

HOMOLOGICAL VANISHING FOR THE STEINBERG REPRESENTATION II: REDUCTIVE GROUPS AND INTEGRAL CONJECTURES

JEREMY MILLER, PETER PATZT, AND ANDREW PUTMAN

ABSTRACT. We prove that the homology groups of any connected reductive group over a field with coefficients in the Steinberg representation vanish in a range. This generalizes work of Ash–Putman–Sam on the classical split groups. We state a connectivity conjecture that would allow us to prove such a vanishing result for $\mathrm{SL}_n(\mathbb{Z})$, as was conjectured by Church–Farb–Putman. We prove some special cases of this conjecture and use it to refine known results about the first and second homology of $\mathrm{SL}_n(\mathbb{Z})$ with Steinberg coefficients.

CONTENTS

1. Introduction	2
Part 1. General theory (Theorem B)	7
2. Reductive groups, Tits buildings, and the Steinberg representation	7
3. Two initial reductions	15
4. Resolution of Steinberg and its associated spectral sequence	17
5. Semisimple rank two	20
6. Reducible root systems	22
7. Reduction to irreducible case	26
Part 2. Vanishing in type A (Theorem B.1)	27
8. Vanishing and surjectivity (type A)	27
9. Vanishing and surjectivity for Levi subgroups (type A)	29
10. Vanishing region (type A)	30
11. Remaining tasks (type A)	31
12. Differentials (type A)	32
Part 3. Vanishing in types B and C and BC (Theorem B.2)	35
13. Vanishing and surjectivity (types B and C and BC)	35
14. Vanishing and surjectivity for Levi subgroups (types B and C and BC)	37
15. Vanishing region (types B and C and BC)	37
16. Remaining tasks (types B and C and BC)	38
17. Differentials (types B and C and BC)	39
Part 4. Vanishing in type D (Theorem B.3)	43
18. Vanishing and surjectivity (type D)	43
19. Vanishing and surjectivity for Levi subgroups (type D)	44
20. Vanishing region (type D)	44
21. Remaining tasks (type D)	45
22. Differentials (type D)	46

JM was supported by NSF grants DMS-2202943 and DMS-2504473 and a Simons Foundation Travel Support for Mathematicians grant. PP was supported by NSF grant DMS-2405310 and a Simons Foundation Collaboration Grant. AP was supported by NSF grant DMS-2305183.

Part 5. Integral vanishing (Theorem C)	48
23. Partial resolution of integral Steinberg representation	48
24. Reducible Levi subgroups over the integers	51
25. Vanishing and surjectivity (integral)	54
26. Vanishing and surjectivity for Levi subgroups (integral)	57
27. Vanishing region (integral)	58
28. Remaining tasks (integral)	59
29. Differentials (integral)	60
Part 6. The double Tits building (Theorem D)	62
30. Outline of connectivity proof	63
31. Three-step partial resolution of Steinberg from modular symbols	64
32. Double complex of resolutions	66
33. Three-step bar resolution of Steinberg	68
References	72

1. INTRODUCTION

Let¹ $\mathbf{G}$ be a (connected)² reductive group³ over a field k . For example, $\mathbf{G}$ might be GL_n or Sp_{2n} . The Steinberg representation of $\mathbf{G}$ encodes the combinatorial structure of the parabolic⁴ k -subgroups of $\mathbf{G}$ and plays a fundamental role in representation theory and algebraic topology. For the classical split groups

$$\mathbf{G} \in \{\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{Sp}_{2n}, \mathrm{SO}_{n,n}, \mathrm{SO}_{n,n+1}\},$$

Ash–Putman–Sam [2] proved that the homology of $\mathbf{G}(k)$ with coefficients in the Steinberg representation vanishes in a range. Using a new approach inspired by Miller–Patz–Wilson’s work on the Rognes connectivity conjecture [22], this paper extends [2] to all reductive groups.⁵ Even for the classical split groups, our approach is technically easier.⁶

Church–Farb–Putman [14] made a vanishing conjecture about the high-dimensional cohomology of $\mathrm{SL}_n(\mathbb{Z})$. By Borel–Serre duality, this is equivalent to the vanishing in a stable range of the homology of $\mathrm{SL}_n(\mathbb{Z})$ with coefficients in the Steinberg representation of $\mathrm{SL}_n(\mathbb{Q})$. This can be viewed as an integral refinement of the SL_n case Ash–Putman–Sam’s vanishing theorem.

We show that our approach to the vanishing of the homology of $\mathrm{SL}_n(\mathbb{Q})$ with coefficients in its Steinberg representation can be extended to $\mathrm{SL}_n(\mathbb{Z})$ if a certain simplicial complex is highly connected, and prove this high connectivity in some simple cases. Using this, we refine the known cases of this conjecture in degrees 1 and 2.

¹Here we are using the functorial language of algebraic groups. For an algebraic group $\mathbf{G}$ over a field k , there is a group $\mathbf{G}(k)$ of k -points of $\mathbf{G}$, and more generally for any commutative k -algebra A there is a group $\mathbf{G}(A)$ of A -points. For $\mathbf{G} = \mathrm{GL}_n$, these are the groups $\mathrm{GL}_n(k)$ and $\mathrm{GL}_n(A)$.

²Here “connected” refers to the Zariski topology. Our convention is that all reductive groups are connected.

³For a reader who has not seen it before, the definition of a reductive group is not particularly enlightening and we refer them to standard sources for it, e.g., [4, 23]. A lot of this paper can be understood by assuming that $\mathbf{G}$ is GL_n or SL_n , or more generally one of the classical split groups $\{\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{Sp}_{2n}, \mathrm{SO}_{n,n}, \mathrm{SO}_{n,n+1}\}$.

⁴A k -subgroup $\mathbf{P}$ of $\mathbf{G}$ is parabolic if $\mathbf{G}/\mathbf{P}$ is a projective variety. For example, for $\mathbf{G} = \mathrm{GL}_{n+1}$ the stabilizer $\mathbf{P}$ of a line L in k^{n+1} is a parabolic subgroup with $\mathbf{G}/\mathbf{P} \cong \mathbb{P}_k^n$; indeed, $\mathbf{G}(k) = \mathrm{GL}_{n+1}(k)$ acts transitively on the set $\mathbb{P}_k^n(k)$ of lines in k^{n+1} and the stabilizer of the point $L \in \mathbb{P}_k^n(k)$ is $\mathbf{P}(k)$. Projective varieties are compact from the viewpoint of algebraic geometry, so parabolic subgroups are “large” subgroups.

⁵For example, we can handle k -forms of GL_n , i.e., algebraic groups $\mathbf{G}$ over k that are not necessarily isomorphic to GL_n , but whose base-change $\mathbf{G}_{\bar{k}}$ to an algebraic closure $\bar{k}$ is isomorphic to GL_n .

⁶To handle general reductive groups we will have to use a lot of their basic structural properties, so this might not be clear on a first reading. One concrete simplification is that unlike [2], our proof is self-contained and does not rely on understanding the topology of the complexes of partial bases and partial isotropic bases.

1.1. Tits building. Let the semisimple k -rank of $\mathbf{G}$ be n . The spherical Tits building for $\mathbf{G}$, denoted $\mathcal{T}(\mathbf{G})$, is an $(n-1)$ -dimensional simplicial complex whose r -simplices are the proper parabolic k -subgroups $\mathbf{P}$ of $\mathbf{G}$ whose semisimple k -rank is $n-1-r$. The simplex corresponding to a proper parabolic k -subgroup $\mathbf{P}'$ is a face of the one corresponding to $\mathbf{P}$ if⁷ $\mathbf{P} \subset \mathbf{P}'$. The conjugation action of $\mathbf{G}(k)$ on itself permutes k -points $\mathbf{P}(k)$ of the different parabolic k -subgroups $\mathbf{P}$, giving an action of $\mathbf{G}(k)$ on $\mathcal{T}(\mathbf{G})$. See [34, 8] for more details.

Example 1.1. If $\mathbf{G}$ is either GL_{n+1} or SL_{n+1} , then the semisimple k -rank of $\mathbf{G}$ is n and the parabolic k -subgroups of $\mathbf{G}$ are the stabilizers of flags

$$(1.1) \quad 0 \subsetneq V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r \subsetneq k^{n+1}$$

of subspaces of k^{n+1} . The quotient of $\mathbf{G}$ by the stabilizer of (1.1) is a flag variety. The semisimple k -rank of the stabilizer of (1.1) is $n-1-r$. The Tits building $\mathcal{T}(\mathbf{G})$ is the simplicial complex whose r -simplices are length- r flags of subspaces of k^{n+1} . $\square$

Remark 1.2. If $\mathbf{G}$ has no proper parabolic k -subgroups, then its semisimple k -rank is 0 and $\mathcal{T}(\mathbf{G}) = \emptyset$. Such $\mathbf{G}$ are called *anisotropic*. Easy examples include GL_0 and GL_1 . A more interesting example is the orthogonal group of a quadratic form q on a k -vector space V that is anisotropic, i.e., such that the only $\vec{v} \in V$ with $q(\vec{v}) = 0$ is $\vec{v} = 0$. $\square$

1.2. Steinberg representation. Let $\mathbb{F}$ be a commutative ring. The Solomon–Tits theorem [28] says that $\mathcal{T}(\mathbf{G})$ is homotopy equivalent to a wedge of $(n-1)$ -dimensional spheres. The Steinberg representation of $\mathbf{G}$ with coefficients in $\mathbb{F}$, denoted $\mathrm{St}(\mathbf{G}; \mathbb{F})$, is its reduced $(n-1)$ -dimensional homology group with coefficients in $\mathbb{F}$, i.e., $\mathrm{St}(\mathbf{G}; \mathbb{F}) = \tilde{H}_{n-1}(\mathcal{T}(\mathbf{G}); \mathbb{F})$. For $\mathbb{F} = \mathbb{Z}$, we will omit $\mathbb{F}$ from our notation and just write $\mathrm{St}(\mathbf{G})$. The action of $\mathbf{G}(k)$ on $\mathcal{T}(\mathbf{G})$ induces an action of $\mathbf{G}(k)$ on $\mathrm{St}(\mathbf{G}; \mathbb{F})$. See [17, 32] for surveys.

These are finite-dimensional representations of $\mathbf{G}(k)$ when k is finite, and Steinberg and Curtis [29, 30, 31, 16] showed that if k is finite and $\mathbb{F}$ is a field, then $\mathrm{St}(\mathbf{G}; \mathbb{F})$ is usually⁸ an irreducible representation of $\mathbf{G}(k)$. Except in degenerate cases, for infinite k the representation $\mathrm{St}(\mathbf{G}; \mathbb{F})$ is infinite-dimensional. Putman–Snowden [24] showed that in this case $\mathrm{St}(\mathbf{G}; \mathbb{F})$ is irreducible for all fields $\mathbb{F}$.

Remark 1.3. If $\mathbf{G}$ is anisotropic, then $n = 0$ and $\mathcal{T}(\mathbf{G}) = \emptyset$. Our convention is $\mathrm{St}(\mathbf{G}; \mathbb{F}) = \tilde{H}_{-1}(\mathcal{T}(\mathbf{G}); \mathbb{F}) = \tilde{H}_{-1}(\emptyset; \mathbb{F}) = \mathbb{F}$. The group $\mathbf{G}(k)$ acts trivially on $\mathrm{St}(\mathbf{G}; \mathbb{F}) = \mathbb{F}$. $\square$

1.3. Weak form of main theorem. We can now state a weak form of our main theorem.

Theorem A (Weak form). *Let $\mathbf{G}$ be a reductive group over a field k with semisimple k -rank n and let $\mathbb{F}$ be a commutative ring. Then $H_i(\mathbf{G}(k); \mathrm{St}(\mathbf{G}; \mathbb{F})) = 0$ for $i \leq b(n)$, where $b: \mathbb{N} \rightarrow \mathbb{Z}$ is a function with $\lim_{n \rightarrow \infty} b(n) = \infty$.*

Remark 1.4. For anisotropic groups $\mathbf{G}$, we have $n = 0$. The function b satisfies $b(0) = -1$, so our theorem says nothing nontrivial about such $\mathbf{G}$. $\square$

Remark 1.5. If k is a finite field and $\mathbf{G}(k)$ is a finite group of Lie type, then $\mathrm{St}(\mathbf{G}; k)$ is a projective $\mathbf{G}(k)$ -module (see [17]) and thus $H_i(\mathbf{G}(k); \mathrm{St}(\mathbf{G}; k)) = 0$ for $i \geq 0$. However, $\mathrm{St}(\mathbf{G}; \mathbb{F})$ is not projective for a general commutative ring like $\mathbb{F} = \mathbb{Z}$, so even for finite groups of Lie type Theorem A has nontrivial content. $\square$

The stronger form of our main theorem gives a specific vanishing range for $\mathbf{G}$ depending on its relative root system, which we now discuss.

⁷It is not obvious that this specifies a simplicial complex. Also, we remark that if $\mathbf{P}$ is a parabolic k -subgroup of $\mathbf{G}$ and $\mathbf{Q}$ is another k -subgroup with $\mathbf{P} \subset \mathbf{Q}$, then $\mathbf{Q}$ is a parabolic k -subgroup.

⁸For example, this holds if $\mathrm{char}(\mathbb{F}) = 0$ or $\mathrm{char}(\mathbb{F}) = \mathrm{char}(k)$.

Φ	$\mathbf{b}(\Phi)$
A_n	$\lfloor \frac{n-1}{2} \rfloor$
B_n, C_n, BC_n	$\lfloor \frac{n-2}{2} \rfloor$
D_n	$\lfloor \frac{n-3}{2} \rfloor$
G_2, F_4, E_6, E_7, E_8	0

Table 1. The bounds $\mathbf{b}(\Phi)$ associated to irreducible relative root systems Φ .

1.4. Relative root system. Let $\mathbf{T}$ be a maximal⁹ k -split torus¹⁰ in $\mathbf{G}$ and let $\mathfrak{g}$ be the Lie algebra of $\mathbf{G}$. The action of $\mathbf{T}$ on $\mathfrak{g}$ decomposes into eigenspaces. The *relative root system* of $\mathbf{G}$, denoted ${}_k\Phi(\mathbf{G})$, is the set of characters of $\mathbf{T}$ that are nontrivial eigenvalues for this action. As we will discuss in more detail in §2, there is a natural inner product on ${}_k\Phi(\mathbf{G})$ for which ${}_k\Phi(\mathbf{G})$ is a root system. All maximal k -split tori in $\mathbf{G}$ are conjugate, so this does not depend on the choice of $\mathbf{T}$.

Example 1.6. Reduced (cf. Remark 1.7) irreducible¹¹ root systems fall into 4 infinite families $\{A_n, B_n, C_n, D_n\}$ and 5 exceptional systems $\{G_2, F_4, E_6, E_7, E_8\}$. For the classical split groups, the relative root systems are as follows:

$$\begin{aligned} {}_k\Phi(\mathrm{GL}_{n+1}) &\cong A_n & {}_k\Phi(\mathrm{SL}_{n+1}) &\cong A_n & {}_k\Phi(\mathrm{SO}_{n,n+1}) &\cong B_n \\ {}_k\Phi(\mathrm{Sp}_{2n}) &\cong C_n & {}_k\Phi(\mathrm{SO}_{n,n}) &\cong D_n \end{aligned} \quad \square$$

Remark 1.7. When k is not algebraically closed, it might be the case that maximal tori¹² in $\mathbf{G}$ are not split over k , so $\mathbf{T}$ is not a maximal torus. In this case, the relative root system ${}_k\Phi(\mathbf{G})$ might not be reduced, i.e., there might be roots $\alpha \in {}_k\Phi(\mathbf{G})$ such that $c\alpha \in {}_k\Phi(\mathbf{G})$ for some $c \neq \pm 1$. It turns out that the only additional possibility here is $c = \pm 2$, and the only nonreduced irreducible root systems are those of type BC_n . $\square$

Remark 1.8. The *absolute root system* of $\mathbf{G}$ is the root system $\Phi(\mathbf{G})$ associated to a maximal torus $\mathbf{S}$. All maximal tori in $\mathbf{G}$ are conjugate, so this does not depend on the choice of $\mathbf{S}$. Choosing $\mathbf{S}$ with $\mathbf{T} \subset \mathbf{S}$, we can restrict characters from $\mathbf{S}$ to $\mathbf{T}$ and get a surjective map

$$\Phi(\mathbf{G}) \sqcup \{0\} \longrightarrow {}_k\Phi(\mathbf{G}) \sqcup \{0\}.$$

If $\mathbf{S}$ splits¹³ and thus $\mathbf{S} = \mathbf{T}$, then $\Phi(\mathbf{G}) = {}_k\Phi(\mathbf{G})$ and we say that $\mathbf{G}$ is *split*. $\square$

1.5. Stronger form of main theorem. The irreducible root systems Φ are listed in Table 1. For each of them, let $\mathbf{b}(\Phi)$ be the bound listed in that table. A general root system Φ can be written in the form $\Phi = \Phi_1 \times \cdots \times \Phi_m$ with each Φ_j irreducible, and we define

$$(1.2) \quad \mathbf{b}(\Phi) = (m-1) + \sum_{j=1}^m \mathbf{b}(\Phi_j).$$

⁹Here “maximal” means among k -split tori.

¹⁰A k -split torus is an algebraic group $\mathbf{T}$ defined over k that is isomorphic to $(\mathrm{GL}_1)^d$ for some $d \geq 0$, so $\mathbf{T}(k) = (k^\times)^d$. If $\mathbf{G}$ is GL_{n+1} or SL_{n+1} , then its subgroup of diagonal matrices is a maximal k -split torus.

¹¹A root system is irreducible if it does not decompose as a nontrivial product of root systems.

¹²An (algebraic) torus is an algebraic group $\mathbf{T}$ defined over k whose base-change $\mathbf{T}_{\bar{k}}$ to an algebraic closure is a $\bar{k}$ -split torus. For example, let $k = \mathbb{R}$ and let $\mathbf{T} = \mathrm{SO}(2)$. For an $\mathbb{R}$ -algebra A , we have $\mathbf{T}(A) = \{(x, y) \in A^2 \mid x^2 + y^2 = 1\}$ with the multiplication $(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$. This is a non-split torus with $\mathbf{T}(\mathbb{R}) \cong \mathbb{S}^1$ and $\mathbf{T}_{\mathbb{C}} \cong \mathrm{GL}_1$. The isomorphism between $\mathbf{T}_{\mathbb{C}}(\mathbb{C}) = \mathbf{T}(\mathbb{C})$ and $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$ takes $(x, y) \in \mathbb{C}^2$ with $x^2 + y^2 = 1$ to $x + iy \in \mathbb{C}^\times$.

¹³This always holds when k is algebraically closed.

If $\Phi = \emptyset$, then $m = 0$ and $\mathbf{b}(\Phi) = -1$. With this notation, our main theorem is as follows:

Theorem B (Stronger form). *Let $\mathbf{G}$ be a reductive group over a field k and let $\mathbb{F}$ be a commutative ring. Then $H_i(\mathbf{G}(k); \text{St}(\mathbf{G}; \mathbb{F})) = 0$ for $i \leq \mathbf{b}_k(\Phi(\mathbf{G}))$.*

Remark 1.9. For the classical split groups, this was proved by Ash–Putman–Sam [2]. Their bound is the same as ours for $\mathbf{G} \in \{\text{GL}_n, \text{SL}_n, \text{SO}_{n,n+1}, \text{Sp}_{2n}\}$. However, for $\mathbf{G} = \text{SO}_{n,n}$ their bound is $\lfloor (n-2)/2 \rfloor$, while ours is $\lfloor (n-3)/2 \rfloor$. $\square$

Remark 1.10. The bounds in Table 1 for the exceptional root systems are not the best that can be achieved via our methods.¹⁴ With more work, one could instead achieve the bounds

$$\mathbf{b}(\text{G}_2) = 0, \quad \mathbf{b}(\text{F}_4) = 0, \quad \mathbf{b}(\text{E}_6) = 1, \quad \mathbf{b}(\text{E}_7) = 1, \quad \mathbf{b}(\text{E}_8) = 2.$$

However, doing this would make our proof less uniform, so we decided not to do it. $\square$

1.6. Arithmetic groups. Our proof of Theorem B suggests an approach to also proving vanishing theorems for certain arithmetic groups, which we now describe. Consider $\mathbf{G} = \text{SL}_n$ over the field $k = \mathbb{Q}$. This algebraic group has a natural $\mathbb{Z}$ -form, so we can talk about the arithmetic group $\mathbf{G}(\mathbb{Z}) = \text{SL}_n(\mathbb{Z})$. Borel–Serre [5] proved that the virtual cohomological dimension of $\text{SL}_n(\mathbb{Z})$ is $\binom{n}{2}$. They also proved that $\text{SL}_n(\mathbb{Z})$ is a virtual duality group with dualizing module the Steinberg representation. This implies that

$$(1.3) \quad H^{\binom{n}{2}-i}(\text{SL}_n(\mathbb{Z}); \mathbb{Q}) \cong H_i(\text{SL}_n(\mathbb{Z}); \text{St}(\text{SL}_n; \mathbb{Q})) \quad \text{for } i \geq 0.$$

Church–Farb–Putman [14] conjectured that

$$H^{\binom{n}{2}-i}(\text{SL}_n(\mathbb{Z}); \mathbb{Q}) = 0 \quad \text{for } n \geq i + 2.$$

By (1.3), this is equivalent to the following:

Conjecture 1.11. *For $n \geq 2$, we have $H_i(\text{SL}_n(\mathbb{Z}); \text{St}(\text{SL}_n; \mathbb{Q})) = 0$ for $i \leq n - 2$.*

This can be viewed as an integral version of Theorem B, though the stable range in it is stronger than Theorem B. Conjecture 1.11 is known to hold for $n \leq 7$ (see [14, Table 1]). It has also been proven for $i = 0$ by Lee–Szczařba [18], for $i = 1$ by Church–Putman [15], and for $i = 2$ by Brück–Miller–Patz–Sroka–Wilson [9].

