]

EXAMPLE 5.2.1. The space $\mathbb{R} \setminus 0$ is disconnected. Indeed, $\mathbb{R} \setminus 0 = (-\infty, 0) \cup (0, \infty)$. □

EXAMPLE 5.2.2. The space $\mathbb{Q}$ is disconnected. Indeed, for $U \subset \mathbb{R}$ let $U_{\mathbb{Q}} = U \cap \mathbb{Q}$. We then have $\mathbb{Q} = (-\infty, \sqrt{2})_{\mathbb{Q}} \cup (\sqrt{2}, \infty)_{\mathbb{Q}}$. □

A space X is *connected* if it is not disconnected. Another way of saying this is that X is connected if whenever $X = U \cup V$ with $U, V \subset X$ open we have $U \cap V \neq \emptyset$. Here are some basic properties of this (see Exercise 5.2):

- The space $I = [0,1]$ is connected.
- If X is connected and $f\colon X \to Y$ is a map, then $f(X)$ is connected.
- Let X be a space and let $\{Y_i\}_{i \in I}$ be a collection of subspaces of X. Assume that:
  - each Y_i is connected; and
  - for all $i, j \in I$, the space $Y_i \cap Y_j$ is nonempty; and
  - $X = \cup_{i \in I} Y_i$.

  Then X is connected.

Together, these three properties imply the following:

LEMMA 5.2.3. *Let X be a path connected space. Then X is connected.*

PROOF. This is trivial if $X = \emptyset$, so assume that $X \neq \emptyset$. Fix a point $p \in X$. For each $q \in X$, pick a path $\gamma_q\colon I \to X$ from p to q. Set $Y_q = \gamma_q(I)$. Since I is connected, so is Y_q. The space X is the union of the Y_q, and for $q, q' \in X$ we have $p \in Y_q \cap Y_{q'}$. It follows that X is connected. □

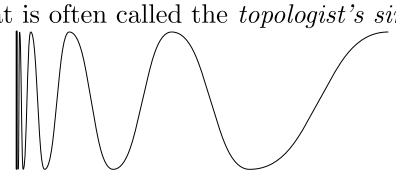
The converse of Lemma 5.2.3 is not true:

[1] This is a terrible term, but is the standard word for this.

Example 5.2.4. Let X be the following subset of $\mathbb{R}^2$:

$$X = \{(0, y) \mid -1 \leq y \leq 1\} \cup \{(x, \sin(1/x) \mid x > 0\}.$$

This is a closed subset of $\mathbb{R}^2$ that is often called the *topologist's sine-curve*:



The space X is *not* path connected; indeed, there is no path connecting $(0, 0)$ and $(x, \sin(1/x))$ for any $x > 0$ (see Exercise 5.3). However, X is connected (see also Exercise 5.3). □

5.3. Path components

Let X be a space. Say that $p, q \in X$ are equivalent if there is a path in X from p to q. This is an equivalence relation on the points of X (see Exercise 5.5), and the equivalence classes are the *path components* of X. It is immediate from the definition that the path components of X are path connected and that X is the disjoint union of its path components.

Example 5.3.1. Let X be the topologist's sine-curve from Example 5.2.4. The path components of X are as follows (see Exercise 5.3):

$$X_1 = \{(0, y) \mid -1 \leq y \leq 1\},$$
$$X_2 = \{(x, \sin(1/x) \mid x > 0\}.$$
□

5.4. Connected components

Continue to let X be a space. Now say that points $p, q \in X$ are equivalent if there is a connected subspace $Y \subset X$ with $p, q \in Y$. This is an equivalence relation on the points of X (see Exercise 5.5), and the equivalence classes are the *connected components* of X. The connected components of X are connected (see Exercise 5.6), and X is the disjoint union of its connected components. Since path connected spaces are connected, each connected component of X is the union of a collection of path components.

5.5. Local connectivity

Since a space X is disconnected if we can write $X = U \cup V$ with $U, V \subset X$ disjoint nonempty clopen subsets, it is natural to hope that the connected components of X are clopen. Unfortunately, this need not hold:

Example 5.5.1. Let $X = \mathbb{Q}$. The connected components of X and the path components of X both consist of the one-points sets $\{q\}$ for $q \in \mathbb{Q}$. □

As this example suggests, the cause of this is pathological local behavior. A space X is *locally connected* at $p \in X$ if for all open neighborhoods U of p, there is a connected open neighborhood V of p with $V \subset U$. The space X is *locally connected* if it is locally connected at all $p \in X$. Similarly, a space X is *locally path connected* at $p \in X$ if for all open neighborhoods U of p, there is a path connected open neighborhood V of p with $V \subset U$. The space X is *locally path connected* if it is locally path connected at all $p \in X$. We then have:

Lemma 5.5.2. *Let X be a space. Then:*

- *If X is locally connected, then all connected components of X are clopen.*
- *If X is locally path connected, then all path components of X are clopen.*

Proof. The two conclusions have similar proofs, so we will prove the first. Assume that X is locally connected. Let Y be a connected component of X. For $p \in Y$, since X is locally connected we can find a connected open neighborhood V of p. Since Y and V are connected and $p \in Y \cap V$, the union $Y \cup V$ is connected (see Exercise 5.2). This implies that $Y = Y \cup V$, i.e., that $V \subset Y$. We deduce that Y is open. Since $X \setminus Y$ is the union of connected components and these connected components are open, it follows that $X \setminus Y$ is open. Thus Y is closed and hence clopen, as desired. □

COROLLARY 5.5.3. *Let X be a locally path connected space. Then the connected components and path components of X coincide.*

PROOF. Let Y be a connected component of X. The subspace Y is the disjoint union of a collection of path components. To prove that it is actually a path component, it is enough to prove that Y is path connected. Assume otherwise. We can then write $Y = Y_1 \cup Y_2$ with each Y_i a nonempty union of path components and $Y_1 \cap Y_2 = \emptyset$. Lemma 5.5.2 implies that each path component is clopen, so both Y_1 and Y_2 are also clopen. Since $Y = Y_1 \cup Y_2$, we deduce that Y is disconnected, contradicting the fact that it is connected. □

REMARK 5.5.4. As our examples show, not all metric spaces (or even subspaces of $\mathbb{R}^n$) are locally connected or locally path connected. However, most spaces that appear in algebraic topology are locally path connected. In particular, when we discuss CW complexes in Volume 1 we will prove that they are always locally path connected. □

5.6. Exercises

EXERCISE 5.1. Let X and Y be spaces. Give $X \times Y$ the product topology (see Exercise 3.4).

(a) If X and Y are connected, prove that $X \times Y$ is connected.
(b) If X and Y are path-connected, prove that $X \times Y$ is path-connected. Be careful to prove that your paths are continuous! □

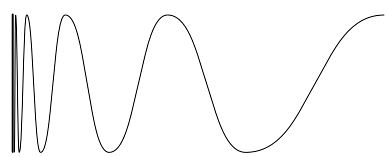
EXERCISE 5.2. Prove the following basic properties of connected spaces:

(a) The space $I = [0, 1]$ is connected.
(b) If X is connected and $f : X \to Y$ is a map, then $f(X)$ is connected.
(c) Let X be a space and let $\{Y_i\}_{i \in I}$ be a collection of subspaces of X. Assume that:
   - each Y_i is connected; and
   - for all $i, j \in I$, the space $Y_i \cap Y_j$ is nonempty; and
   - $X = \cup_{i \in I} Y_i$.

   Then X is connected. □

EXERCISE 5.3. Let X be the topologist's sine curve:

$$X = \{(0, y) \mid -1 \le y \le 1\} \cup \{(x, \sin(1/x) \mid x > 0\} \subset \mathbb{R}^2.$$



Prove that X is connected but not path connected. Also, prove that its path components are

$$X_1 = \{(0, y) \mid -1 \le y \le 1\},$$
$$X_2 = \{(x, \sin(1/x) \mid x > 0\}.$$ □

EXERCISE 5.4. Let $I = [0, 1]$ and let $\le$ be the dictionary order topology on I^2, so $(x, y) \le (z, w)$ if $x \le z$ or if $x = z$ and $y \le w$. Let X the topological space on the set I^2 with the corresponding order topology (see Exercise 3.3). Prove the following:

(a) X is connected.
(b) X is not path connected. □

EXERCISE 5.5. Let X be a space. Prove that the following are equivalence relations on the points of X:

(a) For $p, q \in X$, the relation where p is equivalent to q if there is a path in X from p to q.
(b) For $p, q \in X$, the relation where p is equivalent to q if there is a connected subspace $Y \subset X$ with $p, q \in X$. □

EXERCISE 5.6. Let X be a space and let Y be a path component of X. Prove that Y is connected. □

Exercise 5.7. Let X be a connected space and let $f\colon X \to Y$ be a quotient map. Prove that Y is connected. $\square$

CHAPTER 6

Countability properties

This chapter discussed properties that ensure a topological space is not "too large".

6.1. First countability

Let X be a space. A *neighborhood basis* for X at a point $p \in X$ is a collection $\mathfrak{B}_p$ of open neighborhoods of p such that:

- For all open neighborhoods V of p, we have $U \subset V$ for some $U \in \mathfrak{B}_p$.

The space X is *first countable* if it has a countable neighborhood basis at each point $p \in X$. All metric spaces have this property:

LEMMA 6.1.1. *Let M be a metric space. Then M is first countable.*

PROOF. Recall that $B_r(p)$ is the open ball of radius $r > 0$ around $p \in M$. For $p \in M$, the set $\{B_r(p) \mid r > 0 \text{ rational}\}$ is a countable neighborhood basis for X at p. □

6.2. Sequences

Let X be a space. If X is first countable, then we will show that limits of sequences can be used in X in a manner analogous to the way sequences are used in real analysis. A *sequence* in X is an ordered collection $\{x_n\}_{n \geq 1}$ of points of X. Given such a sequence, a point $y \in X$ is its *limit* if for all open neighborhoods U of y there is some $N \geq 1$ such that $x_n \in U$ for $n \geq N$. If y is a limit of $\{x_n\}_{n \in X}$, then we write $\lim_{n \mapsto \infty} x_n = y$ and say that $\{x_n\}_{n \geq 1}$ *converges* to y. If such a y exists, then we say that $\{x_n\}_{n \geq 1}$ is a *convergent sequence*.

REMARK 6.2.1. Be warned that a sequence can have multiple distinct limits. This only happens for fairly pathological spaces. In the next chapter, we introduce a property of spaces called being Hausdorff that forces convergent sequences to have unique limits. □

6.3. Closure

If X is first countable, then for $A \subset X$ we can construct the closure $\overline{A}$ using limits:

LEMMA 6.3.1. *Let X be a first countable space and let $A \subset X$. Then $\overline{A}$ is the set of all $y \in A$ such that there exists a sequence $\{a_n\}_{n \geq 1}$ of points of A such that* $\lim_{n \mapsto \infty} a_n = y$.

PROOF. Let B be the set of limits of sequences of points of A. We first prove that $B \subset \overline{A}$. Let $b \in B$ and let $C \subset X$ be a closed set with $A \subset C$. We must prove that $b \in C$. Indeed, if $b \notin C$ then we can find an open neighborhood U of b such that $U \subset X \setminus C$. However, since $b \in B$ there must exist points of $A \subset C$ in U, contradicting the fact that U is disjoint from C.

We next prove that $\overline{A} \subset B$. This uses first countability. Consider a point $p \in \overline{A}$. Each open neighborhood V of p must contain a point of A. Let $\mathfrak{B}_p = \{U_1, U_2, \ldots\}$ be a countable neighborhood basis at p. For each $n \geq 1$, choose $x_n \in U_n$ with $x_n \in A$. We then have $\lim_{n \mapsto \infty} x_n = p$, so $p \in B$. □

REMARK 6.3.2. Though metric spaces are first countable, not all spaces that appear in algebraic topology are first countable. In particular, not all CW complexes are first countable. This is why arguments using limits are mostly avoided in this book. There are generalizations of sequences and limits (nets, filters, etc.) that work for spaces that are not first countable (see [1]), but in practice they do not simplify arguments in algebraic topology. □

6.4. Second countability

A space X is *second countable* if there is a countable basis for its topology. It is clear that all second countable spaces are first countable. It is not true that all metric spaces are second countable, but all subspaces of $\mathbb{R}^n$ are second countable:

LEMMA 6.4.1. *Let X be a subspace of $\mathbb{R}^n$. Then X is second countable.*

PROOF. For all $p \in \mathbb{R}^n$ and $r > 0$, let $B_r(p) \subset \mathbb{R}^n$ be the open ball around p. Then X has the countable basis $\{B_r(p) \cap X \mid p \in \mathbb{Q}^n \text{ and } r > 0 \text{ rational}\}$. □

REMARK 6.4.2. Since CW complexes need not be first countable, they definitely do not need to be second countable. The main reason we introduce second countability is that it appears in the definition of a manifold; see Chapter 14 below. □

6.5. Separability

There is one further countability condition that occasionally shows up. For a space X, a set $A \subset X$ is *dense* if its closure $\overline{A}$ equals X. The space X is *separable* if X has a countable dense subset. This is slightly weaker than second countability:

LEMMA 6.5.1. *Let X be a second countable space. Then X is separable.*

PROOF. Let $\mathfrak{B} = \{U_1, U_2, \ldots, \}$ be a countable basis for the topology of X. Pick $x_n \in U_n$. Then the set $\{x_n \mid n \geq 1\}$ is a countable dense set in X. □

For metric spaces, these two notions coincide:

LEMMA 6.5.2. *Let M be a separable metric space. Then M is second countable.*

PROOF. The proof is similar to that of Lemma 6.4.1: if $A \subset M$ is a countable dense set, then $\big\{B_{1/n}(a) \mid a \in A,\, n \geq 1\big\}$ is a countable basis for the topology on M. □

6.6. Exercises

EXERCISE 6.1. Let X be second countable and let $\mathcal{B}$ be any basis for the topology on X. Prove that there is a countable subset $\mathcal{B}' \subset \mathcal{B}$ that is also a basis for the topology on X. □

EXERCISE 6.2. Let X be second countable and let $\mathfrak{U}$ be a collection of disjoint open subsets of X. Prove that $\mathfrak{U}$ is countable. □

EXERCISE 6.3. Let $I = [0, 1]$ and let $\leq$ be the dictionary order topology on I^2, so $(x, y) \leq (z, w)$ if $x \leq z$ or if $x = z$ and $y \leq w$. Let X the topological space on the set I^2 with the corresponding order topology (see Exercise 3.3). Prove the following:

(a) X is first countable.
(b) X is not second countable. □

Bibliography

[1] J. Kelley, *General topology*, D. Van Nostrand Co., Inc., Toronto-New York-London, 1955.

CHAPTER 7

Separation properties and the Tietze extension theorem

This chapter discusses properties that are necessary to ensure that continuous functions have the properties one would expect.

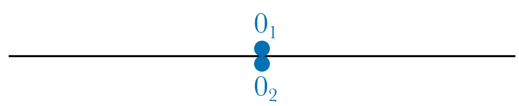
7.1. Pathology

Consider maps $f, g\colon X \to Y$. If $A \subset X$ is dense and $f|_A = g|_A$, then it is natural to expect that $f = g$. Unfortunately, this need not hold:

Example 7.1.1 (Line with two origins). As a set, let $Y = (\mathbb{R} \setminus \{0\}) \sqcup \{0_1, 0_2\}$. For $i = 1, 2$, let $f_i\colon \mathbb{R} \to Y$ be the map defined by $f_i(x) = x$ for $x \in \mathbb{R} \setminus \{0\}$ and $f_i(0) = 0_i$. Give Y the identification space topology, so:

- a set $U \subset Y$ is open if and only if $f_1^{-1}(U)$ and $f_2^{-1}(U)$ are open in $\mathbb{R}$.

Here is a picture of Y:



With this topology, the subspaces $Y \setminus \{0_2\} = f_1(\mathbb{R})$ and $Y \setminus \{0_1\} = f_2(\mathbb{R})$ are both homeomorphic to $\mathbb{R}$. The maps $f_1, f_2\colon \mathbb{R} \to Y$ are continuous and agree on the dense set $\mathbb{R} \setminus \{0\}$. However, $f_1 \neq f_2$. □

7.2. Hausdorff spaces

The issue with the line with two origins from Example 7.1.1 is that the points 0_1 and 0_2 do not have disjoint open neighborhoods. To rule this out, say that a space X is *Hausdorff* if for all distinct points $p, q \in X$, there are open neighborhoods U of p and V of q with $U \cap V = \emptyset$. This has a number of nice consequences (see Exercise 7.3):

- All points in X are closed, i.e., for all $p \in X$ the one-point set $\{p\}$ is closed.
- If Z is another space and $f, g\colon Z \to X$ are two maps, then the subset $\{z \in Z \mid f(z) = g(z)\}$ of points in Z where f and g are equal is closed. In particular, if f and g agree on a dense subset of Z, then $f = g$.
- Limits in X are unique in the following sense. Let $\{x_n\}_{n \geq 1}$ be a sequence in X and let $y_1, y_2 \in X$ be such that $\lim_{n \mapsto \infty} x_n = y_1$ and $\lim_{n \mapsto \infty} x_n = y_2$. Then $y_1 = y_2$.

Most geometrically natural spaces are Hausdorff. In particular:

Lemma 7.2.1. *Let $(M, \mathfrak{d})$ be a metric space. Then M is Hausdorff.*

Proof. For distinct $p, q \in M$, let $\epsilon = \mathfrak{d}(p, q)/2$. The open balls $B_\epsilon(p)$ and $B_\epsilon(q)$ are disjoint. □

Remark 7.2.2. For an infinite field $\mathbf{k}$, an important non-example is given by the Zariski topology on $\mathbf{k}^n$. See Exercise 7.4. □

7.3. Continuity

For first countable Hausdorff spaces, we can characterize continuity with sequences:

Lemma 7.3.1. *Let X be a first countable Hausdorff space, let Y be a Hausdorff space, and let $f\colon X \to Y$ be a map of sets. Then f is continuous if and only if the following holds:*[1]

[1] There is a version of this result that is true without the Hausdorff assumption, but it is awkward to state since in non-Hausdorff spaces limits need not be unique.

- *Let* $\{x_n\}_{n\geq 1}$ *be a convergent sequence in* X*. Then* $\{f(x_n)\}_{n\geq 1}$ *is a convergent sequence in* Y *and* $\lim_{n\mapsto\infty} f(x_n) = f(\lim_{n\mapsto\infty} x_n)$*.*

Proof. See Exercise 7.2. □

7.4. Normal spaces

In fact, most geometrically natural spaces have even stronger separation properties. A space X is *normal* if it satisfies the following two conditions:

- for all disjoint closed sets $C, D \subset X$, there exist open neighborhoods U of C and V of D with $U \cap V = \emptyset$; and
- all points in X are closed.[2]

All normal spaces are Hausdorff. The key example is:

Lemma 7.4.1. *Let* $(M, \mathfrak{d})$ *be a metric space. Then* M *is normal.*

Proof. Since M is Hausdorff, all points in M are closed. Consider disjoint closed sets $C, D \subset M$. For $z \in M$, let

$$r(z) = \inf\{\mathfrak{d}(z,c) \mid c \in C\} \quad \text{and} \quad s(z) = \inf\{\mathfrak{d}(z,d) \mid d \in D\}.$$

Since C and D are disjoint closed sets, we have $r(d) > 0$ for $d \in D$ and $s(c) > 0$ for $c \in C$. Define

$$U = \bigcup_{c\in C} B_{s(c)/3}(c) \quad \text{and} \quad V = \bigcup_{d\in D} B_{r(d)/3}(d).$$

The sets U and V are open, and $C \subset U$ and $D \subset V$. To prove the lemma, it is enough to show that $U \cap V = \emptyset$. Assume this is false, and let $x \in U \cap V$. We can therefore find $c_0 \in C$ and $d_0 \in D$ such that $\mathfrak{d}(c_0, x) < s(c_0)/3$ and $\mathfrak{d}(d_0, x) < r(d_0)/3$. We either have $s(c_0) \leq r(d_0)$ or $r(d_0) \leq s(c_0)$. Both cases lead to a similar contradiction, so we will give the details for $s(c_0) \leq r(d_0)$. This implies that

$$\mathfrak{d}(c_0, d_0) \leq \mathfrak{d}(c_0, x) + \mathfrak{d}(x, d_0) < s(c_0)/3 + r(d_0)/3 \leq r(d_0)/3 + r(d_0)/3 = \frac{2}{3} r(d_0).$$

However, we also have $\mathfrak{d}(c_0, d_0) \geq \inf\{\mathfrak{d}(d_0, c) \mid c \in C\} = r(d_0)$, a contradiction. □

The following characterization of normality is often useful. Recall that $\overline{V}$ is the closure of V.

Lemma 7.4.2. *A space* X *is normal if and only if all points in* X *are closed and:*

(♠) *For all closed sets* $C \subset X$ *and all open neighborhoods* U *of* C*, there exists an open neighborhood* V *of* C *with* $\overline{V} \subset U$*.*

Proof. Assume first that X is normal. To verify (♠), let $C \subset X$ be closed and let U be an open neighborhood of C. The set $D = X \setminus U$ is then a closed set that is disjoint from C, so by normality there exist disjoint open neighborhoods V and W of C and D. Since $X \setminus W$ is a closed subset of U containing V, it follows that $\overline{V} \subset U$.

Assume now that all points in X are closed and (♠) holds. To verify normality, let $C, D \subset X$ be disjoint closed sets. Applying (♠) to the open neighborhood $U = X \setminus D$ of C, we obtain an open neighborhood V of C with $\overline{V} \subset U$. It follows that V and $W = X \setminus \overline{V}$ are disjoint open neighborhoods of C and D. □

7.5. Urysohn's Lemma

A key feature of normal spaces is that they have a rich supply of continuous real-valued functions. For our first example of this, we need a definition. The *support* of a function $f\colon X \to \mathbb{R}$, denoted $\mathrm{supp}(f)$, is the closure of the set $\{p \in X \mid f(p) \neq 0\}$. We then have:

Theorem 7.5.1 (Urysohn's Lemma). *Let* X *be a normal space, let* $C \subset X$ *be closed, and let* $U \subset X$ *be an open neighborhood of* C*. Then there exists a map* $f\colon X \to [0,1]$ *such that* $f|_C = 1$ *and* $\mathrm{supp}(f) \subset U$*.*

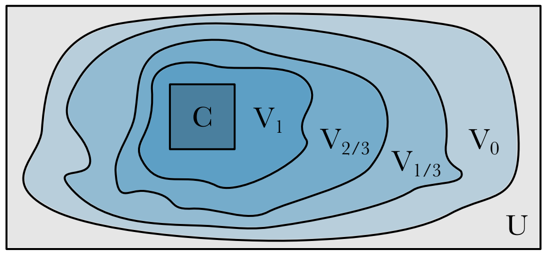
Proof. We must use the open sets provided by normality to construct f. The key is:

[2]This is not always included in the definition of normality, but it ensures that normal spaces are Hausdorff.

Claim. *There exist open sets V_α for all $\alpha \in \mathbb{Q}$ with the following properties:*

(i) For rational $r < s$, we have $\overline{V}_s \subset V_r$.
(ii) $C \subset V_1$ and $\overline{V}_0 \subset U$.
(iii) $V_\alpha = \emptyset$ for $\alpha > 1$ and $V_\alpha = X$ for $\alpha < 0$.

Proof of claim. The picture is as follows:



Define V_α for $\alpha > 1$ and $\alpha < 0$ as in (iii). Next, using Lemma 7.4.2 choose an open neighborhood V_0 of C with $\overline{V}_0 \subset U$ and an open neighborhood V_1 of C with $\overline{V}_1 \subset \overline{V}_0$. Conditions (ii) and (iii) hold, and we inductively construct the remaining V_α satisfying (i) as follows. Enumerate the rational numbers in $[0, 1]$ as $\{\alpha_0, \alpha_1, \ldots\}$ with $\alpha_0 = 0$ and $\alpha_1 = 1$. We have already constructed V_0 and V_1, so assume that $n \geq 2$ and that we have constructed V_{α_m} for $0 \leq m \leq n-1$ satisfying (i). We construct V_{α_n} as follows. Let

$$r = \max\{\alpha_m \mid 0 \leq m \leq n-1,\ \alpha_m < \alpha_n\} \quad \text{and} \quad s = \min\{\alpha_m \mid 0 \leq m \leq n-1,\ \alpha_m > \alpha_n\}.$$

We thus have $r < \alpha_n < s$, and by (iii) we have $\overline{V}_s \subset V_r$. Using Lemma 7.4.2, we can then find an open neighborhood V_{α_n} of $\overline{V}_s$ such that $\overline{V}_{\alpha_n} \subset V_r$. □

We now define a set map $f\colon X \to \mathbb{R}$ via the formula

$$f(p) = \sup\{\alpha \in \mathbb{Q} \mid p \in V_\alpha\}.$$

By (iii) we have $f(p) \in [0, 1]$ for all $p \in X$. Also, by (ii) we have $f(p) = 1$ for $p \in C$ and $\operatorname{supp}(f) \subset U$. All that remains is to check that f is continuous.

Let $W \subset \mathbb{R}$ be open. We must prove that $f^{-1}(W)$ is open. Consider $p \in f^{-1}(W)$. Choose rational $r < s$ such that $[r, s] \subset W$ and $f(p) \in [r, s]$. By (iii), we have $\overline{V}_s \subset V_r$. To prove that $f^{-1}(W)$ is open, it is enough to prove that the open set $V_r \setminus \overline{V}_s$ is contained in $f^{-1}(W)$. To do this, it is enough to prove that f maps $V_r \setminus \overline{V}_s$ into $[r, s] \subset W$. This follows from the following two facts, both of which are immediate from (iii):

- for $q \in V_r$, we have $f(q) \geq r$; and
- for $q \notin V_s$, we have $f(q) \leq s$. □

7.6. Converse to Urysohn

The following lemma shows that the conclusion of Urysohn's lemma characterizes normality:

Lemma 7.6.1. *Let X be a space such that all points in X are closed. For every closed $C \subset X$ and every neighborhood U of C, assume that there exists a continuous map*[3] *$f\colon X \to \mathbb{R}$ with $f|_C = 1$ and $\operatorname{supp}(f) \subset U$. Then X is normal.*

Proof. Let C and D be disjoint closed sets in X. By assumption, there is a continuous map $f\colon X \to \mathbb{R}$ with $f|_C = 1$ and $\operatorname{supp}(f) \subset X \setminus D$. The sets $U = f^{-1}((1/2, \infty))$ and $V = X \setminus \operatorname{supp}(f)$ are then disjoint open neighborhoods of C and D. □

[3] In Urysohn's lemma, the target of f is $[0, 1]$. Here we relax this.

7.7. Strengthening Urysohn

Say that a space X is *perfectly normal* if points in X are closed and for all closed $C \subset X$ and all open neighborhoods U of C, there exists a continuous map $f\colon X \to [0,1]$ such that $f^{-1}(1) = C$ and $\mathrm{supp}(f) \subset U$. Lemma 7.6.1 implies that perfectly normal spaces are normal.

The definition of a perfectly normal space resembles the conclusion of Urysohn's lemma, but there is a small difference: in a perfectly normal space we have $f^{-1}(1) = C$, while in the conclusion of Urysohn's lemma we only have $C \subset f^{-1}(1)$. Most geometrically natural spaces are perfectly normal. In particular:

LEMMA 7.7.1. *Let $(M, \mathfrak{d})$ be a metric space. Then M is perfectly normal.*

PROOF. Lemma 7.4.1 implies that M is normal, and in particular points are closed. Consider $C \subset X$ closed and U an open neighborhood of C. By Urysohn's Lemma, there exists a continuous map $f\colon X \to [0,1]$ such that $f|_C = 1$ and $\mathrm{supp}(f) \subset U$. We want to modify f to ensure it is less than 1 at all points that do not lie in C. Let $g\colon X \to \mathbb{R}$ be the function

$$g(p) = \inf\{\mathfrak{d}(p,c) \mid c \in C\} \quad \text{for } p \in X$$

and let $h\colon X \to [0,1]$ be the function

$$h(p) = \min(g(p), 1) \quad \text{for } p \in X.$$

Both g and h are continuous and satisfy $g^{-1}(0) = h^{-1}(0) = C$. The function $f'\colon X \to [0,1]$ defined by

$$f'(p) = (1 - h(p)) \cdot f(p) \quad \text{for all } p \in M$$

then satisfies $(f')^{-1}(1) = C$ and $\mathrm{supp}(f') \subset U$. □

REMARK 7.7.2. We have introduced the notion of a space being Hausdorff, being normal, and being perfectly normal. These are called *separation axioms*. It is common to call a Hausdorff space a *T_2-space*, a normal space a *T_4-space*, and a perfectly normal space a *T_6-space*. As this terminology suggests, there are many other separation axioms as well.[4]

The vast majority of spaces considered in algebraic topology are perfectly normal. In fact, as we mentioned in Remark 3.6.1 the most natural spaces from the viewpoint of algebraic topology are the so-called CW complexes, and we will prove in Volume 1 that CW complexes are perfectly normal. □

7.8. Uniform limits of functions

Our next goal is to prove the Tietze extension theorem, which says that continuous real-valued functions on closed subsets of normal spaces can be extended to the whole space. The extension we construct will be a limit of functions constructed using Urysohn's Lemma. We therefore need a way to certify that such functions are continuous.

Let X be a space. A sequence of functions $f_n\colon X \to \mathbb{R}$ is said to *converge uniformly* to a function $f\colon X \to \mathbb{R}$ if the following holds:

- for all $\epsilon > 0$, there exists some $N \geq 1$ such that $|f(p) - f_n(p)| < \epsilon$ for all $n \geq N$ and $p \in X$.

We then have the following, which generalizes a familiar fact from real analysis:

LEMMA 7.8.1. *Let X be a space and let $f_n\colon X \to \mathbb{R}$ be a sequence of continuous functions converging uniformly to a function $f\colon X \to \mathbb{R}$. Then f is continuous.*

PROOF. This can be proved using an argument similar to the one used to prove the analogous fact for functions defined on $X = \mathbb{R}$. See Exercise 7.10. □

[4] In fact, not only are there T_k-spaces for $0 \leq k \leq 6$, but there are even $T_{2.5}$-spaces and $T_{3.5}$-spaces.

7.9. Tietze Extension Theorem

We can now prove the Tietze Extension Theorem:

THEOREM 7.9.1 (Tietze Extension Theorem). *Let X be a normal space, let $C \subset X$ be closed, and let $f\colon C \to \mathbb{R}$ be a continuous function. Then f can be extended to a continuous function $F\colon X \to \mathbb{R}$. Moreover, if the image of f lies in a closed interval $[a,b]$ then F can be chosen such that its image also lies in $[a,b]$.*

PROOF. We first prove the case where f is bounded, and then derive the unbounded case.

