Math 60440: Basic Topology II
Problem Set 3

1. Compute the homology of the following chain complex:

$$0 \to \mathbb{Z} < U, L > \xrightarrow{d_2} \mathbb{Z} < a, b, c > \xrightarrow{d_1} \mathbb{Z} < v, w > \to 0,$$

   where

$$d_2(U) = -a + b + c$$
$$d_2(L) = a - b + c$$

   and

$$d_1(a) = w - v$$
$$d_1(b) = w - v$$
$$d_1(c) = 0.$$

2. For each $k \geq 1$, define a chain complex $D^k_\bullet$ by letting

$$D^k_n = \begin{cases} \mathbb{Z} & \text{if } n = k, k-1, \\ 0 & \text{otherwise} \end{cases}$$

   and letting the differential $D^k_k \to D^k_{k-1}$ be the identity (and all other differentials be 0).

   (a) Calculate $\mathrm{H}_n(D^k_\bullet)$.
   (b) Prove that for all chain complexes $C_\bullet$, the set of chain complex maps $D^k_\bullet \to C_\bullet$ is in bijection with C_k.

3. Let $f\colon (C_\bullet, d_\bullet) \to (C'_\bullet, d'_\bullet)$ be a homomorphism between chain complexes. Define $(\mathrm{Con}(f)_\bullet, e_\bullet)$ (the *mapping cone* of f) via the formulas

$$\mathrm{Con}(f)_n = C_{n-1} \oplus C'_n$$

   and

$$e_n\colon \mathrm{Con}(f)_n \to \mathrm{Con}(f)_{n-1} \quad \text{is} \quad e_n(x, y) = (-d_{n-1}(x), f(x) + d'_n(y)).$$

   Prove the following:

   (a) $(\mathrm{Con}(f)_\bullet, e_\bullet)$ is a chain complex.
   (b) The natural inclusion $(C'_\bullet, d'_\bullet) \to (\mathrm{Con}(f)_\bullet, e_\bullet)$ is a homomorphism of chain complexes.

4. Let $A_0, A_1, A_2, \ldots$ be a sequence of finitely generated abelian groups. Construct a chain complex $C_\bullet$ with the following properties:

- Each C_n is a finitely generated free abelian group, i.e. $C_n \cong \mathbb{Z}^{k_n}$ for some $k_n \geq 0$, and $C_n = 0$ for $n < 0$.
- $\mathrm{H}_n(C_\bullet) \cong A_n$ for all $n \geq 0$.

5. Let X be a topological space. Identify S^n with the boundary $\partial\Delta^{n+1}$ of an $(n+1)$-dimensional simplex $\Delta^{n+1} = [v_0, \ldots, v_{n+1}]$. For every continuous map $f \colon S^n \to X$, define $\theta_f \in C_n(X)$ to equal

$$\sum_{i=0}^{n+1} (-1)^i f|_{[v_0,\ldots,\widehat{v_i},\ldots,n+1]}.$$

Prove the following facts:

(a) $\theta_f \in Z_n(X)$, so we have an associated element $[\theta_f] \in \mathrm{H}_n(X)$.

(b) If $f, g \colon S^n \to X$ are homotopic, we have $[\theta_f] = [\theta_g]$.

(c) Fixing a basepoint $x_0 \in X$, the map $\pi_n(X, x_0) \to \mathrm{H}_n(X)$ taking the homotopy class of $f \colon S^n \to X$ to $[\theta_f]$ is a homomorphism. By the way, this homomorphism $\pi_n(X, x_0) \to \mathrm{H}_n(X)$ is called the *Hurewicz map*.

We remark that this is a slight variant of what we did for π_1, where to make things a little easier technically we used maps $I \to X$ taking the endpoints to the same place rather than maps $S^1 \to X$.