Remark 1.12. More generally, it is natural to conjecture that something like Conjecture 1.11 holds for other integral Chevalley groups $\mathbf{G}$ like Sp_{2g} or $\text{SO}_{n,n}$. This is asked explicitly in [11, Question 1.2], which in many cases proves this for $i = 0$. See also [10], which proves this for $i = 1$ in the special case of $\mathbf{G} = \text{Sp}_{2g}$. Our conjectural approach for proving a version of Conjecture 1.11 can also be generalized to this setting, though to keep this paper a reasonable length we chose not to make this explicit. $\square$

1.7. Double Tits building. As part of their work on the Rognes connectivity conjecture [26], Miller–Patz–Wilson [22] gave an approach to Conjecture 1.11. Define $\mathcal{T}(\mathbb{Z}^n)$ to be the simplicial complex whose r -simplices are flags

$$(1.4) \quad 0 \subsetneq V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r \subsetneq \mathbb{Z}^n$$

of direct summands of $\mathbb{Z}^n$. There is a bijection between subspaces of $\mathbb{Q}^n$ and direct summands of $\mathbb{Z}^n$ that takes a subspace $V \subset \mathbb{Q}^n$ to $V \cap \mathbb{Z}^n$. Using this, there is an isomorphism $\mathcal{T}(\mathbb{Z}^n) \cong \mathcal{T}(\text{SL}_n)$. Here just like above SL_n is taken over the field $k = \mathbb{Q}$.

For every flag (1.4), there exists a basis $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ for $\mathbb{Z}^n$ such that each V_i is the span of some subset of B . Say that two flags

$$(1.5) \quad 0 \subsetneq V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r \subsetneq \mathbb{Z}^n \quad \text{and} \quad 0 \subsetneq W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_s \subsetneq \mathbb{Z}^n$$

¹⁴They are not entirely trivial since they say something about H_0 . Also, due to the $(m-1)$ in (1.2) these zeros do not affect the fact that $\mathbf{b}_k(\Phi(\mathbf{G}))$ goes to infinity as the semisimple k -rank of $\mathbf{G}$ goes to infinity.

of direct summands are *compatible* if there exists a basis $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ for $\mathbb{Z}^n$ such that each V_i and W_j is the span of some subset of B . Define the *double building* $\mathcal{T}^2(\mathbb{Z}^n)$ to be the simplicial complex whose p -simplices are pairs of compatible flags (1.5) with $p = r + s + 1$. Here if the V_i (resp. the W_j) form the empty flag we have $r = -1$ (resp. $s = -1$). We can identify $\mathcal{T}^2(\mathbb{Z}^n)$ with a subcomplex of the join $\mathcal{T}(\mathbb{Z}^n) * \mathcal{T}(\mathbb{Z}^n)$. The flags formed by the V_i live in the first term, and the flags formed by the W_j live in the second.

1.8. Connectivity of double Tits building. The complex $\mathcal{T}^2(\mathbb{Z}^n)$ is $(2n - 3)$ -dimensional. Miller–Patz–Wilson [22] proved that if it was $(2n - 4)$ -connected, then the Rognes connectivity conjecture [26, Conjecture 12.3] would hold for the ring $\mathbb{Z}$. Their results show that this would also imply the following slight weakening of Conjecture 1.11:

- For $n \geq 2$ and $\mathbb{F}$ a field of characteristic 0, we have $H_i(\mathrm{GL}_n(\mathbb{Z}); \mathrm{St}(\mathrm{GL}_n; \mathbb{F})) = H_i(\mathrm{SL}_n(\mathbb{Z}); \mathrm{St}(\mathrm{SL}_n; \mathbb{F})) = 0$ for $i \leq \lfloor (n - 2)/2 \rfloor$.

This combines results from [22] (cf. Theorem 23.3 below) with [19, Theorems 7.2 & 7.8].

Remark 1.13. By [2] (which is generalized in Theorem B), for all commutative rings $\mathbb{F}$ we have $H_i(\mathrm{GL}_n(\mathbb{Q}); \mathrm{St}(\mathrm{GL}_n; \mathbb{F})) = H_i(\mathrm{SL}_n(\mathbb{Q}); \mathrm{St}(\mathrm{SL}_n; \mathbb{F})) = 0$ for $i \leq \lfloor (n - 2)/2 \rfloor$. This is the same range Miller–Patz–Wilson’s work conjecturally gives for $\mathrm{GL}_n(\mathbb{Z})$ and $\mathrm{SL}_n(\mathbb{Z})$. $\square$

Remark 1.14. It is natural to hope that $\mathcal{T}^2(\mathbb{Z}^n)$ is $(2n - 4)$ -connected. Indeed, if we were working over a field k rather than $\mathbb{Z}$, then it is easy to see that any two flags are compatible. Thus $\mathcal{T}^2(k^n) = \mathcal{T}(k^n) * \mathcal{T}(k^n)$, which is $(2n - 4)$ -connected since $\mathcal{T}(k^n)$ is $(n - 3)$ -connected. $\square$

1.9. Integral theorems. We refine this. Roughly speaking, the following says that to prove vanishing for $\mathrm{GL}_n(\mathbb{Z})$ and $\mathrm{SL}_n(\mathbb{Z})$ up to degree $i = b$, we must prove that $\mathcal{T}^2(\mathbb{Z}^n)$ is $(n - 2 + b)$ -connected for $n \gg 0$:

Theorem C. *Fix $b \geq 1$. Assume that the following holds for all $n \geq 4$:*

(†) *The space $\mathcal{T}^2(\mathbb{Z}^n)$ is $n - 2 + \min(b, \lfloor (n - 1)/2 \rfloor)$ -connected.*

Then $H_i(\mathrm{GL}_n(\mathbb{Z}); \mathrm{St}(\mathrm{GL}_n; \mathbb{F})) = H_i(\mathrm{SL}_n(\mathbb{Z}); \mathrm{St}(\mathrm{SL}_n; \mathbb{F})) = 0$ for all commutative rings $\mathbb{F}$ and all $n, i \geq 0$ such that:

- $i \leq \min(b, \lfloor (n - 2)/2 \rfloor)$ if 2 and 3 are invertible in $\mathbb{F}$; and
- $i \leq \min(b, \lfloor (n - 3)/3 \rfloor)$ in general.

Remark 1.15. As we will see, for $n \gg 0$ the hypothesis (†) gives a length- b partial resolution of $\mathrm{St}(\mathrm{GL}_n; \mathbb{F})$ from which $H_i(\mathrm{GL}_n(\mathbb{Z}); \mathrm{St}(\mathrm{GL}_n; \mathbb{F}))$ can be inductively calculated, and similarly for $\mathrm{SL}_n(\mathbb{Z})$. Similar resolutions appeared in the proofs of the cases $i = 1$ and $i = 2$ of Conjecture 1.11; however, it was unclear how to extend those resolutions further. $\square$

Remark 1.16. More generally, if $\mathcal{T}^2(\mathbb{Z}^n)$ is $(n - 2 + b)$ -connected for $n \gg 0$ then our proof will show that $H_i(\mathrm{GL}_n(\mathbb{Z}); \mathrm{St}(\mathrm{GL}_n; \mathbb{F})) = H_i(\mathrm{SL}_n(\mathbb{Z}); \mathrm{St}(\mathrm{SL}_n; \mathbb{F})) = 0$ for $i \leq b$ and $n \gg 0$. We remark that $\mathcal{T}^2(\mathbb{Z}^n)$ will not be $(n - 2 + b)$ -connected for small b , which is why our connectivity assumption in Theorem C is phrased the way it is.

The specific connectivity ranges in Theorem C are the minimal ones needed to get the optimal vanishing ranges from our proof strategy. Without further ideas, improvements to them would not yield better vanishing ranges. $\square$

As evidence that $\mathcal{T}^2(\mathbb{Z}^n)$ should be highly-connected, we prove the following, which verifies the hypotheses of Theorem C for¹⁵ $b = 2$.

Theorem D. *The complex $\mathcal{T}^2(\mathbb{Z}^n)$ is n -connected for $n \geq 4$.*

¹⁵In fact, for $n = 4$ it is slightly better since for $b = 2$ Theorem C only requires $\mathcal{T}^2(\mathbb{Z}^4)$ to be 3-connected.

Remark 1.17. The proof of Theorem D uses the same technology as the proofs of the cases $i = 1$ and $i = 2$ of Conjecture 1.11. $\square$

Combining Theorems C and D, we deduce the following:

Corollary E. *We have $H_i(\mathrm{GL}_n(\mathbb{Z}); \mathrm{St}(\mathrm{GL}_n; \mathbb{F})) = H_i(\mathrm{SL}_n(\mathbb{Z}); \mathrm{St}(\mathrm{SL}_n; \mathbb{F})) = 0$ for:*

- $i = 1$ and $n \geq 4$ if 2 and 3 are invertible in the commutative ring $\mathbb{F}$; and
- $i = 1$ and $n \geq 5$ for general commutative rings $\mathbb{F}$; and
- $i = 2$ and $n \geq 6$ if 2 and 3 are invertible in the commutative ring $\mathbb{F}$; and
- $i = 2$ and $n \geq 8$ for general commutative rings $\mathbb{F}$.

Remark 1.18. Previously, $H_i(\mathrm{GL}_n(\mathbb{Z}); \mathrm{St}(\mathrm{GL}_n; \mathbb{F}))$ and $H_i(\mathrm{SL}_n(\mathbb{Z}); \mathrm{St}(\mathrm{SL}_n; \mathbb{F}))$ were known to vanish for the following i and n and $\mathbb{F}$:

i	n	coefficients	reference
$i = 0$	$n \geq 2$	$\mathbb{F}$ arbitrary	[18]
$i = 1$	$n \geq 3$	$\mathbb{F}$ field of characteristic 0	[15]
$i = 1$	$n \geq 6$	$\mathbb{F}$ arbitrary	[19]
$i = 2$	$n \geq 3$	$\mathbb{F}$ field of characteristic 0	[9]

In fact, with only a little more effort the proofs in [9, 15] work for $\mathbb{F}$ a commutative ring in which all primes $p \leq n$ are invertible. See the proof of Claim 2 of Lemma 25.3 below. $\square$

1.10. Outline. We start in Part 1 with some general results about reductive groups. This part closes by reducing Theorem B to the special case where ${}_k\Phi(\mathbf{G})$ is irreducible and non-exceptional, i.e., ${}_k\Phi(\mathbf{G}) \in \{A_n, B_n, C_n, BC_n, D_n\}$. The heart of the paper consists of Parts 2 and 3 and 4, which prove Theorem B for those root systems. The three cases are ${}_k\Phi(\mathbf{G}) = A_n$, and ${}_k\Phi(\mathbf{G}) \in \{B_n, C_n, BC_n\}$, and ${}_k\Phi(\mathbf{G}) = D_n$. In Part 5 we generalize this to $\mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{GL}_n(\mathbb{Z})$ and prove Theorem C. Finally, in Part 6 we prove Theorem D.

1.11. Conventions. Unless otherwise specified, all algebraic groups $\mathbf{G}$ are defined over a field k that is fixed throughout the paper. All subgroups, morphisms, quotients, etc. we discuss are closed and defined over k . In particular, we will just talk about parabolic subgroups rather than parabolic k -subgroups and the semisimple rank of a group rather than the semisimple k -rank. All reductive groups are connected.

Part 1. General theory (Theorem B)

We survey some general properties of reductive groups in §2. Next, in §3 we perform some initial reductions. The main tool for our proofs is a spectral sequence we introduce in §4. To show how this spectral sequence can be used, in §5 we prove a surjectivity result in rank 2. We then develop some tools for understanding reducible root systems in §6, and close in §7 by reducing Theorem B to the case where the relative root system is irreducible and non-exceptional. Those cases will be handled in subsequent parts.

2. REDUCTIVE GROUPS, TITS BUILDINGS, AND THE STEINBERG REPRESENTATION

Let $\mathbf{G}$ be a reductive group (cf. the conventions in §1.11). We thus have a group $\mathbf{G}(k)$ of k -points of $\mathbf{G}$. In this section, we discuss some background about $\mathbf{G}$ needed for the rest of the paper. To help make this understandable to readers less familiar with the general picture, we will also explain what this background means for¹⁶ $\mathbf{G} = \mathrm{GL}_{n+1}$. Unless otherwise specified, the results in this section are due to Borel–Tits ([6, 7], see [4, Chapter V] and [23] for textbook references).

¹⁶We use GL_{n+1} since it has semisimple rank n , so its numerology agrees with that of the general theory.

2.1. Levi factors and unipotent radicals. Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$. We can write $\mathbf{P} = \mathbf{V} \rtimes \mathbf{L}$, with $\mathbf{V}$ a normal unipotent subgroup of $\mathbf{P}$ called its *unipotent radical* and $\mathbf{L}$ a reductive subgroup of $\mathbf{P}$ called a *Levi factor*. All Levi factors of $\mathbf{P}$ are conjugate. The decomposition $\mathbf{P} = \mathbf{V} \rtimes \mathbf{L}$ is the *Levi decomposition* of $\mathbf{P}$.

Example 2.1. Let $\mathbf{G} = \mathrm{GL}_{n+1}$. Write $n+1 = m + m'$ with $m, m' \geq 1$. Let $\mathbf{P} < \mathbf{G}$ be the subgroup fixing the subspace $k^m \oplus 0$ of $k^{n+1} = k^m \oplus k^{m'}$. The Levi decomposition of $\mathbf{P}$ is $\mathbf{V} \rtimes \mathbf{L}$, where:

- the Levi factor $\mathbf{L}$ is the subgroup $\mathrm{GL}_m \times \mathrm{GL}_{m'}$ of $\mathbf{G}$; and
- the unipotent radical $\mathbf{V}$ is the unipotent subgroup consisting of block matrices $\begin{pmatrix} \mathbb{1}_m & V \\ 0 & \mathbb{1}_{m'} \end{pmatrix}$. Here V is an arbitrary $m' \times m$ matrix. $\square$

2.2. Split tori, minimal parabolic subgroups, and relative Weyl groups. Let $\mathbf{B}$ be a minimal parabolic subgroup of $\mathbf{G}$ and let $\mathbf{T}$ be a maximal split torus in $\mathbf{B}$. If k was algebraically closed or more generally if $\mathbf{G}$ was split, then $\mathbf{B}$ would be a Borel subgroup¹⁷ and $\mathbf{T}$ would be a maximal torus, but in general this need not hold. The centralizer $Z_{\mathbf{G}}(\mathbf{T})$ is a Levi factor of $\mathbf{B}$. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. We then have $\mathbf{B} = \mathbf{U} \rtimes Z_{\mathbf{G}}(\mathbf{T})$. All pairs $(\mathbf{B}, \mathbf{T})$ are conjugate in $\mathbf{G}$.

Example 2.2. For $\mathbf{G} = \mathrm{GL}_{n+1}$, we can let $\mathbf{B}$ be the Borel subgroup of upper triangular matrices, $\mathbf{U}$ be the unipotent subgroup of strictly upper triangular matrices, and $\mathbf{T}$ be the group of diagonal matrices. In this case, $\mathbf{T} = Z_{\mathbf{G}}(\mathbf{T})$. The group $\mathbf{T}$ is a Levi factor of $\mathbf{B}$, the group $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$, and $\mathbf{B} = \mathbf{U} \rtimes \mathbf{T}$. $\square$

Let $N_{\mathbf{G}}(\mathbf{T})$ be the normalizer of $\mathbf{T}$ in $\mathbf{G}$. The pair $(\mathbf{B}(k), N_{\mathbf{G}}(\mathbf{T})(k))$ forms a BN-pair in $\mathbf{G}(k)$; see [34, Theorem 5.2] and the references therein for the proof, and [8, Chapter V.2] for a textbook reference about BN-pairs. We have $N_{\mathbf{G}}(\mathbf{T}) \cap \mathbf{B} = Z_{\mathbf{G}}(\mathbf{T})$, and the relative Weyl group is

$$W = N_{\mathbf{G}}(\mathbf{T})/Z_{\mathbf{G}}(\mathbf{T}).$$

This is a finite reflection group (see §2.7 below for more on this). The Bruhat decomposition asserts that representatives of W form a complete set of $(\mathbf{B}(k), \mathbf{B}(k))$ -double cosets of $\mathbf{G}(k)$. More precisely, for each $w \in W$ let $\tilde{w} \in N_{\mathbf{G}}(\mathbf{T})(k)$ be a representative. We then have

$$\mathbf{G}(k) = \bigsqcup_{w \in W} \mathbf{B}(k) \cdot \tilde{w} \cdot \mathbf{B}(k).$$

Example 2.3. For $\mathbf{G} = \mathrm{GL}_{n+1}$, let $\mathbf{B}$ and $\mathbf{U}$ and $\mathbf{T}$ be as in Example 2.2. The normalizer $N_{\mathbf{G}}(\mathbf{T})$ is the group of monomial matrices, and

$$W = N_{\mathbf{G}}(\mathbf{T})/Z_{\mathbf{G}}(\mathbf{T}) = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T} \cong S_{n+1}.$$

In this case, we can identify W with the group of permutation matrices in $N_{\mathbf{G}}(\mathbf{T})(k) \subset \mathbf{G}(k)$, but in general the surjection $N_{\mathbf{G}}(\mathbf{T})(k) \rightarrow W$ need not split. $\square$

2.3. Structure of the building. The Tits building $\mathcal{T}(\mathbf{G})$ is the spherical building associated to the BN-pair $(\mathbf{B}(k), N_{\mathbf{G}}(\mathbf{T})(k))$. See [8] for a textbook reference on buildings. Let n be the semisimple rank of $\mathbf{G}$. The space $\mathcal{T}(\mathbf{G})$ is an $(n-1)$ -dimensional simplicial complex. Its r -dimensional simplices are in bijection with parabolic subgroups of semisimple rank $(n-1-r)$. The group $\mathbf{G}(k)$ acts on $\mathcal{T}(\mathbf{G})$ via its conjugation action on itself, which permutes the parabolic subgroups.

¹⁷A subgroup $\mathbf{B}$ of $\mathbf{G}$ is a Borel subgroup if $\mathbf{B}$ is connected and solvable, and the base-change $\mathbf{B}_{\bar{k}}$ to an algebraic closure is maximal among connected solvable subgroups of $\mathbf{G}_{\bar{k}}$. Borel subgroups are always parabolic, and if they exist then every parabolic subgroup contains a Borel subgroup.

The top-dimensional simplices are called the *chambers*, and are in bijection with the minimal parabolic subgroups. Letting $\mathfrak{P}_{\min}$ be the set of minimal parabolic subgroups of $\mathbf{G}$, we thus have $C_{n-1}(\mathcal{T}(\mathbf{G}); \mathbb{F}) \cong \mathbb{F}[\mathfrak{P}_{\min}]$. The group $\mathbf{G}(k)$ acts transitively on the minimal parabolic subgroups, so every element of $\mathfrak{P}_{\min}$ is of the form $g \cdot \mathbf{B}$ for some $g \in \mathbf{G}(k)$.

By definition, $\text{St}(\mathbf{G}; \mathbb{F}) = \tilde{H}_{n-1}(\mathcal{T}(\mathbf{G}); \mathbb{F})$. Since $C_n(\mathcal{T}(\mathbf{G}); \mathbb{F}) = 0$, we have

$$\begin{aligned} \text{St}(\mathbf{G}; \mathbb{F}) &= \ker(C_{n-1}(\mathcal{T}(\mathbf{G}); \mathbb{F}) \xrightarrow{\partial} C_{n-2}(\mathcal{T}(\mathbf{G}); \mathbb{F})) \\ &= \ker(\mathbb{F}[\mathfrak{P}_{\min}] \xrightarrow{\partial} C_{n-2}(\mathcal{T}(\mathbf{G}); \mathbb{F})). \end{aligned}$$

In particular, $\text{St}(\mathbf{G}; \mathbb{F})$ is a subrepresentation of $\mathbb{F}[\mathfrak{P}_{\min}]$.