CASE 1. *The theorem holds if the image of f lies in a closed interval $[a,b]$.*

Since $[a,b] \cong [-1,1]$, we can assume without loss of generality that $[a,b] = [-1,1]$. For $n \geq 1$, we will construct continuous functions $G_n\colon X \to \mathbb{R}$ such that letting $F_n = G_1 + \cdots + G_n$, we have:

(i) The function F_n satisfies $|f(p) - F_n(p)| \leq (2/3)^n$ for all $p \in C$.
(ii) The function G_n satisfies $|G_n(p)| \leq (1/3)(2/3)^{n-1}$ for all $p \in X$.

Condition (ii) will imply that the functions $F_n = G_1 + \cdots + G_n$ converge uniformly to a function F such that

$$|F(p)| \leq \frac{1}{3}\left(1 + (2/3) + (2/3)^2 + \cdots\right) = \frac{1}{3}\left(\frac{1}{1-2/3}\right) = 1 \quad \text{for all } p \in X.$$

Lemma 7.8.1 implies that $F\colon X \to [-1,1]$ is continuous, and condition (i) implies that $F|_C = f$.

It remains to construct the G_n. Assume that $n \geq 1$ and we have constructed $G_1, \ldots, G_{n-1}$ satisfying (ii) such that letting $F_{n-1} = G_1 + \cdots + G_{n-1}$, we have

$$|f(p) - F_{n-1}(p)| \leq (2/3)^{n-1} \quad \text{for all } p \in C. \tag{7.9.1}$$

This is vacuous for $n = 1$. We will construct G_n as follows. Let

$$L = \left\{p \in C \mid f(p) - F_{n-1}(p) \leq -(1/3)(2/3)^{n-1}\right\}$$
$$R = \left\{p \in C \mid f(p) - F_{n-1}(p) \geq (1/3)(2/3)^{n-1}\right\}.$$

The sets L and R are disjoint closed sets. Using Urysohn's lemma, we can find:

- a continuous map $h_L\colon X \to [0,1]$ with $h_L|_L = 1$ and $\text{supp}(h_L) \subset X \setminus R$; and
- a continuous map $h_R\colon X \to [0,1]$ with $h_R|_R = 1$ and $\text{supp}(h_R) \subset X \setminus L$.

Let $G_n\colon X \to [-(1/3)(2/3)^{n-1}, (1/3)(2/3)^{n-1}]$ be the map

$$G_n = -(1/3)(2/3)^{n-1}h_L + (1/3)(2/3)^{n-1}h_R.$$

By construction, G_n satisfies (ii). To show that $F_n = F_{n-1} + G_n$ satisfies (i), consider some $p \in C$. There are three cases:

- If $p \in L$, then by (7.9.1) we have

$$|f(p) - F_n(p)| = |f(p) - F_{n-1}(p) + (1/3)(2/3)^{n-1}| \leq (2/3)^{n-1} - (1/3)(2/3)^{n-1} = (2/3)^n.$$

- If $p \in R$, then by (7.9.1) we have

$$|f(p) - F_n(p)| = |f(p) - F_{n-1}(p) - (1/3)(2/3)^{n-1}| \leq (2/3)^{n-1} - (1/3)(2/3)^{n-1} = (2/3)^n.$$

- If $p \notin L \cup R$, then by definition we have $|f(p) - F_{n-1}(p)| \leq (1/3)(2/3)^{n-1}$, so since $|G_n(p)| \leq (1/3)(2/3)^{n-1}$ we have

$$|f(p) - F_n(p)| = |f(p) - F_{n-1}(p) - G_n(p)| \leq (1/3)(2/3)^{n-1} + (1/3)(2/3)^{n-1} = (2/3)^n.$$

In all three cases, (ii) is satisfied. The theorem follows.

CASE 2. *The theorem holds in general.*

Since $\mathbb{R} \cong (-1,1)$, it is enough to prove that every continuous function $f\colon C \to (-1,1)$ can be extended to a continuous function $F\colon X \to (-1,1)$. By Case 1, we can extend f to a continuous function $F'\colon X \to [-1,1]$. Our goal is to modify F' such that its image does not contain -1 or 1. Set $U = (F')^{-1}((-1,1))$. Applying Urysohn's Lemma (Theorem 7.5.1), there exists a continuous function $g\colon X \to [0,1]$ with $g|_C = 1$ and $\text{supp}(g) \subset U$. The product $F = g \cdot F'$ then still extends f and satisfies $F(X) \subset (-1,1)$. □

7.10. Exercises

Exercise 7.1. Let X be Hausdorff and let $x_1, \dots, x_n \in X$ be distinct points. Prove that there exist open neighborhoods U_i of the x_i such that $U_i \cap U_j = \emptyset$ for all $1 \le i, j \le n$ distinct. □

Exercise 7.2. Let X be a first countable Hausdorff space, let Y be a Hausdorff space, and let $f\colon X \to Y$ be a map of sets. Then f is continuous if and only if the following holds:

- Let $\{x_n\}_{n\ge 1}$ be a convergent sequence in X. Then $\{f(x_n)\}_{n\ge 1}$ is a convergent sequence in Y and $\lim_{n\mapsto\infty} f(x_n) = f(\lim_{n\mapsto\infty} x_n)$. □

Exercise 7.3. Let X be a Hausdorff space. Prove the following:

(a) All points in X are closed, i.e., for all $p \in X$ the one-point set $\{p\}$ is closed.
(b) If Z is another space and $f, g\colon Z \to X$ are two maps, then the subset $\{z \in Z \mid f(z) = g(z)\}$ of points in Z where f and g are equal is closed. In particular, if f and g agree on a dense subset of Z, then $f = g$.
(c) Let $\{x_n\}_{n\ge 1}$ be a sequence in X and let $y_1, y_2 \in X$ be such that $\lim_{n\mapsto\infty} x_n = y_1$ and $\lim_{n\mapsto\infty} x_n = y_2$. Then $y_1 = y_2$. □

Exercise 7.4. Let $\mathbf{k}$ be a field. Prove that the Zariski topology on $\mathbf{k}^n$ described in Example 3.5.2 is Hausdorff if and only if $\mathbf{k}$ is a finite field. □

Exercise 7.5. As a set, let $X = \mathbb{R}$. Let $\mathcal{B}$ be the set of open subsets of $X = \mathbb{R}$ in the standard topology on $\mathbb{R}$ and let

$$\mathcal{B}' = \mathcal{B} \cup \{U \cap \mathbb{Q} \mid U \in \mathcal{B}\}.$$

Prove the following:

- The set $\mathcal{B}'$ is the basis for a topology on X that we will call the $\mathcal{B}'$-topology.
- The $\mathcal{B}'$-topology on X is Hausdorff and all points are closed.
- The $\mathcal{B}'$-topology on X is not normal. □

Exercise 7.6. Let X be a Hausdorff space and let $Y \subset X$ be a subspace. Prove that Y is Hausdorff. □

Exercise 7.7. Let X be a normal space and let $Y \subset X$ be a closed subspace. Prove that Y is normal. We remark that this need not hold if Y is not closed. See [1] for examples. □

Exercise 7.8. Say that a space X is *completely normal* if every subspace of X is normal. Prove that X is completely normal if and only if for every pair of subsets $A, B \subset X$ with $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$, there exist disjoint open neighborhoods U and V of A and B. □

Exercise 7.9. Let $f\colon X \to Y$ be a quotient map that is also closed. Assume that X is normal. Prove that Y is normal. □

Exercise 7.10. Let X be a space and let $f_n\colon X \to \mathbb{R}$ be a sequence of continuous functions converging uniformly to a function $f\colon X \to \mathbb{R}$. Prove that f is continuous. □

Exercise 7.11. Let X be a connected normal space containing more than one point. Prove that X has uncountably many points. □

Bibliography

[1] L. Steen & J. Seebach Jr., *Counterexamples in topology*, second edition, Springer, New York-Heidelberg, 1978.

CHAPTER 8

Compactness and the Heine–Borel theorem

We now introduce the key concept of compactness, which generalizes the notion of compactness for subsets of $\mathbb{R}$ and $\mathbb{R}^n$ from real analysis.

8.1. Compactness

Let X be a space and let $K \subset X$. An *open cover* of K is a collection $\mathfrak{U}$ of open sets in X such that $K \subset \cup_{U \in \mathfrak{U}} U$. The open cover $\mathfrak{U}$ is *finite* if it consists of finitely many open sets. A *subcover* of an open cover $\mathfrak{U}$ is a subset $\mathfrak{U}' \subset \mathfrak{U}$ that is still a cover. The subspace K is *compact* if every open cover of K has a finite subcover. In particular, X itself is compact if every open cover of X has a finite subcover.

8.2. Compactness and closed sets

Compactness behaves best for Hausdorff spaces. In fact, in some treatments of point-set topology a space is said to be quasi-compact if each open cover has a finite subcover, and a compact space is a space that is Hausdorff and quasi-compact. For Hausdorff spaces, we have:

LEMMA 8.2.1. *Let X be a Hausdorff space and let $K \subset X$ be compact. Then K is closed.*

PROOF. We must prove that $X \setminus K$ is open. Consider $p \in X \setminus K$. Since X is Hausdorff, for each $k \in K$ there are disjoint open neighborhoods U_k and V_k of p and k. Since K is compact, we can find finitely many points $k_1, \ldots, k_n \in K$ such that $\{V_{k_1}, \ldots, V_{k_n}\}$ is an open cover of K. Letting $U = U_{k_1} \cap \cdots \cap U_{k_n}$, the set U is an open neighborhood of p that is disjoint from K, as desired. □

For all spaces, we have:

LEMMA 8.2.2. *Let X be a space, let $K \subset X$ be compact, and let C be a closed subset of X with $C \subset K$. Then C is compact.*

PROOF. Let $\mathfrak{U}$ be an open cover of $C \subset X$. The set $\{X \setminus C\} \cup \mathfrak{U}$ is an open cover of K. Since K is compact, it has a finite subcover. Removing $X \setminus C$ from this finite subcover if necessary, we obtain a finite subcover of $\mathfrak{U}$. □

As another indication of how strong an assumption being compact Hausdorff is, we have:

LEMMA 8.2.3. *Let X be a compact Hausdorff space. Then X is normal.*

PROOF. See Exercise 8.2. □

8.3. Compactness and functions

Continuous maps take compact sets to compact sets:

LEMMA 8.3.1. *Let $f\colon X \to Y$ be a map of spaces and let $K \subset X$ be compact. Then $f(K)$ is compact.*

PROOF. See Exercise 8.4 □

This has the following corollary:

COROLLARY 8.3.2. *Let $f\colon X \to Y$ be a map of spaces with Y Hausdorff. Then f is a closed map.*

Proof. Let $C \subset X$ be closed. Since X is compact, C is compact. It follows that $f(C)$ is compact, so since Y is Hausdorff $f(C)$ is closed. □

Another important property of compact sets is that real-valued functions on them are bounded and attain maximum and minimum values:

Lemma 8.3.3. *Let X be a compact space and let $f\colon X \to \mathbb{R}$ be a map. Then there exist real numbers $m \leq M$ such that:*

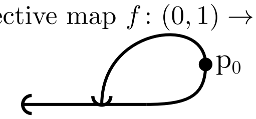
- *for all $p \in X$, we have $m \leq f(p) \leq M$; and*
- *there exists $p_0, q_0 \in X$ such that $m = f(p_0)$ and $M = f(q_0)$.*

Proof. By Lemma 8.3.1, the image $K = f(X)$ is a compact subset of $\mathbb{R}$. The lemma now follows from the following standard fact about compact subsets of $\mathbb{R}$: there exist $m, M \in K$ such that $m \leq k \leq M$ for all $k \in K$ (see Exercise 8.5). □

8.4. Compactness and injective maps

For general spaces X and Y, an injective map $f\colon X \to Y$ need not be an embedding, i.e., a homeomorphism onto its image. Here is an example:

Example 8.4.1. Consider the injective map $f\colon (0,1) \to \mathbb{R}^2$ whose image X is as follows:



This is not an embedding; indeed, for every $p \in (0,1)$ the space $(0,1) \setminus \{p\}$ is disconnected but for the indicated point $p_0 \in X$ we have $X \setminus \{p_0\}$ connected. □

However, if X is compact and Y is Hausdorff this pathology does not occur:

Lemma 8.4.2. *Let X be a compact space, let Y be a Hausdorff space, and let $f\colon X \to Y$ be an injective map. Then f is a closed embedding.*

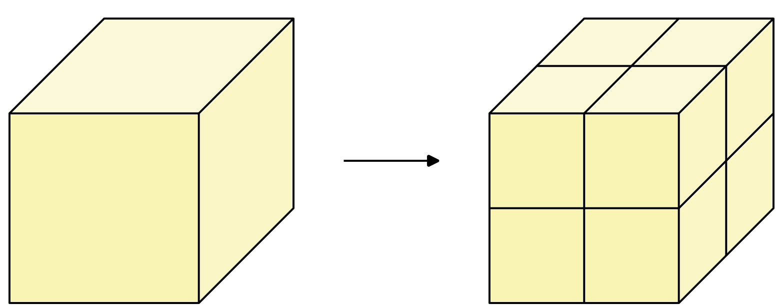
Proof. Immediate from the fact that f is injective and closed (see Corollary 8.3.2). □

8.5. Heine–Borel Theorem

Let $(M, \mathfrak{d})$ be a metric space. A subset $K \subset M$ is *bounded* if there is some $R \geq 0$ such that $\mathfrak{d}(p,q) \leq R$ for all $p, q \in K$. The following theorem gives a large supply of compact spaces:

Theorem 8.5.1 (Heine–Borel Theorem). *Let $K \subset \mathbb{R}^n$ be closed and bounded. Then K is compact.*

Proof. For some $D \gg 0$, the set K is contained in the cube $[-D,D]^n$. By Lemma 8.2.2, it is enough to prove that $[-D,D]^n$ is compact. Since all cubes in $\mathbb{R}^n$, are homeomorphic, it is actually enough to prove that the unit cube $C_1 = [0,1]^n$ is compact. Let $\mathfrak{U}$ be an open cover of C_1. For the sake of contradiction, assume that it has no finite subcover. Divide C_1 into 2^n subcubes with side lengths $1/2$:



The cover $\mathfrak{U}$ is a cover of each of these subcubes. Since no finite subset of $\mathfrak{U}$ covers C_1, it must the case that among these 2^n subcubes there is a subcube C_2 such that no finite subset of $\mathfrak{U}$ covers C_2. This process can then be repeated: C_2 can be divided into 2^n subcubes with side length $1/2^2$, and there among these there must exist a subcube C_3 such that no finite subset of $\mathfrak{U}$ covers it. We then divide C_3 into 2^n subcubes with side lengths $1/2^3$, etc. This procedure gives a nested sequence

$$C_1 \supset C_2 \supset C_3 \supset \cdots$$

of cubes with the following properties:

- the cube C_n has side lengths $1/2^n$; and
- no finite subset of $\mathfrak{U}$ covers any of the C_n.

By the completeness of $\mathbb{R}$, the intersection $\cap_{n=1}^{\infty} C_n$ must consist of a single point p. Pick $U \in \mathfrak{U}$ such that $p \in U$. Since U is open, for some $\epsilon > 0$ the ϵ-ball around p must be contained in U. This implies that for $n \gg 0$ we have $C_n \subset U$, contradicting the fact that no finite subset of $\mathfrak{U}$ covers any C_n. □

REMARK 8.5.2. A metric space in which closed and bounded subsets are compact is called a *proper* metric space. □

8.6. Compactness and intersections of closed sets

The following is a useful rephrasing of the definition of compactness:

LEMMA 8.6.1. *Let X be a space. The X is compact if and only if the following holds for all sets $\mathfrak{C}$ of closed subsets of X:*

(∗) *If for all finite subsets $\mathfrak{C}' \subset \mathfrak{C}$ we have $\bigcap_{C \in \mathfrak{C}'} C \neq \emptyset$, then $\bigcap_{C \in \mathfrak{C}} C \neq \emptyset$.*

PROOF. The condition (∗) is equivalent to:

(∗′) If $\bigcap_{C \in \mathfrak{C}} C = \emptyset$, then there exists a finite subset $\mathfrak{C}' \subset \mathfrak{C}$ such that $\bigcap_{C \in \mathfrak{C}'} C = \emptyset$.

There is a bijection between sets of closed subsets of X and sets of open subsets of X taking a set $\mathfrak{C}$ of closed subsets to $\mathfrak{U}(\mathfrak{C}) = \{X \setminus C \mid C \in \mathfrak{C}\}$. A set $\mathfrak{C}$ of closed subsets of X has empty intersection exactly when $\mathfrak{U}(\mathfrak{C})$ covers X. It follows (∗′) is equivalent to saying that if $\mathfrak{U}(\mathfrak{C})$ is a cover of X, then $\mathfrak{U}(\mathfrak{C})$ has a finite subcover. □

This has the following immediate corollary:

COROLLARY 8.6.2. *Let X be a space and let $C_1 \supset C_2 \supset \cdots$ be a nested sequence of nonempty compact subspaces of X. Then $\cap_{n \geq 1} C_n \neq \emptyset$.*

8.7. Lebesgue number

If M is a metric space and $\mathfrak{U}$ is an open cover of M, then a *Lebesgue number* for $\mathfrak{U}$ is an $\epsilon > 0$ such that for all $p \in M$ there exists some $U \in \mathfrak{U}$ such that the ϵ-ball $B_\epsilon(p)$ is contained in U. The following basic result shows that these always exist if M is compact:

LEMMA 8.7.1 (Lebesgue number lemma). *Let M be a compact metric space and let $\mathfrak{U}$ be an open cover of M. Then $\mathfrak{U}$ has a Lebesgue number.*

PROOF. Since M is compact, we can write M as

$$M = B_{\epsilon_1}(p_1) \cup \cdots \cup B_{\epsilon_n}(p_n) \quad \text{for some } p_1, \ldots, p_n \in M \text{ and } \epsilon_1, \ldots, \epsilon_n > 0$$

such that for each $1 \leq i \leq n$ there is some $U \in \mathfrak{U}$ with $B_{2\epsilon_i}(p_i) \subset U$. Set $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$, and consider $p \in M$. We have $p \in B_{\epsilon_i}(p_i)$ for some $1 \leq i \leq n$. By assumption, there is some $U \in \mathfrak{U}$ with $B_{2\epsilon_i}(p_i) \subset U$. The triangle inequality implies that $B_\epsilon(p) \subset B_{2\epsilon_i}(z_i)$ and thus $B_\epsilon(p) \subset U$. □

8.8. Compactness and limits

If X is a space and $\{x_n\}_{n \geq 1}$ is a sequence in X, then a *subsequence* of $\{x_n\}_{n \geq 1}$ is a sequence of the form $\{x_{n_i}\}_{i \geq 1}$ with $n_1 < n_2 < \cdots$ a strictly increasing sequence of natural numbers. A subspace $K \subset X$ is *sequentially compact* if every sequence in K has a subsequence that converges to a point of K. With appropriate countability assumptions, this is equivalent to compactness. We divide this into two results:

LEMMA 8.8.1. *Let X be a first countable space and let $K \subset X$ be compact. Then K is sequentially compact.*

PROOF. See Exercise 8.3. □

LEMMA 8.8.2. *Let X be a second countable space and let $K \subset X$ be sequentially compact. Then K is compact.*

PROOF. See Exercise 8.3. □

Similarly, for metric spaces compactness and sequential compactness are the same:

LEMMA 8.8.3. *Let $(M, \mathfrak{d})$ be a metric space and let $K \subset X$. Then K is compact if and only if K is sequentially compact.*

PROOF. Since M is first first countable, Lemma 8.8.1 implies that compact subsets of M are sequentially compact. For the converse, we can replace M by the subspace in question and prove that if M is sequentially compact, then M is compact. By Lemma 8.8.2 it is enough to prove that M is second countable, which by Lemma 6.5.2 is equivalent to proving that M is separable, i.e., that M has a countable dense subset.

Since M is sequentially compact, it cannot contain an infinite discrete subspace. In particular, for each $n \geq 1$ there does not exist an infinite subset $T \subset M$ with $\mathfrak{d}(t_1, t_2) \geq 1/n$ for all distinct $t_1, t_2 \in T$. For each $n \geq 1$, we can therefore find a finite set S_n such that for all $p \in M$ there exists some $s \in S_n$ with $\mathfrak{d}(p, s) < 1/n$. The set $\cup_{n \geq 1} S_n$ is then a countable dense subset of M. □

8.9. Exercises

EXERCISE 8.1. Let X be a space and let $C_1, \ldots, C_n \subset X$ be compact subspaces. Prove that $C_1 \cup \cdots \cup C_n$ is compact. □

EXERCISE 8.2. Let X be a compact Hausdorff space. Prove that X is normal. □

EXERCISE 8.3. Let X be a space and $K \subset X$ be a subspace. Prove:

(a) If X is first countable and K is compact, then K is sequentially compact.
(b) If X is second countable and K is sequentially compact, then K is compact. □

EXERCISE 8.4. Let $f \colon X \to Y$ be a map of spaces and let $K \subset X$ be compact. Prove that $f(K)$ is compact. □

EXERCISE 8.5. Let $K \subset \mathbb{R}$ be compact. Prove that there exist $m, M \in K$ such that $m \leq k \leq M$ for all $k \in K$. □

EXERCISE 8.6. Let $(M, \mathfrak{d}_M)$ and $(N, \mathfrak{d}_N)$ be metric spaces and let $f \colon M \to N$ be a map. We say that f is *uniformly continuous* if for all $\epsilon > 0$ there exists some $\delta > 0$ such that if $p, q \in M$ satisfy $\mathfrak{d}_M(p, q) < \delta$ then $\mathfrak{d}_N(f(p), f(q)) < \epsilon$. Use the Lebesgue number lemma to prove that if M is compact then all continuous maps $f \colon M \to N$ are uniformly continuous. □

EXERCISE 8.7. Let X and Y be spaces. Give $X \times Y$ the product topology (see Exercise 3.4). Let $U \subset X \times Y$ be open. Let $A \subset X$ and $K \subset Y$ be such that $A \subset K \subset U$. Assume that K is compact. Prove that there exists an open neighborhood V of A such that $A \times K \subset U$. We remark that this is often called the "tube lemma". □

EXERCISE 8.8. Let $f \colon X \to Y$ be a map of spaces (not necessarily continuous) with Y compact Hausdorff. Give $X \times Y$ the product topology (see Exercise 3.4). Define the *graph* of f to be

$$\Gamma_f = \{(x, f(x)) \in X \times Y \mid x \in X\}.$$

Prove that f is continuous if and only if Γ_f is a closed subset of $X \times Y$. □

EXERCISE 8.9. Let X and Y be spaces with Y compact. Give $X \times Y$ the product topology (see Exercise 3.4). Let $\pi \colon X \times Y \to X$ be the projection onto the first factor. Prove that π is a closed map. □

CHAPTER 9

Local compactness and the Baire category theorem

We now turn to the local version of compactness.

9.1. Local compactness

Let X be a space. Recall that a general neighborhood of $p \in X$ is a set $Z \subset X$ with $p \in \mathrm{Int}(Z)$. The space X is *locally compact* if the following holds for all $p \in X$:

- For all open neighborhoods U of p, there exists a compact neighborhood K of p with $K \subset U$.

For Hausdorff spaces, this is easier to understand:

LEMMA 9.1.1. *Let X be a Hausdorff space. Then X is locally compact if and only if for all $p \in X$, there exists a compact neighborhood K of p. In particular, if X is compact then X is locally compact.*

PROOF. See Exercise 9.2. □

REMARK 9.1.2. Local compactness is poorly behaved for non-Hausdorff spaces, and not all sources agree on the right definition for non-Hausdorff spaces. □

EXAMPLE 9.1.3. If X is either an open or a closed subspace of $\mathbb{R}^n$, then the Heine–Borel Theorem (Theorem 8.5.1) implies that X is locally compact. □

9.2. One-point compactification

Let X be a space. A *compactification* of X is a compact space $\widehat{X}$ containing X as an open dense subspace.

EXAMPLE 9.2.1. The space $\mathbb{S}^n$ is a compactification of $\mathbb{R}^n$. Indeed, for all $p_0 \in \mathbb{S}^n$ we have $\mathbb{S}^n \setminus p_0 \cong \mathbb{R}^n$ (see Exercise 9.6). □

If X is locally compact Hausdorff, there is a natural way to compactify X that generalizes the compactification $\mathbb{S}^n$ of $\mathbb{R}^n$. As a set, let $\widehat{X} = X \sqcup \{\infty\}$ with ∞ a formal symbol that does not lie in X. Say that $U \subset \widehat{X}$ is open if either:

- U is an open subset of X; or
- $U = (X \setminus C) \cup \{\infty\}$, where $C \subset X$ is compact.

This is a topology (see Exercise 9.5), and the space $\widehat{X}$ is called the *one-point compactification* of X. The following shows that it is indeed a compactification:

LEMMA 9.2.2. *Let X be compact Hausdorff and let $\widehat{X}$ be the one-point compactification of X. Then $\widehat{X}$ is compact Hausdorff, and $\widehat{X}$ is a compactification of X.*

PROOF. See Exercise 9.5. □

9.3. σ-compactness

A space X is *σ-compact* if it is the union of countably many compact subspaces. This condition will be important in the next chapter when we discuss paracompactness and partitions of unity. Here we prove:

LEMMA 9.3.1. *Let X be a Hausdorff space that is second countable and locally compact. Then X is σ-compact.*

PROOF. Let $\mathfrak{B}$ be a countable basis for the topology of X. Set $\mathfrak{U} = \{U \in \mathfrak{B} \mid \overline{U} \text{ is compact}\}$, so $\mathfrak{U}$ is a countable collection of open sets of X. It is enough to prove that $\mathfrak{U}$ covers X. Indeed, consider $p \in X$. We must find some $U \in \mathfrak{U}$ with $p \in U$. By Lemma 9.1.1, there is a compact neighborhood K of p. Since $p \in \text{Int}(K)$, we can find $U \in \mathfrak{B}$ such that $p \in U$ and $U \subset K$. Since X is Hausdorff the compact set K is closed, so $\overline{U} \subset K$. Since $\overline{U}$ is a closed subset of the compact set K, it follows that $\overline{U}$ is compact and $U \in \mathfrak{U}$, as desired. □

EXAMPLE 9.3.2. If X is either an open or a closed subspace of $\mathbb{R}^n$, then the Heine–Borel Theorem (Theorem 8.5.1) implies that X is σ-compact. □

9.4. Baire category theorem

The following is a surprisingly powerful tool for proving existence theorems:

THEOREM 9.4.1 (Baire category theorem). *Let X be a locally compact Hausdorff space and let $\{U_n\}_{n\geq 1}$ be a collection of open dense subsets of X. Then $\cap_{n\geq 1} U_n$ is dense.*

PROOF. Let $V_0 \subset X$ be a nonempty open set. We must prove that V_0 intersects $\cap_{n\geq 1} U_n$. Since U_1 is open and dense, the set $V_0 \cap U_1$ is open and nonempty. Since X is locally compact and Hausdorff, we can find a nonempty open set V_1 with $\overline{V}_1$ compact such that $\overline{V}_1 \subset V_0 \cap U_1$. The same argument shows that there exists a nonempty open set V_2 with $\overline{V}_2$ compact such that $\overline{V}_2 \subset V_1 \cap U_2$. Repeating this over and over, we find nonempty open sets $\{V_n\}_{n\geq 1}$ with the following property for all $n \geq 1$:

- $\overline{V}_n$ is compact and $\overline{V}_{n+1} \subset V_n \cap U_{n+1}$.

Applying Corollary 8.6.2 to the nested sequence $\overline{V}_1 \supset \overline{V}_2 \supset \overline{V}_3 \supset \cdots$ of nonempty compact subspaces of X, we see that their intersection must be nonempty, i.e., there exists some p with $p \in \overline{V}_n$ for all $n \geq 1$. By construction, p lies in both V_0 and $\cap_{n\geq 1} U_n$, as desired. □

REMARK 9.4.2. The word "category" in the Baire category theorem has nothing to do with category theory. Instead, it refers to the following archaic terminology: a space X is of the *first category* if it is the union of countably many nowhere dense[1] sets, and is of the *second category* otherwise. The conclusion of the Baire category theorem then is equivalent to saying that every nonempty open set in X is of the second category. □

9.5. Complete metric spaces

A space X is a *Baire space* if all countable intersections of open dense subsets of X are dense. Theorem 9.4.1 says that locally compact Hausdorff spaces are Baire spaces. For another useful class of such spaces, consider a metric space $(M, \mathfrak{d})$. A *Cauchy sequence* in M is a sequence $\{p_n\}_{n\geq 1}$ such that for all $\epsilon > 0$ there exists some $N \geq 1$ such that $\mathfrak{d}(p_n, p_m) < \epsilon$ for all $n, m \geq N$. The metric space M is *complete* if all Cauchy sequences in M have limits. For instance, $\mathbb{R}^n$ is complete (Exercise 9.8). We have:

THEOREM 9.5.1 (Baire category theorem′). *Let M be a complete metric space. Then M is a Baire space.*

PROOF. This is similar to the proof of Theorem 9.4.1, so we leave it as Exercise 9.9. □

9.6. Application: nowhere differentiable functions

To illustrate how the Baire category theorem can be used, we prove the following classic result:

THEOREM 9.6.1. *Let $\mathcal{C}(I, \mathbb{R})$ be the set of continuous functions $f \colon I \to \mathbb{R}$. Let $\mathfrak{d}(f, g) = \max\{|f(x) - g(x)| \mid x \in I\}$ be the standard metric on $\mathcal{C}(I, \mathbb{R})$. Then the set of nowhere-differentiable functions on is dense in $\mathcal{C}(I, R)$.*

[1] A subset A of a topological space is *nowhere dense* if $\overline{A}$ contain no nonempty open sets, i.e., if $\text{Int}(\overline{A}) = \emptyset$.

Proof. For each $n \geq 1$, let U_n be the set of all continuous functions $f\colon I \to \mathbb{R}$ satisfying:

(♠) There exists $0 < \delta < 1/n$ and $\lambda > 0$ such that for all $x \in I$, there exists some $y \in I$ with $\delta < |x-y| < 1/n$ and $\left|\frac{f(x)-f(y)}{x-y}\right| > n + \lambda$.

In the three steps below, we will prove that U_n is open (Step 1), we will construct a family of function in U_n (Step 2), and we will show that U_n is dense (Step 3). Since $\mathcal{C}(I,\mathbb{R})$ is a complete metric space, Theorem 9.5.1 will then apply and show that $\Lambda = \cap_{n\geq 1} U_n$ is dense in $\mathcal{C}(I,\mathbb{R})$. Each $f \in \Lambda$ is nowhere differentiable; indeed, for $x \in I$ the condition (♠) forces $\lim_{y\mapsto x} \frac{f(x)-f(y)}{x-y}$ to not exist.