Example 2.4. For $\mathbf{G} = \text{GL}_{n+1}$, the semisimple rank of $\mathbf{G}$ is n . The parabolic subgroups of $\mathbf{G}$ are the stabilizers of flags

$$(2.1) \quad 0 \subsetneq V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r \subsetneq k^{n+1}$$

of linear subspaces of k^{n+1} . The semisimple rank of the stabilizer of (2.1) is $n - 1 - r$. The Tits building $\mathcal{T}(\mathbf{G})$ can thus be identified with the simplicial complex whose r -simplices are length- r flags of linear subspaces of k^{n+1} . The minimal parabolic subgroups are the stabilizers of complete flags, so the chambers in $\mathcal{T}(\mathbf{G})$ can be identified with the complete flags. A complete flag has $r = n - 1$, so $\mathcal{T}(\mathbf{G})$ is $(n - 1)$ -dimensional. $\square$

2.4. Apartments. The Steinberg representation $\text{St}(\mathbf{G}; \mathbb{F})$ is spanned by apartment classes. These are the homology classes of certain $(n - 1)$ -dimensional subcomplexes of $\mathcal{T}(\mathbf{G})$ that are isomorphic to the Coxeter complexes of the relative Weyl group $W = N_{\mathbf{G}}(\mathbf{T})/Z_{\mathbf{G}}(\mathbf{T})$. They are defined as follows. Consider some $g \in \mathbf{G}(k)$. For each $w \in W$, pick some representative $\tilde{w} \in N_{\mathbf{G}}(\mathbf{T})(k)$. Since $Z_{\mathbf{G}}(\mathbf{T})$ is contained in $\mathbf{B}$, the element $\tilde{w} \cdot \mathbf{B}$ does not depend on the choice of $\tilde{w}$. The apartment class $\mathbb{A}_g$ is then

$$\mathbb{A}_g = \sum_{w \in W} (-1)^w g\tilde{w} \cdot \mathbf{B} \in \text{St}(\mathbf{G}; \mathbb{F}) \subset \mathbb{F}[\mathfrak{P}_{\min}].$$

Here $(-1)^w$ is the sign of w , which is defined since W is a finite reflection group.

Example 2.5. For $\mathbf{G} = \text{GL}_{n+1}$, the Coxeter complex of the relative Weyl group $W \cong S_{n+1}$ is the barycentric subdivision of the boundary of an n -simplex, so in particular it has $(n + 1)!$ top-dimensional simplices. For $1 \leq j \leq n + 1$, let $L_j \subset k^{n+1}$ be the line spanned by the j^{th} coordinate vector. Recall that $\mathbf{B}$ is the subgroup of upper triangular matrices, i.e., the stabilizer of the flag

$$0 \subsetneq L_1 \subsetneq L_1 + L_2 \subsetneq \cdots \subsetneq L_1 + \cdots + L_n \subsetneq k^{n+1}.$$

Consider $g \in \mathbf{G}(k) = \text{GL}_{n+1}(k)$. For $w \in W \cong S_{n+1}$, let $\tilde{w}$ be the associated permutation matrix. We then have $\tilde{w}L_j = L_{w(j)}$ so $g\tilde{w} \cdot \mathbf{B}$ is the stabilizer of the flag

$$0 \subsetneq gL_{w(1)} \subsetneq gL_{w(1)} + gL_{w(2)} \subsetneq \cdots \subsetneq gL_{w(1)} + \cdots + gL_{w(n)} \subsetneq k^{n+1}.$$

Here $gL_{w(j)}$ is the line spanned by the $w(j)^{\text{th}}$ column of g . The stabilizers of these $(n + 1)!$ flags form the chambers in $\mathbb{A}_g$. $\square$

2.5. Basis for Steinberg. The apartment classes $\mathbb{A}_g$ are not linearly independent. Recall that $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$. One version of the Solomon–Tits theorem [8, Theorem 4.73] says that

$$\mathcal{B} = \{\mathbb{A}_u \mid u \in \mathbf{U}(k)\}$$

forms a basis for $\text{St}(\mathbf{G}; \mathbb{F})$. In other words, as an $\mathbb{F}$ -module $\text{St}(\mathbf{G}; \mathbb{F})$ is isomorphic to $\mathbb{F}[\mathbf{U}(k)]$. The action of $\mathbf{B}(k)$ on $\text{St}(\mathbf{G}; \mathbb{F}) \cong \mathbb{F}[\mathbf{U}(k)]$ comes from the conjugation action of $\mathbf{B}(k)$ on $\mathbf{U}(k)$; however, it is not easy to describe the action of $\mathbf{G}(k)$ on $\mathbb{F}[\mathbf{U}(k)]$ coming from this

identification. Another way of thinking about $\mathcal{B}$ is that it is precisely the set of apartment classes that when expressed as an element of $\mathbb{F}[\mathfrak{P}_{\min}]$ have $\mathbf{B}$ -coefficient 1.

Remark 2.6. For $b \in \mathbf{B}(k)$ the $\mathbf{B}$ -coefficient of $\mathbb{A}_b$ is also 1. However, we can write $b = ut$ with $u \in \mathbf{U}(k)$ and $t \in \mathbf{T}(k)$. For $w \in W$, since $W = N_{\mathbf{G}}(\mathbf{T})/Z_{\mathbf{G}}(\mathbf{T})$ the element $t\tilde{w}$ is also an element of $N_{\mathbf{G}}(\mathbf{T})(k)$ lifting w , so in particular $t\tilde{w} \cdot \mathbf{B} = \tilde{w} \cdot \mathbf{B}$. We conclude that

$$\mathbb{A}_b = \sum_{w \in W} (-1)^w b\tilde{w} \cdot \mathbf{B} = \sum_{w \in W} (-1)^w ut\tilde{w} \cdot \mathbf{B} = \sum_{w \in W} (-1)^w u\tilde{w} \cdot \mathbf{B} = \mathbb{A}_u \in \mathcal{B}. \quad \square$$

2.6. Reeder map. Recall that $Z_{\mathbf{G}}(\mathbf{T})$ is a Levi factor of the minimal parabolic subgroup $\mathbf{B}$ and that $\mathbf{B} = \mathbf{U} \rtimes Z_{\mathbf{G}}(\mathbf{T})$. The semisimple rank of $Z_{\mathbf{G}}(\mathbf{T})$ is 0, i.e., it is anisotropic. This implies (cf. Remark 1.3) that $\text{St}(Z_{\mathbf{G}}(\mathbf{T}); \mathbb{F}) \cong \mathbb{F}$ is the trivial representation. Another way of stating the isomorphism $\text{St}(\mathbf{G}; \mathbb{F}) \cong \mathbb{F}[\mathbf{U}(k)]$ is that as a representation of $\mathbf{B}(k)$, we have

$$\text{St}(\mathbf{G}; \mathbb{F}) \cong \mathbb{F}[\mathbf{U}(k)] = \mathbb{F}[\mathbf{B}(k)/Z_{\mathbf{G}}(\mathbf{T})(k)] \cong \text{Ind}_{Z_{\mathbf{G}}(\mathbf{T})(k)}^{\mathbf{B}(k)} \mathbb{F} \cong \text{Ind}_{Z_{\mathbf{G}}(\mathbf{T})(k)}^{\mathbf{B}(k)} \text{St}(Z_{\mathbf{G}}(\mathbf{T}); \mathbb{F}).$$

Reeder gave a beautiful generalization of this:

Theorem 2.7 (Reeder, [25, Proposition 1.1]). *Let $\mathbf{G}$ be a reductive group. Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ and let $\mathbf{L}$ be a Levi factor of $\mathbf{P}$. As a representation of $\mathbf{P}(k)$, we then have*

$$\text{St}(\mathbf{G}; \mathbb{F}) \cong \text{Ind}_{\mathbf{L}(k)}^{\mathbf{P}(k)} \text{St}(\mathbf{L}; \mathbb{F}).$$

Underlying the isomorphism in Theorem 2.7 is a map $\text{St}(\mathbf{L}; \mathbb{F}) \rightarrow \text{St}(\mathbf{G}; \mathbb{F})$ that we will call the *Reeder map*.

2.7. Relative root system. We now return to general properties of reductive groups. Let¹⁸ $X(\mathbf{T}) \cong \mathbb{Z}^m$ be the abelian group of characters of the maximal split torus $\mathbf{T}$. The conjugation action of $N_{\mathbf{G}}(\mathbf{T})$ on $\mathbf{T}$ induces an action of $W = N_{\mathbf{G}}(\mathbf{T})/Z_{\mathbf{G}}(\mathbf{T})$ on $X(\mathbf{T})$. Fix a W -invariant inner product on

$$X_{\mathbb{R}}(\mathbf{T}) = X(\mathbf{T}) \otimes \mathbb{R} \cong \mathbb{R}^m.$$

Let $\mathfrak{g}$ be the Lie algebra of $\mathbf{G}$. The action of $\mathbf{T}$ on $\mathfrak{g}$ can be diagonalized, resulting in a decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in {}_k\Phi(\mathbf{G})} \mathfrak{g}_{\alpha}.$$

Here ${}_k\Phi(\mathbf{G}) \subset X(\mathbf{T}) \subset X_{\mathbb{R}}(\mathbf{T})$ is a finite set of nontrivial characters, and for $\alpha \in \{0\} \cup {}_k\Phi(\mathbf{G})$ the subspace $\mathfrak{g}_{\alpha} \neq 0$ is the α -eigenspace for the action of $\mathbf{T}$ on $\mathfrak{g}$.

The action of W on $X_{\mathbb{R}}(\mathbf{T})$ preserves ${}_k\Phi(\mathbf{G})$, and with respect to the W -invariant inner product on $X_{\mathbb{R}}(\mathbf{T})$ the subset ${}_k\Phi(\mathbf{G})$ is the relative root system¹⁹ of $\mathbf{G}$. The orthogonal reflections of $X_{\mathbb{R}}(\mathbf{T})$ in the roots ${}_k\Phi(\mathbf{G})$ preserve ${}_k\Phi(\mathbf{G})$ and generate W , making W into a finite reflection group. Maximal split tori in $\mathbf{G}$ are all conjugate, so this does not depend on the choice of $\mathbf{T}$. It is also independent of the choice of W -invariant inner product.

Example 2.8. For $\mathbf{G} = \text{GL}_{n+1}$, the group $\mathbf{T}$ is the group of diagonal matrices. For $t_1, \dots, t_{n+1} \in k^{\times}$, let $\text{diag}(t_1, \dots, t_{n+1})$ be the associated diagonal matrix. We then have that $X(\mathbf{T}) \cong \mathbb{Z}^{n+1}$, generated by the elements $E_i \in X(\mathbf{T})$ defined via the formula

$$E_i(\text{diag}(t_1, \dots, t_{n+1})) = t_i \quad \text{for } t_1, \dots, t_{n+1} \in k^{\times}.$$

¹⁸If $\mathbf{G}$ is a semisimple group like SL_{n+1} , then $X(\mathbf{T}) \cong \mathbb{Z}^n$, but in general it is only isomorphic to $\mathbb{Z}^m$ for some $m \geq n$.

¹⁹If $\mathbf{G}$ is a non-semisimple group like GL_{n+1} , then ${}_k\Phi(\mathbf{G})$ does not span $X_{\mathbb{R}}(\mathbf{T})$. In this case, ${}_k\Phi(\mathbf{G})$ is a root system in the subspace of $X_{\mathbb{R}}(\mathbf{T})$ spanned by ${}_k\Phi(\mathbf{G})$.

The relative Weyl group $W \cong S_{n+1}$ permutes the E_i . We have

$${}_k\Phi(\mathbf{G}) = \{E_i - E_j \mid 1 \leq i, j \leq n+1 \text{ distinct}\} \cong A_n,$$

and the subspace $\mathfrak{g}_{E_i - E_j}$ of $\mathfrak{g} \cong \mathfrak{gl}_{n+1}$ consists of matrices whose only nonzero entry is at position (i, j) . The reflection in $E_i - E_j$ is the transposition $(i, j) \in W$. $\square$

Remark 2.9. As we discussed in Remark 1.8, the absolute root system $\Phi(\mathbf{G})$ is the root system associated to a maximal torus $\mathbf{S}$. If maximal tori are split,²⁰ then $\mathbf{G}$ is said to be split and $\Phi(\mathbf{G}) = {}_k\Phi(\mathbf{G})$. For example, $\mathbf{G} = \mathrm{GL}_{n+1}$ is split. While the absolute root system $\Phi(\mathbf{G})$ is a reduced root system, the relative root system ${}_k\Phi(\mathbf{G})$ need not be reduced. See [33] for a survey about the relationship between the absolute and relative root systems. $\square$

2.8. Root subgroups. For $\alpha \in {}_k\Phi(\mathbf{G})$, let (α) be the set of all elements of ${}_k\Phi(\mathbf{G})$ of the form $c\alpha$ for some $c \geq 1$ and let $\mathfrak{g}_{(\alpha)}$ be the direct sum of all $\mathfrak{g}_{\alpha'}$ with $\alpha' \in (\alpha)$. There then exists a unique connected unipotent subgroup $\mathbf{U}_{(\alpha)}$ of $\mathbf{G}$ called a *root subgroup* that is normalized by $\mathbf{T}$ and has Lie algebra $\mathfrak{g}_{(\alpha)}$. There are two cases:

- The root α is *non-multiplicable*, i.e., the only $c \geq 1$ with $c\alpha \in {}_k\Phi(\mathbf{G})$ is $c = 1$. In this case, $\mathbf{U}_{(\alpha)}$ is an abelian unipotent subgroup. We will sometimes omit the parentheses and just write $\mathbf{U}_\alpha$.
- The root α is *multiplicable*, so $(\alpha) = \{\alpha, 2\alpha\}$. In this case, $\mathbf{U}_{(\alpha)}$ is 2-step nilpotent and $\mathbf{U}_{2\alpha}$ is its center.

The root α is *non-divisible* if $\frac{1}{2}\alpha \notin {}_k\Phi(\mathbf{G})$.

Recall that the relative Weyl group $W = N_{\mathbf{G}}(\mathbf{T})/Z_{\mathbf{G}}(\mathbf{T})$ acts on ${}_k\Phi(\mathbf{G})$ via the conjugation action of $N_{\mathbf{G}}(\mathbf{T})$ on $\mathbf{T}$. For $w \in W$ represented by $\tilde{w} \in N_{\mathbf{G}}(\mathbf{T})(k)$, we have

$$\tilde{w}\mathbf{U}_{(\alpha)}(k)\tilde{w}^{-1} = \mathbf{U}_{(w\cdot\alpha)}(k) \quad \text{for } \alpha \in {}_k\Phi(\mathbf{G}).$$

Example 2.10. For $\mathbf{G} = \mathrm{GL}_{n+1}$ and $\alpha = E_i - E_j \in \Phi(\mathbf{G})$, the root subgroup $\mathbf{U}_\alpha$ is the group of matrices that equal the identity matrix except possibly at position (i, j) . $\square$

Remark 2.11. If $\mathbf{G}$ is split, then ${}_k\Phi(\mathbf{G}) = \Phi(\mathbf{G})$ is reduced and hence all the roots are non-multiplicable. Moreover, each root subgroup $\mathbf{U}_\alpha$ is one-dimensional and isomorphic to the additive group²¹ $\mathbb{G}_a$. $\square$

Remark 2.12. In the nonsplit case, for $\alpha \in {}_k\Phi(\mathbf{G})$ let $\{\alpha_1, \dots, \alpha_r\}$ be the preimage of (α) under the surjection $\Phi(\mathbf{G}) \sqcup \{0\} \rightarrow {}_k\Phi(\mathbf{G}) \sqcup \{0\}$ that restricts characters from a maximal torus to the maximal split torus $\mathbf{T}$. The absolute root subgroups $\mathbf{U}_{\alpha_j}$ are defined over an algebraic closure $\bar{k}$, and $\mathbf{U}_{(\alpha)}(\bar{k})$ is the subgroup of $\mathbf{G}(\bar{k})$ generated by the $\mathbf{U}_{\alpha_j}(\bar{k})$. $\square$

2.9. Positive roots and Dynkin diagrams. Recall that $\mathbf{U}$ is the unipotent radical of our fixed minimal parabolic subgroup $\mathbf{B}$. Let ${}_k\Phi^+(\mathbf{G})$ be the set of all $\lambda \in {}_k\Phi(\mathbf{G})$ such that $\mathbf{U}_{(\lambda)} \subset \mathbf{U}$. These are the *positive roots*, and for every $\lambda \in {}_k\Phi(\mathbf{G})$ there exists a unique sign $\epsilon = \pm 1$ such that $\epsilon\lambda \in {}_k\Phi^+(\mathbf{G})$. Letting ${}_k\Phi_{\mathrm{nd}}^+(\mathbf{G})$ be the set of non-divisible elements of ${}_k\Phi^+(\mathbf{G})$, the product map

$$(2.2) \quad \prod_{\alpha \in {}_k\Phi_{\mathrm{nd}}^+(\mathbf{G})} \mathbf{U}_{(\alpha)}(k) \rightarrow \mathbf{U}(k)$$

is a set-theoretic bijection. Inside ${}_k\Phi^+(\mathbf{G})$ is a unique set ${}_k\Delta(\mathbf{G})$ of simple roots, that is, elements of ${}_k\Phi^+(\mathbf{G})$ that cannot be written as the sum of two elements of ${}_k\Phi^+(\mathbf{G})$. The set

²⁰This always holds if k is algebraically closed.

²¹The additive group $\mathbb{G}_a$ is the algebraic group over k with $\mathbb{G}_a(A)$ equal to the additive group of A for any commutative k -algebra A .

Φ	Dynkin Diagram
A_n	
B_n	
C_n	
BC_n	
D_n	

Table 2. The Dynkin diagrams of irreducible non-exceptional root systems.

${}_k\Delta(\mathbf{G})$ is a linearly independent set of characters such that

$${}_k\Phi^+(\mathbf{G}) = \left\{ \lambda \in {}_k\Phi(\mathbf{G}) \mid \lambda = \sum_{\alpha \in {}_k\Delta(\mathbf{G})} c_\alpha \alpha \text{ with } c_\alpha \in \mathbb{Z} \text{ nonnegative} \right\}.$$

Recall from §2.7 that the relative Weyl group W is generated by reflections in the roots from ${}_k\Phi(\mathbf{G})$. In fact, W is generated by the reflections in the simple roots ${}_k\Delta(\mathbf{G})$.

The set ${}_k\Delta$ forms the vertices of the Dynkin diagram for ${}_k\Phi(\mathbf{G})$. A root system is irreducible (that is, not isomorphic to a product) if its Dynkin diagram is connected. The Dynkin diagrams of the irreducible non-exceptional²² root systems are listed in Table 2. A general root system is a product of irreducible root systems.

Example 2.13. For $\mathbf{G} = \mathrm{GL}_{n+1}$, the positive roots are

$${}_k\Phi^+(\mathbf{G}) = \{E_i - E_j \mid 1 \leq i < j \leq n+1\}$$

and the simple roots are

$${}_k\Delta(\mathbf{G}) = \{E_1 - E_2, E_2 - E_3, \dots, E_n - E_{n+1}\}.$$

The root system ${}_k\Phi(\mathbf{G})$ is of type A_n , and in the Dynkin diagram for A_n depicted in Table 2 the simple root $E_i - E_{i+1}$ is the i^{th} node from the left. $\square$

2.10. Other minimal parabolics containing torus. The above choice of positive roots was dictated by our choice of minimal parabolic subgroup $\mathbf{B}$ containing the maximal split torus $\mathbf{T}$. The group $\mathbf{T}$ is the unique maximal split torus in $\mathbf{B}$. Since $\mathbf{G}(k)$ acts transitively by conjugation on the set of minimal parabolic subgroups, the group $N_{\mathbf{G}}(\mathbf{T})(k)$ acts transitively on the set of minimal parabolic subgroups containing $\mathbf{T}$. Just like for all parabolic subgroups, we have $N_{\mathbf{G}}(\mathbf{B}) = \mathbf{B}$. It follows that the $N_{\mathbf{G}}(\mathbf{T})(k)$ -stabilizer of $\mathbf{B}$ is

$$\mathbf{B}(k) \cap N_{\mathbf{G}}(\mathbf{T})(k) = Z_{\mathbf{G}}(\mathbf{T})(k).$$

In other words, the group $W = N_{\mathbf{G}}(\mathbf{T})/Z_{\mathbf{G}}(\mathbf{T})$ acts simply transitively on the set of minimal parabolic subgroups containing $\mathbf{T}$. For $w \in W$, let²³ $\mathbf{B}_w$ be the image of $\mathbf{B}$ under the action of w and let $\mathbf{U}_w$ be the unipotent radical of $\mathbf{B}_w$. We thus have $\mathbf{B}_w = \mathbf{U}_w \rtimes Z_{\mathbf{G}}(\mathbf{T})$. Just like in (2.2), the product map

$$\prod_{\alpha \in {}_k\Phi_{\mathrm{nd}}^+(\mathbf{G})} \mathbf{U}_{(w(\alpha))}(k) \rightarrow \mathbf{U}_w(k)$$

²²The exceptional root systems are those of types $\{G_2, F_4, E_6, E_7, E_8\}$. We will not need their Dynkin diagrams in this paper.

²³With this notation, the apartment $\mathbb{A}_g$ corresponding to $g \in \mathbf{G}(k)$ (cf. §2.4) is $\mathbb{A}_g = \sum_{w \in W} (-1)^w g \cdot \mathbf{B}_w$.

is a set-theoretic bijection. If we used $\mathbf{B}_w$ as our base minimal parabolic instead of $\mathbf{B}$, then the positive roots would be $w({}_k\Phi^+(\mathbf{G}))$.