Step 1. *For all $n \geq 1$, the set U_n is open in $\mathcal{C}(I,\mathbb{R})$.*

Consider $f \in U_n$. Let $0 < \delta < 1/n$ and $\lambda > 0$ be the constants for f from (♠). Let $g \in \mathcal{C}(I,\mathbb{R})$ be such that $\mathfrak{d}(f,g) < \lambda\delta/4$. We claim that $g \in U_n$. Indeed, consider $x \in I$. Choose $y \in I$ such that $\delta < |x-y| < 1/n$ and $\left|\frac{f(x)-f(y)}{x-y}\right| > n+\lambda$. We then have

$$\begin{aligned}\left|\frac{g(x)-g(y)}{x-y}\right| \geq& \left|\frac{f(x)-f(y)}{x-y}\right| - \left|\frac{g(x)-f(x)}{x-y}\right| - \left|\frac{g(y)-f(y)}{x-y}\right| \\ >&(n+\lambda) - 2\frac{\lambda\delta/4}{\delta} = n + \lambda/2.\end{aligned}$$

It follows that g satisfies (♠) with the constants δ and $\lambda/2$, so $g \in U_n$.

Step 2. *For some $n \geq 1$, let $g\colon I \to \mathbb{R}$ be a piecewise-linear continuous function such that $|g'(x)| > n$ for all $x \in I$ where g is differentiable. Then $g \in U_n$.*

Let $0 = a_0 < a_1 < \cdots < a_m = 1$ be a partition of I such that $g|_{[a_i,a_{i+1}]}$ is linear for all $0 \leq i < m$. For each $0 \leq i < m$, let $c_i, d_i \in \mathbb{R}$ be the constants such that $g(x) = c_i x + d_i$ for all $x \in [a_i, a_{i+1}]$. By assumption, $|c_i| > n$ for all $0 \leq i < m$. Pick $\lambda > 0$ such that $|c_i| > n + \lambda$ for all $0 \leq i < m$. Also, pick $0 < \delta < 1/n$ such that $\delta < (a_{i+1} - a_i)/2$ for all $0 \leq i < m$. Consider some $x \in I$. We have $x \in [a_{i_0}, a_{i_0+1}]$ for some $0 \leq i_0 < m$. Since $0 < \delta < (a_{i_0+1} - a_{i_0})/2$, we can choose some $y \in [a_{i_0}, a_{i_0+1}]$ such that $\delta < |x-y| < 1/n$. It follows that

$$\left|\frac{g(x)-g(y)}{x-y}\right| = \left|\frac{(c_i x + d_i) - (c_i y + d_i)}{x-y}\right| = |c_i| > n + \lambda,$$

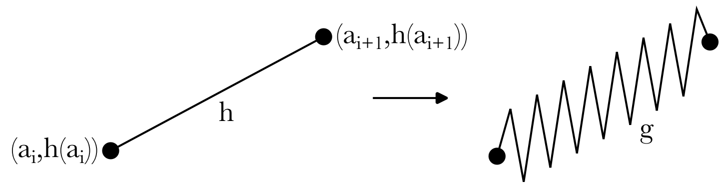
proving that g satisfies (♠) and thus $g \in U_n$.

Step 3. *For all $n \geq 1$, the set U_n is dense in $\mathcal{C}(I,\mathbb{R})$.*

Consider $f \in \mathcal{C}(I,\mathbb{R})$ and $\epsilon > 0$. We must find some $g \in U_n$ such that $\mathfrak{d}(f,g) < \epsilon$. Since f is uniformly continuous on I, we can choose a partition $0 = a_0 < a_1 < \cdots < a_m = 1$ of I such that for all $0 \leq i < m$ and $x \in [a_i, a_{i+1}]$ we have $|f(x) - f(a_i)| < \epsilon/4$. Let $h\colon I \to \mathbb{R}$ be the piecewise-linear continuous function that is linear on each $[a_i, a_{i+1}]$ and satisfies $h(a_i) = f(a_i)$ for $0 \leq i \leq m$. For $x \in [a_i, a_{i+1}]$, we therefore have

$$\begin{aligned}|h(x) - f(x)| &= \left|\frac{f(a_{i+1}) - f(a_i)}{a_{i+1} - a_i}(x - a_i) + f(a_i) - f(x)\right| \\ &\leq \left|\frac{f(a_{i+1}) - f(a_i)}{a_{i+1} - a_i}\right| |x - a_i| + |f(a_i) - f(x)| \\ &\leq |f(a_{i+1}) - f(a_i)| + |f(a_i) - f(x)| < \epsilon/4 + \epsilon/4 = \epsilon/2.\end{aligned}$$

It follows that $\mathfrak{d}(f,h) < \epsilon/2$. As in the following figure, we can find a piecewise-linear continuous function $g\colon I \to \mathbb{R}$ with $\mathfrak{d}(g,h) < \epsilon/2$ and $|g'(x)| > n$ for all $x \in I$ where g is differentiable by changing h on each interval $[a_i, a_{i+1}]$ to a function whose graph is a rapidly osculating sawtooth:



We have $\mathfrak{d}(f, g) \leq \mathfrak{d}(f, h) + \mathfrak{d}(h, g) < \epsilon$, and by Step 2 we have $g \in U_n$. □

9.7. Exercises

EXERCISE 9.1. Find an example of a locally compact Hausdorff space X that has a subspace $Y \subset X$ that is not locally compact. □

EXERCISE 9.2. Let X be a Hausdorff space. Prove that X is locally compact if and only if for all $p \in X$, there exists a compact neighborhood K of p. □

EXERCISE 9.3. Let X be a locally compact space. Let $K \subset X$ be compact and $U \subset X$ be open with $K \subset U$. Prove that there is a compact neighborhood L of K with $L \subset U$. □

EXERCISE 9.4. Let X be a locally compact Hausdorff space, let $K \subset X$ be a compact subspace, and let $f\colon K \to \mathbb{R}$ be a continuous function. Prove that f can be extended to a continuous function $F\colon X \to \mathbb{R}$ with $\operatorname{supp}(F)$ compact. We remark that X might not be normal, so you can't just apply the Tietze extension theorem. □

EXERCISE 9.5. Let X be a compact Hausdorff space. Recall that the one-point compactification of X is the set $\widehat{X} = X \sqcup \{\infty\}$ with the following topology: a set $U \subset \widehat{X}$ is open if either:

- U is an open subset of X; or
- $U = (X \setminus C) \cup \{\infty\}$, where $C \subset X$ is compact.

Prove the following:

(a) This is a topology.
(b) The space $\widehat{X}$ is compact Hausdorff.
(c) The space X is a compactification of X. □

EXERCISE 9.6. Prove the following:

(a) For all $p_0 \in \mathbb{S}^n$ we have $\mathbb{S}^n \setminus p_0 \cong \mathbb{R}^n$.
(b) The space $\mathbb{S}^n$ is a compactification of $\mathbb{R}^n$.
(c) The space $\mathbb{S}^n$ is homeomorphic to the one-point compactification of X. □

EXERCISE 9.7. Let X be the one-point compactification of $\mathbb{Z}$. Prove that X is homeomorphic to $\{0\} \cup \{1/n \mid n \in \mathbb{Z} \text{ nonzero}\} \subset \mathbb{R}$. □

EXERCISE 9.8. Prove that $\mathbb{R}^n$ with it standard metric is complete, i.e., that all Cauchy sequences in $\mathbb{R}^n$ have limits. □

EXERCISE 9.9. Let M be a complete metric space. Prove that M is a Baire space, i.e., that the following holds. Let $\{U_n\}_{n \geq 1}$ be a collection of open dense subsets of X. Then $\cap_{n \geq 1} U_n$ is dense. □

EXERCISE 9.10. Recall that a subset A of a space is nowhere dense if $\overline{A}$ contain no nonempty open sets, i.e., if $\operatorname{Int}(\overline{A}) = \emptyset$. Prove that $\mathbb{R} \setminus \mathbb{Q}$ cannot be written as a coutable union of nowhere dense sets. □

CHAPTER 10

Paracompactness and partitions of unity

We now turn to paracompactness, which is a condition that ensure the existence of what are called partitions of unity. These play a basic role in algebraic topology, especially in the theory of manifolds.

10.1. Locally finite collections of subsets

Let X be a space and let $\mathfrak{Z}$ be a collection of subsets X. We say that $\mathfrak{Z}$ is *locally finite* if for all $p \in X$, there are only finitely many $Z \in \mathfrak{Z}$ such that $p \in Z$. One nice property of locally finite collections of open sets is:

LEMMA 10.1.1. *Let X be a space and let $\mathfrak{Z}$ be a locally finite collection of subsets of X. Then*

$$\overline{\bigcup_{Z \in \mathfrak{Z}} Z} = \bigcup_{Z \in \mathfrak{Z}} \overline{Z}.$$

PROOF. See Exercise 10.1. In that exercise, you will also show that this is false without the local finiteness assumption. □

10.2. Paracompactness

Now let $\mathfrak{U}$ be an open cover of X. A *refinement* of $\mathfrak{U}$ is an open cover $\mathfrak{V}$ such that for all $V \in \mathfrak{V}$, there exists some $U \in \mathfrak{U}$ with $V \subset U$. A space X is *paracompact* if it is Hausdorff and every open cover of X admits a locally finite refinement. We will prove that this has strong consequences for the topology of X. In particular, X must be normal (see Lemma 10.3.1).

REMARK 10.2.1. Most spaces that appear in algebraic topology are paracompact. In particular, we will prove in Volume 1 that CW complexes are paracompact. □

The easiest examples of paracompact spaces are compact Hausdorff spaces, where every open cover admits a finite cover (not just a locally finite one). Our next goal is to prove the following generalization of this:

THEOREM 10.2.2. *Let X be a locally compact Hausdorff space that is σ-compact. Then X is paracompact.*

Before we prove this, we note that in light of Lemma 9.3.1 it implies:

COROLLARY 10.2.3. *Let X be a locally compact Hausdorff space that is second countable. Then X is paracompact. In particular, both open and closed subspaces of $\mathbb{R}^n$ are paracompact.*

We remark that Stone [**4**] proved that every metric space is paracompact. We omit the proof, but good references for it include [**1**, Theorem IX.5.3] and [**2**, Corollary 5.35] and [**3**].

PROOF OF THEOREM 10.2.2. We start by proving:

CLAIM. *There exists a countable open cover $\{W_1, W_2, \ldots\}$ of X such that for all $n \geq 1$ the set $\overline{W}_n$ is compact and satisfies $\overline{W}_n \subset W_{n+1}$.*

PROOF OF CLAIM. Since X is σ-compact, we can write $X = \cup_{n \geq 1} K_n$ with K_n compact. We will inductively construct open sets W_n of X such that $W_0 = \emptyset$ and for all $n \geq 0$ we have:

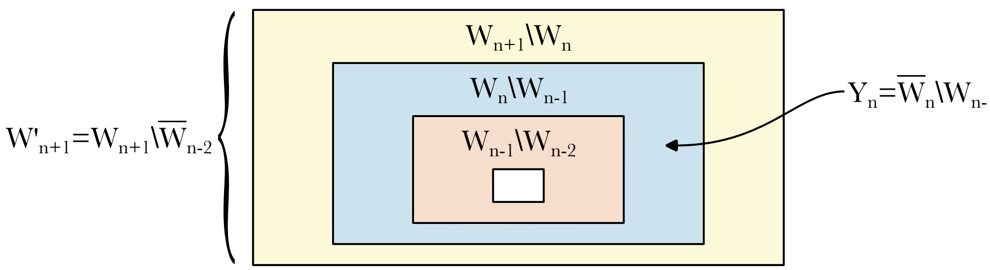
- $\overline{W}_n$ is compact; and
- W_{n+1} contains $\overline{W}_n \cup K_{n+1}$.

Since $X = \cup_{n\geq 1} K_n$, this will be a open open cover of X with the properties indicated in the claim. Start by setting $W_0 = \emptyset$, and assume we have constructed $W_0, \dots, W_n$. For $p \in \overline{W}_n \cup K_{n+1}$, local compactness gives an open neighborhood $W_{n+1}(p)$ of p with $\overline{W}_{n+1}(p)$ compact. Since $\overline{W}_n \cup K_{n+1}$ is compact, we can find $p_1, \dots, p_m \in \overline{W}_n \cup K_{n+1}$ such that $\{W_{n+1}(p_1), \dots, W_{n+1}(p_m)\}$ covers $\overline{W}_n \cup K_{n+1}$. We can then define $W_{n+1} = W_{n+1}(p_1) \cup \dots \cup W_{n+1}(p_m)$. The set $\overline{W}_{n+1}$ is compact since $\overline{W}_{n+1} = \overline{W}_{n+1}(p_1) \cup \dots \cup \overline{W}_{n+1}(p_m)$ (see Lemma 10.1.1). □

We now prove that X is paracompact. Let $\mathfrak{U}$ be an open cover of X. Let $\{W_n\}_{n\geq 1}$ be as in the claim. Set $W_m = \emptyset$ for $m \leq 0$. For $n \in \mathbb{Z}$, define $Y_n = \overline{W}_n \setminus W_{n-1}$ and $W'_{n+1} = W_{n+1} \setminus \overline{W}_{n-2}$. These satisfy:

(i) Y_n is a compact subset of the open set W'_{n+1}; and
(ii) $X = \cup_{n=1}^{\infty} Y_n$; and
(iii) $W'_{n_1} \cap W'_{n_2} = \emptyset$ whenever $|n_1 - n_2| \geq 3$.

See here:



For each $n \geq 1$, the set $\{U \cap W'_{n+1} \mid U \in \mathfrak{U}\}$ is an open cover of compact set Y_n, so there is a finite subset $\mathfrak{U}(n) \subset \mathfrak{U}$ such that $\{U \cap W'_{n+1} \mid U \in \mathfrak{U}(n)\}$ covers Y_n. Let

$$\mathfrak{V} = \left\{U \cap W'_{n+1} \mid n \geq 1 \text{ and } U \in \mathfrak{U}(n)\right\}.$$

The set $\mathfrak{V}$ is an open cover of each Y_n, so by (ii) it follows that $\mathfrak{V}$ is an open cover of X. By construction, $\mathfrak{V}$ refines $\mathfrak{U}$. Using (iii) together with the fact that only finitely many $V \in \mathfrak{V}$ are contained in each W'_n, the open cover $\mathfrak{V}$ is locally finite. The theorem follows. □

10.3. Normality

Our next goal is to prove that paracompact spaces are normal:

LEMMA 10.3.1. *Let X be a paracompact space. Then X is normal.*

PROOF. Recall that paracompact spaces are assumed to be Hausdorff. We start by proving the following weakening of normality which is often called being *regular*:

CLAIM. *For $p \in X$ and $C \subset X$ closed with $p \notin C$, there exist disjoint open neighborhoods of p and C.*

PROOF OF CLAIM. For each $q \in C$, since X is Hausdorff there exist open neighborhoods U_{qp} of q and U'_{qp} of p such that $U_{qp} \cap U'_{qp} = \emptyset$. Since X is paracompact, the open cover $\{X \setminus C\} \cup \{U_{qp} \mid q \in C\}$ admits a locally finite refinement. Let $\mathfrak{V}$ be the open sets in this locally finite refinement that are not contained in $X \setminus C$. For each $V \in \mathfrak{V}$, there is some $q \in C$ such that $V \subset U_{qp}$. Since U'_{qp} is an open neighborhood of p that is disjoint from U_{qp}, we deduce that $p \notin \overline{V}$ for all $V \in \mathfrak{V}$. Set $W = \cup_{V \in \mathfrak{V}} V$. The set W is an open neighborhood of C, and by local finiteness and Lemma 10.1.1 we have

$$\overline{W} = \bigcup_{V \in \mathfrak{V}} \overline{V}.$$

Since $p \notin \overline{V}$ for all $V \in \mathfrak{V}$, we deduce that $p \notin \overline{W}$. It follows that $X \setminus \overline{W}$ and W are disjoint open neighborhoods of p and C. □

To prove that X is normal, let C and D be disjoint closed subsets of X. We can find disjoint open neighborhoods of C and D by the same argument we used to prove the above claim. Simply substitute the above claim for X being Hausdorff and replace every occurrence of the point p by the closed set D. □

10.4. Strong refinements

Let $\mathfrak{U}$ be an open cover of a space X. Enumerate $\mathfrak{U}$ as $\mathfrak{U} = \{U_i\}_{i\in I}$. A *strong refinement* of $\mathfrak{U}$ consists of an open cover $\{V_i\}_{i\in I}$ such that $\overline{V}_i \subset U_i$ for all $i \in I$. We have:

LEMMA 10.4.1. *Let X be a paracompact space and let $\mathfrak{U}$ be an open cover of X. Then there exists a locally finite strong refinement of $\mathfrak{U}$.*

PROOF. Enumerate $\mathfrak{U}$ as $\mathfrak{U} = \{U_i\}_{i\in I}$. Let

$$\mathfrak{W}' = \left\{ W' \mid W' \text{ open set with } \overline{W}' \subset U_i \text{ for some } i \in I \right\}.$$

The set $\mathfrak{W}'$ is an open cover of X; indeed, since X is normal for all $p \in X$ and all $i \in I$ with $p \in U_i$ we can find an open neighborhood W' of p with $\overline{W}' \subset U_i$. Since X is paracompact, we can find a locally finite refinement $\mathfrak{W}$ of $\mathfrak{W}'$. For each $W \in \mathfrak{W}$, there is some $i \in I$ with $\overline{W} \subset U_i$. For $i \in I$, let $\mathfrak{W}(i) = \left\{ W \in \mathfrak{W} \mid \overline{W} \subset U_i \right\}$ and $V_i = \cup_{W\in\mathfrak{W}(i)} W$. Since $\mathfrak{W}(i)$ is a locally finite collection of open sets, Lemma 10.1.1 implies that

$$\overline{V}_i = \bigcup_{W\in\mathfrak{W}(i)} \overline{W} \subset U_i.$$

The open cover $\mathfrak{V} = \{V_i\}_{i\in I}$ is thus a locally finite strong refinement of $\mathfrak{U} = \{U_i\}_{i\in I}$. □

10.5. Partitions of unity

We now come to the most important property of paracompact spaces. Let X be a space. Recall that for a continuous function $f\colon X \to \mathbb{R}$, the support of f is $\text{supp}(f) = \overline{\{p \in X \mid f(p) \neq 0\}}$. A *partition of unity* subordinate to an open cover $\mathfrak{U}$ of X consists of continuous functions $f_U\colon X \to [0, 1]$ for each $U \in \mathfrak{U}$ satisfying the following three conditions:

(a) For all $U \in \mathfrak{U}$, we have $\text{supp}(f) \subset U$.
(b) The set $\{\text{supp}(f_U) \mid U \in \mathfrak{U}\}$ is locally finite.
(c) For all $p \in X$, we have $\sum_{U\in\mathfrak{U}} f_U(p) = 1$. Note that (b) implies that only finitely many terms of this sum are nonzero, so this sum makes sense.

We have:

THEOREM 10.5.1. *Let X be a paracompact space and let $\mathfrak{U}$ be an open cover of X. Then there exists a partition of unity subordinate to $\mathfrak{U}$.*

PROOF. Enumerate $\mathfrak{U}$ as $\mathfrak{U} = \{U_i\}_{i\in I}$. By Lemma 10.4.1, we can find a locally finite strong refinement $\{V_i\}_{i\in I}$ of $\{U_i\}_{i\in I}$. Applying this lemma again, we obtain a locally finite strong refinement $\{W_i\}_{i\in I}$ of $\{V_i\}_{i\in I}$. Lemma 10.3.1 says that X is normal, so we can apply Urysohn's Lemma (Theorem 7.5.1) to X. For $i \in I$, since $\overline{W}_i \subset V_i$ Urysohn's Lemma (Theorem 7.5.1) implies that there is a continuous function $f_i'\colon X \to [0, 1]$ such that $f_i'|_{\overline{W}_i} = 1$ and $\text{supp}(f_i') \subset V_i$. Since $\{V_i\}_{i\in I}$ is locally finite and $\text{supp}(f_i') \subset \overline{W}_i \subset V_i$ for each $i \in I$, we can define $g\colon X \to [0, \infty)$ via the formula

$$g(p) = \sum_{i\in I} f_i'(p) \quad \text{for } p \in X.$$

The function $g\colon X \to [0, \infty)$ is continuous (see Exercise 10.5). Each $p \in X$ lies in some W_i, so since $f_i'|_{\overline{W}_i} = 1$ it follows that $g(p) > 0$ for all $p \in X$. For $i \in I$, we can therefore define $f_i\colon X \to [0, \infty)$ via the formula

$$f_i(p) = \frac{1}{g(p)} f_i'(p) \quad \text{for } p \in X.$$

For $p \in X$, we have

$$\sum_{i\in I} f_i(p) = \frac{1}{g(p)} \sum_{i\in I} f_i'(p) = \frac{1}{g(p)} g(p) = 1.$$

Since $f_i(p) \in [0, \infty)$ for all $i \in I$, this implies that the image of each f_i lies in $[0, 1]$ and that the f_i form a partition of unity subordinate to $\mathfrak{U} = \{U_i\}_{i\in I}$. □

10.6. Application: extending functions

Here is a typical application of partitions of unity:

LEMMA 10.6.1. *Let X be a paracompact space, let $A \subset X$ be a subspace, and let $f\colon A \to \mathbb{R}$ be continuous. For all $a \in A$, assume that there is a neighborhood U_a of a and an extension of $f|_{U_a \cap A}$ to $F_a\colon U_a \to \mathbb{R}$. Set $U = \cup_{a\in A} U_a$. Then f can be extended to a continuous function $F\colon U \to \mathbb{R}$.*

REMARK 10.6.2. If A is closed, then the Tietze extension theorem (Theorem 7.9.1) says that f can be extended to the whole space X. This can fail for non-closed subspaces. For instance, consider the subspace $\mathbb{Q}$ of $\mathbb{R}$. The continuous function $f\colon \mathbb{Q} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} -1 & \text{if } x < \sqrt{2}, \\ 1 & \text{if } x > \sqrt{2} \end{cases} \quad \text{for } x \in \mathbb{Q}.$$

can be extended to a continuous function on the open set $\mathbb{R} \setminus \{\sqrt{2}\}$, but cannot be extended to a continuous function on $\mathbb{R}$. □

PROOF. Replacing X by U, we can assume that $\mathfrak{U} = \{U_a \mid a \in A\}$ is an open cover of X. Let $\{\phi_{U_a}\colon X \to \mathbb{R} \mid a \in A\}$ be a partition of unity subordinate to $\mathfrak{U}$. Since $\operatorname{supp}(\phi_{U_a}) \subset U_a$, the function $F_a\phi_{U_a}\colon U_a \to \mathbb{R}$ can be extended to a continuous function $G_a\colon X \to \mathbb{R}$ by letting $G_a(x) = 0$ for $x \in X \setminus U_a$. We have $\operatorname{supp}(G_a) \subset \operatorname{supp}(\phi_a)$ for $a \in A$, so since the set of supports of the ϕ_a are locally finite we can define $F\colon X \to \mathbb{R}$ via the formula $F = \sum_{a\in A} G_a$. For $a \in A$, we have

$$F(a) = \sum_{a\in A} F_a(a)\phi_{U_a}(a) = f(a) \sum_{a\in A} \phi_{U_a}(a) = f(a),$$

so F is an extension of f. □

10.7. Exercises

EXERCISE 10.1. Let X be a space and let $\mathfrak{Z}$ be a collection of subsets of X.

(a) If $\mathfrak{Z}$ is locally finite, prove that

$$\overline{\bigcup_{Z\in\mathfrak{Z}} Z} = \bigcup_{Z\in\mathfrak{Z}} \overline{Z}.$$

(b) Give an example to show that local finiteness is needed in the previous part. □

EXERCISE 10.2. Let X be a space, let $\mathfrak{U}$ be an open cover of X, and let $\mathfrak{V}$ be an open cover of X that refines $\mathfrak{U}$. Assume that $\mathfrak{V}$ has a finite subcover. Prove that $\mathfrak{U}$ has a finite subcover. □

EXERCISE 10.3. Let X be paracompact and let $A \subset X$ be closed. Topologize X/A using the quotient topology (see Example 4.2.8). Prove that X/A is paracompact. □

EXERCISE 10.4. Let X and Y be paracompact spaces, let $A \subset X$ be closed, and let $\phi\colon A \to Y$ be a closed map. Let Z be the result of gluing X to Y using ϕ (see Example 4.2.1). Prove that Z is paracompact. □

EXERCISE 10.5. Let X be a space and let $\{V_i\}_{i\in I}$ be a locally finite collection of open subsets of X. For each $i \in I$, let $h_i\colon X \to \mathbb{R}$ be a continuous function such that $\operatorname{supp}(h_i) \subset V_i$. Define $h\colon X \to \mathbb{R}$ via the formula

$$h(p) = \sum_{i\in I} h_i(p) \quad \text{for } p \in X.$$

Prove that $h\colon X \to \mathbb{R}$ is continuous. □

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CHAPTER 11

Products and Tychonoff's theorem

We now discuss products of spaces, generalizing the topology on $X \times Y$ from Exercise 3.4.

11.1. Finite products

Let $X_1, \ldots, X_n$ be spaces. As a set, $X_1 \times \cdots \times X_n$ consists of tuples $(x_1, \ldots, x_n)$ with $x_i \in X_i$ for $1 \leq i \leq n$. Give this the topology with the basis consisting of products $U_1 \times \cdots \times U_n$ with $U_i \subset X_i$ open for $1 \leq i \leq n$. We will call these the *basic open sets* of the product. A general open set $V \subset X_1 \times \cdots \times X_n$ can therefore be written a union of basic open sets. Equivalently, $V \subset X_1 \times \cdots \times X_n$ is open if and only if for all $(p_1, \ldots, p_n) \in V$, there exist open neighborhoods $U_i \subset X_i$ of each p_i such that $U_1 \times \cdots \times U_n \subset V$.

EXAMPLE 11.1.1. This gives the usual topology on $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ (see Exercise 11.1). □

11.2. Finite universal property

Let $\pi_i \colon X_1 \times \cdots \times X_n \to X_i$ be the projection. This is continuous: if $U_i \subset X_i$ is open, then

$$\pi_i^{-1}(U_i) = X_1 \times \cdots \times X_{i-1} \times U_i \times X_{i+1} \times \cdots \times X_n.$$

Now let Y be another space, and for $1 \leq i \leq n$ let $f_i \colon Y \to X_i$ be a continuous map. Let $f_1 \times \cdots \times f_n \colon Y \to X_1 \times \cdots \times X_n$ be the map where $f_1 \times \cdots \times f_n(y) = (f_1(y), \ldots, f_n(y))$ for $y \in Y$. This is continuous; indeed, if $U_i \subset X_i$ is open for $1 \leq i \leq n$ then

$$(f_1 \times \cdots \times f_n)^{-1}(U_1 \times \cdots \times U_n) = U_1 \cap \cdots \cap U_n.$$

Conversely, if $F \colon Y \to X_1 \times \cdots \times X_n$ is a continuous map, then letting $f_i = \pi_i \circ F$ we have $F = f_1 \times \cdots \times f_n$. We summarize this informally as:

- A continuous map $F \colon Y \to X_1 \times \cdots \times X_n$ is the same thing as a collection of continuous maps $f_i \colon Y \to X_i$ for all $1 \leq i \leq n$.

Just like for quotient spaces in §4.4, this is an example of a universal mapping property and it characterizes product spaces (see Exercise 11.2).

11.3. Homotopies, products, and quotient maps

One place where products show up in algebraic topology is in the definition of a homotopy. Roughly speaking, a homotopy is a continuous deformation of a map. The precise definition is as follows. Let $f_0, f_1 \colon Y \to Z$ be maps. A *homotopy* from f_0 to f_1 is a map $H \colon Y \times I \to Z$ such that $H(y, 0) = f_0(y)$ and $H(y, 1) = f_1(y)$ for $y \in Y$. If such a homotopy exists, we say that f_0 and f_1 are *homotopic*

For $t \in I$, we can let $f_t \colon Y \to Z$ be the map defined by $f_t(y) = H(y, t)$ for $y \in Y$. The maps $f_t \colon Y \to Z$ can be viewed informally as a continuous family of maps connecting f_0 to f_1. See §12.7 for how to topologize the space of maps $Y \to Z$ and make this precise.

EXAMPLE 11.3.1. Any two maps $f_0, f_1 \colon Y \to \mathbb{R}^n$ are homotopic via the homotopy $H \colon Y \times I \to \mathbb{R}^n$ defined by $H(y, t) = (1 - t) f_0(y) + t f_1(y)$. □

EXAMPLE 11.3.2. Let $Y = \{*\}$ be a one-point space. Two maps $f_0, f_1 \colon Y \to Z$ are homotopic if and only if $f_0(*)$ and $f_1(*)$ lie in the same path component of Z. □

Now assume that $q\colon X \to Y$ is a quotient map (see §4.3), so q is surjective and $U \subset Y$ is open if and only if $q^{-1}(U) \subset X$ is open. Given $f_0, f_1\colon Y \to Z$, it is natural to try to construct a homotopy from f_0 to f_1 as follows:

- Define $g_0 = f_0 \circ q$ and $g_1 = f_1 \circ q$. Construct a homotopy $\widetilde{H}\colon X \times I \to Z$ from g_0 to g_1.
- Next, use the universal property of the quotient map from §4.4 to show that $\widetilde{H}$ descends to a homotopy $H\colon Y \times I \to Z$.

Here are an example of how this might work:

EXAMPLE 11.3.3. We have $\mathbb{D}^n/\partial\mathbb{D}^n \cong \mathbb{S}^n$ (see Example 4.2.10). A map $f\colon \mathbb{S}^n \to Z$ is thus the same as a map $g\colon \mathbb{D}^n \to Z$ such that $g|_{\partial\mathbb{D}^n}$ is constant. Given $f_0, f_1\colon \mathbb{S}^n \to Z$, let $g_0, g_1\colon \mathbb{D}^n \to Z$ be the corresponding maps. To construct a homotopy from f_0 to f_1, it is natural to instead try to construct a homotopy g_t from g_0 to g_1 such that $g_t|_{\partial\mathbb{D}^n}$ is constant for all t. □

However, there is a flaw in the above reasoning: if $q\colon X \to Y$ is a quotient map, it not clear that $q \times \mathbb{1}\colon X \times I \to Y \times I$ is a quotient map. Indeed, there are counterexamples if I is replaced by a more complicated space. However, for nice spaces like I this is not a problem. More generally:

LEMMA 11.3.4. *Let $q\colon X \to Y$ be a quotient map and let Z be a locally compact space. Then the map $q \times \mathbb{1}\colon X \times Z \to Y \times Z$ is a quotient map.*

PROOF.[1] The map $q \times \mathbb{1}\colon X \times Z \to Y \times Z$ is continuous, so for every open set $U \subset Y \times Z$ we have $q^{-1}(U)$ open. We must prove the converse. In other words, letting $U \subset Y \times Z$ be a set such that $q^{-1}(U)$ is open, we must prove that U is open. Letting $(y, z) \in U$, it is enough to find an open neighborhood of (y, z) that is contained in U.