2.11. Opposite minimal parabolic. Let ${}_k\Phi^-(\mathbf{G}) = \{-\lambda \mid \lambda \in {}_k\Phi^+(\mathbf{G})\}$, so ${}_k\Phi(\mathbf{G}) = {}_k\Phi^+(\mathbf{G}) \sqcup {}_k\Phi^-(\mathbf{G})$. Let S be the generating set for W consisting of reflections in the simple roots ${}_k\Delta(\mathbf{G})$. For $w \in W$, the length of the minimal word in S representing w equals the number of $\lambda \in {}_k\Phi^+(\mathbf{G})$ with $w(\lambda) \in {}_k\Phi^-(\mathbf{G})$. The group W contains a unique element w_0 whose S -word length is maximal. This element satisfies $w_0({}_k\Phi^+(\mathbf{G})) = {}_k\Phi^-(\mathbf{G})$. Define

$$\mathbf{B}^- = \mathbf{B}_{w_0} \quad \text{and} \quad \mathbf{U}^- = \mathbf{U}_{w_0}.$$

The group $\mathbf{B}^-$ is the *opposite* minimal parabolic subgroup to $\mathbf{B}$. It satisfies $\mathbf{B} \cap \mathbf{B}^- = Z_{\mathbf{G}}(\mathbf{T})$. In particular, $\mathbf{B} \cap \mathbf{U}^- = 1$.

Example 2.14. For $\mathbf{G} = \mathrm{GL}_{n+1}$, the group $\mathbf{B}$ is the group of upper triangular matrices and $\mathbf{B}^-$ is the group of lower triangular matrices. $\square$

2.12. Intersecting minimal parabolics. Consider $\Lambda \subset {}_k\Phi(\mathbf{G})$. The set Λ is *closed* if $\lambda_1 + \lambda_2 \in \Lambda$ for all $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 + \lambda_2 \in {}_k\Phi(\mathbf{G})$. It is *one-sided* if $\Lambda \subset w({}_k\Phi^+(\mathbf{G}))$ for some $w \in W$, i.e., if Λ lies in the set of positive roots associated to some minimal parabolic containing $\mathbf{T}$.

Assume that Λ is closed and one-sided. Let $\mathbf{U}_{\Lambda}$ be the subgroup of $\mathbf{G}$ generated by the $\mathbf{U}_{(\lambda)}$ for $\lambda \in \Lambda$. Our assumptions imply that $\mathbf{U}_{\Lambda}$ is a unipotent subgroup of $\mathbf{G}$. Let Λ_{nd} be the set of all $\lambda \in \Lambda$ with $\frac{1}{2}\lambda \notin \Lambda$. Our assumptions also imply that the product map

$$\prod_{\lambda \in \Lambda_{\mathrm{nd}}} \mathbf{U}_{(\lambda)}(k) \rightarrow \mathbf{U}_{\Lambda}(k)$$

is a set-theoretic bijection. We then have the following:

Lemma 2.15. *Let $\mathbf{G}$ be a reductive group. Fix a minimal parabolic subgroup $\mathbf{B}$ with unipotent radical $\mathbf{U}$. Let $\Lambda \subset {}_k\Phi(\mathbf{G})$ be closed and one-sided. Set $\Lambda' = \Lambda \cap {}_k\Phi^+(\mathbf{G})$. We then have $\mathbf{B}(k) \cap \mathbf{U}_{\Lambda}(k) = \mathbf{U}_{\Lambda'}(k)$.*

Proof. We clearly have $\mathbf{U}_{\Lambda'}(k) \subset \mathbf{B}(k) \cap \mathbf{U}_{\Lambda}(k)$, so we must prove the other inclusion. Consider $u \in \mathbf{B}(k) \cap \mathbf{U}_{\Lambda}(k)$. Let $\Lambda'' = \Lambda \setminus \Lambda'$. The product map

$$\mathbf{U}_{\Lambda''}(k) \times \mathbf{U}_{\Lambda'}(k) \rightarrow \mathbf{U}_{\Lambda}(k)$$

is a set-theoretic bijection, so we can write $u = u''u'$ with $u'' \in \mathbf{U}_{\Lambda''}(k)$ and $u' \in \mathbf{U}_{\Lambda'}(k)$. To prove that $u \in \mathbf{U}_{\Lambda'}(k)$, it is enough to prove that $u'' = 1$. Since $\mathbf{U}_{\Lambda'}(k) \subset \mathbf{U}(k) \subset \mathbf{B}(k)$, we have $u' \in \mathbf{B}(k)$. Since u is assumed to be in $\mathbf{B}(k)$, it follows that $u'' \in \mathbf{B}(k)$ and thus that $u'' \in \mathbf{B}(k) \cap \mathbf{U}_{\Lambda''}(k)$. Since $\Lambda'' \subset {}_k\Phi^-(\mathbf{G})$ we have $\mathbf{U}_{\Lambda''}(k) \subset \mathbf{U}^-(k)$. Since $\mathbf{U}^-(k) \cap \mathbf{B}(k) = 1$, this implies that $\mathbf{U}_{\Lambda''}(k) \cap \mathbf{B}(k) = 1$ and thus that $u'' = 1$, as desired. $\square$

The following corollary will play a key role in one of the base cases of our proof:

Corollary 2.16. *Let $\mathbf{G}$ be a reductive group. Fix a minimal parabolic subgroup $\mathbf{B}$ with maximal split torus $\mathbf{T}$ and unipotent radical $\mathbf{U}$. Letting $W = N_{\mathbf{G}}(\mathbf{T})/Z_{\mathbf{G}}(\mathbf{T})$ be the relative Weyl group, for each $w \in W$ fix some representative $\tilde{w} \in N_{\mathbf{G}}(\mathbf{T})(k)$. There then exists some $u \in \mathbf{U}(k)$ such that the only $w_1, w_2 \in W$ with $\tilde{w}_1 u \tilde{w}_2 \in \mathbf{B}(k)$ are $w_1 = w_2 = 1$.*

Example 2.17. For $\mathbf{G} = \mathrm{GL}_{n+1}$, the group $\mathbf{U}$ is the subgroup of strictly upper triangular matrices. Pick $u \in \mathbf{U}(k)$ such that all of its upper triangular entries are nonzero. For $w \in W \cong S_{n+1}$, let $\tilde{w}$ be the associated permutation matrix. For $w_1, w_2 \in W$, the matrix $\tilde{w}_1 u \tilde{w}_2$ is obtained from u by permuting the rows via w_1 and the columns via w_2 . It is an easy exercise to see that this will have a nonzero entry below the diagonal unless $w_1 = w_2 = 1$. $\square$

Proof of Corollary 2.16. Enumerate ${}_k\Phi_{\text{nd}}^+$ as ${}_k\Phi_{\text{nd}}^+ = \{\alpha_1, \dots, \alpha_r\}$. For $1 \leq j \leq r$, pick $u_j \in \mathbf{U}_{(\alpha_j)}(k)$ with $u_j \neq 1$. Set $u = u_1 \cdots u_r \in \mathbf{U}(k)$, and consider $w_1, w_2 \in W$ with $\tilde{w}_1 u \tilde{w}_2 \in \mathbf{B}(k)$. We must prove that $w_1 = w_2 = 1$. Set $v = \tilde{w}_1 u \tilde{w}_2 \in \mathbf{B}(k)$. We have $\tilde{w}_1^{-1} = u \tilde{w}_2 v^{-1}$, so $\tilde{w}_2$ and $\tilde{w}_1^{-1}$ lie in the same $\mathbf{B}(k)$ -double coset. By the Bruhat decomposition, we have $\tilde{w}_2 = \tilde{w}_1^{-1}$. Set $w = w_1$, so $v = \tilde{w} u \tilde{w}^{-1}$.

Let $\Lambda = w({}_k\Phi^+(\mathbf{G}))$ and $\Lambda' = \Lambda \cap {}_k\Phi^+(\mathbf{G})$. Lemma 2.15 implies that

$$\mathbf{B}(k) \cap \mathbf{U}_w(k) = \mathbf{B}(k) \cap \mathbf{U}_{\Lambda}(k) = \mathbf{U}_{\Lambda'}(k).$$

For $1 \leq j \leq r$, we have

$$\tilde{w} u_j \tilde{w}^{-1} \in \mathbf{U}_{(w(\alpha_j))}(k) \subset \mathbf{U}_w(k).$$

We therefore have

$$(2.3) \quad \tilde{w} u \tilde{w}^{-1} = (\tilde{w} u_1 \tilde{w}^{-1}) \cdots (\tilde{w} u_r \tilde{w}^{-1}) \in \mathbf{B}(k) \cap \mathbf{U}_w(k) = \mathbf{U}_{\Lambda'}.$$

By construction, we have $\Lambda'_{\text{nd}} \subset {}_k\Phi_{\text{nd}}^+$. Reordering the α_j if necessary, there thus exists some $1 \leq s \leq r$ with $\Lambda'_{\text{nd}} = \{\alpha_1, \dots, \alpha_s\}$. The product maps

$$\begin{aligned} \mathbf{U}_{(w(\alpha_1))}(k) \times \cdots \times \mathbf{U}_{(w(\alpha_s))}(k) \times \mathbf{U}_{(w(\alpha_{s+1}))}(k) \times \cdots \times \mathbf{U}_{(w(\alpha_r))}(k) &\longrightarrow \mathbf{U}_w(k) \\ \mathbf{U}_{(\alpha_1)}(k) \times \cdots \times \mathbf{U}_{(\alpha_s)}(k) &\longrightarrow \mathbf{U}_{\Lambda'}(k) \end{aligned}$$

are set-theoretic bijections, and for each $1 \leq j \leq s$ there exists a unique $1 \leq \ell_j \leq r$ with $\alpha_j = w(\alpha_{\ell_j})$. Since each u_j is nontrivial and (2.3) holds, we conclude that in fact we have must have $r = s$, and thus $w({}_k\Phi^+(\mathbf{G})) = {}_k\Phi^+(\mathbf{G})$. In other words, the element w of the relative Weyl group W takes each positive root to another positive root. As we discussed in §2.11, this implies that $w = 1$, as desired. $\square$

2.13. Standard parabolic and Levi subgroups. Recall that ${}_k\Delta(\mathbf{G}) \subset {}_k\Phi^+(\mathbf{G})$ is the set of simple roots. Consider some $\Delta \subset {}_k\Delta(\mathbf{G})$. Let $[\Delta] \subset {}_k\Phi^+(\mathbf{G})$ be the closed subset generated by Δ and let $\Psi(\Delta) = {}_k\Phi^+(\mathbf{G}) \setminus [\Delta]$. Both $[\Delta]$ and $\Psi(\Delta)$ are closed and one-sided. Each $\alpha \in \Delta$ is a character of $\mathbf{T}$, i.e., a homomorphism $\alpha: \mathbf{T} \rightarrow \text{GL}_1$. Define²⁴

$$\mathbf{T}_{\Delta} = \left(\bigcap_{\alpha \in \Delta} \ker(\alpha) \right)^0 \subset \mathbf{T}.$$

Let $\mathbf{L}_{\Delta}$ be the centralizer of $\mathbf{T}_{\Delta}$ in $\mathbf{G}$ and let $\mathbf{P}_{\Delta}$ be the subgroup of $\mathbf{G}$ generated by $\mathbf{L}_{\Delta}$ and $\mathbf{U}_{\Psi(\Delta)}$. The group $\mathbf{P}_{\Delta}$ is a parabolic subgroup of $\mathbf{G}$ with Levi factor $\mathbf{L}_{\Delta}$ and unipotent radical $\mathbf{U}_{\Psi(\Delta)}$, so

$$\mathbf{P}_{\Delta} = \mathbf{U}_{\Psi(\Delta)} \rtimes \mathbf{L}_{\Delta}.$$

The parabolic subgroups of the form $\mathbf{P}_{\Delta}$ for some $\Delta \subset {}_k\Delta(\mathbf{G})$ are the *standard parabolic subgroups* and their Levi factors $\mathbf{L}_{\Delta}$ are the *standard Levi subgroups*. Every parabolic subgroup is conjugate to a unique standard parabolic subgroup. The standard Levi subgroup $\mathbf{L}_{\Delta}$ is a reductive group of semisimple rank $|\Delta|$ whose relative Dynkin diagram is the full subgraph of the relative Dynkin diagram of $\mathbf{G}$ with vertex set Δ .

Example 2.18. Consider $\mathbf{G} = \text{GL}_8$, and let $\Delta = \{E_1 - E_2, E_3 - E_4, E_4 - E_5, E_5 - E_6\} \subset {}_k\Delta(\mathbf{G})$. This corresponds to the following subgraph of the Dynkin diagram for $\Phi(\mathbf{G}) \cong A_7$:



²⁴Here the 0 indicates that we are taking the connected component of the identity.

We should have $\Phi(\mathbf{L}_\Delta) \cong A_1 \times A_3$, and indeed keeping in mind that $\Phi(\mathrm{GL}_1) = \emptyset$ we have

$$\mathbf{T}_\Delta(k) = \{\mathrm{diag}(a, a, b, b, b, b, c, d) \mid a, b, c, d \in k^\times\} \subset \mathbf{T}(k),$$

$$\mathbf{L}_\Delta(k) = Z_{\mathrm{GL}_8(k)}(\mathbf{T}_\Delta(k)) = \mathrm{GL}_2(k) \times \mathrm{GL}_4(k) \times \mathrm{GL}_1(k) \times \mathrm{GL}_1(k) \subset \mathrm{GL}_8(k).$$

More generally, if $\mathbf{G} = \mathrm{GL}_{n+1}$ and $\Delta \subset {}_k\Delta(\mathbf{G})$ is such that ${}_k\Delta(\mathbf{G}) \setminus \Delta$ contains r roots, then

$$\mathbf{L}_\Delta = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_{r+1}} \subset \mathrm{GL}_{n+1} \quad \text{with } n_1 + \cdots + n_{r+1} = n + 1. \quad \square$$

2.14. Conjugacy of Levi subgroups. The following observation will play an important role in our proof:

Lemma 2.19. *Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = A_n$. Let $\Delta, \Delta' \subset {}_k\Delta(\mathbf{G})$ be such that ${}_k\Phi(\mathbf{L}_\Delta) = {}_k\Phi(\mathbf{L}_{\Delta'}) = A_m$ for some $m \geq 1$. Then $\mathbf{L}_\Delta(k)$ and $\mathbf{L}_{\Delta'}(k)$ are conjugate²⁵ subgroups of $\mathbf{G}(k)$.*

Example 2.20. Let $\mathbf{G} = \mathrm{GL}_{n+1}$, so $\Phi(\mathbf{G}) \cong A_n$. Let $\Delta, \Delta' \subset {}_k\Delta(\mathbf{G})$ be as in Lemma 2.19. Since $\Phi(\mathbf{L}_\Delta) \cong A_m$, the set Δ consists of m consecutive vertices in the Dynkin diagram A_n for $\mathbf{G} = \mathrm{GL}_{n+1}$, for instance:



Keeping in mind that $\Phi(\mathrm{GL}_{m+1}) \cong A_m$, just like in Example 2.18 we can write

$$\mathbf{L}_\Delta = (\mathrm{GL}_1)^a \times \mathrm{GL}_{m+1} \times (\mathrm{GL}_1)^b \subset \mathrm{GL}_{n+1} \quad \text{with } a + (m + 1) + b = n + 1.$$

Similarly,

$$\mathbf{L}_{\Delta'} = (\mathrm{GL}_1)^{a'} \times \mathrm{GL}_{m+1} \times (\mathrm{GL}_1)^{b'} \subset \mathrm{GL}_{n+1} \quad \text{with } a' + (m + 1) + b' = n + 1.$$

Let $w \in \mathbf{G}(k)$ be an $(n + 1) \times (n + 1)$ permutation matrix representing a permutation taking the subset $\{a + 1, \dots, a + m + 1\}$ of $\{1, \dots, n\}$ to $\{a' + 1, \dots, a' + m + 1\}$. We then have $w\mathbf{L}_\Delta(k)w^{-1} = \mathbf{L}_{\Delta'}(k)$, as desired. $\square$

Proof of Lemma 2.19. The full subgraphs of the Dynkin diagram of ${}_k\Phi(\mathbf{G})$ with vertex sets Δ and Δ' are isomorphic the Dynkin diagram of A_m . Our assumptions imply that there exists some w in the relative Weyl group $W = N_{\mathbf{G}}(\mathbf{T})/Z_{\mathbf{G}}(\mathbf{T}) \cong \mathfrak{S}_n$ with $w \cdot \Delta = \Delta'$. The sets of subgroups

$$\{\ker(w \cdot \alpha) \mid \alpha \in \Delta\} \quad \text{and} \quad \{\ker(\alpha) \mid \alpha \in \Delta'\}$$

of $\mathbf{T}$ are the same. Letting $\tilde{w} \in N_{\mathbf{G}}(\mathbf{T})(k)$ be a representative of w , it follows that $\tilde{w}\mathbf{L}_{\Delta_1}(k)\tilde{w}^{-1}$ equals

$$\tilde{w}Z_{\mathbf{G}(k)}\left(\bigcap_{\alpha \in \Delta} \ker(\alpha)(k)\right)^0 \tilde{w}^{-1} = Z_{\mathbf{G}(k)}\left(\bigcap_{\alpha \in \Delta} \ker(w \cdot \alpha)(k)\right)^0 = Z_{\mathbf{G}(k)}\left(\bigcap_{\alpha \in \Delta'} \ker(\alpha)(k)\right)^0.$$

This is $\mathbf{L}_{\Delta_2}(k)$, as desired. $\square$

3. TWO INITIAL REDUCTIONS

We now begin developing the tools needed to prove Theorem B, which says that for a reductive group $\mathbf{G}$ we have $H_i(\mathbf{G}(k); \mathrm{St}(\mathbf{G}; \mathbb{F})) = 0$ for i in a range depending on the relative root system ${}_k\Phi(\mathbf{G})$.

²⁵Note that our assumptions do not imply that $\mathbf{P}_\Delta(k)$ and $\mathbf{P}_{\Delta'}(k)$ are conjugate subgroups of $\mathbf{G}(k)$.

3.1. Reduction to the integers. We first show that it is enough to prove this for $\mathbb{F} = \mathbb{Z}$. For a reductive group $\mathbf{G}$, we will write $\text{St}(\mathbf{G})$ for $\text{St}(\mathbf{G}; \mathbb{Z})$.

Lemma 3.1. *Let $\mathbf{G}$ be a reductive group. Let $b \geq 0$ be such that $H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ for $i \leq b$. Then for all commutative rings $\mathbb{F}$ we have $H_i(\mathbf{G}(k); \text{St}(\mathbf{G}; \mathbb{F})) = 0$ for $i \leq b$.*

Proof. Letting $F_\bullet \rightarrow \mathbb{Z}$ be a free resolution of $\mathbf{G}(k)$ -modules, the homology groups of $F_\bullet \otimes_{\mathbf{G}(k)} \text{St}(\mathbf{G})$ are $H_i(\mathbf{G}(k); \text{St}(\mathbf{G}))$. Since $\text{St}(\mathbf{G})$ is a free abelian group equipped with an action of $\mathbf{G}(k)$, each $F_i \otimes_{\mathbf{G}(k)} \text{St}(\mathbf{G})$ is also a free abelian group. We can therefore apply the universal coefficients theorem to

$$F_\bullet \otimes_{\mathbf{G}(k)} \text{St}(\mathbf{G}) \otimes_{\mathbb{Z}} \mathbb{F} = F_\bullet \otimes_{\mathbf{G}(k)} \text{St}(\mathbf{G}; \mathbb{F}),$$

which computes $H_i(\mathbf{G}(k); \text{St}(\mathbf{G}; \mathbb{F}))$. The result is a short exact sequence

$$0 \rightarrow H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) \otimes \mathbb{F} \rightarrow H_i(\mathbf{G}(k); \text{St}(\mathbf{G}; \mathbb{F})) \rightarrow \text{Tor}(H_{i-1}(\mathbf{G}(k); \text{St}(\mathbf{G})), \mathbb{F}) \rightarrow 0.$$

The lemma follows. $\square$

We can therefore omit the $\mathbb{F}$ from our calculations, though at one point when handling reducible root systems we will need more general coefficients.

3.2. Zeroth homology. We next prove our vanishing theorem for H_0 :

Lemma 3.2. *Let $\mathbf{G}$ be a reductive group of semisimple rank $n \geq 1$. Then $H_0(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$.*

Proof. The homology group $H_0(\mathbf{G}(k); \text{St}(\mathbf{G}))$ equals the $\mathbf{G}(k)$ -coinvariants $\text{St}(\mathbf{G})_{\mathbf{G}(k)}$, so we must show that these coinvariants vanish. We divide this into two cases.