Pick $x \in X$ with $q(x) = y$. We have $(x, z) \in q^{-1}(U)$. Since $q^{-1}(U) \subset X \times Z$ is open and Z is locally compact, we can find an open neighborhood $V_1 \subset X$ of x and a compact neighborhood $K \subset Z$ of z such that $V_1 \times K \subset q^{-1}(U)$. We have

$$(y, z) \in q(V_1 \times \mathrm{Int}(K)) = q(V_1) \times \mathrm{Int}(K) \subset U.$$

If $q(V_1) \subset Y$ were open, then $q(V_1) \times \mathrm{Int}(K)$ would be an open neighborhood of (y, z) contained in U and we would be done.

Unfortunately, $q(V_1)$ might not be open since $q^{-1}(q(V_1))$ might be larger than V_1. We do have $q^{-1}(q(V_1)) \times K \subset q^{-1}(U)$. Since K is compact and $q^{-1}(U)$ is open, we can find an open neighborhood V_2 of $q^{-1}(q(V_1))$ with $V_2 \times K \subset q^{-1}(U)$ (see Exercise 8.7; this is often called the "tube lemma"). Just like for V_1, there is no reason to expect $q(V_2) \subset Y$ to be open since $q^{-1}(q(V_2))$ might be larger than V_2. However, we can iterate the procedure we used to find V_2. The result is an increasing sequence $V_1 \subset V_2 \subset \cdots$ of open subsets of Y such that for all $n \geq 1$ we have:

- $V_n \times K \subset q^{-1}(U)$ and $q^{-1}(q(V_n)) \subset V_{n+1}$.

The set $V = \cup_{n \geq 1} V_n$ is then an open subset of X with $V \times K \subset q^{-1}(U)$ and $q^{-1}(q(V)) = V$. It follows that $q(V)$ is an open subset of Y, so $q(V) \times \mathrm{Int}(K)$ is an open neighborhood of (y, z) with $q(V) \times \mathrm{Int}(K) \subset U$, as desired. □

11.4. Tychonoff's theorem, finite case

We have the following basic result:

THEOREM 11.4.1 (Tychonoff's theorem, finite case). *Let $X_1, \ldots, X_n$ be compact spaces. Then $X_1 \times \cdots \times X_n$ is compact.*

PROOF. By induction, it is enough to prove this for $n = 2$. Let $\mathfrak{U}$ be an open cover of $X_1 \times X_2$. We must prove that $\mathfrak{U}$ has a finite subcover. In fact, it is enough to prove that some refinement of $\mathfrak{U}$ has a finite subcover (see Exercise 10.2). Each element of $\mathfrak{U}$ is a union of basic open sets. Letting $\mathfrak{V}$ be the set of all basic open sets V such that there exists some $U \in \mathfrak{U}$ with $V \subset U$, it is therefore enough to prove that $\mathfrak{V}$ has a finite subcover.

[1]We will give an alternate proof in the next chapter which is shorter but more abstract.

For $p \in X_1$, let $Z(p) = p \times X_2$. By assumption, $Z(p) \cong X_2$ is compact. We can therefore find a finite subset $\mathfrak{V}(p)$ of $\mathfrak{V}$ that covers $Z(p)$. Since $\mathfrak{V}$ consists of basic open sets, we can write

$$\mathfrak{V}(p) = \{V_1(p) \times V_1'(p), \ldots, V_{m_p}(p) \times V_{m_p}'(p)\}$$

with $V_i(p) \subset X_1$ and $V_i'(p) \subset X_2$ for $1 \leq i \leq m_p$. Discarding unneeded terms if necessary, we can assume that $p \in V_i(p)$ for all $1 \leq i \leq m_p$. Letting $V(p) = V_1(p) \cap \cdots \cap V_{m_p}(p)$, it follows that $V(p)$ is an open neighborhood of p and $\mathfrak{V}(p)$ covers $V(p) \times X_2$.

The set $\{V(p) \mid p \in X_1\}$ is an open cover of the compact space X_1, so we can find $p_1, \ldots, p_d \in X_1$ such that $X_1 = V(p_1) \cup \cdots \cup V(p_d)$. Since $\mathfrak{V}(p_i)$ is a finite cover of $V(p_i) \times X_2$ for $1 \leq i \leq d$, we conclude that $\mathfrak{V}(p_1) \cup \cdots \mathfrak{V}(p_d)$ is a finite subset of $\mathfrak{V}$ that covers $X_1 \times X_2$. □

11.5. Infinite products

Now let $\{X_i\}_{i \in I}$ be an arbitrary collection of spaces. As a set, the product $\prod_{i \in I} X_i$ consists of tuples $(x_i)_{i \in I}$ with $x_i \in X_i$ for $i \in I$. The obvious first guess for a topology on $\prod_{i \in I} X_i$ is the one with basis the collection of products $\prod_{i \in I} U_i$ with $U_i \subset X_i$ open for all $i \in I$. However, this topology turns out to be pathological. The issue is that it has too many open sets, and there are maps into it that should be continuous but are not. Here is a key example:

EXAMPLE 11.5.1. Let X be a space and let I be an infinite indexing set. Consider the diagonal map $\Delta\colon X \to \prod_{i \in I} X$, so $\Delta(x) = (x)_{i \in I}$ for all $x \in X$. If $U_i \subset X$ is an open set for all $i \in I$, then

$$\Delta^{-1}(\prod_{i \in I} U_i) = \bigcap_{i \in I} U_i.$$

Since the collection of open sets is *not* closed under infinite intersections, this is not always open. It follows that Δ will generally not be continuous if all such sets of the form $\prod_{i \in I} U_i$ are open. □

To eliminate this pathology, we must avoid infinite intersections of open sets. This can be done as follows. A *basic open set* in $\prod_{i \in I} X_i$ is a product $\prod_{i \in I} U_i$ such that:

- $U_i \subset X_i$ is open for all $i \in I$; and
- $U_i = X_i$ for all but finitely many $i \in I$.

The *product topology* on $\prod_{i \in I} X_i$ is the topology with basis the basic open sets, so a subset of $\prod_{i \in I} X_i$ is open if and only if it is a union of basic open sets. To simplify our notation when talking about these infinite products, we introduce the following convention:

CONVENTION 11.5.2. We regard the indexing set I as being unordered, and thus if $I = J \sqcup K$ we identify

$$\left(\prod_{j \in J} X_j\right) \times \left(\prod_{k \in K} X_k\right) \quad \text{and} \quad \prod_{i \in I} X_i$$

in the obvious way. □

With this notational convention, the basic open sets in $\prod_{i \in I} X_i$ are those that for some distinct $j_1, \ldots, j_n \in I$ can be written as

$$U_{j_1} \times \cdots \times U_{j_n} \times \prod_{i \in I \setminus \{j_1, \ldots, j_n\}} X_i \quad \text{with } U_{j_k} \subset X_{j_k} \text{ open for } 1 \leq k \leq n.$$

REMARK 11.5.3. The topology on $\prod_{i \in I} X_i$ with basis arbitrary products $\prod_{i \in I} U_i$ with $U_i \subset X_i$ open is sometimes called the *box topology*. It is rarely useful. □

11.6. Infinite universal property

Continue to let $\{X_i\}_{i \in I}$ be an arbitrary collection of spaces. For $j \in I$, let $\pi_j\colon \prod_{i \in I} X_i \to X_j$ be the projection. The map π_j is continuous; indeed, if $U_j \subset X_i$ is open, then

$$\pi_j^{-1}(U_j) = U_j \times \prod_{i \in I \setminus \{j\}} X_i.$$

Now let Y be another space, and for $i \in I$ let $f_i \colon Y \to X_i$ be a continuous map. Let $\prod_{i \in I} f_i \colon Y \to \prod_{i \in I} X_i$ be the map

$$\left(\prod_{i \in I} f_i\right)(y) = (f_i(y))_{i \in I} \quad \text{for } y \in Y.$$

This map is continuous; indeed, if $\prod_{i \in I} U_i$ is a basic open set then

$$(\left(\prod_{i \in I} f_i\right)^{-1}(\prod_{i \in I} U_i) = \bigcap_{i \in I} f_i^{-1}(U_i).$$

This is open since $f_i^{-1}(U_i) = f_i^{-1}(X_i) = Y$ for all but finitely many $i \in I$, so this intersection is actually a finite intersection. Conversely, if $F \colon Y \to \prod_{i \in I} X_i$ is a continuous map, then letting $f_i = \pi_i \circ F$ we have $F = \prod_{i \in I} f_i$. We summarize this informally as:

- A continuous map $F \colon Y \to \prod_{i \in I} X_i$ is the same thing as a collection of continuous maps $f_i \colon Y \to X_i$ for all $i \in I$.

This universal property characterizes product spaces (see Exercise 11.2), and having it is one of the reasons we defined the product topology like we did.

REMARK 11.6.1. In more categorical language, what the above shows is that $\prod_{i \in I} X_i$ is the product of the X_i in the category of topological spaces. There is also a notion of a sum of objects in a category, and as we discussed in Example 4.4.1 the disjoint union $\sqcup_{i \in I} X_i$ with the disjoint union topology discussed in Example 4.1.1 is the categorical sum of the X_i. See Exercise 11.3 for related constructions in the category of abelian group. □

11.7. Metrics on countable products

Arbitrary products of metric spaces need not be metric spaces. However, it turns out that countable products of metric spaces can be given metrics. This would not be true if we used the box topology.

LEMMA 11.7.1. *For each $n \geq 1$, let $(M_n, \mathfrak{d}_n)$ be a metric space. There is then a metric on $\prod_{n=1}^{\infty} M_n$ inducing the product topology.*

PROOF. Let $\mathfrak{d}'_n$ be the metric on M_n defined by $\mathfrak{d}'_n(p,q) = \min\{\mathfrak{d}_n(p,q), 1\}$. This induces the same topology on M_n as $\mathfrak{d}_n$ (see Exercise 2.1). We can then define a two-variable real-valued function on $\prod_{n=1}^{\infty} M_n$ via the formula

$$\mathfrak{d}((p_n)_{n \geq 1}, (q_n)_{n \geq 1}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \mathfrak{d}'_n(p_n, q_n).$$

This is a metric on $\prod_{n=1}^{\infty} M_n$ that induces the product topology (see Exercise 11.5). □

11.8. Sequences in infinite products

Another nice property of the product topology is the following, which would also not be true if we used the box topology:

LEMMA 11.8.1. *Let $\{X_i\}_{i \in I}$ be a collection of spaces. For each $i \in I$, let $\{p(i)_n\}_{n \geq 1}$ be a sequence of points in X_i that converges to $p(i) \in X_i$. For $n \geq 1$, let $p_n = (p(i)_n)_{i \in I} \in \prod_{i \in I} X_i$. Then $\{p_n\}_{n \geq 1}$ converges to $(p(i))_{i \in I} \in \prod_{i \in I} X_i$.*

PROOF. See Exercise 11.4. □

EXAMPLE 11.8.2. For $n \geq 1$, let $p_n \in \prod_{i \geq 1} \mathbb{Z}$ be the tuple of integers $p_n = (1, \ldots, 1, 0, \ldots)$ with n initial 1's and then 0's. Let $p_\infty = (1, 1, 1, \ldots)$ be the tuple all of whose entries are 1. Then $\lim_{n \mapsto \infty} p_n = p_\infty$. □

11.9. Tychonoff's theorem, countable case

Tychonoff's theorem generalizes to arbitrary products of compact spaces. We start by proving this for countable products. The proof of the general case is similar, but requires more set theoretic technology.

THEOREM 11.9.1 (Tychonoff's theorem, countable case). *Let* $\{X_i\}_{i \geq 1}$ *be a countable collection of compact spaces. Then* $\prod_{i \geq 1} X_i$ *is compact.*

PROOF. Unlike in the finite case, we cannot prove this by induction. However, we will see that the argument we gave in the finite case is almost enough. Only one new idea is needed. Let $\mathfrak{U}$ be an open cover of $\prod_{i \geq 1} X_i$. We must prove that $\mathfrak{U}$ has a finite subcover. In fact, it is enough to prove that some refinement of $\mathfrak{U}$ has a finite subcover (see Exercise 10.2). Each element of $\mathfrak{U}$ is a union of basic open sets. Letting $\mathfrak{V}$ be the set of all basic open sets V such that there exists some $U \in \mathfrak{U}$ with $V \subset U$, it is therefore enough to prove that $\mathfrak{V}$ has a finite subcover.

Assume for the sake of contradiction that $\mathfrak{V}$ has no finite subcover. The proof now has two steps:

STEP 1. *For all* $i \geq 1$*, there exists some* $p_i \in X_i$ *such that no finite subset of* $\mathfrak{V}$ *covers* $p_1 \times \cdots \times p_n \times \prod_{i \geq n+1} X_i$ *for any* $n \geq 1$*.*

We construct the p_i inductively. Assume that for some $n \geq 1$ we have found $p_i \in X_i$ for $1 \leq i \leq n-1$ such that no finite subset of $\mathfrak{V}$ covers $p_1 \times \cdots \times p_{n-1} \times \prod_{i \geq n} X_i$. For $n = 1$, this is simply our assumption that the open cover $\mathfrak{V}$ of $\prod_{i \geq 1} X_i$ has no finite subcover. We find $p_n \in X_n$ as follows. For $p \in X_n$, let

$$Z(p) = p_1 \times \cdots \times p_{n-1} \times p \times \prod_{i \geq n+1} X_i.$$

Assume for the sake of contradiction that for all $p \in X_n$, there exists a finite subset $\mathfrak{V}(p)$ of $\mathfrak{V}$ that covers $Z(p)$. Since $\mathfrak{V}$ consists of basic open sets, we can write

$$\mathfrak{V}(p) = \left\{ \prod_{i \geq 1} V_{i,j}(p) \mid 1 \leq j \leq m_p \right\}$$

with $V_{i,j}(p) \subset X_i$ for all $i \geq 1$ and $1 \leq j \leq m_p$. Discarding unneeded terms if necessary, we can assume that $p_i \in V_{i,j}(p)$ for all $1 \leq i \leq n-1$ and $1 \leq j \leq m_p$, and also that $p \in V_{n,j}(p)$ for all $1 \leq j \leq m_p$. Define

$$V_i(p) = \bigcap_{j=1}^{m_p} V_{i,j}(p) \quad \text{for } 1 \leq i \leq n,$$
$$V(p) = V_1(p) \times \cdots \times V_n(p).$$

It follows that $V(p)$ is an open neighborhood of $(p_1, \ldots, p_{n-1}, p) \in X_1 \times \cdots \times X_n$ and that $\mathfrak{V}(p)$ covers $V(p) \times \prod_{i \geq n+1} X_i$.

The set $\{V(p) \mid p \in X_n\}$ is an open cover of the compact space $p_1 \times \cdots \times p_{n-1} \times X_n$, so we can find $q_1, \ldots, q_d \in X_n$ such that

$$p_1 \times \cdots \times p_{n-1} \times X_n \subset V(q_1) \cup \cdots \cup V(q_d).$$

Since $\mathfrak{V}(q_k)$ is a finite cover of $V(q_k) \times \prod_{i \geq n+1} X_i$ for $1 \leq k \leq d$, we conclude that $\mathfrak{V}(q_1) \cup \cdots \mathfrak{V}(q_d)$ is a finite subset of $\mathfrak{V}$ that covers $p_1 \times \cdots \times p_{n-1} \times \prod_{i \geq n} X_i$, contradicting the fact that no such finite cover exists.

STEP 2. *No finite subset of* $\mathfrak{V}$ *covers* $\prod_{i \geq 1} X_i$*.*

Pick $V \in \mathfrak{V}$ such that $(p_i)_{i \geq 1} \in V$. Since $\mathfrak{V}$ consists of basic open sets, we can write $V = \prod_{i \geq 1} V_i$ with $V_i \subset X_i$ open for all $i \geq 1$. Moreover, we have $V_i = X_i$ for all but finitely many $i \geq 1$. This implies that there exists some $n \geq 1$ such that $V_i = X_i$ for $i \geq n+1$. It follows that

$$p_1 \times \cdots \times p_n \times \prod_{i \geq n+1} X_i \subset V \in \mathfrak{V}.$$

This contradicts the fact that no finite subset of $\mathfrak{V}$ covers $p_1 \times \cdots \times p_n \times \prod_{i \geq n+1} X_i$. □

11.10. Well-ordered sets

To generalize the above proof of Tychonoff's theorem to arbitrary products, we need some set-theoretic technology. A *well-ordered set* is a set I equipped with a total ordering $\leq$ such that every nonempty subset $S \subset I$ has a minimal element. The canonical example is $\mathbb{N} = \{1, 2, \ldots\}$ with the usual ordering. A remarkable consequence of the axiom of choice is that every set can be equipped with a well-ordering.

If I is a well-ordered set with ordering $\leq$, then an *initial segment* of I is a subset $J \subset I$ such that for all $j \in J$ and $i \in I$ with $i \leq j$ we have $i \in J$. If $J_1, J_2 \subset I$ are initial segments, then either $J_1 \subset J_2$ or $J_2 \subset J_1$. Indeed, assume that J_1 is *not* a subset of J_2 and pick $j_1 \in J_1 \setminus J_2$. For $j_2 \in J_2$, we cannot have $j_1 \leq j_2$ since $j_1 \notin J_2$. It follows that $j_2 \leq j_1$, so $j_2 \in J_1$ and thus $J_2 \subset J_1$. The initial segments of I are thus totally ordered under inclusion. They fall into three classes:

- The empty set $\emptyset$, which is the unique initial segment that is contained in all initial segments.
- The *successor segments*, which are initial segments $J \subset I$ of the form $J = J' \sqcup \{n\}$ for some initial segment $J' \subsetneq J$ and some $n \in J \setminus J'$.
- The *limit segments*, which are nonempty initial segments $J \subset I$ that are not successor segments. These J are the union of the initial segments $J' \subsetneq J$.

For $\mathbb{N}$, the successor segments are of the form $\{1, \ldots, n\}$ and the whole set $\mathbb{N}$ is the only limit segment.

11.11. Transfinite induction

Assume now that I is a well-ordered set and for each $i \in I$ we have a set X_i. Our goal is to construct some $p_i \in X_i$ for all $i \in I$. For each initial segment $J \subset I$, we want some property $\mathcal{P}(J)$ to hold that only refers to the $p_i \in X_i$ for $i \in J$. To simplify our exposition, assume that if $\mathcal{P}(J)$ holds then so does $\mathcal{P}(J')$ for all initial segments $J' \subset J$.

We can construct the $p_i \in X_i$ by *transfinite induction*.[2] For this, we must prove three things:

(0) The property $\mathcal{P}(\emptyset)$ holds. This makes sense since $\mathcal{P}(\emptyset)$ makes no reference to any p_i.
(1) Let J be a successor segment of the form $J = J' \sqcup \{n\}$ for some initial segment $J' \subsetneq J$. Assume that we have already constructed $p_i \in X_i$ for all $i \in J'$ such that $\mathcal{P}(J')$ holds. We must show how to construct $p_n \in X_n$ such that $\mathcal{P}(J)$ holds.
(2) Let J be a limit segment. Assume that we have constructed $p_i \in X_i$ for all $i \in J$ such that $\mathcal{P}(J')$ holds for all initial segments $J' \subsetneq J$. We must prove that $\mathcal{P}(J)$ holds.

We can then construct $p_i \in X_i$ for all $i \in I$ such that $\mathcal{P}(J)$ holds for all initial segments $J \subset I$. Indeed, let $\mathfrak{J}$ be the set of all initial segments $J \subset I$ for which we can construct $p_i \in X_i$ for each $i \in J$ such that $\mathcal{P}(J)$ holds. The set $\mathfrak{J}$ is linearly ordered by inclusion and nonempty since $\emptyset \in \mathfrak{J}$. Let $J_0 = \cup_{J \in \mathfrak{J}} J$. By (1) and (2), we have $J_0 \in \mathfrak{J}$. We must prove that $J_0 = I$. Indeed, assume that $J_0 \subsetneq I$. Since I is well-ordered, there is a minimal $n \in I \setminus J_0$. It follows that $J_0 \sqcup \{n\}$ is an initial segment, and by (1) we have $J_0 \sqcup \{n\} \in \mathfrak{J}$, contradicting the fact that $J \subset J_0$ for all $J \in \mathfrak{J}$.

Remark 11.11.1. Isomorphism classes of well-ordered sets are called *ordinals*. Any set of ordinals has a well-ordering where $\mathcal{O}_1 \leq \mathcal{O}_2$ when $\mathcal{O}_1$ is isomorphic to an initial segment of $\mathcal{O}_2$. Transfinite induction is typically discussed using ordinals. □

11.12. Tychonoff's theorem, general case

The above was a little abstract. We now use it to prove the general case of Tychonoff's theorem:

Theorem 11.12.1 (Tychonoff's theorem). *Let $\{X_i\}_{i \in I}$ be a collection of compact spaces. Then $\prod_{i \in I} X_i$ is compact.*

Proof. The proof will be almost identical to proof in the countable case, but with some small complications due to the need for transfinite induction. Let $\mathfrak{U}$ be an open cover of $\prod_{i \in I} X_i$. We must prove that $\mathfrak{U}$ has a finite subcover. In fact, it is enough to prove that some refinement of $\mathfrak{U}$ has a finite subcover (see Exercise 10.2). Each element of $\mathfrak{U}$ is a union of basic open sets. Letting $\mathfrak{V}$ be the set of all basic open sets V such that there exists some $U \in \mathfrak{U}$ with $V \subset U$, it is therefore enough to prove that $\mathfrak{V}$ has a finite subcover.

[2] Since we constructing things, this is sometimes called *transfinite recursion*.

Assume for the sake of contradiction that $\mathfrak{V}$ has no finite subcover. Choose a well-ordering on the indexing set I. By transfinite induction, for each $i \in I$ we will construct some $p_i \in X_i$ such that the following holds for all initial segments $J \subset I$:

($\spadesuit_J$) No finite subset of $\mathfrak{V}$ covers $Y(J) = \prod_{j \in J} p_j \times \prod_{i \in I \setminus J} X_i$.

The special case ($\spadesuit_I$) says that no finite subset of $\mathfrak{V}$ covers the one-point set $Y(I) = \prod_{i \in I} p_i$, which will be our contradiction. We have ($\spadesuit_\emptyset$) from our assumption that no finite subset of $\mathfrak{V}$ covers $Y(\emptyset) = \prod_{i \in I} X_i$. According to the transfinite induction scheme discussed in §11.11, to prove that ($\spadesuit_J$) holds for all initial segments $J \subset I$ we must prove:

STEP 1. *Let $J \subset I$ be a successor segment, so $J = J' \sqcup \{n\}$ for some initial segment $J' \subset J$ and $n \in J \setminus J'$. Assume that we have constructed $p_i \in X_i$ for all $i \in J'$ such that ($\spadesuit_{J'}$) holds. We can then construct $p_n \in X_n$ such that ($\spadesuit_J$) holds.*

For $p \in X_n$, let

$$Z(p) = p \times \prod_{j' \in J'} p_{j'} \times \prod_{i \in I \setminus J} X_i.$$

Assume for the sake of contradiction that for all $p \in X_n$, there exists a finite subset $\mathfrak{V}(p)$ of $\mathfrak{V}$ that covers $Z(p)$. Since $\mathfrak{V}$ consists of basic open sets, we can write

$$\mathfrak{V}(p) = \left\{ \prod_{i \in I} V_{i,k}(p) \mid 1 \leq k \leq m_p \right\}$$

with $V_{i,k}(p) \subset X_i$ for all $i \in I$ and $1 \leq k \leq m_p$. Discarding unneeded terms if necessary, we can assume that $p_{j'} \in V_{j',k}(p)$ for all $j' \in J'$ and $1 \leq k \leq m_p$, and also that $p \in V_{n,k}(p)$ for all $1 \leq k \leq m_p$. Keeping in mind that $J = J' \sqcup \{n\}$, define

$$V_j(p) = \bigcap_{k=1}^{m_p} V_{j,k}(p) \quad \text{for } j \in J,$$
$$V(p) = V_n(p) \times \prod_{j' \in J'} V_{j'}(p).$$

It follows that $V(p)$ is an open neighborhood of $p \times \prod_{j' \in J} p_{j'}$ and that $\mathfrak{V}(p)$ covers $V(p) \times \prod_{i \geq I \setminus J} X_i$.

The set $\{V(p) \mid p \in X_n\}$ is an open cover of the compact space $X_n \times \prod_{j' \in J'} p_{j'}$, so we can find $q_1, \ldots, q_d \in X_n$ such that

$$X_n \times \prod_{j' \in J'} p_{j'} \subset V(q_1) \cup \cdots \cup V(q_d).$$

Since $\mathfrak{V}(q_\ell)$ is a finite cover of $V(q_\ell) \times \prod_{i \in I \setminus J} X_i$ for $1 \leq \ell \leq d$, we conclude that $\mathfrak{V}(q_1) \cup \cdots \mathfrak{V}(q_d)$ is a finite subset of $\mathfrak{V}$ that covers

$$X_n \times \prod_{j' \in J'} p_{j'} \times \prod_{i \in I \setminus J} X_i = \prod_{j' \in J'} p_{j'} \times \prod_{i \in I \setminus J'} X_i = Y(J'),$$

contradicting the fact that no such finite cover exists.

STEP 2. *Let $J \subset I$ be a limit segment. Assume that we have constructed p_i for all $i \in J$ such that ($\spadesuit_{J'}$) holds for all initial segments $J' \subsetneq J$. Then ($\spadesuit_J$) holds.*

Assume for the sake of contradiction that a finite subset $\{V_1, \ldots, V_d\}$ of $\mathfrak{V}$ covers $Y(J)$. Each V_k is a basic open set, so we can write

$$V_k = \prod_{i \in I} V_{k,i} \quad \text{with } V_{k,i} \subset X_i \text{ open for all } i \in I.$$

Moreover, we have $V_{k,i} = X_i$ for all but finitely many $i \in I$. For $1 \leq k \leq d$, let $J(k) = \{j \in J \mid V_{k,j} \neq X_j\}$. Set $\widehat{J} = J(1) \cup \cdots\cdots \cup J(d)$. Let J' be the smallest initial segment containing $\widehat{J}$. Since $\widehat{J}$ is a finite subset of J, we have $J' \subsetneq J$. Since $V_{k,j} = X_j$ for all $1 \leq k \leq d$ and $j \in J \setminus J'$, the fact that $\{V_1, \ldots, V_d\}$ covers $Y(J)$ implies that it also covers $Y(J')$. This contradicts the fact that no finite subset of $\mathfrak{V}$ covers $Y(J')$. □

11.13. Exercises

Exercise 11.1. Prove that the product topology on $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ is the same as the metric space topology. □

Exercise 11.2. Let $\{X_i\}_{i \in I}$ be a collection of spaces. Let $Y = \prod_{i \in I} X_i$ and for $i \in I$ let $\pi_i \colon Y \to X_i$ be the projection. Let Y' be a space equipped with continuous maps $\pi_i' \colon Y' \to X_i$ for each $i \in I$ such that the following holds:

- For all spaces Z and all collections of continuous maps $f_i \colon Z \to X_i$ for $i \in I$, there exists a unique continuous map $F \colon Z \to Y'$ such that $f_i = \pi_i' \circ F$ for all $i \in I$.

Prove that there is a homeomorphism $g \colon Y \to Y'$ such that $\pi_i = \pi_i' \circ g$ for all $i \in I$. In other words, the above universal mapping property characterizes the product space. In category theory, a product in a category is something satisfying a universal property of the above form. A category theorist would therefore say that $\prod_{i \in I} X_i$ is the product of the X_i in the category of topological spaces. □

Exercise 11.3. Let $\{A_i\}_{i \in I}$ be a collection of abelian groups. Let $\prod_{i \in I} A_i$ be the product of the A_i and let $\oplus_{i \in I} A_i$ be the sum of the A_i, so

$$\bigoplus_{i \in I} A_i = \left\{ (a_i)_{i \in I} \in \prod_{i \in I} A_i \mid a_i = 0 \text{ for all but finitely many } i \in I \right\}.$$

Imitate the definitions from Example 4.4.1 and Exercise 11.2 to formulate what it should mean to have a product and a sum in the category of abelian groups, and prove that $\prod_{i \in I} A_i$ and $\oplus_{i \in I} A_i$ are the product and sum of the A_i. We remark that unlike for topological spaces, the product and sum coincide for finite collections of abelian group. □

Exercise 11.4. Let $\{X_i\}_{i \in I}$ be a collection of spaces. For each $i \in I$, let $\{p(i)_n\}_{n \geq 1}$ be a sequence of points in X_i that converges to $p(i) \in X_i$. For $n \geq 1$, let $p_n = (p(i)_n)_{i \in I} \in \prod_{i \in I} X_i$. Prove that $\{p_n\}_{n \geq 1}$ converges to $(p(i))_{i \in I} \in \prod_{i \in I} X_i$. □

Exercise 11.5. For each $n \geq 1$, let $(M_n, \mathfrak{d}_n)$ be a metric space. For each $n \geq 1$, assume that $\mathfrak{d}_n(p, q) \leq 1$ for all $p, q \in M_n$. Define a two-variable real-valued function on $\prod_{n=1}^{\infty} M_n$ via the formula

$$\mathfrak{d}((p_n)_{n \geq 1}, (q_n)_{n \geq 1}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \mathfrak{d}_n'(p_n, q_n).$$

Prove that this is a metric on $\prod_{n=1}^{\infty} M_n$ that induces the product topology. □

Exercise 11.6. Recall that the classical *Cantor set* is the following subspace C of $I = [0, 1]$:

- Define $L \colon I \to I$ and $R \colon I \to I$ via the formulas $L(x) = x/3$ and $R(x) = (x + 2)/3)$. Recursively define closed sets $C_1 \supset C_2 \supset C_3 \supset \cdots$ by letting $C_1 = I$ and $C_{n+1} = L(C_n) \cup R(C_n)$. We then set $C = \cap_{n=1}^{\infty} C_n$.

Since C is an intersection of closed subsets of I, it is closed and hence compact. Let X be the discrete 2-point space $X = \{0, 2\}$. Prove the following:

(a) Define a set map $\Psi \colon \prod_{n=1}^{\infty} X \to I$ via the formula

$$\Psi((x_n)_{n \geq 1}) = \sum_{n=1}^{\infty} \frac{x_n}{3^n}.$$

Prove that Ψ is a continuous embedding with image the Cantor set C.

(b) Define a set map $\Phi \colon \prod_{n=1}^{\infty} X \to I$ via the formula

$$\Phi((x_n)_{n \geq 1}) = \sum_{n=1}^{\infty} \frac{x_n}{2^n}.$$

Prove that Φ is a continuous surjection.