Case 1. *The semisimple rank n of $\mathbf{G}$ is 1, i.e., ${}_k\Phi(\mathbf{G}) \cong A_1$.*

The Tits building $\mathcal{T}(\mathbf{G})$ is $(n-1)$ -dimensional, and the Steinberg representation $\text{St}(\mathbf{G})$ equals $\tilde{H}_{n-1}(\mathcal{T}(\mathbf{G}))$. Since $n = 1$, we deduce that $\mathcal{T}(\mathbf{G})$ is 0-dimensional, i.e., a discrete set.²⁶ The points of $\mathcal{T}(\mathbf{G})$ are the proper parabolic subgroups of $\mathbf{G}$. The group $\mathbf{G}(k)$ acts double-transitively²⁷ on the vertices of $\mathcal{T}(\mathbf{G})$.

For $\theta \in \text{St}(\mathbf{G})$, let $[\theta]$ be its image in the $\mathbf{G}(k)$ -coinvariants. We can identify the abelian group $\text{St}(\mathbf{G}) = \tilde{H}_0(\mathcal{T}(\mathbf{G}))$ with the collection of formal sums

$$c_1 P_1 + \cdots + c_m P_m \quad \text{with } P_1, \dots, P_m \in \mathcal{T}(\mathbf{G})^{(0)} \text{ and } c_1, \dots, c_m \in \mathbb{Z} \text{ such that } \sum c_i = 0.$$

This is generated by elements of the form $P - P'$ with $P, P' \in \mathcal{T}(\mathbf{G})^{(0)}$ distinct, so it is enough to show that $[P - P'] = 0$. Letting $Q \in \mathcal{T}(\mathbf{G})^{(0)}$ be a third vertex,²⁸ we have

$$(3.1) \quad [P - P'] = [P - Q] + [Q - P'].$$

By the double transitivity of the action of $\mathbf{G}(k)$ on the vertices, we can find $g_1, g_2 \in \mathbf{G}(k)$ such that

$$g_1 \cdot P - g_1 \cdot P' = P - Q \quad \text{and} \quad g_2 \cdot P - g_2 \cdot P' = Q - P'.$$

It follows that in the $\mathbf{G}(k)$ -coinvariants we have

$$(3.2) \quad [P - P'] = [P - Q] = [Q - P'].$$

²⁶In the case $\mathbf{G} = \text{GL}_2$, the Tits building $\mathcal{T}(\mathbf{G})$ is the discrete set of lines in k^2 .

²⁷The vertices of $\mathcal{T}(\mathbf{G})$ are its chambers and the apartments are Coxeter complexes of type A_1 , i.e., 0-spheres. This double-transitivity therefore follows from the following standard properties of the building associated to a reductive group: any two chambers lie in an apartment, the group $\mathbf{G}(k)$ acts transitively on apartments, and the $\mathbf{G}(k)$ -stabilizer of an apartment acts transitively on its chambers.

²⁸Whether or not this exists for all buildings of type A_1 depends on your conventions, but this always exists for such buildings if they come from reductive groups. In this case the building is what is called a “thick building”; see, e.g., [8, Remark IV.1.1].

Combining (3.1) and (3.2), we conclude that $[P - P'] = 2[P - P']$, so $[P - P'] = 0$, as desired.

Case 2. *The semisimple rank of $\mathbf{G}$ is greater than 1.*

Let $\Delta \subset {}_k\Delta(\mathbf{G})$ be a set consisting of a single simple root. The standard Levi subgroup $\mathbf{L}_\Delta$ of the standard parabolic subgroup $\mathbf{P}_\Delta$ thus has semisimple rank $|\Delta| = 1$, i.e., ${}_k\Phi(\mathbf{L}_\Delta) \cong A_1$. Theorem 2.7 says that as a $\mathbf{P}_\Delta(k)$ -representation, we have

$$\mathrm{St}(\mathbf{G}) = \mathrm{Ind}_{\mathbf{L}_\Delta(k)}^{\mathbf{P}_\Delta(k)} \mathrm{St}(\mathbf{L}_\Delta).$$

It follows that

$$\mathrm{St}(\mathbf{G})_{\mathbf{P}_\Delta(k)} = \mathrm{St}(\mathbf{L}_\Delta)_{\mathbf{L}_\Delta},$$

which vanishes by Case 1. This implies that $\mathrm{St}(\mathbf{G})_{\mathbf{G}(k)} = 0$, as desired. $\square$

4. RESOLUTION OF STEINBERG AND ITS ASSOCIATED SPECTRAL SEQUENCE

Let $\mathbf{G}$ be a reductive group of semisimple rank n . In this section, we explain how to relate $H_i(\mathbf{G}(k); \mathrm{St}(\mathbf{G}))$ to the homology groups of Levi factors of parabolic subgroups of $\mathbf{G}$ with coefficients in their Steinberg representations. This relationship is expressed as a spectral sequence that will form the foundation for our proof of Theorem B.

4.1. Resolution. As in §2.13, for each $\Delta \subset {}_k\Delta(\mathbf{G})$ there is a standard Levi subgroup $\mathbf{L}_\Delta$. For $i \geq 0$, let $\mathcal{L}_i(\mathbf{G})$ be the set of all subsets $\Delta \subset {}_k\Delta(\mathbf{G})$ with $|\Delta| = n - 1 - i$. For these Δ , the semisimple rank $n - 1 - i$ of $\mathbf{L}_\Delta$ is smaller than that of $\mathbf{G}$, so in some sense $\mathbf{L}_\Delta$ is a simpler group than $\mathbf{G}$. The following (non-projective) resolution of $\mathrm{St}(\mathbf{G})$ relates it to the Steinberg representations of the $\mathbf{L}_\Delta$:

Proposition 4.1. *Let $\mathbf{G}$ be a reductive group of semisimple rank n . We then have an exact sequence*

$$0 \rightarrow \mathbf{R}_n \rightarrow \mathbf{R}_{n-1} \rightarrow \cdots \rightarrow \mathbf{R}_0 \rightarrow \mathrm{St}(\mathbf{G}) \rightarrow 0$$

of $\mathbb{Z}[\mathbf{G}(k)]$ -modules with

$$\mathbf{R}_i \cong \bigoplus_{\Delta \in \mathcal{L}_i(\mathbf{G})} \mathrm{Ind}_{\mathbf{L}_\Delta(k)}^{\mathbf{G}(k)} \mathrm{St}(\mathbf{L}_\Delta) \quad \text{for } 0 \leq i \leq n-1$$

and with $\mathbf{R}_n \cong \mathrm{St}(\mathbf{G})^{\otimes 2}$.

Example 4.2. The resolution of $\mathrm{St}(\mathrm{GL}_{n+1})$ from Proposition 4.1 has the following concrete form. For a finite-dimensional k -vector space V , let $\mathcal{T}(V)$ be the complex of flags of nonzero proper subspaces of V , so $\mathcal{T}(k^{n+1})$ is the Tits building for GL_{n+1} . For $d = \dim(V)$, let $\mathrm{St}(V) = \widetilde{H}_{d-2}(\mathcal{T}(V))$, so $\mathrm{St}(\mathrm{GL}_{n+1}) = \mathrm{St}(k^{n+1})$. For $\mathbf{G} = \mathrm{GL}_{n+1}$, as we discussed in Example 2.18 the groups $\mathbf{L}_\Delta$ associated to $\Delta \in \mathcal{L}_i(\mathbf{G})$ are those of the form

$$\mathbf{L}_\Delta = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_{i+2}} \quad \text{with } n_1, \dots, n_{i+2} \geq 1 \text{ and } n_1 + \cdots + n_{i+2} = n + 1.$$

For reductive groups $\mathbf{H}_1$ and $\mathbf{H}_2$, we have $\mathrm{St}(\mathbf{H}_1 \times \mathbf{H}_2) \cong \mathrm{St}(\mathbf{H}_1) \otimes \mathrm{St}(\mathbf{H}_2)$ (cf. Lemma 6.3 below). It follows that

$$\mathbf{R}_i \cong \bigoplus_{n_1 + \cdots + n_{i+2} = n+1} \mathrm{Ind}_{\mathrm{GL}_{n_1}(k) \times \cdots \times \mathrm{GL}_{n_{i+2}}(k)}^{\mathrm{GL}_n(k)} \mathrm{St}(k^{n_1}) \otimes \cdots \otimes \mathrm{St}(k^{n_{i+2}}).$$

Here and throughout this discussion the sum is over expressions $n_1 + \cdots + n_{i+2} = n + 1$ with $n_1, \dots, n_{i+2} \geq 1$. Fixing such an expression, the group $\mathrm{GL}_{n+1}(k)$ acts transitively on decompositions $k^{n+1} = V_1 \oplus \cdots \oplus V_{i+2}$ with $\dim(V_j) = n_j$ for $1 \leq j \leq i+2$. Moreover, the

stabilizer of the standard decomposition $k^{n+1} = k^{n_1} \oplus \cdots \oplus k^{n_{i+2}}$ is $\mathrm{GL}_{n_1}(k) \times \cdots \times \mathrm{GL}_{n_{i+2}}(k)$. It follows that

$$\begin{aligned} \mathbf{R}_i &\cong \bigoplus_{n_1 + \cdots + n_{i+2} = n+1} \mathrm{Ind}_{\mathrm{GL}_{n_1}(k) \times \cdots \times \mathrm{GL}_{n_{i+2}}(k)}^{\mathrm{GL}_n(k)} \mathrm{St}(k^{n_1}) \otimes \cdots \otimes \mathrm{St}(k^{n_{i+2}}) \\ &\cong \bigoplus_{n_1 + \cdots + n_{i+2} = n+1} \left(\bigoplus_{\substack{k^{n+1} = V_1 \oplus \cdots \oplus V_{i+2} \\ \dim(V_j) = n_j}} \mathrm{St}(V_1) \otimes \cdots \otimes \mathrm{St}(V_{i+2}) \right) \\ &\cong \bigoplus_{V_1 \oplus \cdots \oplus V_{i+2} = k^{n+1}} \mathrm{St}(V_1) \otimes \cdots \otimes \mathrm{St}(V_{i+2}). \end{aligned}$$

Here the final direct sum is over such expressions with $\dim(V_j) \geq 1$ for all $1 \leq j \leq i+2$. It follows that aside from the final term $\mathbf{R}_n = \mathrm{St}(k^{n+1})^{\otimes 2}$, our resolution for $\mathrm{St}(\mathrm{GL}_{n+1})$ is a version of the bar resolution for the Steinberg representation introduced in [19].

For the special case $\mathbf{G} = \mathrm{GL}_{n+1}$ we are considering, [19] proves the exactness of the truncated sequence

$$\mathbf{R}_{n-1} \rightarrow \cdots \rightarrow \mathbf{R}_0 \rightarrow \mathrm{St}(k^{n+1}) \rightarrow 0$$

from Proposition 4.1. The reference [19] has a technical error that was corrected in [20]. Another proof of Proposition 4.1 for $\mathbf{G} = \mathrm{GL}_{n+1}$ that also includes the initial $\mathrm{St}(k^{n+1})^{\otimes 2}$ term can be found in [22, Theorem 6.3]. Our proof of Proposition 4.1 is different from and more direct than either of these approaches. In the special case $\mathbf{G} = \mathrm{GL}_{n+1}$, our proof was discovered independently by Charlton–Radchenko–Rudenko [13]. $\square$

Remark 4.3. For other $\mathbf{G}$, it is hard to interpret the resolution from Proposition 4.1 as a bar resolution. The issue is that the standard Levi subgroups are often different from what you expect. For instance, consider $\mathbf{G} = \mathrm{Sp}_{2g}$. For a finite-dimensional k -vector space W equipped with a symplectic form, let $\mathcal{T}_{\mathrm{Sp}}(W)$ be the complex of flags of nonzero proper isotropic subspaces of W , so $\mathcal{T}_{\mathrm{Sp}}(k^{2g})$ is the Tits building for Sp_{2g} . For $d = \dim(W)$, let $\mathrm{St}_{\mathrm{Sp}}(W) = \tilde{H}_{d/2-1}(\mathcal{T}_{\mathrm{Sp}}(W))$, so $\mathrm{St}_{\mathrm{Sp}}(k^{2g}) = \mathrm{St}(\mathrm{Sp}_{2g})$.

An analysis similar to the one we did for GL_{n+1} in Example 4.2 shows that the resolution $\mathbf{R}_\bullet$ of $\mathrm{St}(\mathrm{Sp}_{2g})$ from Proposition 4.1 has the following description. For $0 \leq i \leq g-1$, the term $\mathbf{R}_i$ is the direct sum of terms of the following two forms:

- $\mathrm{St}(V_1) \otimes \cdots \otimes \mathrm{St}(V_{i+1})$, where $U = V_1 \oplus \cdots \oplus V_{i+1}$ is a Lagrangian, i.e., a maximal isotropic subspace of k^{2g} . Necessarily $\dim(U) = g$.
- $\mathrm{St}(V_1) \otimes \cdots \otimes \mathrm{St}(V_{i+1}) \otimes \mathrm{St}_{\mathrm{Sp}}(W)$, where:
 - W is a nonzero proper symplectic subspace of k^{2g} ; and
 - $U = V_1 \oplus \cdots \oplus V_{i+1}$ is a Lagrangian in $W^\perp$.

Here we emphasize that the $\mathrm{St}(V_j)$ terms are the Steinberg representations of $\mathrm{GL}(V_j)$, not of symplectic groups. $\square$

Proof of Proposition 4.1. Since $\mathcal{T}(\mathbf{G})$ is an $(n-1)$ -dimensional $(n-2)$ -connected complex and $\mathrm{St}(\mathbf{G}) = \tilde{H}_{n-2}(\mathcal{T}(\mathbf{G}))$, we have an exact sequence

$$(4.1) \quad 0 \rightarrow \mathrm{St}(\mathbf{G}) \rightarrow C_{n-1}(\mathcal{T}(\mathbf{G})) \rightarrow \cdots \rightarrow C_0(\mathcal{T}(\mathbf{G})) \rightarrow \mathbb{Z} \rightarrow 0.$$

Here the final map $C_0(\mathcal{T}(\mathbf{G})) \rightarrow \mathbb{Z}$ is the augmentation. The term $C_i(\mathcal{T}(\mathbf{G}))$ is a free $\mathbb{Z}$ -module with basis the parabolic subgroups of $\mathbf{G}$ of semisimple rank $n-1-i$, which are permuted by the conjugation action of $\mathbf{G}(k)$. Every such parabolic subgroup of $\mathbf{G}$ is conjugate to a unique standard parabolic subgroup of the form $\mathbf{P}_\Delta$ for some $\Delta \in \mathcal{L}_i(\mathbf{G})$.

Moreover, a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ satisfies $N_{\mathbf{G}}(\mathbf{P}) = \mathbf{P}$. It follows that

$$C_i(\mathcal{T}(\mathbf{G})) \cong \bigoplus_{\Delta \in \mathcal{L}_i(\mathbf{G})} \mathbb{Z}[\mathbf{G}(k)/\mathbf{P}_{\Delta}(k)] \quad \text{for } 0 \leq i \leq n-1.$$

Now tensor (4.1) with $\text{St}(\mathbf{G})$:

$$0 \rightarrow \text{St}(\mathbf{G})^{\otimes 2} \rightarrow C_{n-1}(\mathcal{T}(\mathbf{G})) \otimes \text{St}(\mathbf{G}) \rightarrow \cdots \rightarrow C_0(\mathcal{T}(\mathbf{G})) \otimes \text{St}(\mathbf{G}) \rightarrow \text{St}(\mathbf{G}) \rightarrow 0.$$

Since $\text{St}(\mathbf{G})$ is a free $\mathbb{Z}$ -module, this chain complex is also exact. By the above, we have

$$C_i(\mathcal{T}(\mathbf{G})) \otimes \text{St}(\mathbf{G}) \cong \bigoplus_{\Delta \in \mathcal{L}_i(\mathbf{G})} \mathbb{Z}[\mathbf{G}(k)/\mathbf{P}_{\Delta}(k)] \otimes \text{St}(\mathbf{G}) \quad \text{for } 0 \leq i \leq n-1.$$

For $\Delta \in \mathcal{L}_i(\mathbf{G})$, we have

$$\mathbb{Z}[\mathbf{G}(k)/\mathbf{P}_{\Delta}(k)] \otimes \text{St}(\mathbf{G}) \cong \text{Ind}_{\mathbf{P}_{\Delta}(k)}^{\mathbf{G}(k)} \text{Res}_{\mathbf{P}_{\Delta}(k)}^{\mathbf{G}(k)} \text{St}(\mathbf{G}).$$

By Theorem 2.7,

$$\text{Res}_{\mathbf{P}_{\Delta}(k)}^{\mathbf{G}(k)} \text{St}(\mathbf{G}) \cong \text{Ind}_{\mathbf{L}_{\Delta}(k)}^{\mathbf{P}_{\Delta}(k)} \text{St}(\mathbf{L}_{\Delta}).$$

Combining all of this, we see that the above exact sequence has all the properties claimed in the proposition. $\square$

4.2. Spectral sequence. The following spectral sequence will play a key role in our proof:

Corollary 4.4. *Let $\mathbf{G}$ be a reductive group of semisimple rank n . There is then a spectral sequence E_{pq}^r converging to $H_{p+q}(\mathbf{G}(k); \text{St}(\mathbf{G}))$ with*

$$E_{pq}^1 \cong \begin{cases} \bigoplus_{\Delta \in \mathcal{L}_p(\mathbf{G})} H_q(\mathbf{L}_{\Delta}(k); \text{St}(\mathbf{L}_{\Delta})) & \text{if } 0 \leq p \leq n-1 \\ H_q(\mathbf{G}(k); \text{St}(\mathbf{G})^{\otimes 2}) & \text{if } p = n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\mathbf{R}_{\bullet} \rightarrow \text{St}(\mathbf{G})$ be the resolution of $\mathbb{Z}[\mathbf{G}(k)]$ -modules given by Proposition 4.1. As is standard,²⁹ there is a spectral sequence of the form

$$E_{pq}^1 \cong H_q(\mathbf{G}(k); \mathbf{R}_p) \Rightarrow H_{p+q}(\mathbf{G}(k); \text{St}(\mathbf{G})).$$

We can apply the description of $\mathbf{R}_p$ from Proposition 4.1 to see that

$$E_{pq}^1 \cong \begin{cases} \bigoplus_{\Delta \in \mathcal{L}_p(\mathbf{G})} H_q(\mathbf{G}(k); \text{Ind}_{\mathbf{L}_{\Delta}(k)}^{\mathbf{G}(k)} \text{St}(\mathbf{L}_{\Delta})) & \text{if } 0 \leq p \leq n-1 \\ H_q(\mathbf{G}(k); \text{St}(\mathbf{G})^{\otimes 2}) & \text{if } p = n, \\ 0 & \text{otherwise.} \end{cases}$$

For $\Delta \in \mathcal{L}_p(\mathbf{G})$, Shapiro's lemma implies that

$$H_q(\mathbf{G}(k); \text{Ind}_{\mathbf{L}_{\Delta}(k)}^{\mathbf{G}(k)} \text{St}(\mathbf{L}_{\Delta})) \cong H_q(\mathbf{L}_{\Delta}(k); \text{St}(\mathbf{L}_{\Delta})).$$

The corollary follows. $\square$

²⁹Note that if the $\mathbf{R}_q$ were all projective, this would reduce to the usual recipe for computing $H_{\bullet}(\mathbf{G}(k); \text{St}(\mathbf{G}))$ from a projective resolution.

5. SEMISIMPLE RANK TWO

As a first application of the spectral sequence from Corollary 4.4, we prove the following lemma. Later we will only use it when ${}_k\Phi(\mathbf{G}) = A_2$, but making that assumption would not simplify the proof.

Lemma 5.1. *Let $\mathbf{G}$ be a reductive group of semisimple rank $n = 2$. Then the map*

$$\bigoplus_{\Delta \in \mathcal{L}_0(\mathbf{G})} H_1(\mathbf{L}_\Delta(k); \text{St}(\mathbf{L}_\Delta)) \rightarrow H_1(\mathbf{G}(k); \text{St}(\mathbf{G}))$$

is surjective.

Proof. Recall that for $p \geq 0$ the set $\mathcal{L}_p(\mathbf{G})$ is the collection of subsets $\Delta \subset {}_k\Delta(\mathbf{G})$ of simple roots with $|\Delta| = n - 1 - p = 1 - p$. Corollary 4.4 gives a spectral sequence converging to $H_{p+q}(\mathbf{G}(k); \text{St}(\mathbf{G}))$ with

$$E_{pq}^1 \cong \begin{cases} \bigoplus_{\Delta \in \mathcal{L}_p(\mathbf{G})} H_q(\mathbf{L}_\Delta(k); \text{St}(\mathbf{L}_\Delta)) & \text{if } 0 \leq p \leq 1 \\ H_q(\mathbf{G}(k); \text{St}(\mathbf{G})^{\otimes 2}) & \text{if } p = 2, \\ 0 & \text{otherwise.} \end{cases}$$

In particular,

$$E_{01}^1 = \bigoplus_{\Delta \in \mathcal{L}_0(\mathbf{G})} H_1(\mathbf{L}_\Delta(k); \text{St}(\mathbf{L}_\Delta)).$$

Our goal is to prove that this surjects onto $H_1(\mathbf{G}(k); \text{St}(\mathbf{G}))$. The part of the E^1 -page of the spectral sequence needed to compute $H_1(\mathbf{G}(k); \text{St}(\mathbf{G}))$ is

$$\begin{array}{ccccc} E_{01}^1 & \leftarrow & E_{11}^1 & & \\ E_{00}^1 & \leftarrow & E_{10}^1 & \xleftarrow{\partial} & E_{20}^1 \end{array}$$

To prove that E_{01}^1 surjects onto $H_1(\mathbf{G}(k); \text{St}(\mathbf{G}))$, it is enough to prove that the labeled differential ∂ is surjective.³⁰ We divide this into four steps.

Step 1. *We give a concrete description of $\partial: E_{20}^1 \rightarrow E_{10}^1$.*

The spectral sequence in Corollary 4.4 is constructed using the resolution of $\text{St}(\mathbf{G})$ from Proposition 4.1. Ignoring the identifications made in that proposition, that resolution is obtained in two steps:

- Start with the augmented chain complex for the Tits building $\mathcal{T}(\mathbf{G})$. In our case, since the semisimple rank of $\mathbf{G}$ is 2 the complex $\mathcal{T}(\mathbf{G})$ is 1-dimensional, so this chain complex is

$$C_1(\mathcal{T}(\mathbf{G})) \rightarrow C_0(\mathcal{T}(\mathbf{G})) \rightarrow \mathbb{Z} \rightarrow 0.$$

Just like in §2.3, we identify $C_1(\mathcal{T}(\mathbf{G}))$ with $\mathbb{Z}[\mathfrak{P}_{\min}]$, where $\mathfrak{P}_{\min}$ is the set of minimal parabolic subgroups of $\mathbf{G}$. The kernel of $C_1(\mathcal{T}(\mathbf{G})) \rightarrow C_0(\mathcal{T}(\mathbf{G}))$ is $\text{St}(\mathbf{G})$, so we can extend the above to an exact sequence

$$0 \rightarrow \text{St}(\mathbf{G}) \xrightarrow{\iota} \mathbb{Z}[\mathfrak{P}_{\min}] \rightarrow C_0(\mathcal{T}(\mathbf{G})) \rightarrow \mathbb{Z} \rightarrow 0.$$

- Tensor this with $\text{St}(\mathbf{G})$ to get a resolution

$$0 \rightarrow \text{St}(\mathbf{G})^{\otimes 2} \xrightarrow{\iota \otimes \text{id}} \mathbb{Z}[\mathfrak{P}_{\min}] \otimes \text{St}(\mathbf{G}) \rightarrow C_0(\mathcal{T}(\mathbf{G})) \otimes \text{St}(\mathbf{G}) \rightarrow \text{St}(\mathbf{G}) \rightarrow 0.$$

³⁰Note that Lemma 3.2 implies that $E_{00}^1 = 0$, so this surjectivity is the only way to kill the E_{10}^1 -term.

We then have

$$\begin{aligned} E_{20}^1 &= H_0(\mathbf{G}(k); \mathrm{St}(\mathbf{G})^{\otimes 2}) = (\mathrm{St}(\mathbf{G})^{\otimes 2})_{\mathbf{G}(k)}, \\ E_{10}^1 &= H_0(\mathbf{G}(k); \mathbb{Z}[\mathfrak{P}_{\min}] \otimes \mathrm{St}(\mathbf{G})) = (\mathbb{Z}[\mathfrak{P}_{\min}] \otimes \mathrm{St}(\mathbf{G}))_{\mathbf{G}(k)}, \end{aligned}$$

and the differential $\partial: E_{20}^1 \rightarrow E_{10}^1$ is exactly the map on these coinvariants induced by the map $\iota \otimes \mathrm{id}: \mathrm{St}(\mathbf{G})^{\otimes 2} \rightarrow \mathbb{Z}[\mathfrak{P}_{\min}] \otimes \mathrm{St}(\mathbf{G})$.

Step 2. We construct an element $\theta \in E_{20}^1$ and give a concrete description of $\partial(\theta) \in E_{10}^1$.

As in §2.2, let $\mathbf{B}$ be a minimal parabolic subgroup of $\mathbf{G}$, let $\mathbf{T}$ be a maximal split torus in $\mathbf{B}$, and let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. We then have the relative Weyl group $W = N_{\mathbf{G}}(\mathbf{T})/Z_{\mathbf{G}}(\mathbf{T})$. For each $w \in W$, pick a representative $\tilde{w} \in N_{\mathbf{G}}(\mathbf{T})(k)$. Apply Corollary 2.16 to find some $u \in \mathbf{U}(k)$ with the following property:

$$(5.1) \quad \text{the only } w_1, w_2 \in W \text{ with } \tilde{w}_1 u \tilde{w}_2 \in \mathbf{B}(k) \text{ is } w_1 = w_2 = 1.$$

Recall from §2.4 that for $g \in \mathbf{G}(k)$ we have the apartment class $\mathbb{A}_g \in \mathrm{St}(\mathbf{G})$, which is defined via the formula

$$\mathbb{A}_g = \sum_{w \in W} (-1)^w g \tilde{w} \cdot \mathbf{B} \in \mathrm{St}(\mathbf{G}) \subset \mathbb{Z}[\mathfrak{P}_{\min}].$$

The element

$$\theta \in E_{20}^1 = (\mathrm{St}(\mathbf{G})^{\otimes 2})_{\mathbf{G}(k)}$$

we will use is the image in the coinvariants of

$$\mathbb{A}_{\mathrm{id}} \otimes \mathbb{A}_u \in \mathrm{St}(\mathbf{G})^{\otimes 2}.$$

For $x \in \mathbb{Z}[\mathfrak{P}_{\min}] \otimes \mathrm{St}(\mathbf{G})$, denote by

$$[x] \in E_{10}^1 = (\mathbb{Z}[\mathfrak{P}_{\min}] \otimes \mathrm{St}(\mathbf{G}))_{\mathbf{G}(k)}$$

its image in the coinvariants. The image $\partial(\theta) \in E_{10}^1$ is $[\iota(\mathbb{A}_{\mathrm{id}}) \otimes \mathbb{A}_u]$. We expand this out as follows:

$$\begin{aligned} [\iota(\mathbb{A}_{\mathrm{id}}) \otimes \mathbb{A}_u] &= \left[\left(\sum_{w \in W} (-1)^w \tilde{w} \cdot \mathbf{B} \right) \otimes \mathbb{A}_u \right] = \left[\sum_{w \in W} (-1)^w \tilde{w} (\mathbf{B} \otimes \tilde{w}^{-1} \cdot \mathbb{A}_u) \right] \\ &= \left[\sum_{w \in W} (-1)^w (\mathbf{B} \otimes \tilde{w}^{-1} \cdot \mathbb{A}_u) \right] = [\mathbf{B} \otimes \sum_{w \in W} (-1)^w \tilde{w} \cdot \mathbb{A}_u]. \end{aligned}$$

Here the third $=$ comes from the fact that we are working in the $\mathbf{G}(k)$ -coinvariants and the fourth from the fact that $(-1)^w = (-1)^{w^{-1}}$.

Step 3. We prove that $E_{10}^1 \cong \mathbb{Z}$ and then give a way to calculate the image of certain elements of E_{10}^1 under the isomorphism $E_{10}^1 \cong \mathbb{Z}$.

By definition, the minimal parabolic $\mathbf{B}$ equals the standard parabolic $\mathbf{P}_{\emptyset}$ associated to $\emptyset \subset {}_k \Delta(\mathbf{G})$. Every parabolic subgroup is conjugate to a unique standard parabolic subgroup, so every minimal parabolic subgroup is conjugate to $\mathbf{B}$. Moreover, as is the case for every parabolic subgroup we have $N_{\mathbf{G}}(\mathbf{B}) = \mathbf{B}$. It follows that $\mathbb{Z}[\mathfrak{P}_{\min}] \cong \mathbb{Z}[\mathbf{G}(k)/\mathbf{B}(k)]$. This implies that

$$(5.2) \quad E_{10}^1 = (\mathbb{Z}[\mathfrak{P}_{\min}] \otimes \mathrm{St}(\mathbf{G}))_{\mathbf{G}(k)} \cong (\mathbb{Z}[\mathbf{G}(k)/\mathbf{B}(k)] \otimes \mathrm{St}(\mathbf{G}))_{\mathbf{G}(k)} \cong \mathrm{St}(\mathbf{G})_{\mathbf{B}(k)}.$$

Recall from §2.5 that $\mathrm{St}(\mathbf{G}) \cong \mathbb{Z}[\mathbf{U}(k)]$. Since $\mathbf{B}(k)$ acts transitively on $\mathbf{U}(k)$, we conclude that

$$(5.3) \quad E_{10}^1 \cong \mathbb{Z}[\mathbf{U}(k)]_{\mathbf{B}(k)} \cong \mathbb{Z}.$$

This isomorphism is induced by the augmentation $\epsilon: \mathbb{Z}[\mathbf{U}(k)] \rightarrow \mathbb{Z}$. For $\theta \in \mathbb{Z}[\mathfrak{P}_{\min}] \otimes \text{St}(\mathbf{G})$, write $[\theta]$ for the image of θ in $E_{10}^1 \cong \mathbb{Z}$ under the identifications (5.2) and (5.3).

Consider $x \in \text{St}(\mathbf{G})$. We claim that $[\mathbf{B} \otimes x] \in \mathbb{Z}$ is the $\mathbf{B}$ -coordinate of x when we express x as an element of $\mathbb{Z}[\mathfrak{P}_{\min}]$. As we discussed in §2.5, the abelian group $\text{St}(\mathbf{G})$ is generated by apartment classes $\mathbb{A}_u$ with $u \in \mathbf{U}(k)$. By linearity, it is therefore enough to prove the claim for $x = \mathbb{A}_u$ with $u \in \mathbf{U}(k)$. Examining (5.2) and (5.3), what we must prove is that the image $\theta \in \mathbb{Z}[\mathbf{U}(k)]$ of $x = \mathbb{A}_u$ has $\epsilon(\theta)$ equal to the $\mathbf{B}$ -coordinate of $x = \mathbb{A}_u$. To see this, note that:

- $x = \mathbb{A}_u$ goes to $\theta = u$ in $\mathbb{Z}[\mathbf{U}(k)]$, so $\epsilon(\theta) = \epsilon(u) = 1$; and
- as was noted in §2.5, the $\mathbf{B}$ -coordinate of $x = \mathbb{A}_u$ is also 1.

Step 4. *For the element θ from Step 2, we prove that $\partial(\theta)$ generates $E_{10}^1 \cong \mathbb{Z}$. This will imply that ∂ is surjective, completing the proof.*

Combining the isomorphism $E_{10}^1 \cong \mathbb{Z}$ from Step 3 with what we did in Step 2, we see that what we must prove is that the $\mathbf{B}$ -coordinate of

$$x = \sum_{w \in W} (-1)^w \tilde{w} \cdot \mathbb{A}_u \in \text{St}(\mathbf{G}) \subset \mathbb{Z}[\mathfrak{P}_{\min}]$$

is 1. Expanding out $\mathbb{A}_u$, we see that

$$x = \sum_{w_1 \in W} (-1)^{w_1} \tilde{w}_1 \cdot \sum_{w_2 \in W} (-1)^{w_2} u \tilde{w}_2 \cdot \mathbf{B} = \sum_{w_1, w_2 \in W} (-1)^{w_1 w_2} \tilde{w}_1 u \tilde{w}_2 \cdot \mathbf{B}.$$

The only elements $g \in \mathbf{G}(k)$ with $g \cdot \mathbf{B} = \mathbf{B}$ are $g \in \mathbf{B}(k)$, and the defining feature of u from (5.1) in Step 2 is that the only $w_1, w_2 \in W$ with $\tilde{w}_1 u \tilde{w}_2 \in \mathbf{B}(k)$ are $w_1 = w_2 = 1$. The fact that the $\mathbf{B}$ -coordinate of x is 1 follows. $\square$

6. REDUCIBLE ROOT SYSTEMS

Let $\mathbf{G}$ be a reductive group. Recall that our goal is to prove Theorem B, which says that $H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ for i in a range depending on the relative root system ${}_k\Phi(\mathbf{G})$. A root system is irreducible when it cannot be decomposed as a direct product, or equivalently if its Dynkin diagram is connected. Even if you only care about $\mathbf{G}$ with ${}_k\Phi(\mathbf{G})$ irreducible, reducible root systems will arise when you study Levi factors of parabolic subgroups. This section gives tools for handling reducible root systems.

6.1. Semisimplicity. We start with a preliminary reduction. For a reductive group $\mathbf{G}$, its derived subgroup $\mathbf{G}_{\text{der}}$ is³¹ semisimple [23, Proposition 19.21]. The inclusion $\mathbf{G}_{\text{der}} \hookrightarrow \mathbf{G}$ induces a bijection between parabolic subgroups, and thus isomorphisms $\mathcal{T}(\mathbf{G}) \cong \mathcal{T}(\mathbf{G}_{\text{der}})$ and $\text{Res}_{\mathbf{G}_{\text{der}}(k)}^{\mathbf{G}(k)} \text{St}(\mathbf{G}) \cong \text{St}(\mathbf{G}_{\text{der}})$. The following lemma will allow us to reduce many questions to the semisimple case:

Lemma 6.1. *Let $\mathbf{G}$ be a reductive group and $b \geq -1$. Assume that $H_i(\mathbf{G}_{\text{der}}(k); \text{St}(\mathbf{G}_{\text{der}})) = 0$ for $i \leq b$. Then $H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ for $i \leq b$ and the map*

$$H_{b+1}(\mathbf{G}_{\text{der}}(k); \text{St}(\mathbf{G}_{\text{der}})) \longrightarrow H_{b+1}(\mathbf{G}(k); \text{St}(\mathbf{G}))$$

is a surjection.

³¹Just like the definition of a reductive group, the definition of a semisimple group is not particularly enlightening and we refer the reader to standard sources for it, e.g., [4, 23]. The key example to keep in mind is that GL_{n+1} is reductive and its derived subgroup SL_{n+1} is semisimple.

Proof. We have a short exact sequence

$$1 \longrightarrow \mathbf{G}_{\text{der}}(k) \longrightarrow \mathbf{G}(k) \longrightarrow A \longrightarrow 1$$

with A abelian. The associated Hochschild–Serre spectral sequence with coefficients in $\text{St}(\mathbf{G})$ is of the form

$$E_{pq}^2 = H_p(A; H_q(\mathbf{G}_{\text{der}}(k); \text{St}(\mathbf{G}_{\text{der}}))) \Rightarrow H_{p+q}(\mathbf{G}(k); \text{St}(\mathbf{G})).$$

Our vanishing assumption implies that $E_{pq}^2 = 0$ for $q \leq b$, so $H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ for $i \leq b$ and $H_{b+1}(\mathbf{G}(k); \text{St}(\mathbf{G})) = E_{0,b+1}^2$. We deduce that

$$H_{b+1}(\mathbf{G}(k); \text{St}(\mathbf{G})) = H_0(A; H_{b+1}(\mathbf{G}_{\text{der}}(k); \text{St}(\mathbf{G}_{\text{der}}))) = H_{b+1}(\mathbf{G}_{\text{der}}(k); \text{St}(\mathbf{G}_{\text{der}}))_A,$$

where the subscript indicates that we are taking coinvariants. For any group Γ and any Γ -module M , the map $M \rightarrow M_\Gamma$ is a surjection. In light of the above identity, we deduce that the map $H_{b+1}(\mathbf{G}_{\text{der}}(k); \text{St}(\mathbf{G}_{\text{der}})) \rightarrow H_{b+1}(\mathbf{G}(k); \text{St}(\mathbf{G}))$ is a surjection, as desired. $\square$

6.2. Products. For a semisimple group $\mathbf{G}$ with ${}_k\Phi(\mathbf{G})$ reducible, the following lemma will allow us to reduce questions about $H_i(\mathbf{G}; \text{St}(\mathbf{G}))$ to questions about a product of semisimple groups with irreducible relative root systems:

Lemma 6.2. *Let $\mathbf{G}$ be a semisimple group. Write*

$${}_k\Phi(\mathbf{G}) = \Phi_1 \times \cdots \times \Phi_m$$

with each Φ_i irreducible. There then exists a semisimple group $\mathbf{H}$ equipped with a surjection $\mathbf{H} \rightarrow \mathbf{G}$ such that:

- (i) *the surjection $\mathbf{H} \rightarrow \mathbf{G}$ induces a bijection between parabolic subgroups, and thus isomorphisms $\mathcal{T}(\mathbf{H}) \cong \mathcal{T}(\mathbf{G})$ and $\text{St}(\mathbf{H}) \cong \text{Res}_{\mathbf{H}(k)}^{\mathbf{G}(k)} \text{St}(\mathbf{G})$; and*
- (ii) *$\mathbf{H} = \mathbf{H}_1 \times \cdots \times \mathbf{H}_m$ with ${}_k\Phi(\mathbf{H}_j) = \Phi_j$ for $1 \leq j \leq m$.*

If in addition for some $b \geq -1$ we have $H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ for $i \leq b$, then:

- (iii) *the map $H_{b+1}(\mathbf{H}(k); \text{St}(\mathbf{H})) \rightarrow H_{b+1}(\mathbf{G}(k); \text{St}(\mathbf{G}))$ is an isomorphism.*

Proof. Let $\mathbf{G}_1, \dots, \mathbf{G}_r$ be the minimal elements among the connected normal subgroups of $\mathbf{G}$ that have strictly positive dimension. By [4, Theorem 22.10], the following hold:

- Letting $\mathbf{H} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_r$, the map $\mathbf{H} \rightarrow \mathbf{G}$ is surjective with finite central kernel.
- Exactly $m \leq r$ of the $\mathbf{G}_j$ have positive semisimple rank. Moreover, we can order the $\mathbf{G}_j$ such that ${}_k\Phi(\mathbf{G}_j) = \Phi_j$ for $1 \leq j \leq m$.

That the map $\mathbf{H} \rightarrow \mathbf{G}$ satisfies (i) is immediate, and for the decomposition $\mathbf{H} = \mathbf{H}_1 \times \cdots \times \mathbf{H}_m$ from (ii) we can take

$$\mathbf{H}_j = \begin{cases} \mathbf{G}_j & \text{if } 1 \leq j \leq m-1, \\ \mathbf{G}_m \times \mathbf{G}_{m+1} \times \cdots \times \mathbf{G}_r & \text{if } j = m. \end{cases}$$

Now assume that for some $b \geq -1$ we have $H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ for $i \leq b$. We must prove (iii). By construction, we have a central extension

$$1 \longrightarrow C \longrightarrow \mathbf{H}(k) \longrightarrow \mathbf{G}(k) \longrightarrow 1.$$

Since the map $\mathbf{H} \rightarrow \mathbf{G}$ induces an isomorphism $\text{St}(\mathbf{H}) \cong \text{Res}_{\mathbf{H}(k)}^{\mathbf{G}(k)} \text{St}(\mathbf{G})$, the group C acts trivially on $\text{St}(\mathbf{H}) \cong \text{Res}_{\mathbf{H}(k)}^{\mathbf{G}(k)} \text{St}(\mathbf{G})$. Consider the associated Hochschild–Serre spectral sequence with coefficients in $\text{St}(\mathbf{G})$. Since $\text{St}(\mathbf{G})$ is a free abelian group on which C acts

trivially, this spectral sequence has

$$\begin{aligned} E_{pq}^2 &= H_p(\mathbf{G}(k); H_q(C; \text{St}(\mathbf{G}))) \\ &= H_p(\mathbf{G}(k); H_q(C) \otimes \text{St}(\mathbf{G})) \\ &= H_p(\mathbf{G}(k); \text{St}(\mathbf{G}; H_q(C))). \end{aligned}$$

By our vanishing assumption and Lemma 3.1, we have $E_{pq}^2 = 0$ for $p \leq b$. We conclude that

$$H_{b+1}(\mathbf{H}(k); \text{St}(\mathbf{H})) = E_{b+1,0}^2 = H_{b+1}(\mathbf{G}(k); \text{St}(\mathbf{G}; H_0(C))) = H_{b+1}(\mathbf{G}(k); \text{St}(\mathbf{G})),$$

as desired. $\square$

To use Lemma 6.2, we need to know the Steinberg representation of a product:

Lemma 6.3. *Let $\mathbf{G}_1$ and $\mathbf{G}_2$ be reductive groups and let $\mathbb{F}$ be a commutative ring. Then $\text{St}(\mathbf{G}_1 \times \mathbf{G}_2; \mathbb{F}) \cong \text{St}(\mathbf{G}_1; \mathbb{F}) \otimes \text{St}(\mathbf{G}_2; \mathbb{F})$.*

Proof. We have $\mathcal{T}(\mathbf{G}_1 \times \mathbf{G}_2) = \mathcal{T}(\mathbf{G}_1) * \mathcal{T}(\mathbf{G}_2)$, where $*$ means join. If n_i is the semisimple rank of $\mathbf{G}_i$, then the Solomon–Tits theorem says that $\mathcal{T}(\mathbf{G}_i)$ is $n_i - 2$ connected and $\text{St}(\mathbf{G}_i; \mathbb{F}) = \tilde{H}_{n_i-1}(\mathcal{T}(\mathbf{G}_i); \mathbb{F})$. The semisimple rank of $\mathbf{G}_1 \times \mathbf{G}_2$ is $n_1 + n_2$, so

$$\begin{aligned} \text{St}(\mathbf{G}_1 \times \mathbf{G}_2; \mathbb{F}) &= \tilde{H}_{(n_1+n_2)-1}(\mathcal{T}(\mathbf{G}_1) * \mathcal{T}(\mathbf{G}_2); \mathbb{F}) \\ &= \tilde{H}_{n_1-1}(\mathcal{T}(\mathbf{G}_1); \mathbb{F}) \otimes \tilde{H}_{n_2-1}(\mathcal{T}(\mathbf{G}_2); \mathbb{F}) = \text{St}(\mathbf{G}_1; \mathbb{F}) \otimes \text{St}(\mathbf{G}_2; \mathbb{F}). \quad \square \end{aligned}$$

6.3. Reducible root systems and vanishing. Recall that our goal is to prove that for a reductive group $\mathbf{G}$ we have $H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ for i in some range. The following result will allow us to reduce this to $\mathbf{G}$ with ${}_k\Phi(\mathbf{G})$ irreducible:

Lemma 6.4 (Reducible vanishing). *Let $\Phi_1, \dots, \Phi_m$ be nontrivial irreducible root systems. For each $1 \leq j \leq m$, assume that there is some $b_j \geq -1$ such that the following holds:*

(♥) *If $\mathbf{H}_j$ is a reductive group with ${}_k\Phi(\mathbf{H}_j) = \Phi_j$, then $H_i(\mathbf{H}_j(k); \text{St}(\mathbf{H}_j)) = 0$ for $i \leq b_j$.*

Then for all reductive groups $\mathbf{G}$ with ${}_k\Phi(\mathbf{G}) = \Phi_1 \times \dots \times \Phi_m$, we have $H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ for $i \leq (m-1) + b_1 + \dots + b_m$.