(c) For each $d \geq 1$, construct a homeomorphism $\lambda_d \colon \prod_{n=1}^{\infty} X \to (\prod_{n=1}^{\infty} X)^{\times d}$.

(d) For each $d \geq 1$, let $f_d \colon C \to I^d$ be the composition

$$C \xrightarrow{\Psi^{-1}} \prod_{n=1}^{\infty} X \xrightarrow{\lambda_d} \left(\prod_{n=1}^{\infty} X\right)^{\times d} \xrightarrow{\prod_{i=1}^{d} \Phi} I^d.$$

Prove that f_d is a continuous surjection that can be extended to a continuous surjection $g_d \colon I \to I^d$ (a "space-filling curve"). $\square$

CHAPTER 12

Function spaces and the compact-open topology

Let X and Y be spaces and let[1] $\mathcal{C}(X,Y)$ be the set of all continuous maps $f\colon X \to Y$. In this chapter we explain how to turn $\mathcal{C}(X,Y)$ into a space.

12.1. Subbasis

Let X be a set and let $\mathfrak{B}$ be a set of subsets of X. We would like to topologize X with the smallest collection of open sets possible to make each $U \in \mathfrak{B}$ open. If for all $U, V \in \mathfrak{B}$ the intersection $U \cap V$ could be written as a union of sets in $\mathfrak{B}$, then $\mathfrak{B}$ would be a basis for a topology as in §3.4.3. In that case, we could topologize X by saying that $U \subset X$ is open precisely when U is the union of sets in $\mathfrak{B}$.

However, if $\mathfrak{B}$ does not form a basis then this does not work since in the resulting "topology" the collection of open sets is not closed under finite intersections. To fix this, let $\mathfrak{B}'$ be the set of all finite intersections of elements of $\mathfrak{B}$. Here we interpret the intersection of zero sets as X, so $X \in \mathfrak{B}'$. The set $\mathfrak{B}'$ does form a basis for a topology on X. In this case, we say that $\mathfrak{B}$ is a *subbasis* for this topology.

12.2. Compact-open topology

For sets $A, B \subset X$, define

$$B(A,B) = \{f\colon X \to Y \mid f(K) \subset U\} \subset \mathcal{C}(X,Y).$$

The *compact-open topology* on $\mathcal{C}(X,Y)$ is the topology with subbasis the collection of all $B(K,U)$ with $K \subset X$ compact and $U \subset Y$ open. In other words, a set $V \subset \mathcal{C}(X,Y)$ is open if for all $f \in V$ there exist $K_1, \ldots, K_n \subset X$ compact and $U_1, \ldots, U_n \subset Y$ open such that

$$f \in B(K_1,U_1) \cap \cdots \cap B(K_n,U_n) \subset V.$$

12.3. Metrics

If $(Y,\mathfrak{d})$ is a metric space, then it is also natural to try to topologize $\mathcal{C}(X,Y)$ using $\mathfrak{d}$. This is easiest for X compact, in which case we can define a metric $\mathfrak{D}$ on $\mathcal{C}(X,Y)$ by letting

$$\mathfrak{D}(f,g) = \max\{\mathfrak{d}(f(x_1), f(x_1)) \mid x_1, x_2 \in X\} \quad \text{for } f,g\colon X \to Y. \tag{12.3.1}$$

This makes sense since X is compact, which implies that $f(X)$ and $g(X)$ are compact subsets of the metric space Y and thus that the above maximum is finite and realized. We have:

LEMMA 12.3.1. *Let X be a compact space and let $(Y,\mathfrak{d})$ be a metric space. The compact-open topology on $\mathcal{C}(X,Y)$ and the metric topology on $\mathcal{C}(X,Y)$ coming from (12.3.1) are the same.*

PROOF. We divide the proof into two steps:

STEP 1. *Every open set in the compact-open topology is open in the metric topology.*

Let $K \subset X$ be compact and $U \subset Y$ be open. We must prove that $B(K,U)$ is open in the metric topology. Indeed, consider $f \in B(K,U)$, so $f(K) \subset U$. Since $f(K)$ is a compact subset of U, we can find some $\epsilon > 0$ such that the ϵ-neighborhood of $f(K)$ is contained in U. For $g \in \mathcal{C}(X,Y)$ with $\mathfrak{D}(f,g) < \epsilon$, since $\mathfrak{d}(g(k), f(k)) < \epsilon$ for all $k \in K$ it follows that $g(K)$ is contained in the ϵ-neighborhood of $f(K)$. We thus have $g(K) \subset U$, so $g \in B(K,U)$. We conclude that the ϵ-ball around f is contained in $B(K,U)$, so $B(K,U)$ is open in the metric topology.

[1] It is also common to call this space Y^X, but we think the notation $\mathcal{C}(X,Y)$ is easier to understand.

STEP 2. *Every open set in the metric topology is open in the compact-open topology.*

Let $f \in \mathcal{C}(X, Y)$ and let $\epsilon > 0$. Let

$$B_\epsilon(f) = \{g \in \mathcal{C}(X, Y) \mid d(g(x), f(x)) < \epsilon \text{ for all } x \in X\}$$

be the open ball around f in the metric topology. It is enough to find compact sets $K_1, \ldots, K_n \subset X$ and open sets $U_1, \ldots, U_n \subset Y$ such that

$$f \in B(K_1, U_1) \cap \cdots \cap B(K_n, U_n) \subset B_\epsilon(f).$$

Since $f(X)$ is a compact subset of Y, we can find $x_1, \ldots, x_n \in X$ such that

$$f(X) \subset B_{\epsilon/3}(f(x_1)) \cup \cdots \cup B_{\epsilon/3}(f(x_n)). \tag{12.3.2}$$

For $1 \leq i \leq n$, let $K_i = f^{-1}(\overline{B_{\epsilon/3}(f(x_i))})$ and $U_i = B_{\epsilon/2}(f(x_i))$. Since K_i is a closed subset of the compact space X, it follows that K_i is closed. By (12.3.2), the sets K_i cover X. Finally, by construction

$$f \in B(K_1, U_1) \cap \cdots \cap B(K_n, U_n).$$

Now consider some $g \in B(K_1, U_1) \cap \cdots \cap B(K_n, U_n)$. We must prove that $g \in B_\epsilon(f)$. In other words, letting $x \in X$ we must prove that $\mathfrak{d}(f(x), g(x)) < \epsilon$. We have $x \in K_i$ for some $1 \leq i \leq n$, so $f(x), g(x) \in U_i$. It follows that $\mathfrak{d}(f(x), g(x))$ is at most the diameter ϵ of $U_i = B_{\epsilon/2}(f(x_i))$. □

REMARK 12.3.2. If $(Y, \mathfrak{d})$ is a metric space but X is not compact, then the metric $\mathfrak{d}$ induces a topology on $\mathcal{C}(X, Y)$ as follows. For $f \in \mathcal{C}(X, Y)$ and a compact subset $K \subset X$ and $\epsilon > 0$, let

$$B(f, K, \epsilon) = \{g \in \mathcal{C}(X, Y) \mid \mathfrak{d}(f(x), g(x)) < \epsilon \text{ for all } x, y \in K\}.$$

These sets form the basis for a topology on $\mathcal{C}(X, Y)$ called the *topology of compact convergence*, and this is the same as the compact-open topology (see Exercise 12.1). □

12.4. Composition

For spaces X and Y and Z, there is a composition map $\mathfrak{c}\colon \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$ defined by $\mathfrak{c}(g, f) = g \circ f$ for $g \in \mathcal{C}(Y, Z)$ and $f \in \mathcal{C}(X, Z)$. It is natural to hope that this is continuous. Unfortunately, this does not hold in general. However, it does hold if Y is locally compact:

LEMMA 12.4.1. *Let X and Y and Z be spaces with Y locally compact. Then the composition map $\mathfrak{c}\colon \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$ is continuous.*

PROOF. Let $K \subset X$ be compact and $U \subset Z$ be open. We must prove that $\mathfrak{c}^{-1}(B(K, U))$ is open. Let $(g, f) \in \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y)$ satisfy $\mathfrak{c}(g, f) \in B(K, U)$. It is enough to find an open neighborhood of (g, f) that is mapped by $\mathfrak{c}$ into $B(K, U)$. Since $g \circ f \in B(K, U)$, we have $f(K) \subset g^{-1}(U)$. Since $f(K)$ is a compact subset of the open subset $g^{-1}(U) \subset Y$ and Y is locally compact, there is a compact neighborhood L of $f(K)$ with $L \subset g^{-1}(U)$ (see Exercise 9.3). It follows that $\mathfrak{c}$ takes the open neighborhood $B(L, U) \times B(K, g^{-1}(U))$ of (g, f) into $B(K, U)$, as desired. □

12.5. Evaluation

For spaces X and Y, there is an evaluation map $\mathfrak{e}\colon \mathcal{C}(X, Y) \times X \to Y$ defined by $\mathfrak{e}(f, x) = f(x)$ for $f \in \mathcal{C}(X, y)$ and $x \in X$. Just like for the composition map, to ensure this is continuous we need to assume that X is locally compact:

LEMMA 12.5.1. *Let X and Y be spaces with X locally compact. Then the evaluation map $\mathfrak{e}\colon \mathcal{C}(X, Y) \times X \to Y$ is continuous.*

PROOF. Let p_0 be a one-point space. We have $\mathcal{C}(p_0, X) = X$ and $\mathcal{C}(p_0, Y) = Y$. Applying these identities, the evaluation map becomes the composition map $\mathcal{C}(X, Y) \times \mathcal{C}(p_0, X) \to \mathcal{C}(p_0, Y)$, which is continuous by Lemma 12.4.1. □

12.6. Parameterized maps

Let X and Y and Z be spaces. It is natural to expect maps $\phi\colon X \times Z \to Y$ and $\Phi\colon Z \to \mathcal{C}(X,Y)$ to be closely related. Indeed, if we were working with sets rather than spaces then such maps would be in bijection with each other: a map $\Phi\colon Z \to \mathcal{C}(X,Y)$ would correspond to the map $\phi\colon X \times Z \to Y$ defined by $\phi(x,z) = \Phi(z)(x)$. The following shows that this holds topologically if X is locally compact:

LEMMA 12.6.1. *Let X and Y and Z be spaces. The following holds:*

- *(i) Let $\phi\colon X \times Z \to Y$ be continuous. Define $\Phi\colon Z \to \mathcal{C}(X,Y)$ to be the map that takes $z \in Z$ to the map $X \to Y$ taking $x \in X$ to $\phi(x,z) \in Y$. Then Φ is continuous.*
- *(ii) Assume that X is locally compact. Let $\Psi\colon Z \to \mathcal{C}(X,Y)$ be continuous. Define $\psi\colon X \times Z \to Y$ to be the map taking $(x,z) \in X \times Z$ to $\Psi(z)(x) \in Y$. Then ψ is continuous.*

PROOF. For (i), let $\phi\colon X \times Z \to Y$ be continuous and define $\Phi\colon Z \to \mathcal{C}(X,Y)$ as in (i). Let $K \subset X$ be compact and $U \subset Y$ be open. We must prove that $\Phi^{-1}(B(K,U)) \subset Z$ is open. Let $z_0 \in \Phi^{-1}(B(K,U))$, so $K \times z_0 \subset \phi^{-1}(U)$. Since $K \subset X$ is compact and $\phi^{-1}(U)$ is an open neighborhood of $K \times z_0$, Exercise 8.7 (the "tube lemma") gives an open neighborhood $V \subset Z$ of z_0 with $K \times V \subset \phi^{-1}(U)$. It follows that V is an open neighborhood of z_0 with $V \subset \Phi^{-1}(B(K,U))$, as desired.

We now prove (ii). Assume that X is locally compact and that $\Psi\colon Z \to \mathcal{C}(X,Y)$ is continuous. The map $\psi\colon X \times Z \to Y$ defined in (ii) is the composition

$$X \times Z \xrightarrow{\mathbb{1} \times \Psi} X \times \mathcal{C}(X,Y) \xrightarrow{\mathfrak{e}} Y,$$

where $\mathfrak{e}\colon X \times \mathcal{C}(X,Y) \to Y$ is the evaluation map $\mathfrak{e}(x,f) = f(x)$. Lemma 12.5.1 implies that $\mathfrak{e}$ is continuous, so we conclude that ψ is continuous. □

12.7. Homotopies and the compact-open topology

Let $f_0, f_1\colon X \to Y$ be maps. Recall that a homotopy from f_0 to f_1 is a continuous map $H\colon X \times I \to Y$ with $H(x,0) = f_0(x)$ and $H(x,1) = f_1(x)$ for all $x \in X$. Lemma 12.6.1 implies that such a homotopy gives a map $h\colon I \to \mathcal{C}(X,Y)$. This map h can be viewed as a path from $h(0) = f_0$ to $h(1) = f_1$. Conversely, if X is locally compact then Lemma 12.6.1 implies that a path in $\mathcal{C}(X,Y)$ from f_0 to f_1 gives a homotopy from f_0 to f_1.

12.8. Quotient maps and the compact-open topology

As an application of our results, we give another proof of the following result from §11.3:

LEMMA 11.3.4. *Let $q\colon X \to Y$ be a quotient map and let Z be a locally compact space. Then the map $q \times \mathbb{1}\colon X \times Z \to Y \times Z$ is a quotient map.*

PROOF. As we discussed in §4.4, the quotient map $q\colon X \to Y$ satisfies the following universal property. Let $\sim$ be the equivalence relation on X where $x_1 \sim x_2$ if $q(x_1) = q(x_2)$. A map $F\colon X \to W$ is $\sim$-invariant if $F(x_1) = F(x_2)$ whenever $x_1 \sim x_2$. For all spaces W, the following holds:

- There is a bijection between maps $f\colon Y \to W$ and $\sim$-invariant maps $F\colon X \to W$ taking $f\colon Y \to W$ to $f \circ q\colon X \to W$.

In fact, this universal properties characterizes the quotient topology (see Exercise 4.6). We must therefore verify the analogue of it for the map $q \times \mathbb{1}\colon X \times Z \to Y \times Z$.

Consider a space W and a map $G\colon X \times Z \to W$ that is $\sim$-invariant in the sense that $G(x_1,z) = G(x_2,z)$ for all $x_1, x_2 \in X$ and $z \in Z$ with $x_1 \sim x_2$. We must construct a map $g\colon Y \times Z \to W$ such that $G = g \circ (q \times \mathbb{1})$. Let $F\colon X \to \mathcal{C}(Z,W)$ be the map defined by

$$F(x)(z) = G(x,z) \quad \text{for all } x \in X \text{ and } z \in Z.$$

By Lemma 12.6.1, the map F is continuous. Since G is $\sim$-invariant, so is F. It follows that there is a map $f\colon Y \to \mathcal{C}(Z,W)$ with $F = f \circ q$. Let $g\colon Y \times Z \to W$ be the map defined by

$$g(y,z) = f(y)(z) \quad \text{for all } y \in Y \text{ and } z \in Z.$$

Since Z is locally compact, Lemma 12.6.1 says that g is continuous. By construction we have $G = g \circ (q \times \mathbb{1})$, as desired. □

12.9. Parameterized maps, II

Let X and Y and Z be spaces with X locally compact. Lemma 12.6.1 gives a bijection between $\mathcal{C}(X \times Z, Y)$ and $\mathcal{C}(Z, \mathcal{C}(X, Y))$. The following lemma says that this bijection is a homeomorphism if X and Z are Hausdorff:

Lemma 12.9.1. *Let X and Y and Z be spaces with X locally compact Hausdorff and Z Hausdorff. Let $\lambda\colon \mathcal{C}(X \times Z, Y) \to \mathcal{C}(Z, \mathcal{C}(X, Y))$ be the map taking $\phi\colon X \times Z \to Y$ to the map $\Phi\colon Z \to \mathcal{C}(X, Y)$ defined by*

$$\Phi(z)(x) = \phi(x, z) \in Y \quad \text{for all } z \in Z \text{ and } x \in X.$$

Then λ is a homeomorphism.

Proof. Lemma 12.6.1 says that λ is a bijection. For $K \subset X$ and $L \subset Z$ compact and $U \subset Y$ open the map λ restricts to a bijection between $B(K \times L, U)$ and $B(L, B(K, U))$. To prove the lemma, it is enough to prove that open sets of these forms are subbases for the topologies on $\mathcal{C}(X \times Z, Y)$ and $\mathcal{C}(Z, \mathcal{C}(X, Y))$:

- For $\mathcal{C}(X \times Z, Y)$, we prove this in Lemma 12.9.2 below.
- For $\mathcal{C}(Z, \mathcal{C}(X, Y))$, in Lemma 12.9.3 below we prove more generally that if $\mathcal{B}$ is any subbasis for the topology on a space W, then sets of the form $B(L, V)$ with $L \subset Z$ compact and $V \in \mathcal{B}$ form a subbasis for $\mathcal{C}(Z, W)$. □

The above proof used the following two results:

Lemma 12.9.2. *Let X and Y and Z be spaces with X and Z Hausdorff. Then the set of all $B(K \times L, U)$ with $K \subset X$ compact and $L \subset Z$ compact and $U \subset Y$ open forms a subbasis for the compact-open topology on $\mathcal{C}(X \times Z, Y)$.*

Proof. Let $C \subset X \times Z$ be compact and $U \subset Y$ be open. We must prove that $B(C, U)$ is open in the topology with the indicated subbasis. Consider $f \in B(C, U)$. It is enough to find $K_1, \ldots, K_n \subset X$ compact and $L_1, \ldots, L_n \subset Z$ compact such that

$$f \in B(K_1 \times L_1, U) \cap \cdots \cap B(K_n \times L_n, U) \subset B(C, U).$$

Unwrapping this, we need the K_i and L_i to satisfy the following:

- $C \subset \cup_{i=1}^n K_i \times L_i$; and
- $K_i \times L_i \subset f^{-1}(U)$ for all $1 \leq i \leq n$.

Let $C(X) \subset X$ and $C(Z) \subset Z$ be the projections of $C \subset X \times Z$. Both $C(X)$ and $C(Z)$ are compact Hausdorff spaces, and $C \subset C(X) \times C(Z)$. Replacing X with $C(X)$ and Z with $C(Z)$, we can therefore assume without loss of generality that X and Z are compact Hausdorff spaces. The space $X \times Z$ is thus also a compact Hausdorff space, and in particular is normal (see Lemma 8.2.3).

The set $f^{-1}(U)$ is an open neighborhood of C. Since $X \times Z$ is normal, for each $c \in C$ we can find open sets $V_c \subset X$ and $W_c \subset Z$ such that $c \in V_c \times W_c$ and $\overline{V}_c \times \overline{W}_c \subset f^{-1}(U)$. Since C is compact, we can find $c_1, \ldots, c_n$ such that $C \subset \cup_{i=1}^n V_{c_i} \times W_{c_i}$. Let $K_i = \overline{V}_{c_i} \subset X$ and $L_i = \overline{W}_{c_i} \subset Z$, so $K_i \times L_i \subset f^{-1}(U)$. Since X and Z are compact, the closed sets K_i and L_i are also compact. By construction we have $C \subset \cup_{i=1}^n K_i \times L_i$, as desired. □

Lemma 12.9.3. *Let Z and W be spaces with Z Hausdorff and let $\mathcal{B}$ be a subbasis for the topology on W. Then the set of all $B(K, V)$ with $K \subset Z$ compact and $V \in \mathcal{B}$ forms a subbasis for the compact-open topology on $\mathcal{C}(Z, W)$.*

Proof. See Exercise 12.2. □

Remark 12.9.4. It is a little annoying that the above results require local compactness. Unfortunately, they are false in general. There is a way around this using the theory of compactly generated spaces. Rather than try to describe this, we refer the interested reader to Steenrod's classic paper [**2**], where he describes conditions that make a category of spaces a "convenient category" for homotopy theory. A nice textbook reference is [**1**, Chapter 8]. □

12.10. Exercises

EXERCISE 12.1. Let X be a space and let $(Y, \mathfrak{d})$ be a metric space. For $f \in Y^X$ and a compact subset $K \subset X$ and $\epsilon > 0$, let

$$B(f, K, \epsilon) = \left\{ g \in Y^X \mid \mathfrak{d}(f(x), g(x)) < \epsilon \text{ for all } x, y \in K \right\}.$$

Prove that these sets form the basis for a topology on Y^X, and this topology is the same as the compact-open topology. □

EXERCISE 12.2. Let Z and W be spaces with Z Hausdorff and let $\mathcal{B}$ be a subbasis for the topology on W. Prove that the set of all $B(K, V)$ with $K \subset Z$ compact and $V \in \mathcal{B}$ forms a subbasis for the compact-open topology on $\mathcal{C}(Z, W)$. □

EXERCISE 12.3. Topologize $\mathrm{GL}_n(\mathbb{R})$ by identifying it as a subspace of $\mathrm{Mat}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. For each $M \in \mathrm{GL}_n(\mathbb{R})$, multiplication by M gives a linear map $\phi_M : \mathbb{R}^n \to \mathbb{R}^n$. Prove that the map $\iota\colon \mathrm{GL}_n(\mathbb{R}) \to \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n)$ defined by $\iota(M) = \phi_M$ is a closed embedding. □

EXERCISE 12.4. Let $(M, \mathfrak{d})$ be a compact metric space. An isometry of M is a bijection $f\colon M \to M$ such that $\mathfrak{d}(f(p), f(q)) = \mathfrak{d}(p, q)$ for all $p, q \in M$. Let $\mathrm{Isom}(M)$ be the group of isometries of M. Topologize $\mathrm{Isom}(M)$ using the compact-open topology, i.e., by identifying $\mathrm{Isom}(M)$ with a subspace of $\mathcal{C}(M, M)$. Prove that $\mathrm{Isom}(M)$ is compact. □

EXERCISE 12.5. Let $\mathrm{Homeo}(\mathbb{S}^1)$ be the set of homeomorphisms $f\colon \mathbb{S}^1 \to \mathbb{S}^1$. Topologize $\mathrm{Homeo}(\mathbb{S}^1)$ using the compact-open topology, i.e., by identifying $\mathrm{Homeo}(\mathbb{S}^1)$ with a subspace of $\mathcal{C}(\mathbb{S}^1, \mathbb{S}^1)$. Prove that $\mathrm{Homeo}(\mathbb{S}^1)$ is not locally compact. □

Bibliography

[1] B. Gray, *Homotopy theory: an introduction to algebraic topology*, Academic Press, New York-London, 1975.
[2] N. Steenrod, A convenient category of topological spaces, Michigan Math. J. 14 (1967) 133–152.

CHAPTER 13

Quotients by group actions

CHAPTER 14

Topological manifolds

In this final chapter, we use the tools we have developed to study manifolds, which are perhaps the most important class of spaces in algebraic topology.

14.1. Basic definitions

An *n-dimensional manifold* (or simply an n-manifold) is a second countable Hausdorff space M^n that is locally homeomorphic to $\mathbb{R}^n$ in the following sense:

- For all $p \in M^n$, there exists an open neighborhood U of p that is homeomorphic to an open subset of $\mathbb{R}^n$.

A *chart* on M^n is a homeomorphism $\phi\colon U \to V$ with $U \subset M^n$ and $V \subset \mathbb{R}^n$ open sets. If U is an open neighborhood of $p \in M^n$, we call this chart $\phi\colon U \to V$ a chart around p. An *atlas* for M^n is a collection of charts $\{\phi_i\colon U_i \to V_i\}_{i \in I}$ such that the U_i cover M^n.

Here are several basic examples:

EXAMPLE 14.1.1. The whole space $\mathbb{R}^n$ is an n-manifold with an atlas consisting of a single chart $\mathbb{1}\colon \mathbb{R}^n \to \mathbb{R}^n$. More generally, an open set $U \subset \mathbb{R}^n$ is an n-manifold, again with an atlas consisting of a single chart $\mathbb{1}\colon U \to U$. □

EXAMPLE 14.1.2. More generally, if M^n is an n-manifold and $W \subset M^n$ is open, then W is an n-manifold. Indeed, for $p \in W$ let $\phi\colon U \to V$ be a chart around p for M^n. Letting $U' = U \cap W$ and $V' = \phi(U')$, the homeomorphism $\phi|_{U'}\colon U' \to V'$ is a chart around p for W. □

EXAMPLE 14.1.3. Let $\mathbb{S}^n$ be the n-sphere, so

$$\mathbb{S}^n = \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1\right\}.$$

This is an n-manifold. Indeed, for $1 \leq k \leq n+1$ let

$$U_{x_k > 0} = \{(x_1, \ldots, x_{n+1}) \in \mathbb{S}^n \mid x_k > 0\},$$
$$U_{x_k < 0} = \{(x_1, \ldots, x_{n+1}) \in \mathbb{S}^n \mid x_k < 0\}.$$

Letting $B = B_1(0) \subset \mathbb{R}^n$ be the open unit ball, we have homeomorphisms $\phi_{x_k>0}\colon U_{x_k>0} \to B$ and $\phi_{x_k<0}\colon U_{x_k<0} \to B$ taking a point $(x_1, \ldots, x_{n+1})$ to $(x_1, \ldots, \widehat{x_k}, \ldots, x_{n+1}) \in B$, where the hat in $\widehat{x_j}$ indicates that this coordinate is being omitted. The set

$$\{\phi_{x_k>0}\colon U_{x_k>0} \to B,\ \phi_{x_k<0}\colon U_{x_k<0} \to B \mid 1 \leq k \leq n+1\}$$

is an atlas for $\mathbb{S}^n$. □

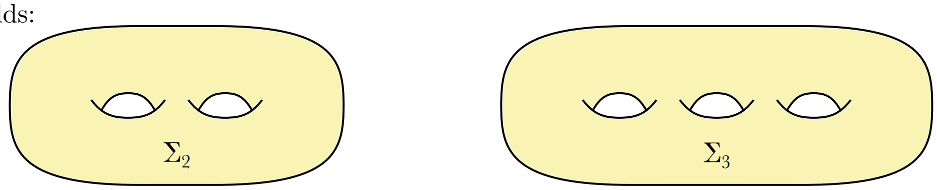
EXAMPLE 14.1.4. Let $\mathbb{RP}^n$ be the set of lines though the origin in $\mathbb{R}^{n+1}$. There is a projection map $q\colon \mathbb{R}^{n+1} \setminus 0 \to \mathbb{RP}^n$ taking $x \in \mathbb{R}^{n+1} \setminus 0$ to the line through 0 and x. We endow $\mathbb{RP}^n$ with the quotient topology from this projection, so $U \subset \mathbb{RP}^n$ is open if and only if $q^{-1}(U) \subset \mathbb{R}^{n+1} \setminus 0$ is open. The space $\mathbb{RP}^n$ is known as the n-dimensional real projective space. As notation, for $(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \setminus 0$ we write $[x_1, \ldots, x_{n+1}]$ for the corresponding point of $\mathbb{RP}^n$, so for $\lambda \in \mathbb{R}$ nonzero we have $[\lambda x_1, \ldots, \lambda x_{n+1}] = [x_1, \ldots, x_{n+1}]$.

The space $\mathbb{RP}^n$ is an n-manifold. Unlike our previous examples, it is not totally obvious that it is second countable and Hausdorff, so we leave this as an exercise (Exercise 14.1). We prove it is locally Euclidean by exhibiting an atlas as follows. For $1 \leq k \leq n+1$, let $U_k = \{[x_1, \ldots, x_{n+1}] \in \mathbb{RP}^n \mid x_k \neq 0\}$. This set is well-defined, and the map $\phi_k\colon U_k \to \mathbb{R}^n$ defined by

$$\phi_k([x_1, \ldots, x_{n+1}]) = (x_1/x_k, \ldots, \widehat{x_k/x_k}, \ldots, x_{n+1}/x_k) \quad \text{for } [x_1, \ldots, x_{n+1}] \in \mathbb{RP}^n$$

is a well-defined homeomorphism (see Exercise 14.1). The set $\{\phi_k : U_k \to \mathbb{R}^n \mid 1 \le k \le n+1\}$ is an atlas for $\mathbb{RP}^n$. □

REMARK 14.1.5. It is clear that the only connected 0-dimensional manifold is a single point. It turns out that $\mathbb{R}$ and $\mathbb{S}^1$ are the only connected 1-dimensional manifolds. There is also a very beautiful classification of compact connected 2-dimensional manifolds. Here are two examples of such 2-manifolds:



We describe the classification of 2-manifolds in Essay A. The exercises in that essay also outline a proof of the classification of 1-dimensional manifolds. In higher dimensions, things are much more complicated. □

REMARK 14.1.6. The requirement that manifolds be second countable and Hausdorff is needed to rule out various pathological examples. Without them, even 1-manifolds would not have a simple classification. We describe some of these pathological examples later in this chapter. □

14.2. Basic properties

The following summarizes some of the basic point-set topological properties of manifolds:

LEMMA 14.2.1. *Let M^n be an n-dimensional manifold. Then:*

- *M^n is normal.*
- *M^n is locally compact.*
- *M^n is paracompact.*
- *M^n is locally path connected, so its path components and connected components coincide and are clopen.*

PROOF. Since M^n is locally homeomorphic to $\mathbb{R}^n$, the fact that M^n is locally compact and locally path connected follows immediately from the fact that $\mathbb{R}^n$ is locally compact and locally path connected. Since M^n is second countable, Hausdorff, and locally compact, it follows that M^n is paracompact (see Corollary 10.2.3). This implies that M^n is normal (see Lemma 10.3.1). □

REMARK 14.2.2. One basic property of manifolds we do not list above is that their dimension is well-defined. In fact, it is true that if M is both an n-manifold and an m-manifold then $n = m$, but this is a difficult theorem called the *invariance of domain*. The most natural proof of invariance of domain uses homology. □

14.3. Embedding manifolds into Euclidean space

Many n-manifolds are constructed as subspaces of some $\mathbb{R}^d$, but some manifolds like $\mathbb{RP}^n$ do not have obvious embeddings into any Euclidean space. However, it turns out that all manifolds can be embedded in some $\mathbb{R}^d$:

THEOREM 14.3.1. *Let M^n be an n-dimensional manifold. Then for some $d \gg 0$ there exists an embedding $\iota : M^n \hookrightarrow \mathbb{R}^d$.*

We remark that using dimension theory, one can embed M^n into $\mathbb{R}^{2n+1}$. See [**2**, Theorem V3]. To avoid technical complications, we only prove Theorem 14.3.1 when M^n is compact. See the remark after the proof for how to extend our argument to the non-compact case.