Proof. By Lemma 6.1, it is enough to prove this for $\mathbf{G}$ semisimple. The proof will be by induction on m . The base case $m = 1$ follows immediately from (♥), so assume that $m \geq 2$ and that the lemma is true whenever m is smaller. For all $b \leq (m-1) + b_1 + \dots + b_m$, we will prove that:

(♠_b) For all semisimple groups $\mathbf{G}$ with ${}_k\Phi(\mathbf{G}) = \Phi_1 \times \dots \times \Phi_m$, we have $H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ for $i \leq b$.

The proof will be by induction on b . The base case $b = -1$ is trivial, so assume that $-1 \leq b < (m-1) + b_1 + \dots + b_m$ and that (♠_b) holds. We will prove that (♠_{b+1}) holds. Let $\mathbf{G}$ be a semisimple group with ${}_k\Phi(\mathbf{G}) = \Phi_1 \times \dots \times \Phi_m$. Since (♠_b) holds, we have $H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ for $i \leq b$. To prove (♠_{b+1}), we must prove that $H_{b+1}(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$. Applying Lemma 6.2, we see that there exists a semisimple group $\mathbf{H}$ equipped with a surjection $\mathbf{H} \rightarrow \mathbf{G}$ such that:³²

- (i) the surjection $\mathbf{H} \rightarrow \mathbf{G}$ induces a bijection between parabolic subgroups, and thus isomorphisms $\mathcal{T}(\mathbf{H}) \cong \mathcal{T}(\mathbf{G})$ and $\text{St}(\mathbf{H}) \cong \text{Res}_{\mathbf{H}(k)}^{\mathbf{G}(k)} \text{St}(\mathbf{G})$; and
- (ii) $\mathbf{H} = \mathbf{H}_1 \times \dots \times \mathbf{H}_m$ with ${}_k\Phi(\mathbf{H}_j) = \Phi_j$ for $1 \leq j \leq m$; and
- (iii) the map $H_{b+1}(\mathbf{H}(k); \text{St}(\mathbf{H})) \rightarrow H_{b+1}(\mathbf{G}(k); \text{St}(\mathbf{G}))$ is an isomorphism.

³²We get (iii) because of (♠_b).

By (iii), to prove that $H_{b+1}(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ it is enough to prove that $H_{b+1}(\mathbf{H}(k); \text{St}(\mathbf{H})) = 0$.

As notation, set $\mathbf{A} = \mathbf{H}_1 \times \cdots \times \mathbf{H}_{m-1}$ and $\mathbf{B} = \mathbf{H}_m$, so $\mathbf{H} = \mathbf{A} \times \mathbf{B}$. By Lemma 6.3, we have $\text{St}(\mathbf{A} \times \mathbf{B}) = \text{St}(\mathbf{A}) \otimes \text{St}(\mathbf{B})$. Since the abelian groups underlying $\text{St}(\mathbf{A})$ and $\text{St}(\mathbf{B})$ are free abelian, the Künneth formula applies and shows that $H_i(\mathbf{A}(k) \times \mathbf{B}(k); \text{St}(\mathbf{A} \times \mathbf{B}))$ fits into a short exact sequence with the following kernel and cokernel:

- $\bigoplus_{i_1+i_2=i} H_{i_1}(\mathbf{A}(k); \text{St}(\mathbf{A})) \otimes H_{i_2}(\mathbf{B}(k); \text{St}(\mathbf{B})).$
- $\bigoplus_{i_1+i_2=i-1} \text{Tor}(H_{i_1}(\mathbf{A}(k); \text{St}(\mathbf{A})), H_{i_2}(\mathbf{B}(k); \text{St}(\mathbf{B}))).$

Recall that we are inducting on m . By this induction hypothesis and (♥), we have

- $H_{i_1}(\mathbf{A}(k); \text{St}(\mathbf{A})) = 0$ for $i_1 \leq (m-2) + b_1 + \cdots + b_{m-1}$; and
- $H_{i_2}(\mathbf{B}(k); \text{St}(\mathbf{B})) = 0$ for $i_2 \leq b_m$.

Note that if

$$i_1 + i_2 \leq ((m-2) + b_1 + \cdots + b_{m-1}) + b_m + 1 = (m-1) + b_1 + \cdots + b_m,$$

then either

$$i_1 \leq (m-2) + b_1 + \cdots + b_{m-1} \quad \text{or} \quad i_2 \leq b_m.$$

It follows that $H_i(\mathbf{A}(k) \times \mathbf{B}(k); \text{St}(\mathbf{A} \times \mathbf{B})) = 0$ for $i \leq (m-1) + b_1 + \cdots + b_m$. In particular,³³ this holds for $i = b+1$, as desired. $\square$

6.4. Reducible root systems and surjectivity. Our proof that for some b we have $H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ for $i \leq b$ will be by induction, and to make the induction work we will have to also prove a surjectivity statement for H_{b+1} . The following will allow us to reduce this surjectivity statement to $\mathbf{G}$ with ${}_k\Phi(\mathbf{G})$ irreducible:

Lemma 6.5 (Reducible surjectivity). *Let $\Phi_1, \dots, \Phi_m$ be nontrivial irreducible root systems. For each $1 \leq j \leq m$, assume that there is some $b_j \geq -1$ such that the following holds:*

- (♥) *If $\mathbf{H}_j$ is a reductive group with ${}_k\Phi(\mathbf{H}_j) = \Phi_j$, then $H_i(\mathbf{H}_j(k); \text{St}(\mathbf{H}_j)) = 0$ for $i \leq b_j$.*

Additionally, for some $1 \leq j_0 \leq m$ assume there exists a subset Δ' of the set of simple roots of Φ_{j_0} such that the following holds:

- (♥♥) *If $\mathbf{H}_{j_0}$ is a reductive group with ${}_k\Phi(\mathbf{H}_{j_0}) = \Phi_{j_0}$ and $\mathbf{L}_{\Delta'}^{\mathbf{H}_{j_0}}$ is the corresponding standard Levi subgroup of $\mathbf{H}$, then the map $H_{b_{j_0}+1}(\mathbf{L}_{\Delta'}^{\mathbf{H}_{j_0}}(k); \text{St}(\mathbf{L}_{\Delta'}^{\mathbf{H}_{j_0}})) \rightarrow H_{b_{j_0}+1}(\mathbf{H}_{j_0}(k); \text{St}(\mathbf{H}_{j_0}))$ is surjective.*

Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = \Phi_1 \times \cdots \times \Phi_m$. Let

$$\overline{\Delta}' = \Delta' \sqcup \bigsqcup_{\substack{1 \leq j \leq m \\ j \neq j_0}} {}_k\Delta(\Phi_j),$$

and let $\mathbf{L}_{\overline{\Delta}'}^{\mathbf{G}}$ be the corresponding standard Levi subgroup of $\mathbf{G}$. Then for

$$b = (m-1) + b_1 + \cdots + b_m$$

the map $H_{b+1}(\mathbf{L}_{\overline{\Delta}'}^{\mathbf{G}}(k); \text{St}(\mathbf{L}_{\overline{\Delta}'}^{\mathbf{G}})) \rightarrow H_{b+1}(\mathbf{G}(k); \text{St}(\mathbf{G}))$ is surjective.

Proof. Lemma 6.4 (reducible vanishing) implies that $H_i(\mathbf{G}_{\text{der}}(k); \text{St}(\mathbf{G}_{\text{der}})) = 0$ for $i \leq b$. We can therefore apply Lemma 6.1 and see that the map

$$(6.1) \quad H_{b+1}(\mathbf{G}_{\text{der}}; \text{St}(\mathbf{G}_{\text{der}})) \rightarrow H_{b+1}(\mathbf{G}; \text{St}(\mathbf{G}))$$

³³We are inducting on b so we can reduce to a group that splits as a product of groups with relative root systems Φ_j . If our group $\mathbf{G}$ was already a product, then there would be no need to induct on b .

is surjective. We can also apply Lemma 6.2 to find a semisimple group $\mathbf{H}$ equipped with a surjection $\mathbf{H} \rightarrow \mathbf{G}_{\text{der}}$ such that:

- (i) the surjection $\mathbf{H} \rightarrow \mathbf{G}_{\text{der}}$ induces a bijection between parabolic subgroups, and thus isomorphisms $\mathcal{T}(\mathbf{H}) \cong \mathcal{T}(\mathbf{G}_{\text{der}})$ and $\text{St}(\mathbf{H}) \cong \text{Res}_{\mathbf{H}(k)}^{\mathbf{G}_{\text{der}}(k)} \text{St}(\mathbf{G}_{\text{der}})$; and
 - (ii) $\mathbf{H} = \mathbf{H}_1 \times \cdots \times \mathbf{H}_m$ with ${}_k\Phi(\mathbf{H}_j) = \Phi_j$ for all $1 \leq j \leq m$; and
 - (iii) the map $\text{H}_{b+1}(\mathbf{H}(k); \text{St}(\mathbf{H})) \rightarrow \text{H}_{b+1}(\mathbf{G}_{\text{der}}(k); \text{St}(\mathbf{G}_{\text{der}}))$ is an isomorphism.
- In particular, combining (iii) and the fact that (6.1) is surjective, we see that the map

$$\text{H}_{b+1}(\mathbf{H}_1(k) \times \cdots \times \mathbf{H}_m(k); \text{St}(\mathbf{H}_1 \times \cdots \times \mathbf{H}_m)) \rightarrow \text{H}_{b+1}(\mathbf{G}; \text{St}(\mathbf{G}))$$

is surjective.

Let $\mathbf{L}_{\bar{\Delta}}^{\mathbf{H}}$ and $\mathbf{L}_{\Delta'}^{\mathbf{H}_{j_0}}$ be the standard Levi subgroups of $\mathbf{H}$ and $\mathbf{H}_{j_0}$ corresponding to the simple roots $\bar{\Delta}'$ and Δ' . We therefore have

$$\mathbf{L}_{\bar{\Delta}'}^{\mathbf{H}} = \mathbf{H}_1 \times \cdots \times \mathbf{L}_{\Delta'}^{\mathbf{H}_{j_0}} \times \cdots \times \mathbf{H}_m.$$

Examining the proof of Lemma 6.2, we see that $\mathbf{L}_{\bar{\Delta}'}^{\mathbf{H}}$ is precisely the group obtained by applying Lemma 6.2 to the derived subgroup of $\mathbf{L}_{\bar{\Delta}'}^{\mathbf{G}}$, though in this case we do not have conclusion (iii) of Lemma 6.2. We therefore have a commutative diagram

$$\begin{array}{ccc} \text{H}_{b+1}(\mathbf{L}_{\bar{\Delta}'}^{\mathbf{H}}(k); \text{St}(\mathbf{L}_{\bar{\Delta}'}^{\mathbf{H}})) & \rightarrow & \text{H}_{b+1}(\mathbf{L}_{\bar{\Delta}'}^{\mathbf{G}}(k); \text{St}(\mathbf{L}_{\bar{\Delta}'}^{\mathbf{G}})) \\ \downarrow \tilde{f} & & \downarrow f \\ \text{H}_{b+1}(\mathbf{H}(k); \text{St}(\mathbf{H})) & \twoheadrightarrow & \text{H}_{b+1}(\mathbf{G}(k); \text{St}(\mathbf{G})) \end{array}$$

From this, we see that to prove that f is surjective, it is enough to prove that $\tilde{f}$ is surjective.

In light of (♥), we can use the Künneth formula just like we did in the proof of Lemma 6.4 (reducible vanishing) to see that

$$\text{H}_{b+1}(\mathbf{H}(k); \text{St}(\mathbf{H})) = \bigotimes_{j=1}^m \text{H}_{b_j+1}(\mathbf{H}_j(k); \text{St}(\mathbf{H}_j)).$$

We thus see that it is enough to prove that the map

$$\begin{aligned} & \left(\bigotimes_{j=1}^{j_0-1} \text{H}_{b_j+1}(\mathbf{H}_j(k); \text{St}(\mathbf{H}_j)) \right) \otimes \text{H}_{b_{j_0}+1}(\mathbf{L}_{\Delta'}^{\mathbf{H}_{j_0}}(k); \text{St}(\mathbf{L}_{\Delta'}^{\mathbf{H}_{j_0}})) \\ & \otimes \left(\bigotimes_{j=j_0+1}^m \text{H}_{b_j+1}(\mathbf{H}_j(k); \text{St}(\mathbf{H}_j)) \right) \rightarrow \bigotimes_{j=1}^m \text{H}_{b_j+1}(\mathbf{H}_j(k); \text{St}(\mathbf{H}_j)) \end{aligned}$$

is surjective. Since the maps on all but one tensor factor are the identity, this is equivalent to the surjectivity of the map

$$\text{H}_{b_{j_0}+1}(\mathbf{L}_{\Delta'}^{\mathbf{H}_{j_0}}(k); \text{St}(\mathbf{L}_{\Delta'}^{\mathbf{H}_{j_0}})) \rightarrow \text{H}_{b_{j_0}+1}(\mathbf{H}_{j_0}(k); \text{St}(\mathbf{H}_{j_0}))$$

on the remaining tensor factor, which is exactly (♥♥). $\square$

7. REDUCTION TO IRREDUCIBLE CASE

In this section, we reduce Theorem B to the case of irreducible root systems. Recall that for a reductive group $\mathbf{G}$ we defined the bound $\mathbf{b}({}_k\Phi(\mathbf{G}))$ for our vanishing range in Table 1 and equation (1.2). We will recall the formula for $\mathbf{b}({}_k\Phi(\mathbf{G}))$ for various ${}_k\Phi(\mathbf{G})$ as it is needed.

7.1. Irreducible cases. The heart of our proof is verifying Theorem B for groups $\mathbf{G}$ whose relative root system ${}_k\Phi(\mathbf{G})$ is one of the non-exceptional types $\{A_n, B_n, C_n, BC_n, D_n\}$:

Theorem B.1 (Type A). *Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = A_n$ for some $n \geq 1$. Then $H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ for $i \leq \mathbf{b}({}_k\Phi(\mathbf{G})) = \lfloor (n-1)/2 \rfloor$.*

Theorem B.2 (Type B / C / BC). *Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) \in \{B_n, C_n, BC_n\}$ for some³⁴ $n \geq 2$. Then $H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ for $i \leq \mathbf{b}({}_k\Phi(\mathbf{G})) = \lfloor (n-2)/2 \rfloor$.*

Theorem B.3 (Type D). *Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = D_n$ for some³⁵ $n \geq 4$. Then $H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ for $i \leq \mathbf{b}({}_k\Phi(\mathbf{G})) = \lfloor (n-3)/2 \rfloor$.*

We will prove these three theorems in Parts 2, 3, and 4.

7.2. General case. Before starting our work on Theorems B.1, B.2, and B.3, we show that they imply the general case:

Theorem B. *Let $\mathbf{G}$ be a reductive group and $\mathbb{F}$ be a commutative ring. Then we have $H_i(\mathbf{G}(k); \text{St}(\mathbf{G}; \mathbb{F})) = 0$ for $i \leq \mathbf{b}({}_k\Phi(\mathbf{G}))$.*

Proof of Theorem B, assuming Theorems B.1, B.2, and B.3. By Lemma 3.1, we can assume that $\mathbb{F} = \mathbb{Z}$, and thus omit $\mathbb{F}$ from our notation. If ${}_k\Phi(\mathbf{G})$ is irreducible, then it either lies in $\{A_n, B_n, C_n, BC_n, D_n\}$ or is one of the exceptional root systems $\{G_2, F_4, E_6, E_7, E_8\}$. If ${}_k\Phi(\mathbf{G})$ is non-exceptional, then the result follows from Theorems B.1, B.2, and B.3. If it is exceptional, then by definition we have $\mathbf{b}({}_k\Phi(\mathbf{G})) = 0$, so the theorem asserts that $H_0(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$. This is exactly the content of Lemma 3.2.

It remains to handle the case where ${}_k\Phi(\mathbf{G})$ is reducible. For some $m \geq 2$, we can write

$${}_k\Phi(\mathbf{G}) = \Phi_1 \times \cdots \times \Phi_m$$

with each Φ_i irreducible. By definition, we have

$$\mathbf{b}({}_k\Phi(\mathbf{G})) = (m-1) + \sum_{j=1}^m \mathbf{b}(\Phi_j).$$

Since we have already proved our theorem for groups with irreducible relative root systems, Lemma 6.4 applies and shows that $H_i(\mathbf{G}(k); \text{St}(\mathbf{G}; \mathbb{F})) = 0$ for $i \leq \mathbf{b}({}_k\Phi(\mathbf{G}))$. $\square$

Part 2. Vanishing in type A (Theorem B.1)

This part of the paper proves our vanishing result in type A (Theorem B.1). For our induction, we prove a stronger result (Theorem B.1') that incorporates a surjectivity statement. We state this in §8. After the preliminary §9 about Levi subgroups, in §10 we show that our inductive hypotheses gives a vanishing region in the spectral sequence from Corollary 4.4. As we show in §11, this almost implies Theorem B.1'. The only missing ingredients are three computations of differentials in our spectral sequence, which are in §12.

8. VANISHING AND SURJECTIVITY (TYPE A)

In this section, we first introduce some notation for the standard Levi factors of groups of type A_n . We then state a stronger version of Theorem B.1 and prove it for ranks at most 2.

³⁴The root systems B_n and C_n and BC_n are only defined for $n \geq 2$.

³⁵The root system D_n is only defined for $n \geq 4$.

8.1. Levi factor notation. Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = A_n$. Let $\Delta = {}_k\Delta(\mathbf{G})$ be the set of simple roots of ${}_k\Phi(\mathbf{G}) = A_n$. Number the elements of Δ from left to right as in the usual Dynkin diagram:



For $1 \leq j_1, \dots, j_\ell \leq n$, let $\Delta[j_1, \dots, j_\ell]$ be the result of removing the simple roots labeled $j_1, \dots, j_\ell$ from Δ . We thus have a standard Levi subgroup $\mathbf{L}_{\Delta[j_1, \dots, j_\ell]}$ of $\mathbf{G}$.

Example 8.1. We have ${}_k\Phi(\mathbf{L}_{\Delta[1]}) = {}_k\Phi(\mathbf{L}_{\Delta[n]}) = A_{n-1}$, while for $2 \leq j \leq n-1$ we have ${}_k\Phi(\mathbf{L}_{\Delta[j]}) = A_{j-1} \times A_{n-j}$. In general, for distinct $1 \leq j_1, \dots, j_\ell \leq n$ we have

$${}_k\Phi(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]}) = A_{n_1} \times \cdots \times A_{n_m} \quad \text{with } n_1 + \cdots + n_m + \ell = n. \quad \square$$

We have a Reeder map (cf. §2.6) of the form $\text{St}(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]}) \rightarrow \text{St}(\mathbf{G})$, and thus maps $H_i(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]}(k); \text{St}(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]})) \rightarrow H_i(\mathbf{G}(k); \text{St}(\mathbf{G}))$.

8.2. Strong vanishing. The main result we will prove in this part of the paper is:

Theorem B.1'. *Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = A_n$. Then:*

- $H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ for $i \leq \lfloor (n-1)/2 \rfloor$; and
- letting $\Delta = {}_k\Delta(\mathbf{G})$, the maps

$$\begin{aligned} H_i(\mathbf{L}_{\Delta[1]}(k); \text{St}(\mathbf{L}_{\Delta[1]})) &\rightarrow H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) \quad \text{and} \\ H_i(\mathbf{L}_{\Delta[n]}(k); \text{St}(\mathbf{L}_{\Delta[n]})) &\rightarrow H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) \end{aligned}$$

are both surjective for $i \leq \lfloor n/2 \rfloor$.