PROOF OF THEOREM 14.3.1 FOR M^n COMPACT. Since M^n is compact, it has a finite atlas $\{\phi_k : U_k \to V_k \mid 1 \le k \le m\}$. Since M^n is paracompact, there is a partition of unity $\{f_1, \ldots, f_m\}$ subordinate to $\{U_1, \ldots, U_n\}$. Recall that this means that each f_k is a function $f_k : M^n \to [0,1]$ with $\text{supp}(f_k) \subset U_k$, and $f_1(p) + \cdots + f_m(p) = 1$ for all $p \in M^n$. Multiplying ϕ_k by f_k, we get a map $f_k\phi_k\colon U_k \to \mathbb{R}^n$. Since $\operatorname{supp}(f_k) \subset U_k$, we can extend $f_k\phi_k\colon U_k \to \mathbb{R}^n$ to a continuous map $G_k\colon M^n \to \mathbb{R}^n$ with $G_k(p) = 0$ for $p \notin U_k$. Let $\iota\colon M^n \to \mathbb{R}^{nm+m}$ be the map defined by

$$\iota(p) = (G_1(p), f_1(p), \ldots, G_m(p), f_m(p)) \in (\mathbb{R}^n \times \mathbb{R}^1)^{\times m} = \mathbb{R}^{nm+m} \quad \text{for } p \in M^n.$$

Since M^n is compact, to prove that ι is an embedding it is enough to prove that ι is injective (see Lemma 8.4.2). For this, consider $p, q \in M^n$ with $\iota(p) = \iota(q)$. Since $f_1(p) + \cdots + f_m(p) = 1$, there is some $1 \leq k \leq m$ with $f_k(p) > 0$. Since $\iota(p) = \iota(q)$, we have $f_k(q) = f_k(p)$. This implies that $p, q \in \operatorname{supp}(f_k) \subset U_k$. Since $\iota(p) = \iota(q)$, the points

$$G_k(p) = f_k(p)\phi_k(p) \quad \text{and} \quad G_k(q) = f_k(q)\phi_k(q)$$

must be equal, so $\phi_k(p) = \phi_k(q)$. Since $\phi_k\colon U_k \to V_k$ is a homeomorphism, it follows that $p = q$. □

Remark 14.3.2. One way to extend Theorem 14.3.1 to noncompact manifolds M^n is to prove that there is still a finite atlas $\{\phi_k\colon U_k \to V_k \mid 1 \leq k \leq m\}$, which allows you to run the above proof (though with a little more care since injective maps need not be embeddings in the noncompact setting). That a finite atlas exists might sound surprising, but the key insight is that the U_k need not be connected, and in fact can have countably many components. We omit the details. □

14.4. Metrics

Theorem 14.3.1 implies the following:

Corollary 14.4.1. *Let M^n be an n-dimensional manifold. Then M^n can be given the structure of a metric space.*

Since we did not prove Theorem 14.3.1 for noncompact manifolds, we give a proof of Corollary 14.4.1 that works in general:

Proof of Corollary 14.4.1. Since $\prod_{k=1}^{\infty} \mathbb{R}^n \times \mathbb{R}^1$ can be given the structure of a metric space (see Lemma 11.7.1), it is enough to embed M^n into this countable product. Since M^n is second countable, it has a countable atlas $\{\phi_k\colon U_k \to V_k \mid k \geq 1\}$. Since M^n is paracompact, there is a partition of unity $\{f_k\colon M^n \to [0,1] \mid k \geq 1\}$ subordinate to $\{U_k \mid k \geq 1\}$. Multiplying ϕ_k by f_k, we get a map $f_k\phi_k\colon U_k \to \mathbb{R}^n$. Since $\operatorname{supp}(f_k) \subset U_k$, we can extend $f_k\phi_k\colon U_k \to \mathbb{R}^n$ to a continuous map $G_k\colon M^n \to \mathbb{R}^n$ with $G_k(p) = 0$ for $p \notin U_k$. Let $\iota\colon M^n \to \prod_{k=1}^{\infty} \mathbb{R}^n \times \mathbb{R}^1$ be the map defined by

$$\iota(p) = (G_k(p), f_k(p))_{k \geq 1} \in \prod_{k=1}^{\infty} \mathbb{R}^n \times \mathbb{R}^1 \quad \text{for } p \in M^n.$$

The proof that ι is injective is the same as in the proof of Theorem 14.3.1, so we omit it. Letting $X = \operatorname{Im}(\iota)$, to prove that ι is an embedding we must prove that $\iota^{-1}\colon X \to M^n$ is continuous. Consider some $p_0 \in M^n$. We prove that ι^{-1} is continuous at $\iota(p_0)$ as follows. Choose $d \geq 1$ such that $f_d(p_0) > 0$. Let

$$U'_d = \{p \in U_d \mid f_d(p) > 0\} \quad \text{and} \quad V'_d = \phi_d(U'_d).$$

Set

$$W = X \cap \left\{ (x_k, \lambda_k)_{k \geq 1} \in \prod_{k=1}^{\infty} \mathbb{R}^n \times \mathbb{R}^1 \mid \lambda_d > 0 \right\},$$

so W is an open neighborhood of $\iota(p_0)$ in X. The map ι^{-1} takes W to U'_d. On W, the map ι^{-1} can be written as a composition of a sequence of continuous maps:

- First, the projection

$$W \hookrightarrow \prod_{k=1}^{\infty} \mathbb{R}^n \times \mathbb{R}^1 \xrightarrow{\pi} \mathbb{R}^n \times \mathbb{R}^1$$

  onto the d^{th} factor, whose image is contained in $\{(\lambda v, \lambda) \mid \lambda > 0 \text{ and } v \in V'_d\}$.
- Next, the map

$$\{(\lambda v, \lambda) \mid \lambda > 0 \text{ and } v \in V'_d\} \longrightarrow V'_d$$

  that takes $(\lambda v, \lambda)$ to v.
- Finally, the inverse of the map $\phi_d\colon U'_d \to V'_d$.

We deduce that the restriction of ι^{-1} to W is continuous, and thus that ι^{-1} is continuous at $\iota(p_0)$. □

Remark 14.4.2. There are various metrization theorems giving conditions that imply that a topological space can be given a metric. Most of them are proved using arguments related to the one we gave for Corollary 14.4.1. See [**3**, Chapter 6] for a discussion of this. □

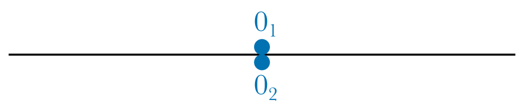
14.5. Non-Hausdorff manifolds

Recall that we require manifolds to be Hausdorff and second countable. Removing these hypotheses gives many exotic generalized manifolds, even in dimension 1. We have already seen one example of a non-Hausdorff 1-manifold, namely the line with two origins from Example 7.1.1. We recall the construction:

Example 14.5.1. As a set, let $Y = (\mathbb{R} \setminus \{0\}) \sqcup \{0_1, 0_2\}$. For $i = 1, 2$, let $f_i \colon \mathbb{R} \to Y$ be the map defined by $f_i(x) = x$ for $x \in \mathbb{R} \setminus \{0\}$ and $f_i(0) = 0_i$. Give Y the identification space topology, so:

- a set $U \subset Y$ is open if and only if $f_1^{-1}(U)$ and $f_2^{-1}(U)$ are open in $\mathbb{R}$.

Here is a picture of this:



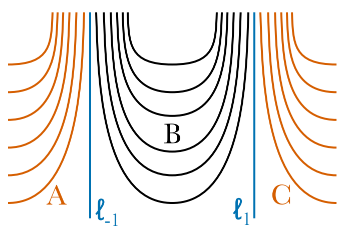
With this topology, the subspaces $Y \setminus \{0_2\} = f_1(\mathbb{R})$ and $Y \setminus \{0_1\} = f_2(\mathbb{R})$ are open subsets of Y that are both homeomorphic to $\mathbb{R}$. It follows that Y a second-countable non-Hausdorff 1-manifold. □

This example might not seem very geometrically interesting. The theory of foliations of the plane gives non-Hausdorff 1-manifolds with a closer connection to geometry. See [**1**] for a beautiful discussion of this. We content ourselves here with one example:

Example 14.5.2. For $c \in \mathbb{R}$, let $X_c = \{(x, y) \mid (x^2 - 1)e^y = c\} \subset \mathbb{R}^2$. Define

$$\mathfrak{F} = \{L \mid L \text{ is a connected component of } X_c \text{ for some } c \in \mathbb{R}\}.$$

The set $\mathfrak{F}$ is what is called a foliation of $\mathbb{R}^2$. Each $L \in \mathfrak{L}$ is called a *leaf* of the foliation. Here is a picture of $\mathfrak{F}$:

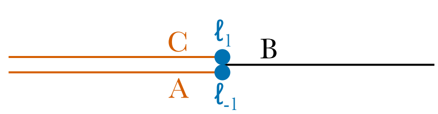


Each leaf L is homeomorphic to $\mathbb{R}$, and $\mathbb{R}^2$ is the disjoint union of the $L \in \mathfrak{F}$. The set X_0 consists of two vertical lines ℓ_{-1} and ℓ_1 where $x = \pm 1$. For $c > 0$, the set X_c consists of two arcs, one lying in the region to the left of ℓ_{-1} labeled A and one lying in the region to the right of ℓ_1 labeled C. For $c < 0$, the set X_c consists of a single arc in the region between ℓ_{-1} and ℓ_1 labeled B.

Let $\mathcal{L}$ be the quotient space of $\mathbb{R}^2$ obtained by collapsing each $L \in \mathfrak{F}$ to a point. This is called the leaf space of the foliation $\mathfrak{F}$. The space $\mathcal{L}$ is a non-Hausdorff 1-manifold. To describe it, let R_1 and R_2 be copies of $\mathbb{R}$. The space $\mathcal{L}$ is obtained by gluing R_1 to R_2 so as to identify each $t \in R_1$ with $t > 0$ with the corresponding $t \in R_2$. The various types of leaves correspond to the following points:

- The points $0 \in R_1$ and $0 \in R_2$ correspond to ℓ_{-1} and ℓ_1.
- The points $t \in R_1$ with $t < 0$ correspond to the arcs in the region A.
- The points $t \in R_2$ with $t < 0$ correspond to the arcs in region C.
- The points $t \in R_1$ and $t \in R_2$ with $t > 0$ that are glued together correspond to the arcs in the region B.

The picture is as follows:



This space is non-Hausdorff since the points corresponding to ℓ_{-1} and ℓ_1 do not have disjoint neighborhoods. You will verify all of this in Exercise 14.2. □

14.6. Non-second countable manifolds

The theory of non-second countable manifolds has a set-theoretic flavor. It turns out that in dimension one there is a single example of a connected non-second countable Hausdorff 1-manifold called the *long line* L. We close this chapter with a brief discussion of it. The space L has the following seemingly paradoxical properties:

- L is a path-connected Hausdorff non-second-countable 1-manifold.
- Like $\mathbb{R}$, the points of L are endowed with a total ordering.
- For $x, y \in L$ with $x < y$, the "interval"
$$[x, y] = \{z \in L \mid x \leq z \leq y\}$$
is homeomorphic to the closed interval $I = [0, 1]$. This accounts for L being path connected.
- On the other hand, since L is not second countable it contains uncountably many subspaces homeomorphic to the open interval $(0, 1)$.

Before we can construct L, we need to discuss some more details about well-ordered sets, which we introduced in §11.10 to set up the process of transfinite induction.

14.6.1. Minimal uncountable well-ordered set. Let S be an uncountable set. Pick a well-ordering on S. Let $\mathfrak{C}$ be the set of all initial segments of S that are either finite or countably infinite. The set $\mathfrak{C}$ is nonempty since $\emptyset \in \mathfrak{C}$. In fact, by starting with $\emptyset$ and repeatedly adding the minimal element we have not yet chosen we see that there exists a countably infinite set in $\mathfrak{C}$. As we discussed in §11.10, the initial segments of S are totally ordered under inclusion. Let
$$S_\Omega = \bigcup_{J \in \mathfrak{C}} J.$$
The set S_Ω is an initial segment of S. By construction, all initial segments J with $J \subsetneq S_\Omega$ are countable. We claim that S_Ω is not countable. Indeed, let s_0 be the minimal element of $S \setminus S_\Omega$. The initial segment $S_\Omega \sqcup \{s_0\}$ cannot lie in $\mathfrak{C}$, so $S_\Omega \sqcup \{s_0\}$ is uncountable. This implies that S_Ω is uncountable. The totally ordered set S_Ω is called the *minimal uncountable well-ordered set*.[1] It is unique up to isomorphism, but we will not need this. All we need to know about S_Ω is that it is uncountable but all proper initial segments of S_Ω are finite or countably infinite.

14.6.2. Constructing the long line. Let $\widehat{L} = S_\Omega \times [0, 1)$. Both S_Ω and $[0, 1)$ have total orderings. Give $\widehat{L}$ the dictionary ordering, so $(s, x) \leq (s', x')$ if $s < s'$ or if $s = s'$ and $x < x'$. An *open interval* in $\widehat{L}$ is a set of the form $(\theta_1, \theta_2) = \{\nu \mid \theta_1 < \nu < \theta_2\}$ for some $\theta_1, \theta_2 \in \widehat{L}$ with $\theta_1 < \theta_2$. This is a basis for a topology (see Exercise 3.3) called the *order topology*. We endow $\widehat{L}$ with the order topology.

To form the long line L, let $s_0 \in S_\omega$ be the minimal element. It follows that $(s_0, 0) \in \widehat{L}$ is the minimal element of $\widehat{L}$. Define $L = \widehat{L} \setminus \{(s_0, 0)\}$. As you will verify in Exercise 14.3, this has the properties claimed in §14.6.

14.7. Exercises

EXERCISE 14.1. Prove the following:

(a) The space $\mathbb{RP}^n$ is Hausdorff and second countable.
(b) Letting $U_k = \{[x_1, \ldots, x_{n+1}] \in \mathbb{RP}^n \mid x_k \neq 0\}$, the map $\phi_k \colon U_k \to \mathbb{R}^n$ defined by
$$\phi_k([x_1, \ldots, x_{n+1}]) = (x_1/x_k, \ldots, \widehat{x_k/x_k}, \ldots, x_{n+1}/x_k) \quad \text{for } [x_1, \ldots, x_{n+1}] \in \mathbb{RP}^n$$
is a well-defined homeomorphism. □

EXERCISE 14.2. Verify the description of $\mathcal{L}$ in Example 14.5.2. □

EXERCISE 14.3. Let L be the long line constructed in §14.6.2. Prove the following:

[1] Or the minimal uncountable ordinal, but we have chosen not to use that terminology.

(a) For $x, y \in L$ with $x < y$, the closed interval
$$[x, y] = \{z \in L \mid x \leq z \leq y\}$$
is homeomorphic to the closed interval $I = [0, 1]$.
(b) The space L is path-connected.
(c) The space L contains uncountably many subspaces homeomorphic to the open interval $(0, 1)$.
(d) The space L is a Hausdorff non-second-countable 1-manifold. □

Bibliography

[1] A. Haefliger & G. Reeb, One dimensional non-Hausdorff manifolds and foliations of the plane, in *Geometric methods in group theory—papers dedicated to Ruth Charney*, 225–241, Sémin. Congr., 34, Soc. Math. France, Paris.
[2] W. Hurewicz & H. Wallman, *Dimension Theory*, Princeton University Press, Princeton, NJ, 1941.
[3] J. Munkres, *Topology*, 2nd edition, Prentice Hall, Upper Saddle River, NJ, 2000.

Part 2

Essays

ESSAY A

Classification of surfaces

A.1. Introduction

An enormous amount of algebraic topology was developed to help classify manifolds up to homeomorphism. This classification is easy in dimensions 0 and 1, where the only connected examples are a point, a circle $\mathbb{S}^1$, and the real line $\mathbb{R}$ (see Exercise A.9). The first interesting dimension is 2, i.e., surfaces. Here there are infinitely many examples, but there is an elegant and easy-to-state classification (at least in the compact case) whose origins go back to 19th century work of Möbius.

A.1.1. Our goal. In this essay, we prove the classification of surfaces. Our goal is to emphasize geometric reasoning. There is a large expository gulf between the geometric topology literature and accounts of the classification of surfaces, which are typically aimed at beginning students and involve elaborate manipulations of triangulations. We include many examples and pictures, but some of our proofs and definitions are a little informal. Making them rigorous will (hopefully) be routine to readers who are experienced with smooth manifolds.

A.1.2. History and sources. The idea of our proof goes back to Zeeman [**13**]. Here are other accounts geared to students earlier in their education:

- See [**9**] or [**12**] for the classical combinatorial proof. I first learned this material from [**9**] when I was an undergraduate.
- See [**2**] for a proof similar to the one we give.

There are other possible proofs of this result. One that is particularly charming is Conway's "ZIP" proof, which can be found in [**4**]. For a history of the classification, see [**5**].

A.1.3. Assumed results. To avoid getting bogged down with point-set topology and foundational results about Euclidean space,[1] we will carefully state but not prove two important results:

- The existence of triangulations of surfaces. Actually, we will use the more flexible notion of "polygonal decompositions".
- The fact that the Euler characteristic of a surface is a topological invariant. This result is very easy once the theory of homology is introduced, so we see little point in giving a combinatorial proof that uses special features of surfaces.

We will also freely use standard results about smooth manifolds, often without mentioning them explicitly.

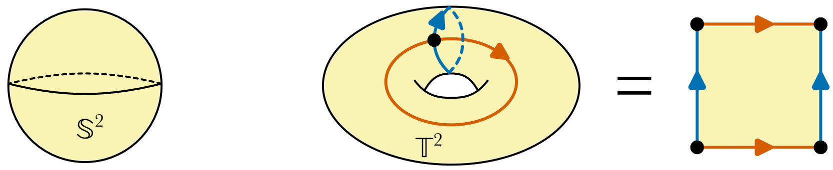
A.1.4. Outline. In §A.2 we give examples of surfaces and state a first version of the classification theorem. Next, as a warm-up to the proof in §A.3 we discuss graphs and their Euler characteristics. We then introduce polygonal decompositions and prove some basic results about the Euler characteristic in §A.4, which ends with a refined version of the classification. We prove the classification in the next two sections: §A.5 proves the "Poincaré conjecture" characterizing the sphere, and §A.6 proves the rest of the classification. Finally, §A.7 gives some extensions and generalizations of the classification.

A.2. Examples of surfaces

A *surface* is a 2-dimensional manifold, possibly with boundary. Our focus will be on surfaces that are connected and *closed*, that is, compact and without boundary. This section focuses on examples.

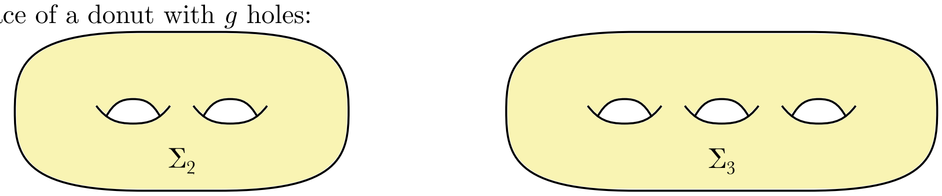
[1]Here are thinking of results like the Jordan curve theorem or more generally the Schoenflies Theorem, which says that any simple closed curve in $\mathbb{R}^2$ bounds a disk.

A.2.1. Basic examples. The most familiar surfaces are the 2-sphere $\mathbb{S}^2$ and the 2-torus $\mathbb{T}^2 = (\mathbb{S}^1)^{\times 2}$:



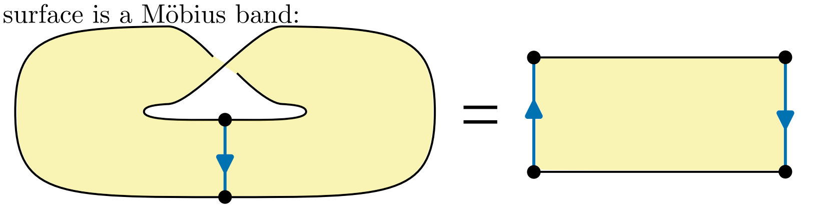
As is shown here, $\mathbb{T}^2$ can be obtained from $\mathbb{D}^2$ by identifying $\mathbb{D}^2$ with a square and identifying parallel sides. The four vertices of the square are all identified to a single point. The sphere $\mathbb{S}^2$ can also be obtained from $\mathbb{D}^2$ by identifying the entire boundary $\partial\mathbb{D}^2 = \mathbb{S}^1$ to a single point.

The torus is the surface of an ordinary donut. More generally, a *genus-g surface*, denoted Σ_g is the surface of a donut with g holes:



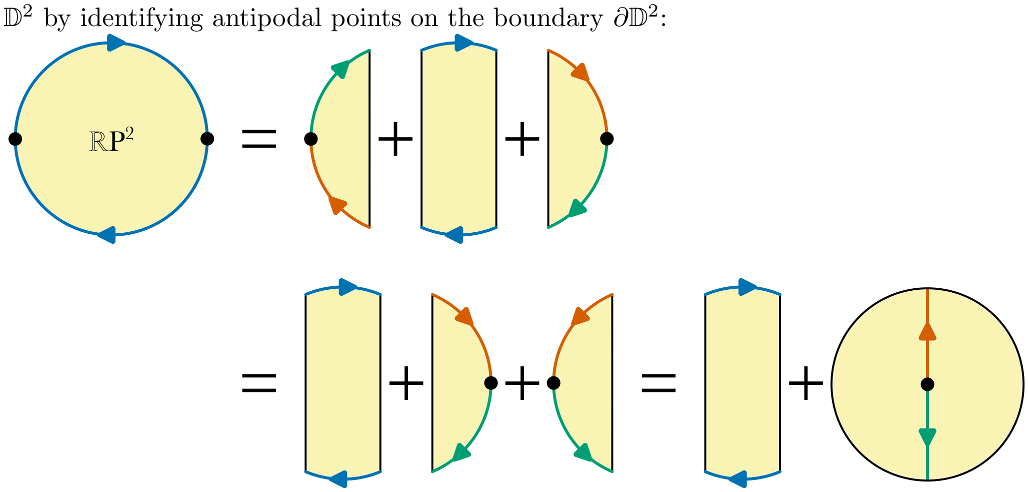
We therefore have $\Sigma_0 \cong \mathbb{S}^2$ and $\Sigma_1 \cong \mathbb{T}^2$. As we will discuss in §A.4 below, for $g \geq 1$ the surface Σ_g can be obtained from a $4g$-gon by identifying sides in an appropriate way.

A.2.2. Möbius band and real projective plane. The surfaces Σ_g are all orientable. We will assume that this notion is familiar from theory of manifolds. The most basic example of a non-orientable surface is a Möbius band:



The Möbius band has one boundary component. To obtain a closed surface, we glue a disk $\mathbb{D}^2$ to this boundary component to form the *real projective plane* $\mathbb{RP}^2$. You might worry that the result depends on the choice of a homeomorphism between $\partial\mathbb{D}^2 = \mathbb{S}^1$ and the boundary component of the Möbius band, but it turns out that the result is independent of the gluing. This holds in great generality; see Exercise A.11. We will use this fact silently throughout this essay.

Pictures of $\mathbb{RP}^2$ are not particularly enlightening,[2] but as the following shows it can be obtained from $\mathbb{D}^2$ by identifying antipodal points on the boundary $\partial\mathbb{D}^2$:



Another way of viewing $\mathbb{RP}^2$ is as the space of lines through the origin in $\mathbb{R}^3$. To connect this with

[2]It cannot be embedded in $\mathbb{R}^3$, but only in $\mathbb{R}^4$. There is a way of drawing it in $\mathbb{R}^3$ with self-intersections called the "Boy's Surface", but this picture does not shed much light on its nature.

the above picture, note that every such line intersects the upper hemisphere

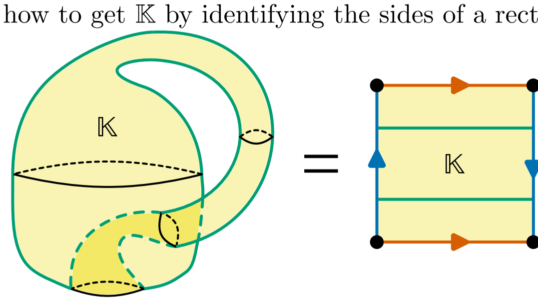
$$U = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1,\ z \geq 0\} \cong \mathbb{D}^2.$$

This intersection is unique except for lines lying in the xy-plane, which intersect

$$\partial U = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\} \cong \partial \mathbb{D}^2$$

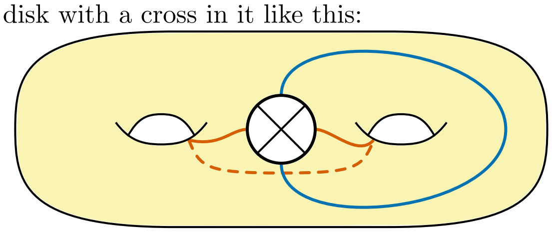
in two antipodal points. The space of lines through the origin can thus be identified with $U \cong \mathbb{D}^2$, but with antipodal points on $\partial U \cong \partial \mathbb{D}^2$ identified.

A.2.3. Klein bottle. Another important example of a non-orientable surface is the Klein bottle $\mathbb{K}$, which is obtained by gluing two Möbius bands together along their boundary. Unlike $\mathbb{RP}^2$, there is a somewhat enlightening way of drawing the $\mathbb{K}$, though necessarily this picture has self-intersections. See here for this and also how to get $\mathbb{K}$ by identifying the sides of a rectangle:

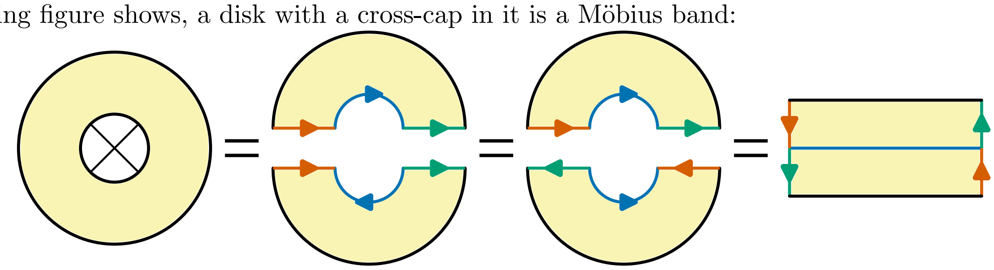


The green curve in both figures is a circle, and when you cut either open along it you get two Möbius bands. This shows that these two surfaces are indeed homeomorphic.

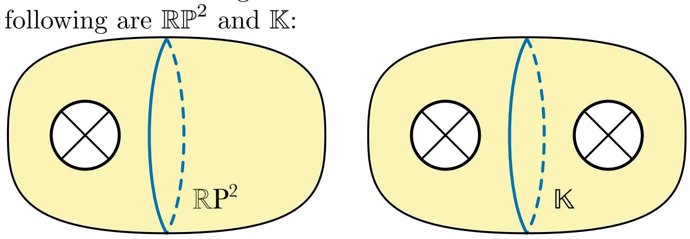
A.2.4. Cross caps. The surfaces $\mathbb{RP}^2$ and $\mathbb{K}$ are the first two elements of an infinite family of non-orientable surfaces. To explain this, we must introduce the notion of a *cross-cap*. A cross-cap on a surface is obtained by removing the interior of a disk and then identifying antipodal points. We denote this by drawing a disk with a cross in it like this:



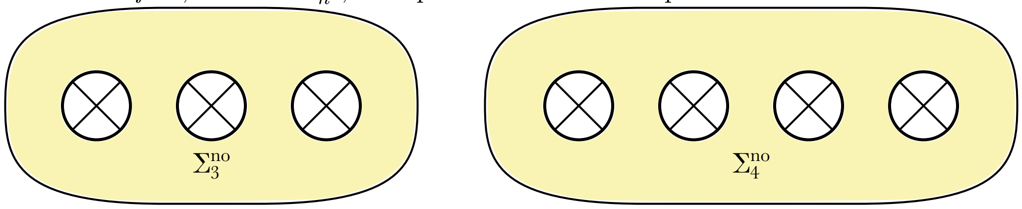
In this figure, the blue and orange arcs are actually disjoint circles embedded in the surface. As the following figure shows, a disk with a cross-cap in it is a Möbius band:



Since $\mathbb{RP}^2$ is a Möbius band with a disk glued to it and $\mathbb{K}$ is two Möbius bands glued together along their boundary, the following are $\mathbb{RP}^2$ and $\mathbb{K}$:



On the left the blue loop divides $\mathbb{RP}^2$ into a Möbius band and a disk, and on the right the blue loop divides $\mathbb{K}$ into two Möbius bands. These pictures suggest the general pattern: the *genus-n* *nonorientable surface*, denoted Σ_n^{no}, is a sphere with n cross-caps on it:



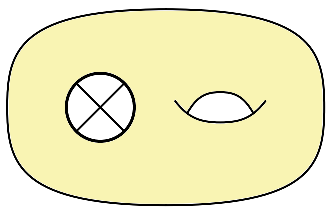
Thus $\Sigma_1^{\text{no}} \cong \mathbb{RP}^2$ and $\Sigma_2^{\text{no}} \cong \mathbb{K}$.

A.2.5. Classification theorem, first version. We can now state a first version of the classification theorem:

THEOREM A.2.1 (Classification of surfaces, weak). *Let Σ be a closed connected surface. Then:*

- *If Σ is orientable, then $\Sigma \cong \Sigma_g$ for a unique $g \geq 0$.*
- *If Σ is non-orientable, then $\Sigma \cong \Sigma_n^{no}$ for a unique $n \geq 1$.*

In some ways this is a very satisfying result, but one weakness is that it does not give an effective way to recognize a given surface. Since it is easy to write down surfaces that do not fit into the above classification in an obvious way, this is a real problem. For instance, consider the following non-orientable surface:

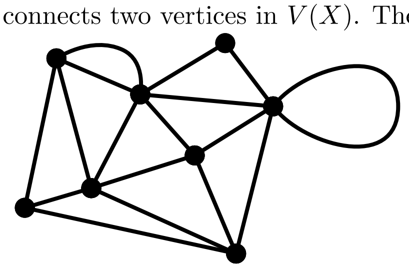


By Theorem A.2.1, this must be homeomorphic to Σ_n^{no} for some $n \geq 1$. However, it is not at all obvious which Σ_n^{no} it is. We will later give a refined classification theorem that will make it clear that the above surface is Σ_3^{no}; see Theorem A.4.12. Before reading this, it is worth trying to prove it for yourself.

A.3. Graphs and their Euler characteristics

As a warm-up before proving the classification of surfaces, this section discusses aspects of graph theory that can be viewed as a one-dimensional analogue of this classification.

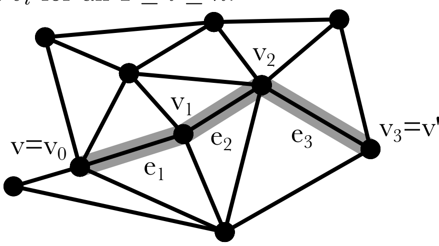
A.3.1. Basic definitions. Recall that a *graph* X is a collection of vertices $V(X)$ and a collection of edges $E(X)$. Each $e \in E(X)$ connects two vertices in $V(X)$. These vertices need not be distinct:



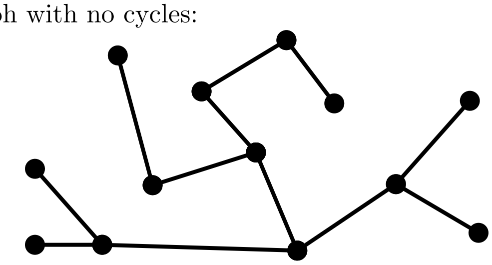
In this essay, we will only consider graphs with finitely many vertices and edges, which we call *finite graphs*. A finite graph is a topological space in a straightforward way. An *edge-path* in a graph from a vertex $v \in V(X)$ to a vertex $v' \in V(X)$ is a sequence of edges $e_1, \ldots, e_n$ such that there exist vertices

$$v = v_0, v_1, \ldots, v_n = v'$$

such that e_i connects v_{i-1} and v_i for all $1 \le i \le n$:

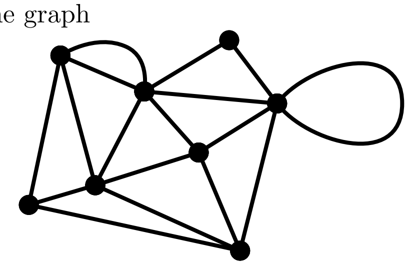


Associated to an edge-path is a continuous map $\gamma\colon I \to X$, and we will often confuse an edge-path with the associated map. The edge-path is *closed* if $v = v'$, in which case the associated path is a loop that we can regard as a continuous map $\gamma\colon \mathbb{S}^1 \to X$. The edge-path is a *cycle* if it is closed, $n \ge 1$, and all the e_i are distinct. A graph is *connected* if all distinct $v, v' \in V(X)$ are connected by an edge-path. This is equivalent to the graph being path-connected as a topological space. A *tree* is a nonempty connected graph with no cycles:



A.3.2. Euler characteristic of graphs. If X is a finite graph, then the *Euler characteristic* of X is $\chi(X) = |V(X)| - |E(X)|$.

Example A.3.1. If X is the graph

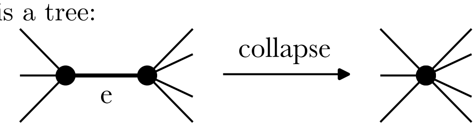


then the Euler characteristic of X is $\chi(X) = |V(X)|/ - |E(X)| = 8 - 17 = -9$. □

A.3.3. Poincaré conjecture for graphs. The importance of the Euler characteristic is illustrated by the following result, which we think of as the "Poincaré conjecture"[3] for graphs:

Lemma A.3.2 (Poincaré conjecture for graphs). *Let X be a finite nonempty connected graph. Then $\chi(X) \le 1$, with equality if and only if X is a tree.*

Proof. If e is an edge of X connecting two distinct vertices, then we can collapse e without changing whether or not X is a tree:



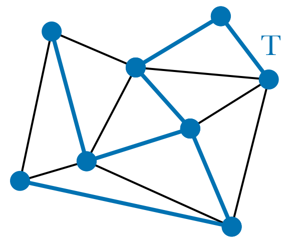
Such a collapse decreases the number of edges and vertices by 1, and thus does not change the Euler characteristic. Collapsing such edges repeatedly, we can therefore assume without loss of generality that all edges of X are loops. Since X is nonempty and connected, this implies that X has one vertex. We therefore have

$$\chi(X) = |V(X)| - |E(X)| = 1 - |E(X)| \le 1$$

[3]For manifolds, the Poincaré conjecture is a topological characterization of a sphere. Once we have defined the Euler characteristic for surfaces, the two-dimensional Poincaré conjecture will say that a compact connected surface Σ is homeomorphic to $\mathbb{S}^2$ if and only if its Euler characteristic is 2.

with equality if and only if $|E(X)| = 0$, i.e., if and only if X is a tree.[4] □

A.3.4. Maximal trees. If X is a connected nonempty graph, then a maximal tree in X is a subtree T of X containing all the vertices. See here:



These always exist:

LEMMA A.3.3. *Let X be a connected nonempty graph. Then X has a maximal tree.*

The proof for finite graphs X is a little easier, and this is the only case we need. We therefore restrict to this case:

PROOF OF LEMMA A.3.3 FOR FINITE GRAPHS. Assume that X has n vertices. Inductively define subtrees

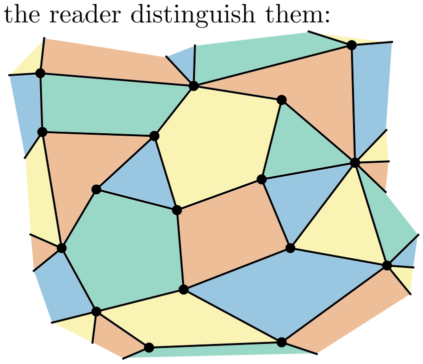
$$T_1 \subset T_2 \subset \cdots \subset T_n$$

of X in the following way. Start by choosing a vertex v_0 of X and letting $T_1 = v_0$. Now assume that T_k has been constructed for some $k < n$. Since X has n vertices, it must be the case that T_k does not contain all the vertices of X. Since X is connected, this implies that there must be an edge e of X that connects a vertex of T_k to a vertex that does not lie in T_k. Let T_{k+1} be the result of adding e to T_k. This process stops at T_n, which contains n vertices and hence is a maximal tree in X. □

A.4. Polygonal decompositions and the Euler characteristic

Our proof of the classification of surfaces will depend on a decomposition of the surface that is a sort of two-dimensional analogue of the decomposition of a graph into vertices and edges.

A.4.1. Basic definitions. A surface equipped with a *polygonal decomposition* is a compact surface Σ (possibly with boundary) together with a finite graph X embedded in Σ such that each path component F of $\Sigma \setminus X$ is homeomorphic to an open disk $\text{Int}(\mathbb{D}^2)$. We will call such an F a *face* of the polygonal decomposition.[5] Here is a picture of part of a polygonal decomposition, with the faces in different colors to help the reader distinguish them:



Here is some terminology for polygonal decompositions:

- The graph X will be called the 1-*skeleton*.
- The vertices and edges of X will be called the vertices and edges of the polygonal decomposition, and the sets of vertices and edges will be written $V(\Sigma)$ and $E(\Sigma)$, respectively.
- The set of faces of the polygonal decomposition will be written $F(\Sigma)$.
- The *Euler characteristic* of the polygonal decomposition is $\chi(\Sigma) = |V(\Sigma)| - |E(\Sigma)| + |F(\Sigma)|$.

[4]Since X has only one vertex, the only way it can be a tree is if it has no edges.

[5]For non-compact surfaces, one would also need to require that the closure of each face is compact.

A.4.2. Existence. The following theorem will play a basic role in our proof:

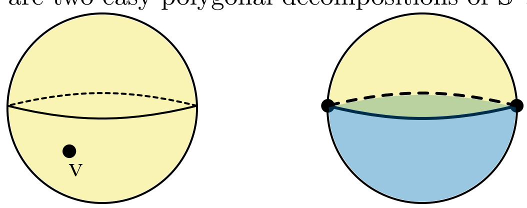
THEOREM A.4.1. *Let Σ be a compact surface, possibly with boundary. Then Σ has a polygonal decomposition.*

See Exercise A.7 for the analogous fact in dimension 1. We will not prove Theorem A.4.1, which requires a long detour into point-set topology. It was originally proved by Radó [**11**]. See [**1**] and [**10**] for modern versions of Radó's proof. I remark that I first learned this proof from [**1**]. A recent and elegant proof along very different lines can be found in [**6**]. Amazingly, the proof in [**6**] uses smooth manifold techniques (even though the surface is not assumed to be smooth), and avoids doing any serious point-set work.

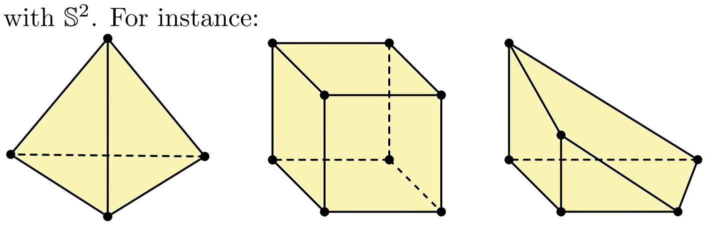
REMARK A.4.2. All of the above sources actually prove the slightly stronger fact that Σ has a triangulation, that is, a polygonal decomposition where the boundaries of each face are length-3 edge-paths in the 1-skeleton. It is easy to subdivide a general polygonal decomposition and turn it into a triangulation. □

A.4.3. Examples of polygonal decompositions. Here are a number of examples:

EXAMPLE A.4.3. Here are two easy polygonal decompositions of $\mathbb{S}^2$:

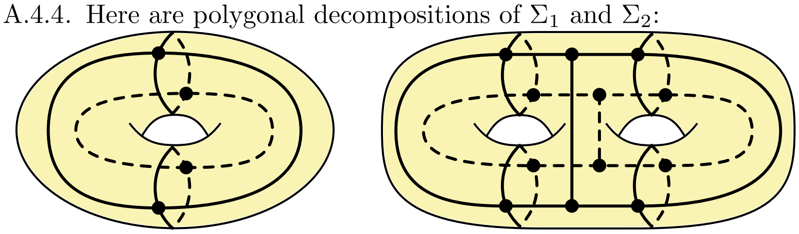


The first has a single vertex v, no edges, and one face $\mathbb{S}^2 \setminus v \cong \mathbb{R}^2 \cong \text{Int}(\mathbb{D}^2)$. Its Euler characteristic is $1 - 0 + 1 = 2$. The second has two vertices, two edges, and two faces. Its Euler characteristic is $2 - 2 + 2 = 2$. Other polygonal decompositions of $\mathbb{S}^2$ can be obtained by identifying the boundaries of polyhedra in $\mathbb{R}^3$ with $\mathbb{S}^2$. For instance:



Here we have stopped trying to draw the faces in different colors. As the reader will check, in each of these cases the Euler characteristic is 2. For instance, the left-most polygonal decomposition has 4 vertices, 6 edges, and 4 faces, so its Euler characteristic is $4 - 6 + 4 = 2$. All these examples reflect a theorem we will discuss below that says that all polygonal decompositions of the same surface have the same Euler characteristic. □

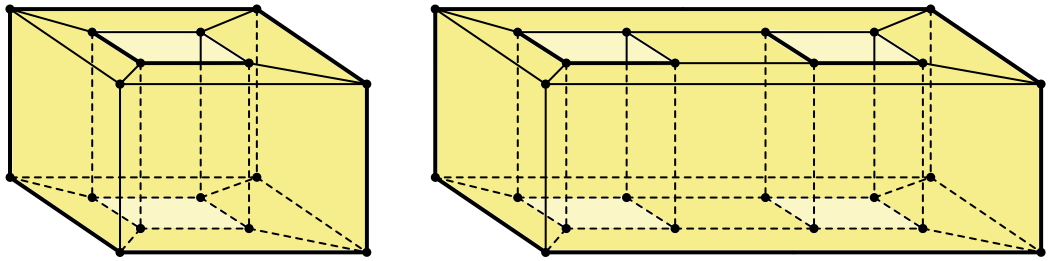
EXAMPLE A.4.4. Here are polygonal decompositions of Σ_1 and Σ_2:



The Euler characteristics of these polygonal decompositions are

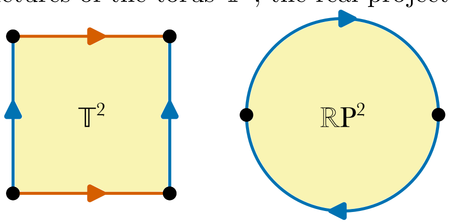
$$\chi(\Sigma_1) = 4 - 8 + 4 = 0,$$
$$\chi(\Sigma_2) = 12 - 22 + 8 = -2.$$

By regarding Σ_1 and Σ_2 as cubes with holes drilled through their centers, we can obtain two other polygonal decompositions for which it is a little easier to see that the faces are open disks:



Again, the reader can verify that these have Euler characteristic 0 and −2. □

Example A.4.5. When we gave examples of surfaces in §A.2, many of our surface came presented as polygons or disks with sides identified. This gives a polygonal decomposition of the surface. For instance, here are pictures of the torus $\mathbb{T}^2$, the real projective plane $\mathbb{RP}^2$, and the Klein bottle $\mathbb{K}$:



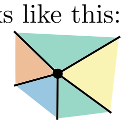
When we identify the boundary points as shown, the boundary becomes a graph and the interior of the disk/polygon gives a single face. In the above examples:

- $\mathbb{T}^2$ has one vertex and two edges and one face, so $\chi(\mathbb{T}^2) = 1 - 2 + 1 = 0$; and
- $\mathbb{RP}^2$ has one vertex and one edge and one face, so $\chi(\mathbb{RP}^2) = 1 - 1 + 1 = 1$; and
- $\mathbb{K}$ has one vertex and two edges and one face, so $\chi(\mathbb{K}) = 1 - 2 + 1 = 0$.

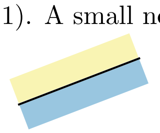
We will give more examples of this later in §A.4.10. □

A.4.4. Local structure of polygonal decompositions. Let Σ be a closed surface equipped with a polygonal decomposition. We now discuss the local structure of this polygonal decomposition. This local structure follows from the definition of a polygonal decomposition. However, the proof is a little technical and we will omit it.[6] For a reader who cannot prove it on their own, we suggest adding these local results to the definition. The various proofs that polygonal decompositions exist (Theorem A.4.1) give polygonal decompositions where this local structure definitely holds.

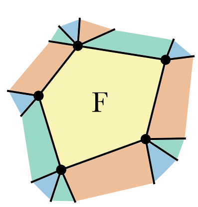
Our statements will be informal, but will be sufficient to understand the proof of the classification. First, a small neighborhood of a vertex looks like this:



Each of the shaded regions is part of a face. These faces need not all be distinct. Next, consider an edge e. We have $e \cong [0, 1]$ and $\text{Int}(e) \cong (0, 1)$. A small neighborhood of $\text{Int}(e)$ looks like this:



Again, the two shaded regions are parts of two faces, though these faces might be the same. We now come to a face F. Recall that $F \cong \text{Int}(\mathbb{D}^2)$. There exists a homeomorphism $\phi\colon \text{Int}(\mathbb{D}^2) \to F$ that extends to a continuous map $\phi\colon \mathbb{D}^2 \to F$. The restriction of ϕ to $\partial\mathbb{D}^2 = \mathbb{S}^1$ is usually a closed edge-path in the 1-skeleton:



[6] See Exercise A.8 for analogous results in dimension 1.

The edges in the 1-skeleton traversed by this closed edge path need not be distinct. There is one case where this does not hold: for the polygonal decomposition of $\mathbb{S}^2$ with a single vertex and face and no edges (cf. Example A.4.3), the restriction of ϕ to $\partial\mathbb{D}^2 = \mathbb{S}^1$ is the constant map to this single vertex.

REMARK A.4.6. Polygonal decompositions also are useful for surfaces with boundary, but the local structure described above needs to be modified for vertices and edges that are contained in the boundary. □

A.4.5. Well-definedness of Euler characteristic. In our examples above, the Euler characteristics of different polygonal decompositions of the same surface were always the same. This always holds:

THEOREM A.4.7. *Let Σ be a compact surface, possibly with boundary. Then the Euler characteristics of any two polygonal decompositions of Σ are the same.*

The most conceptual proof of this theorem uses homology. For any reasonable compact space X, that theory produces integers $b_i(X) \geq 0$ for each $i \geq 0$ called the *Betti numbers* of X. The Betti number of X are manifestly invariants of X, and for any polygonal decomposition of a compact surface Σ we have

$$\chi(\Sigma) = b_0(\Sigma) - b_1(\Sigma) + b_2(\Sigma).$$

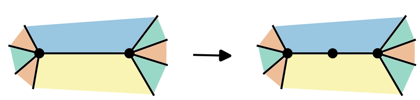
In fact, there are higher-dimensional versions of polygonal decompositions called *CW complex structures*. For a compact space X equipped with a CW complex structure, we have $b_i(X) = 0$ for $i \gg 0$, so the a priori infinite alternating sum

$$\chi(X) = b_0(X) - b_1(X) + b_2(X) - \cdots + (-1)^i b_i(X) + \cdots$$

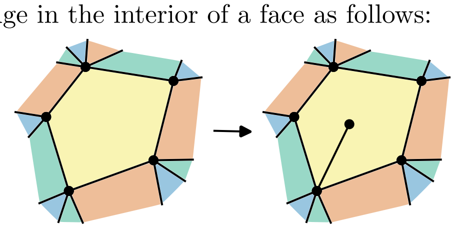
is a finite sum, called the Euler characteristic of X. One of the basic structural theorems about Betti numbers is that $\chi(X)$ also equals an alternating sum of the i-dimensional cells of X. We will prove this more general version in Volume 2 when we discuss homology.

For surfaces, there is also an alternate approach as follows. Along with the existence of polygonal decompositions (Theorem A.4.1), there is also a uniqueness statement saying that any two polygonal decompositions of a compact surface Σ have what is called a *common subdivision*. What this means is that after applying a sequence of the following three moves, any two polygonal decompositions differ by a homeomorphism of Σ:

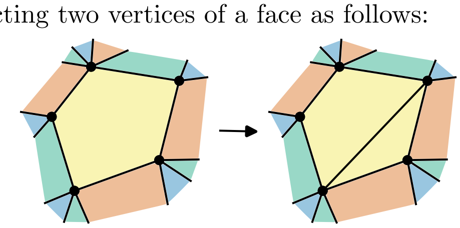
- Subdivide an edge as follows:



- Add a vertex and edge in the interior of a face as follows:



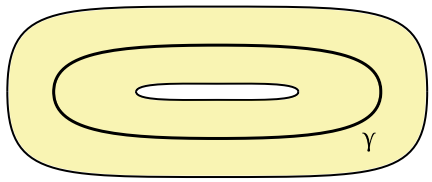
- Add an edge connecting two vertices of a face as follows:



To prove that the Euler characteristic does not depend on the polygonal decomposition, it is thus enough to observe that the above three moves do not change it. For instance, subdividing an edge add one vertex and one edge, and these cancel out when calculating the Euler characteristic.

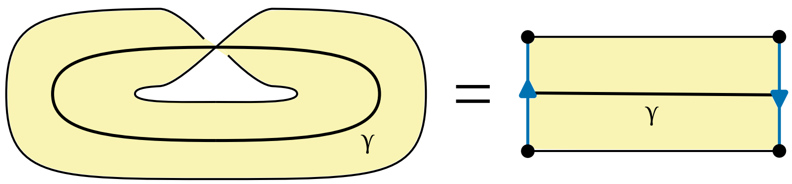
A.4.6. Cutting. We now explore the effect on the Euler characteristic of cutting along simple closed curves. Let Σ be a compact surface, possibly with boundary. Let γ be a simple closed curve lying in the interior of Σ. From the theory of manifolds, γ has what is called a *tubular neighborhood*. There are two cases:

- This tubular neighborhood is an annulus with γ in its interior like this:



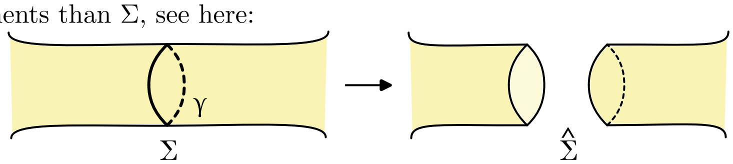
In this case, we say that γ is a *two-sided curve*. If Σ is orientable, all γ are two-sided.

- This tubular neighborhood is a Möbius band with γ in its interior like this:



In this case, we say that γ is a *one-sided curve*.

Cutting Σ open along γ turns Σ into a surface $\widehat{\Sigma}$. If γ is two-sided, the surface $\widehat{\Sigma}$ has two more boundary components than Σ, see here:

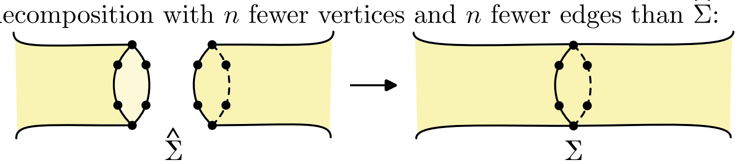


On the other hand, if γ is one-sided then $\widehat{\Sigma}$ has only more more boundary component. We remark that $\widehat{\Sigma}$ might be disconnected.

We will prove:

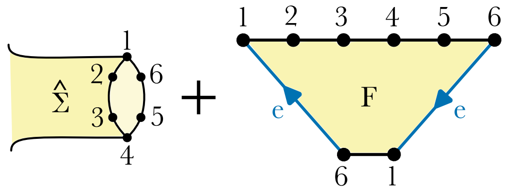
Lemma A.4.8. *Let Σ be a compact surface, possibly with boundary. Let γ be a simple closed curve in the interior of Σ and let $\widehat{\Sigma}$ be the result of cutting Σ open along γ. Then $\chi(\widehat{\Sigma}) = \chi(\Sigma)$.*

Proof. Fix a polygonal decomposition of $\widehat{\Sigma}$. Assume first that γ is two-sided. In this case, there are boundary components ∂_1 and ∂_2 of $\widehat{\Sigma}$ such that Σ can be obtained by gluing ∂_1 to ∂_2. Necessarily ∂_1 and ∂_2 are cycles in the 1-skeleton of $\widehat{\Sigma}$. Subdividing edges in the ∂_i if necessary, we can assume that ∂_1 and ∂_2 contain the same number of n of vertices. As the following shows, Σ then has a polygonal decomposition with n fewer vertices and n fewer edges than $\widehat{\Sigma}$:



This figure does not include the part of the polygonal decomposition lying in the interior of $\widehat{\Sigma}$, which is irrelevant for this calculation. These changes in the numbers of vertices and edges cancel out when we calculate the Euler characteristic, giving that $\chi(\widehat{\Sigma}) = \chi(\Sigma)$.

Now assume that γ is one-sided. In this case, there is a boundary component ∂ of $\widehat{\Sigma}$ such that Σ can be obtained by gluing a Möbius band to ∂. Necessarily ∂ lies in the 1-skeleton of $\widehat{\Sigma}$. Assume that ∂ has n vertices. As the following shows, with respect to an appropriate polygonal decomposition Σ has 1 more edge (labeled e) and 1 more face (labeled F) than $\widehat{\Sigma}$:



Here we have drawn the Möbius band in a skewed way to emphasize that like on ∂ its vertices are equally spaced around its single boundary component, and again we did not include the part of the polygonal decomposition lying in the interior of $\widehat{\Sigma}$. These changes in the numbers of vertices and edges again cancel out when we calculate the Euler characteristic, giving that $\chi(\widehat{\Sigma}) = \chi(\Sigma)$. □

A.4.7. Capping. We now study the effect on the Euler characteristic of gluing a disk to a boundary component. Recall that the result does not depend on the gluing map (Exercise A.11).

LEMMA A.4.9. *Let Σ be a compact surface with boundary and let ∂ be a boundary component of Σ. Let $\overline{\Sigma}$ be the result of gluing a disk to ∂. Then $\chi(\overline{\Sigma}) = \chi(\Sigma) + 1$.*

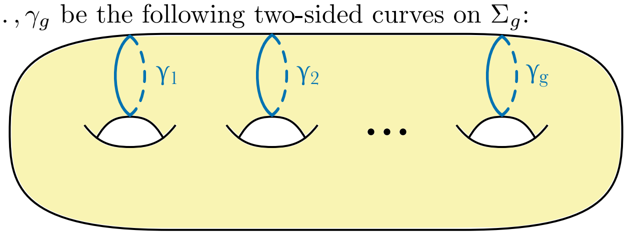
PROOF. Choose a polygonal decomposition of Σ. The disk we glued to ∂ can serve as a face of a polygonal decomposition of $\overline{\Sigma}$, giving a polygonal decomposition of $\overline{\Sigma}$ with the same number of vertices and edges as Σ and one more face. It follows that

$$\chi(\overline{\Sigma}) = |V(\overline{\Sigma})| - |E(\overline{\Sigma})| + |F(\overline{\Sigma})| = |V(\Sigma)| - |E(\Sigma)| + |F(\Sigma)| + 1 = \chi(\Sigma) + 1.$$ □

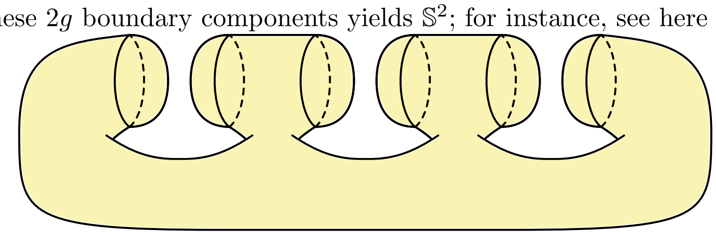
A.4.8. Euler characteristic calculations. Our results about cutting and capping make it easy to calculate the Euler characteristics of surfaces without needing to find explicit polygonal decompositions of them. As examples of this, we calculate the Euler characteristics of the genus-g surface Σ_g and the nonorientable genus-n surface Σ_n^{no}:

LEMMA A.4.10. *For $g \geq 0$, we have $\chi(\Sigma_g) = 2 - 2g$.*

PROOF. Let $\gamma_1, \ldots, \gamma_g$ be the following two-sided curves on Σ_g:



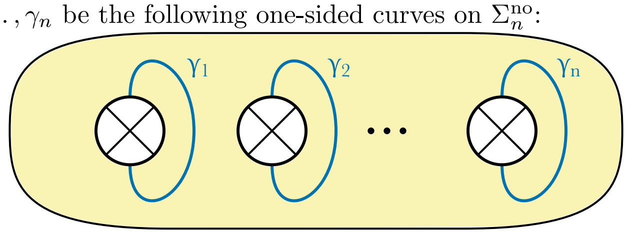
Cutting Σ_g open along the γ_i yields a connected surface $\widehat{\Sigma}$ with $2g$ boundary components. Gluing disks to each of these $2g$ boundary components yields $\mathbb{S}^2$; for instance, see here for the case $g = 3$:



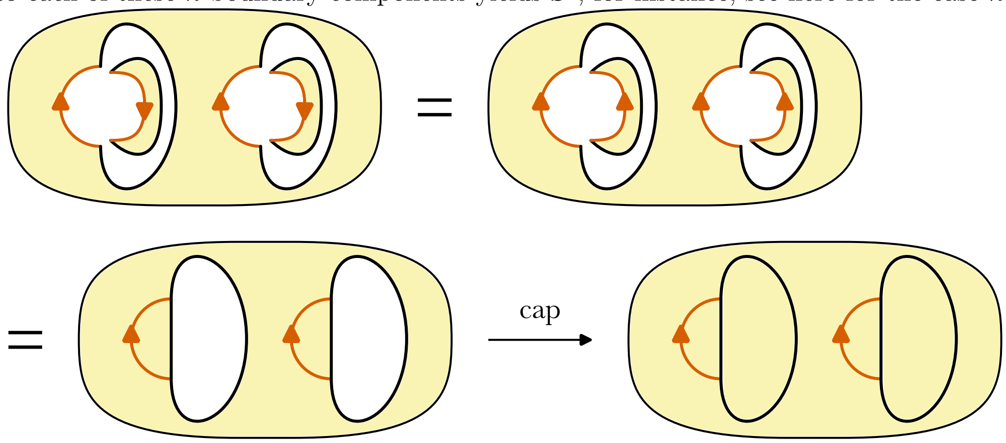
Applying Lemmas A.4.8 and A.4.9, we deduce that $\chi(\Sigma_g) = \chi(\widehat{\Sigma}) = \chi(\mathbb{S}^2) - 2g = 2 - 2g$. □

LEMMA A.4.11. *For $n \geq 1$, we have $\chi(\Sigma_n^{no}) = 2 - n$.*

PROOF. Let $\gamma_1, \ldots, \gamma_n$ be the following one-sided curves on Σ_n^{no}:



Cutting Σ_n^{no} open along the γ_i yields a connected surface $\widehat{\Sigma}$ with n boundary components. Gluing disks to each of these n boundary components yields $\mathbb{S}^2$; for instance, see here for the case $n = 2$:



Applying Lemmas A.4.8 and A.4.9, we deduce that $\chi(\Sigma_n^{\text{no}}) = \chi(\widehat{\Sigma}) = \chi(\mathbb{S}^2) - n = 2 - n$. □

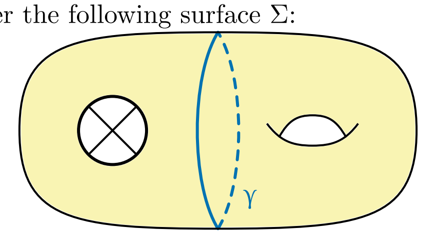
A.4.9. Refined classification. Recall that the classification theorem for surfaces says that every closed connected surface is homeomorphic to some Σ_g or Σ_n^{no}. Since $\chi(\Sigma_g) = 2 - 2g$ and $\chi(\Sigma_n^{\text{no}}) = 2 - n$, this will imply that the homeomorphism type of a closed connected surface is completely determined by its Euler characteristic and whether or not it is orientable. We state this refined version of the classification as follows:

Theorem A.4.12 (Classification of surfaces). *Let Σ be a closed connected surface. Then:*

- *If Σ is orientable then $\Sigma \cong \Sigma_g$, where $g \geq 0$ satisfies $\chi(\Sigma) = 2 - 2g$. In particular, $\chi(\Sigma)$ is even.*
- *If Σ is non-orientable then $\Sigma \cong \Sigma_n^{no}$, where $n \geq 1$ satisfies $\chi(\Sigma) = 2 - n$.*

Using this, we can answer the question we posed after stating the first version of the classification (Theorem A.2.1):

Example A.4.13. Consider the following surface Σ:



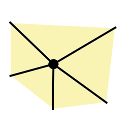
This surface is non-orientable since it contains a cross-cut. To calculate its Euler characteristic, let γ be the curve drawn above. The curve γ is two-sided, and when we cut Σ open along it and cap off the resulting two boundary components we get Σ_1 and Σ_1^{no}. It follows that

$$\chi(\Sigma) = \chi(\Sigma_1) + \chi(\Sigma_1^{\text{no}}) - 2 = 0 + 1 - 2 = -1.$$

By Theorem A.4.12, we deduce that $\Sigma \cong \Sigma_3^{\text{no}}$. □

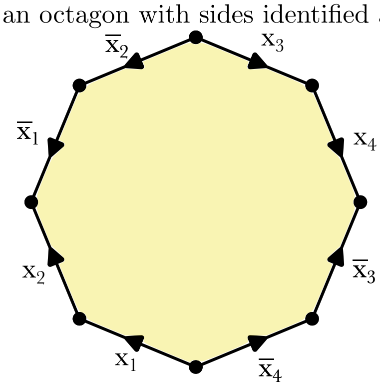
Remark A.4.14. Many proofs of the classification of surfaces require the homeomorphism $\Sigma \cong \Sigma_3^{\text{no}}$ from Example A.4.13, which must be proved by hand. As we will see, our proof does not need this fact, so the above argument is not circular. □

A.4.10. Polygons with sides identified. We will start the proof of Theorem A.4.12 soon, but first we give a few more examples of how it can be used. One natural way to build a surface is to take a polygon (or several polygons) in $\mathbb{R}^2$ and glue its sides together. We saw several examples of this already in Example A.4.5. If each side is glued to exactly one other side, then the result will always be a surface. Indeed, the only place where it is not obviously a surface is at the vertices, and a neighborhood of a vertex looks like this:

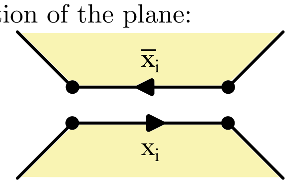


This surface has a natural polygonal decomposition where the glued-together boundary forms the 1-skeleton and the polygon is a single face (or multiple faces if there are multiple polygons). Here are some example of how to use Theorem A.4.12 to identify the resulting surface:

EXAMPLE A.4.15. Let Σ be an octagon with sides identified as follows:



Here we have labeled the sides with letters and oriented them to show how they should be glued. The bars over the letters indicate a reversed orientation on the edge. The surface Σ is orientable since the gluing respects the orientation of the plane:



All the vertices are identified to a single vertex, and the boundary edges glue to 4 edges. Since there is a single face, we see that $\chi(\Sigma) = 1 - 4 + 1 = -2$. It follows that $\Sigma \cong \Sigma_2$. □

REMARK A.4.16. A nice way to give a gluing pattern on the boundary of a $2n$-gon is to label the paired edges by letters $x_1, \ldots, x_n$. You then give a word in letters $\{x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n\}$ obtained by going around the polygon clockwise starting from some vertex and recording which edge-labels appear, with a bar indicating that the edge is traversed in the opposite orientation. For instance, in Example A.4.15 the word would be $x_1x_2\overline{x}_1\overline{x}_2x_3x_4\overline{x}_3\overline{x}_4$. For each $1 \leq i \leq n$, the letter x_i should appear twice (each time possibly with a bar over it). □

EXAMPLE A.4.17. Generalizing Example A.4.15, let Σ be a $4g$-gon with sides identified according to the pattern

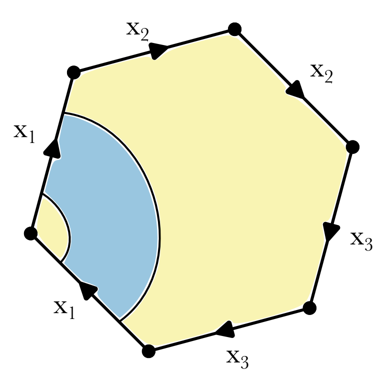
$$x_1x_2\overline{x}_1\overline{x}_2 \cdots x_{2g-1}x_{2g}\overline{x}_{2g-1}\overline{x}_{2g}.$$

Again, all the vertices get identified to a single vertex and there are $2g$ edges and one face, so $\chi(\Sigma) = 1 - 2g + 1 = 2 - 2g$. We thus have $\Sigma \cong \Sigma_g$. □

EXAMPLE A.4.18. Let Σ be a $2n$-gon with sides identified according to the pattern

$$x_1x_1x_2x_2 \cdots x_nx_n.$$

For instance, for $n = 3$ this is



The blue strip here glues up to a Möbius band, so Σ is non-orientable. All the vertices get identified to a single vertex and there are n edges and one face, so $\chi(\Sigma) = 1 - n + 1 = 2 - n$. We thus have $\Sigma \cong \Sigma_n^{\text{no}}$. □

A.5. The two-dimensional Poincaré conjecture

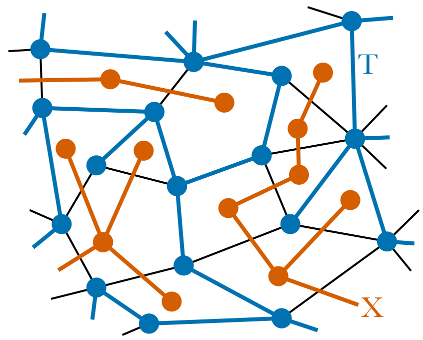
We now start the proof of the classification by proving the two-dimensional Poincaré conjecture, which says that $\mathbb{S}^2$ is the unique connected closed surface with Euler characteristic 2:

THEOREM A.5.1 (Two-dimensional Poincaré conjecture). *Let Σ be a closed connected surface. Then $\chi(\Sigma) \leq 2$, with equality if and only if $\Sigma \cong \mathbb{S}^2$.*

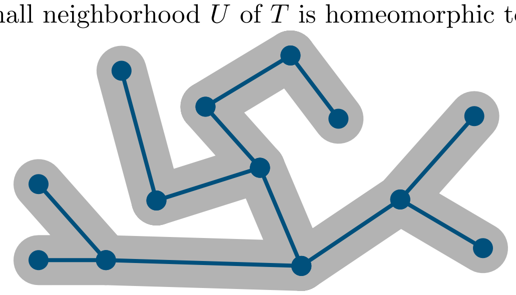
PROOF. Fix a polygonal decomposition of Σ. Its 1-skeleton is a finite graph embedded in Σ. Let T be a maximal tree in the 1-skeleton. Next, let X the following graph, which one can view as a sort of "dual graph" to T:

- Put a vertex of X in the center of each polygon of our polygonal decomposition.
- Connect two vertices of X if their associated polygons share an edge that does not lie in T.

See here:



We claim that the graph X is connected. Equivalently, removing T does not disconnect Σ. The key to this is the fact that a small neighborhood U of T is homeomorphic to $\mathbb{D}^2$:

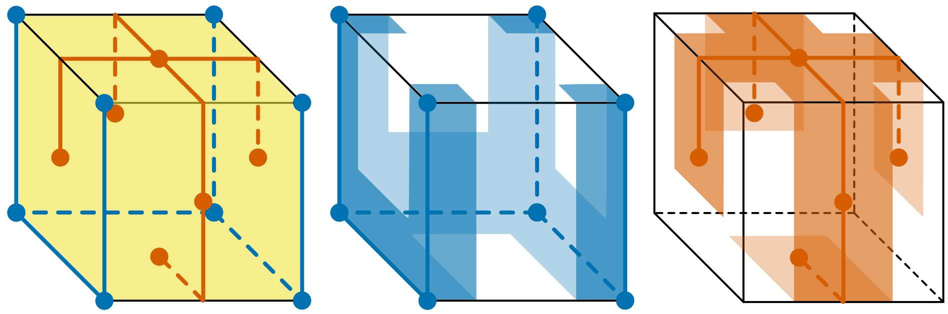


From this, we see that a path in Σ that crosses T can be re-routed to turn and follow the boundary of U until it gets to the other side of T, proving that T does not disconnect the surface.

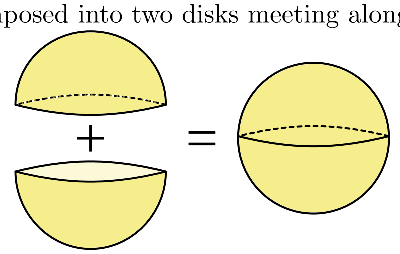
Lemma A.3.2 implies that $\chi(T) = 1$ and that $\chi(X) \leq 1$ with equality if and only if X is a tree. Since each vertex of Σ is a vertex of T, each polygon of Σ contains a unique vertex of X, and each edge of Σ is either an edge of T or is crossed by a unique edge of X, we have

$$\begin{aligned}\chi(\Sigma) &= |V(\Sigma)| - |E(\Sigma)| + |F(\sigma)| = |V(T)| - (|E(T)| + |E(X)|) + |V(X)| \\ &= \chi(T) + \chi(X) \leq 1 + 1 = 2.\end{aligned}$$

This proves half of the theorem. Equality holds if and only if $\chi(X) = 1$, i.e, if X is a tree. Assume, therefore, that X is a tree. We must prove that $\Sigma \cong \mathbb{S}^2$. To see this, note that just like above if we slightly thicken T and X we get disks D_1 and D_2. Choosing these thickenings carefully, we can ensure that D_1 and D_2 intersect in their boundaries. To help the reader understand this, here is an example of a polynomial decomposition of a surface where T and X are trees, along with D_1 and D_2:



We deduce that Σ can be decomposed into two disks meeting along their boundaries:



It follows that $\Sigma \cong \mathbb{S}^2$. □

Before continuing with the classification, we pause to extract the following from the above proof:

Corollary A.5.2. *Let Σ be a closed connected surface such that $\chi(\Sigma) < 2$. Then there exists a simple closed curve γ on Σ that is nonseparating, i.e., such that $\Sigma \setminus \gamma$ is connected.*

Proof. Fix a polygonal decomposition of Σ, and let T and X be as in the proof of Theorem A.5.1. Since $\chi(\Sigma) < 2$, it follows from the proof of Theorem A.5.1 that X is a connected graph that is *not* a tree. It therefore contains a cycle γ. We claim that γ is nonseparating.

In fact, even more is true: $\Sigma \setminus X$ is connected. To see this, note that any point of $\Sigma \setminus X$ can be connected by a path in $\Sigma \setminus X$ to a vertex of the polygonal decomposition. This vertex lies in the maximal tree T, and since T is connected we can follow a path in T to any other vertex of the polygonal decomposition. The claim follows. □

A.6. The classification of surfaces in general

We now come to the proof of the classification of surfaces, whose statement we recall:

Theorem A.4.12 (Classification of surfaces). *Let Σ be a closed connected surface. Then:*

- *If Σ is orientable then $\Sigma \cong \Sigma_g$, where $g \geq 0$ satisfies $\chi(\Sigma) = 2 - 2g$. In particular, $\chi(\Sigma)$ is even.*
- *If Σ is non-orientable then $\Sigma \cong \Sigma_n^{no}$, where $n \geq 1$ satisfies $\chi(\Sigma) = 2 - n$.*

Proof. Theorem A.5.1 says that $\chi(\Sigma) \leq 2$. The proof will be by reverse induction on $\chi(\Sigma)$. The base case is when $\chi(\Sigma) = 2$, in which case Theorem A.5.1 says that $\Sigma \cong \mathbb{S}^2 \cong \Sigma_0$. In particular, Σ must be orientable in this case, as claimed in the theorem.

Assume now that $\chi(\Sigma) < 2$ and that the theorem is true for closed connected surfaces with larger Euler characteristics. Corollary A.5.2 implies that there is a nonseparating simple closed curve γ on Σ. As we discussed in §A.4.6, the curve γ is either two-sided or one-sided. Let $\widehat{\Sigma}$ be the connected surface with boundary obtained by cutting Σ open along γ. There are four cases, with the first case being the only one that occurs for Σ orientable:

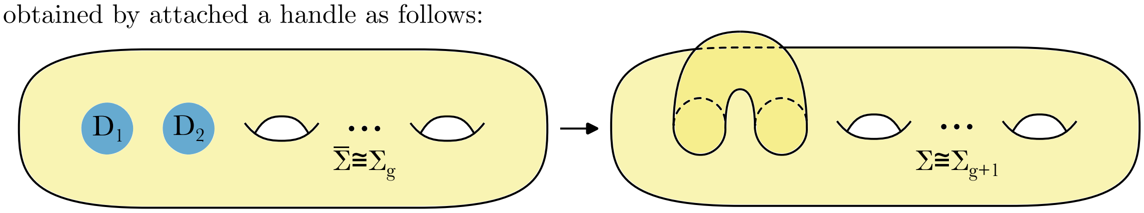
Case 1. *γ is two-sided and $\widehat{\Sigma}$ is orientable.*

Since γ is two-sided, $\widehat{\Sigma}$ has two boundary components. Let $\overline{\Sigma}$ be the closed connected surface obtained from $\widehat{\Sigma}$ by gluing disks to both of its boundary components. Using Lemmas A.4.8 and A.4.9, we have

$$\chi(\overline{\Sigma}) = \chi(\widehat{\Sigma}) + 2 = \chi(\Sigma) + 2.$$

We can therefore apply our inductive hypothesis to $\overline{\Sigma}$. Since $\widehat{\Sigma}$ is orientable, so is $\overline{\Sigma}$. It follows that $\overline{\Sigma} \cong \Sigma_g$, where $g \geq 0$ satisfies $\chi(\overline{\Sigma}) = 2 - 2g$. Since $\chi(\Sigma) = \chi(\Sigma) - 2$, we have $\chi(\Sigma) = 2 - 2(g+1)$. Our goal, therefore, is to prove that $\Sigma \cong \Sigma_{g+1}$.

To see this, note that Σ is obtained from $\overline{\Sigma} \cong \Sigma_g$ by removing two open disks D_1 and D_2 whose closures are disjoint and gluing together the resulting boundary components. In other words, Σ is obtained by attached a handle as follows:



It follows that $\Sigma \cong \Sigma_{g+1}$, as desired.

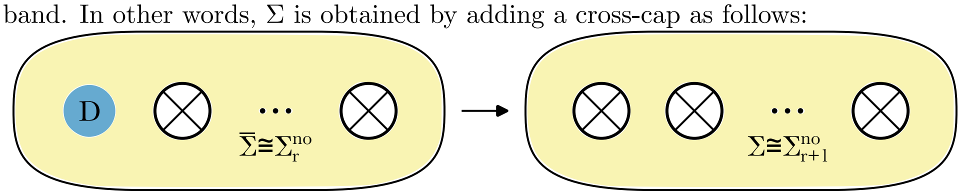
CASE 2. *γ is one-sided and $\widehat{\Sigma}$ is non-orientable.*

Since γ is one-sided, $\widehat{\Sigma}$ has one boundary component. Let $\overline{\Sigma}$ be the closed connected surface obtained from $\widehat{\Sigma}$ by gluing disks to its boundary component. Using Lemmas A.4.8 and A.4.9, we have

$$\chi(\overline{\Sigma}) = \chi(\widehat{\Sigma}) + 1 = \chi(\Sigma) + 1.$$

We can therefore apply our inductive hypothesis to $\overline{\Sigma}$. Since $\widehat{\Sigma}$ is non-orientable, so is $\overline{\Sigma}$. It follows that $\overline{\Sigma} \cong \Sigma_n^{\text{no}}$, where $n \geq 1$ satisfies $\chi(\overline{\Sigma}) = 2 - n$. Since $\chi(\Sigma) = \chi(\Sigma) - 1$, we have $\chi(\Sigma) = 2 - (n+1)$. Our goal, therefore, is to prove that $\Sigma \cong \Sigma_{n+1}^{\text{no}}$.

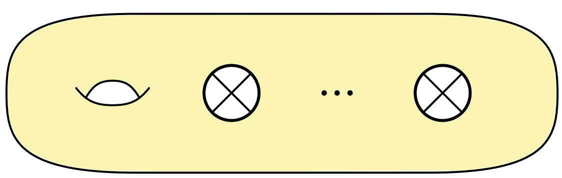
To see this, note that Σ is obtained from $\overline{\Sigma} \cong \Sigma_n^{\text{no}}$ by removing an open disk D and gluing in a Möbius band. In other words, Σ is obtained by adding a cross-cap as follows:



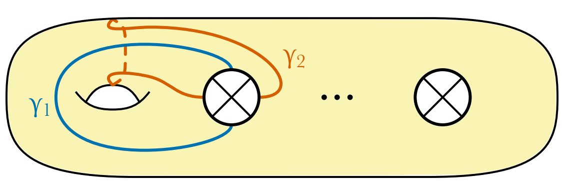
It follows that $\Sigma \cong \Sigma_{n+1}^{\text{no}}$, as desired.

CASE 3. *γ is two-sided and $\widehat{\Sigma}$ is non-orientable.*

Following the argument in the previous two cases, the surface Σ has one genus and $n \geq 1$ cross-caps as follows:



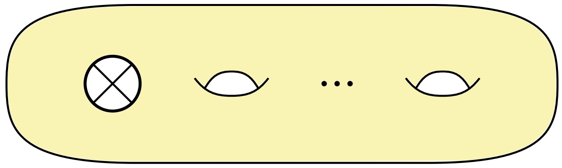
What we did wrong in this case was choose the wrong curve γ to cut along. Let γ_1 and γ_2 be as follows:



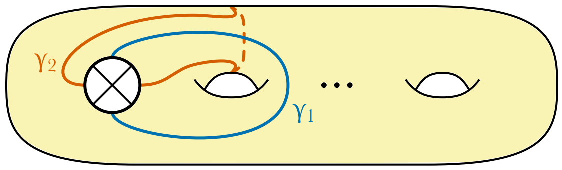
The curves γ_1 and γ_2 are both one-sided. Since γ_2 is one-sided and disjoint from γ_1, it follows that the surface $\widehat{\Sigma}'$ obtained by cutting along γ_1 is non-orientable. This reduces us to Case 2.

CASE 4. *γ is one-sided and $\widehat{\Sigma}$ is orientable.*

This time, if we follow the argument from Cases 1 and 2 we get that Σ has 1 cross-cap and $g \geq 0$ genus as follows:



If $g = 0$ then $\Sigma \cong \Sigma_1^{\mathrm{no}}$ and we are done. Otherwise, we can use the same trick we used in Case 3. Namely, let γ_1 and γ_2 be as follows:

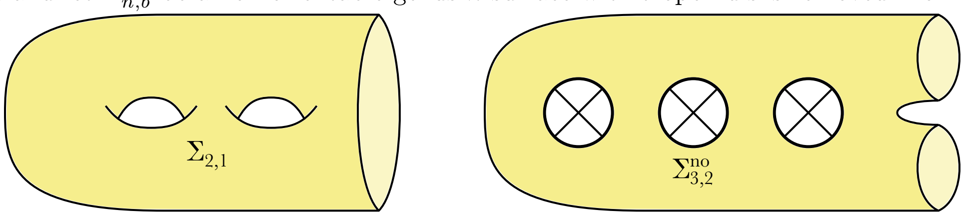


The curves γ_1 and γ_2 are both one-sided. Since γ_2 is one-sided and disjoint from γ_1, it follows that the surface $\widehat{\Sigma}'$ obtained by cutting along γ_1 is non-orientable. This reduces us to Case 2. □

A.7. Extensions of the classification of surfaces

We close this essay by describing two extensions of the classification of surfaces: to compact surfaces with boundary, and to non-compact surfaces.

A.7.1. Compact surfaces with boundary. Let $\Sigma_{g,b}$ be genus-g surface Σ_g with b open disks removed and let $\Sigma_{n,b}^{\mathrm{no}}$ be a non-orientable genus-n surface with b open disks removed. For instance,



Both $\Sigma_{g,b}$ and $\Sigma_{n,b}$ are compact surfaces with b boundary components. These are the only surfaces with boundary:

THEOREM A.7.1 (Classification of surfaces with boundary). *Let Σ be a compact connected surface with $b \geq 0$ boundary components. Then:*

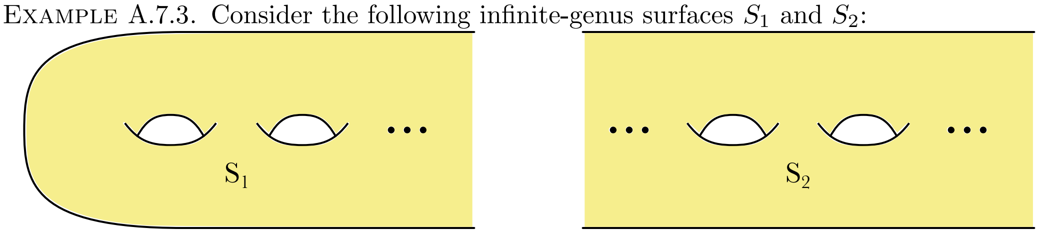
- *If Σ is orientable then $\Sigma \cong \Sigma_{g,b}$, where $g \geq 0$ satisfies $\chi(\Sigma) = 2 - 2g - b$.*
- *If Σ is non-orientable then $\Sigma \cong \Sigma_{n,b}^{no}$, where $n \geq 1$ satisfies $\chi(\Sigma) = 2 - n - b$.*

PROOF. Gluing disks to all the boundary components of Σ gives a closed connected surface $\overline{\Sigma}$ with $\chi(\overline{\Sigma}) = \chi(\Sigma) + b$. By the classification of surfaces (Theorem A.4.12), we either have $\overline{\Sigma} \cong \Sigma_g$ with $\chi(\overline{\Sigma}) = 2 - 2g$ or $\overline{\Sigma} \cong \Sigma_n^{\mathrm{no}}$ with $\chi(\overline{\Sigma}) = 2 - n$ depending on whether or not $\overline{\Sigma}$ (and hence Σ) is orientable. The theorem follows. □

A.7.2. Noncompact surfaces. One way to get a noncompact surface is to remove a finite number of points from the interior of a compact surface with boundary. This gives what is called a surface of *finite type*. However, non-compact surfaces can be much more complicated than this. Here are some examples.

EXAMPLE A.7.2. Let C be a Cantor set embedded in $\mathbb{S}^2$. Then $\mathbb{S}^2 \setminus C$ is a very complicated non-compact surface. □

EXAMPLE A.7.3. Consider the following infinite-genus surfaces S_1 and S_2:



The difference between them is that S_1 has genus going off to infinity only to the right, while S_2 has genus going off to infinity in both directions. These surfaces are not homeomorphic. One way to distinguish them is to note that for every compact subset $K_1 \subset S_1$, there is only one component C of $S_1 \setminus K_1$ such that $\overline{C}$ is noncompact. However, there exist compact subsets $K_2 \subset S_2$ such that $S_2 \setminus K_2$ has two such components. This can be formalized using the theory of what are called "ends" of a space. □

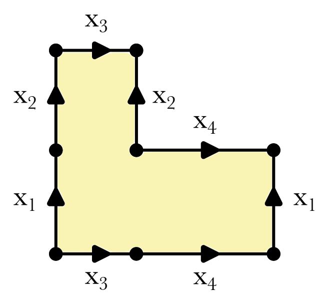
This might lead the reader to think that there is no hope of classifying noncompact surfaces. However, there is such a classification making use of "end data". It was first stated by Kerékjártó [**7**, Chapter 5]. His proof had gaps, and the first correct proof was found by Richards [**8**].

Remark A.7.4. Noncompact surfaces with boundary are even more complicated, especially if they have noncompact boundary components. However, a classification of them was found by Brown–Messer [**3**]. □

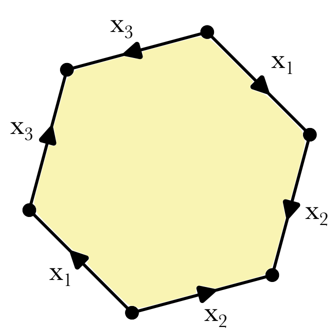
A.8. Exercises

Exercise A.1. Determine the surfaces by identifying sides of polygons as follows:

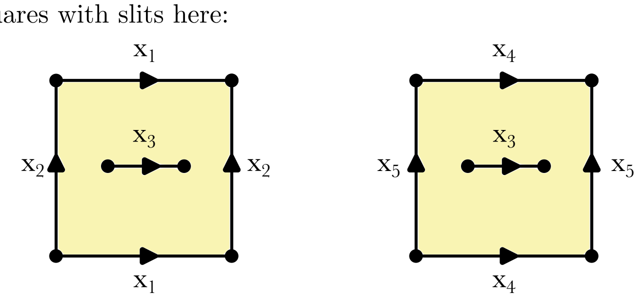
(a) The L-shaped polygon with opposite sides identified here:



(b) The hexagon with sides identified here:

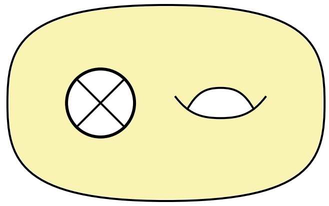


(c) The two squares with slits here:



For this, make sure you are working with a polygonal decomposition. □

Exercise A.2. As we discussed after stating Theorem A.4.12, the following surface is homeomorphic to Σ_3^{no}:

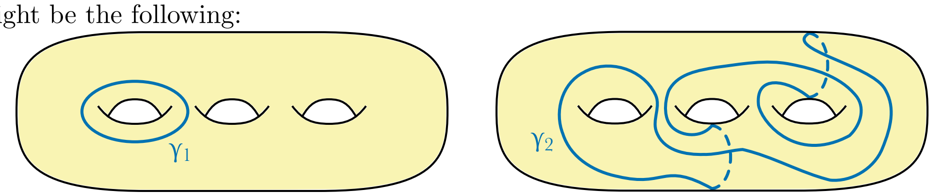


Prove this directly by decomposing both Σ_3^{no} and the above surface into the union of three Möbius bands and $\Sigma_{0,3}$ glued along their boundaries. □

Exercise A.3. Using the fact that the Euler characteristic of $\mathbb{S}^2$ is 2, prove that any regular polyhedron in $\mathbb{R}^3$ is either a tetrahedron, a cube, an octahedron, a dodecahedron, or an icosahedron. This goes back to the ancient greeks, and appears in Euclid's Elements. □

Exercise A.4. Using the fact that the Euler characteristic of $\mathbb{S}^2$ is 2, determine the number of components of the complement of n great circles on $\mathbb{S}^2$ no three of which pass though a common point. □

Exercise A.5. Let γ_1 and γ_2 be two nonseparating simple closed curves on Σ_g. For instance, the γ_i might be the following:



Prove that there is a homeomorphism $f\colon \Sigma_g \to \Sigma_g$ such that $f(\gamma_1) = \gamma_2$. Hint: use the classification of surfaces with boundary to prove that the surfaces you get by cutting Σ_g open along the γ_i are homeomorphic. □

Exercise A.6. Let $f\colon \widetilde{\Sigma} \to \Sigma$ be a degree-d cover between closed connected surfaces. Prove that $\chi(\widetilde{\Sigma}) = d\chi(\Sigma)$. Hint: Figure out how to lift a polygonal decomposition of Σ to one of $\widetilde{\Sigma}$. □

Exercise A.7. Let M be a 1-manifold. A *triangulation* of M is a closed discrete subset $V \subset M$ call the *vertices* such that each path-component E of $M \setminus V$ is homeomorphic to $(0,1)$ and has compact closure. We will call these E the *edges* of the triangulation. Prove that M has a triangulation by following these steps:

(a) First prove that there is a countable open cover $\{U_i \mid i \in I\}$ of M with the following properties:
   - For each $i \in I$, the closure $\overline{U}_i$ is a closed subset of M homeomorphic to $[0,1]$ via a homeomorphism taking U_i to $(0,1)$.
   - The set $\left\{\overline{U}_i \mid i \in I\right\}$ is locally finite, i.e., for each compact subset $K \subset M$ the set $\left\{i \in I \mid \overline{U}_i \cap K \neq \emptyset\right\}$ is finite.

   We remark that this will use the fact that M is second countable and Hausdorff.

(b) Letting $\{U_i \mid i \in I\}$ be as in (a), set
$$V = \bigcup_{i \in I} \overline{U}_i \setminus U_i.$$
   Prove that V is the set of vertices of a triangulation of M. We remark that this will also use the fact that M is Hausdorff. □

Exercise A.8. Let M be a 1-manifold. Fix a triangulation of M as in Exercise A.7. We will verify that M has local properties similar to those of polygonal decompositions of surfaces; cf. §A.4.4.

(a) Prove that for each vertex v of the triangulation, there exists an open neighborhood U of v along with a homeomorphism $f\colon (-1,1) \to U$ such that $f(0) = v$ and such that f takes both $(-1,0)$ and $(0,1)$ into open subsets of edges of the triangulation. These edges need not be distinct.

(b) Let E be an edge of the triangulation. Prove that there is a homeomorphism $g\colon (0,1) \to E$ that extends to a continuous map $G\colon [0,1] \to M$ with $G(0)$ and $G(1)$ vertices of the triangulation. Hint: This exercise will require you to use the fact that manifolds are Hausdorff. □

Exercise A.9. Let M be a connected 1-manifold. Prove that M is homeomorphic to either $\mathbb{S}^1$ or $\mathbb{R}$. Hint: use a triangulation as in Exercises A.7 and A.8. We remark that this relies on the fact that manifolds are second countable and Hausdorff. Without these conditions there are many more examples of connected 1-manifolds. Even assuming second countability, as far as I am aware there is no reasonable classification of non-Hausdorff 1-manifolds. □

Exercise A.10. Let Σ be either $\Sigma_{g,b}$ or $\Sigma^{\text{no}}_{n,b}$ with $b \geq 1$, let ∂ be a boundary component of Σ, and let $f\colon \partial \to \partial$ be a homeomorphism. In this exercise, you will prove that f extends to a homeomorphism $F\colon \Sigma \to \Sigma$.

(a) Prove that if $\phi\colon \mathbb{S}^1 \to \mathbb{S}^1$ is an orientation-preserving homeomorphism, then there is a homotopy $\phi_t\colon \mathbb{S}^1 \to \mathbb{S}^1$ such that $\phi_0 = \phi$ and $\phi_1 = \mathbb{1}_{\mathbb{S}^1}$ and such that each ϕ_t is a homeomorphism. Hint: First reduce to the case where $\phi(1) = 1$. Identify ϕ with a path $\psi\colon I \to \mathbb{S}^1$ with $\psi(0) = \psi(1) = 1$, and then try lifting ϕ to the universal cover $p\colon \mathbb{R} \to \mathbb{S}^1$ of $\mathbb{S}^1$.

(b) Using (a), prove the exercise for f orientation-preserving. Hint: Use the fact[7] that ∂ has a *collar neighborhood*, i.e., an embedding $\iota\colon \partial \times [0,1] \to \Sigma$ such that $\iota(x,0) = x$ for all $x \in \partial$.

(c) Conclude by proving the exercise for f orientation-reversing. Hint: using (b), show that it is enough to exhibit a single orientation-reversing homeomorphism $\phi\colon \Sigma \to \Sigma$ with $\phi(\partial) = \partial$. □

Exercise A.11. Let Σ and Σ' be compact connected surfaces with boundary. Assume that Σ is either $\Sigma_{g,b}$ or $\Sigma^{\text{no}}_{n,b}$ with $b \geq 1$. Let ∂ be a boundary component of Σ and let ∂' be a boundary component of Σ'. Let $f_1, f_2\colon \partial \to \partial'$ be two homeomorphisms and let S_i be the result of gluing $\partial \subset \Sigma$ to $\partial' \subset \Sigma'$ using f_i. Prove that S_1 is homeomorphic to S_2. Hint: Exercise A.10 will be helpful. □

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[7] This fact clearly holds for the Σ in the exercise. In fact, a standard theorem in manifold topology shows that collar neighborhoods exist for boundary components of arbitrary compact manifolds.