This strengthens Theorem B.1 by adding the indicated surjectivity statement. This surjectivity statement will be used to understand differentials in the spectral sequence from Corollary 4.4. To connect Theorem B.1' to what we have already proven, we show that the spectral sequence argument from §5 implies Theorem B.1' for $n \leq 2$. This requires a lemma.

Lemma 8.2. *Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = A_n$ for some $n \geq 2$. Then the images of the two maps*

$$\begin{aligned} H_i(\mathbf{L}_{\Delta[1]}(k); \text{St}(\mathbf{L}_{\Delta[1]})) &\rightarrow H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) \quad \text{and} \\ H_i(\mathbf{L}_{\Delta[n]}(k); \text{St}(\mathbf{L}_{\Delta[n]})) &\rightarrow H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) \end{aligned}$$

are the same.

Proof. Lemma 2.19 says that $\mathbf{L}_{\Delta[1]}(k)$ and $\mathbf{L}_{\Delta[n]}(k)$ are conjugate subgroups of $\mathbf{G}(k)$. This conjugation matches up the parabolic subgroups of $\mathbf{L}_{\Delta[1]}$ and $\mathbf{L}_{\Delta[n]}$, and thus induces an isomorphism from $\text{St}(\mathbf{L}_{\Delta[1]})$ to $\text{St}(\mathbf{L}_{\Delta[n]})$. The lemma now follows from the fact that inner automorphisms act trivially on group homology. $\square$

Lemma 8.3. *Theorem B.1' holds for $n \leq 2$.*

Proof. Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = A_n$ for some $n \leq 2$. For $n = 0$, Theorem B.1' asserts nothing. For $n = 1$, since a map to 0 is surjective Theorem B.1' only asserts that $H_0(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ when ${}_k\Phi(\mathbf{G}) = A_1$, which follows from Lemma 3.2.

For $n = 2$, Theorem B.1' asserts that $H_0(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$, which follows from Lemma 3.2. Theorem B.1' also asserts a surjectivity statement for H_0 and H_1 . The surjectivity statement for H_0 is vacuous since $H_0(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$, so the only nontrivial thing to prove is surjectivity for H_1 . Recall that $\mathcal{L}_p(\mathbf{G})$ consists of all subsets³⁶ R of Δ with $|R| = n - p - 1$.

³⁶Before we used the notation Δ instead of R , but we use R here to avoid confusion between the boldface Δ and the non-boldface Δ .

Lemma 5.1 says that

$$(8.1) \quad \bigoplus_{R \in \mathcal{L}_0(\mathbf{G})} H_1(\mathbf{L}_R(k); \text{St}(\mathbf{L}_R)) = H_1(\mathbf{L}_{\Delta[1]}(k); \text{St}(\mathbf{L}_{\Delta[1]})) \oplus H_1(\mathbf{L}_{\Delta[2]}(k); \text{St}(\mathbf{L}_{\Delta[2]}))$$

surjects onto $H_1(\mathbf{G}(k); \text{St}(\mathbf{G}))$. Lemma 8.2 says that both terms of (8.1) have the same image in $H_1(\mathbf{G}(k); \text{St}(\mathbf{G}))$. This implies that both surject onto $H_1(\mathbf{G}(k); \text{St}(\mathbf{G}))$, as desired. $\square$

Because of Lemma 8.3, for the rest of this part of the paper we can assume that $n \geq 3$. We will also assume as an inductive hypothesis that we have already proved Theorem B.1' in smaller ranks. For this, we make the following definition:

Definition 8.4. For $r \geq 0$, the r -surjectivity and vanishing hypothesis in type A is as follows. Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = A_n$ for some $n \leq r$. Then:

- $H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ for $i \leq \lfloor (n-1)/2 \rfloor$; and
- letting $\Delta = {}_k\Delta(\mathbf{G})$, the maps

$$\begin{aligned} H_i(\mathbf{L}_{\Delta[1]}(k); \text{St}(\mathbf{L}_{\Delta[1]})) &\rightarrow H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) \quad \text{and} \\ H_i(\mathbf{L}_{\Delta[n]}(k); \text{St}(\mathbf{L}_{\Delta[n]})) &\rightarrow H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) \end{aligned}$$

are both surjective for $i \leq \lfloor n/2 \rfloor$. $\square$

9. VANISHING AND SURJECTIVITY FOR LEVI SUBGROUPS (TYPE A)

In this section, we show how to use the r -surjectivity and vanishing hypothesis in type A to analyze the homology of standard Levi subgroups. Our main result is as follows. Its statement uses the ordering on the simple roots of $A_{n_{j_0}}$ discussed in §8.1.

Lemma 9.1 (Levi vanishing and surjectivity). *Assume the $(n-1)$ -surjectivity and vanishing hypothesis in type A (Definition 8.4). Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = A_n$. Let $\Delta \subset {}_k\Delta(\mathbf{G})$ be a set of simple roots with $\Delta \neq {}_k\Delta(\mathbf{G})$. Write*

$${}_k\Phi(\mathbf{L}_\Delta) = A_{n_1} \times \cdots \times A_{n_m}.$$

Set

$$b = \mathbf{b}({}_k\Phi(\mathbf{L}_\Delta)) = (m-1) + \lfloor (n_1-1)/2 \rfloor + \cdots + \lfloor (n_m-1)/2 \rfloor.$$

Then the following hold:

- (i) We have $H_i(\mathbf{L}_\Delta(k); \text{St}(\mathbf{L}_\Delta)) = 0$ for $i \leq b$.
- (ii) For some $1 \leq j_0 \leq m$, assume that n_{j_0} is even and nonzero. Let $\Delta' \subset \Delta$ be the set of simple roots obtained by removing either the first or last simple root from $A_{n_{j_0}}$, so

$${}_k\Phi(\mathbf{L}_{\Delta'}) = A_{n_1} \times \cdots \times A_{n_{j_0}-1} \times \cdots \times A_{n_m}.$$

Then the map $H_{b+1}(\mathbf{L}_{\Delta'}(k); \text{St}(\mathbf{L}_{\Delta'})) \rightarrow H_{b+1}(\mathbf{L}_\Delta(k); \text{St}(\mathbf{G}))$ is surjective.

Proof. Since $\Delta \neq {}_k\Delta(\mathbf{G})$, we have $n_j \leq n-1$ for $1 \leq j \leq m$. The $(n-1)$ -surjectivity and vanishing hypothesis in type A thus applies to all reductive groups $\mathbf{H}_j$ with ${}_k\Phi(\mathbf{H}_j) = A_{n_j}$. This gives the hypothesis (♥) in Lemma 6.4 (reducible vanishing). Applying Lemma 6.4, we deduce (i). Similarly, for Δ' as in (ii) it gives the hypotheses (♥) and (♥♥) in Lemma 6.5 (reducible surjectivity). Applying Lemma 6.5, we deduce (ii). $\square$

10. VANISHING REGION (TYPE A)

Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = \mathbf{A}_n$ for some $n \geq 3$. Corollary 4.4 gives a spectral sequence E_{pq}^r converging to $H_{p+q}(\mathbf{G}(k); \text{St}(\mathbf{G}))$ with

$$E_{pq}^1 \cong \begin{cases} \bigoplus_{R \in \mathcal{L}_p(\mathbf{G})} H_q(\mathbf{L}_R(k); \text{St}(\mathbf{L}_R)) & \text{if } 0 \leq p \leq n-1 \\ H_q(\mathbf{G}(k); \text{St}(\mathbf{G})^{\otimes 2}) & \text{if } p = n, \\ 0 & \text{otherwise.} \end{cases}$$

The following lemma shows that our inductive hypothesis implies that many terms of this spectral sequence vanish.

Lemma 10.1. *Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = \mathbf{A}_n$ for some $n \geq 3$. Assume the $(n-1)$ -surjectivity and vanishing hypothesis in type A (Definition 8.4). Let E_{pq}^1 be the spectral sequence from Corollary 4.4. Then the following hold:*

- For $n = 2d + 1$ with $d \geq 1$, we have $E_{pq}^1 = 0$ for $p + q \leq d$ except for possibly E_{0d}^1 .
- For $n = 2d$ with $d \geq 2$, we have $E_{pq}^1 = 0$ for $p + q \leq d$ except for possibly E_{0d}^1 and $E_{1,d-1}^1$.

Proof. Our goal is to prove a vanishing result for E_{pq}^1 . The terms in question all have $p \leq \lfloor n/2 \rfloor$, so they all satisfy³⁷ $p \leq n-1$ and are therefore of the form

$$E_{pq}^1 = \bigoplus_{R \in \mathcal{L}_p(\mathbf{G})} H_q(\mathbf{L}_R(k); \text{St}(\mathbf{L}_R)).$$

Consider $R \in \mathcal{L}_p(\mathbf{G})$. We will prove that our assumptions imply that $H_q(\mathbf{L}_R(k); \text{St}(\mathbf{L}_R)) = 0$ for the p and q where the lemma claims that $E_{pq}^1 = 0$.

Since R is obtained by deleting $p+1$ simple roots from Δ , we have

$${}_k\Phi(\mathbf{L}_R) = \mathbf{A}_{n_1} \times \cdots \times \mathbf{A}_{n_m} \quad \text{with } n_1 + \cdots + n_m + p + 1 = n.$$

Lemma 9.1 (Levi vanishing and surjectivity) implies that $H_q(\mathbf{L}_R(k); \text{St}(\mathbf{L}_R)) = 0$ for

$$q \leq (m-1) + \lfloor (n_1-1)/2 \rfloor + \cdots + \lfloor (n_m-1)/2 \rfloor.$$

For $a, b \in \mathbb{Z}$, Lemma 10.2 below implies that $1 + \lfloor a/2 \rfloor + \lfloor b/2 \rfloor \geq \lfloor (a+b+1)/2 \rfloor$. Applying this repeatedly, we deduce that

$$\mathbf{b}(\mathbf{A}_{n_1} \times \cdots \times \mathbf{A}_{n_m}) \geq \lfloor (n_1 + \cdots + n_m - 1)/2 \rfloor = \lfloor (n - p - 2)/2 \rfloor.$$

It follows that

$$E_{pq}^1 = 0 \quad \text{for } p \leq n-1 \text{ and } q \leq \lfloor (n - p - 2)/2 \rfloor.$$

For $p \leq n-1$, we deduce that $E_{pq}^1 = 0$ for

$$p + q \leq p + \lfloor (n - p - 2)/2 \rfloor = \lfloor (n + p - 2)/2 \rfloor.$$

We now separate the cases where n is odd and even:

- If $n = 2d + 1$ is odd, then $E_{pq}^1 = 0$ for $p \leq n-1$ such that

$$p + q \leq \lfloor (n + p - 2)/2 \rfloor = \lfloor (2d + p - 1)/2 \rfloor = d + \lfloor (p-1)/2 \rfloor.$$

In particular, $E_{pq}^1 = 0$ for $p + q \leq d$ except for possibly E_{0d}^1 .

- If $n = 2d$ is even, then $E_{pq}^1 = 0$ for $p \leq n-1$ such that

$$p + q \leq \lfloor (n + p - 2)/2 \rfloor = \lfloor (2d + p - 2)/2 \rfloor = d - 1 + \lfloor p/2 \rfloor.$$

In particular, $E_{pq}^1 = 0$ for $p + q \leq d$ except for possibly E_{0d}^1 and $E_{1,d-1}^1$. □

The above proof used:

³⁷In fact, since $n \geq 3$ they even satisfy $p \leq n-2$.

Lemma 10.2. *For $a, b, d \in \mathbb{Z}$ with $d \geq 2$, we have $1 + \lfloor a/d \rfloor + \lfloor b/d \rfloor \geq \lfloor (a + b + 1)/d \rfloor$.*

Proof. Write $a = q_1d + r_1$ and $b = q_2d + r_2$ with $0 \leq r_i \leq d - 1$. We then have

$$\lfloor (a + b + 1)/d \rfloor = q_1 + q_2 + \lfloor (r_1 + r_2 + 1)/d \rfloor \leq q_1 + q_2 + \lfloor (2d - 1)/d \rfloor = \lfloor a/d \rfloor + \lfloor b/d \rfloor + 1. \quad \square$$

11. REMAINING TASKS (TYPE A)

Lemma 10.1 implies many cases of Theorem B.1'. To prove the remaining cases, we need to compute some differentials in our spectral sequence. We now explain the structure of the argument, postponing three calculations to the next section. Recall that Theorem B.1' is:

Theorem B.1'. *Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = A_n$. Then:*

- $H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ for $i \leq \lfloor (n - 1)/2 \rfloor$; and
- letting $\Delta = {}_k\Delta(\mathbf{G})$, the maps

$$\begin{aligned} H_i(\mathbf{L}_{\Delta[1]}(k); \text{St}(\mathbf{L}_{\Delta[1]})) &\rightarrow H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) \quad \text{and} \\ H_i(\mathbf{L}_{\Delta[n]}(k); \text{St}(\mathbf{L}_{\Delta[n]})) &\rightarrow H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) \end{aligned}$$

are both surjective for $i \leq \lfloor n/2 \rfloor$.

Proof. The proof is by induction on n . We proved the base cases $n \leq 2$ in Lemma 8.3, so we can assume that $n \geq 3$ and that the result is true for smaller ranks, i.e., that the $(n - 1)$ -surjectivity and vanishing hypothesis in type A holds.

Corollary 4.4 gives a spectral sequence E_{pq}^r converging to $H_{p+q}(\mathbf{G}(k); \text{St}(\mathbf{G}))$, and Lemma 10.1 implies that $E_{pq}^1 = 0$ for $p + q \leq \lfloor n/2 \rfloor - 1$. This implies that $H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ for $i \leq \lfloor n/2 \rfloor - 1$. Since our surjectivity claim is trivial when the target is 0, all that remains to prove are the following two claims:

Claim 1. *Assume that $n = 2d + 1$ with $d \geq 1$. Then $H_d(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$.*

In this case, Lemma 10.1 says that the only potentially nonzero term E_{pq}^1 in our spectral sequence with $p + q = d$ is E_{0d}^1 . We will prove in Lemma 12.2 below that the differential $E_{1d}^1 \rightarrow E_{0d}^1$ is surjective, so $E_{0d}^2 = 0$. This implies that $H_d(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$, as desired.

Claim 2. *Assume that $n = 2d$ with $d \geq 2$. Then the maps*

$$\begin{aligned} H_d(\mathbf{L}_{\Delta[1]}(k); \text{St}(\mathbf{L}_{\Delta[1]})) &\rightarrow H_d(\mathbf{G}(k); \text{St}(\mathbf{G})) \quad \text{and} \\ H_d(\mathbf{L}_{\Delta[n]}(k); \text{St}(\mathbf{L}_{\Delta[n]})) &\rightarrow H_d(\mathbf{G}(k); \text{St}(\mathbf{G})) \end{aligned}$$

are both surjective.

Lemma 8.2 says that these maps have the same image, so it is enough to prove that

$$(11.1) \quad H_d(\mathbf{L}_{\Delta[1]}(k); \text{St}(\mathbf{L}_{\Delta[1]})) \oplus H_d(\mathbf{L}_{\Delta[n]}(k); \text{St}(\mathbf{L}_{\Delta[n]}))$$

surjects onto $H_d(\mathbf{G}(k); \text{St}(\mathbf{G}))$. Lemma 10.1 says that the only potentially nonzero terms E_{pq}^1 in our spectral sequence with $p + q = d$ are E_{0d}^1 and $E_{1,d-1}^1$. We will prove in Lemma 12.3 below that the differential $E_{2,d-1}^1 \rightarrow E_{1,d-1}^1$ is surjective, so $E_{1,d-1}^2 = 0$. We will also prove in Lemma 12.1 below that the summand (11.1) of

$$E_{0d}^1 = \bigoplus_{R \in \mathcal{L}_0(\mathbf{G})} H_d(\mathbf{L}_R(k); \text{St}(\mathbf{L}_R)) = \bigoplus_{j=1}^n H_d(\mathbf{L}_{\Delta[j]}(k); \text{St}(\mathbf{L}_{\Delta[j]}))$$

surjects onto the cokernel of the differential $E_{1d}^1 \rightarrow E_{0d}^1$. It follows that E_{0d}^2 is a quotient of (11.1). Since E_{0d}^2 is the only potentially nonzero term of the form E_{pq}^2 with $p + q = d$, it follows that (11.1) surjects onto $H_d(\mathbf{G}(k); \text{St}(\mathbf{G}))$, as desired. $\square$

12. DIFFERENTIALS (TYPE A)

This final section of this part of the paper determines the images of three differentials whose calculations were needed in the previous section.

12.1. Differentials, I (type A). Our first differential calculation is:

Lemma 12.1. *Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = \mathbf{A}_{2d}$ for some $d \geq 2$. Assume the $(2d-1)$ -surjectivity and vanishing hypothesis in type A (Definition 8.4). Let E_{pq}^1 be the spectral sequence from Corollary 4.4. Then the summand*

$$H_d(\mathbf{L}_{\Delta[1]}(k); \text{St}(\mathbf{L}_{\Delta[1]})) \oplus H_d(\mathbf{L}_{\Delta[2d]}(k); \text{St}(\mathbf{L}_{\Delta[2d]}))$$

of E_{0d}^1 surjects onto the cokernel of the differential $E_{1d}^1 \rightarrow E_{0d}^1$.

Proof. As notation, for $1 \leq j_1, \dots, j_\ell \leq 2d$ let

$$M[j_1, \dots, j_\ell] = H_d(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]}(k); \text{St}(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]})).$$

We have

$$E_{0d}^1 = \bigoplus_{1 \leq j \leq 2d} M[j] \quad \text{and} \quad E_{1d}^1 = \bigoplus_{1 \leq j_1 < j_2 \leq 2d} M[j_1, j_2].$$

Consider some $1 < j < 2d$. We must prove that when we quotient E_{0d}^1 by the image of the differential $E_{1d}^1 \rightarrow E_{0d}^1$, the summand $M[j]$ of E_{0d}^1 is identified with a subspace of $M[1] \oplus M[2d]$. In fact, we will show that it is identified with a subspace of $M[1]$. Our argument will use the fact that $j < 2d$, so it does not show that $M[2d]$ is identified with a subspace of $M[1]$. Note that

$$(12.1) \quad {}_k\Phi(\mathbf{L}_{\Delta[j]}) = \mathbf{A}_{j-1} \times \mathbf{A}_{2d-j}$$

$$(12.2) \quad \mathbf{b}({}_k\Phi(\mathbf{L}_{\Delta[j]})) = 1 + \lfloor (j-2)/2 \rfloor + \lfloor (2d-j-1)/2 \rfloor = d-1.$$

There are two cases.

The first is that j is odd. Recall that by assumption $j > 1$. Let $f: M[1, j] \rightarrow M[j]$ and $g: M[1, j] \rightarrow M[1]$ be the obvious maps. It is then immediate from the definitions that the differential $E_{1d}^1 \rightarrow E_{0d}^1$ takes the summand $M[1, j]$ of E_{1d}^1 to E_{0d}^1 via the map

$$M[1, j] \xrightarrow{f \oplus (-g)} M[j] \oplus M[1] \hookrightarrow E_{0d}^1.$$

Since the $\mathbf{A}_{j-1}$ -factor in (12.1) has $j-1$ even and positive, we can use Lemma 9.1 (Levi vanishing and surjectivity) to see that f is surjective. Here we are using the fact that $\mathbf{b}({}_k\Phi(\mathbf{L}_{\Delta[j]})) + 1 = d$; cf. (12.2). Thus when we quotient E_{0d}^1 by the image of the differential, $M[j]$ is identified with a subspace of $M[1]$, as desired.

Now assume that j is even. Recall that by assumption $j < 2d$. Let $f': M[j, j+1] \rightarrow M[j+1]$ and $g': M[j, j+1] \rightarrow M[j]$ be the obvious maps. Just like above, the differential $E_{1d}^1 \rightarrow E_{0d}^1$ takes the summand $M[j, j+1]$ of E_{1d}^1 to E_{0d}^1 via the map

$$M[j, j+1] \xrightarrow{f' \oplus (-g')} M[j+1] \oplus M[j] \hookrightarrow E_{0d}^1.$$

Since the $\mathbf{A}_{2d-j}$ -factor in (12.1) has $2d-j$ even and positive, we can use Lemma 9.1 (Levi vanishing and surjectivity) to see that g' is surjective. Again, we are using the fact that $\mathbf{b}({}_k\Phi(\mathbf{L}_{\Delta[j]})) + 1 = d$; cf. (12.2). Thus when we quotient E_{0d}^1 by the image of the differential, $M[j]$ is identified with a subspace of $M[j+1]$. Since $j+1$ is odd, the previous paragraph shows that this quotient identifies $M[j+1]$ with a subspace of $M[1]$, completing the proof. $\square$

12.2. Differentials, II (type A). Our second differential calculation is:

Lemma 12.2. *Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = \mathbf{A}_{2d+1}$ for some $d \geq 1$. Assume the $2d$ -surjectivity and vanishing hypothesis in type A (Definition 8.4). Let E_{pq}^1 be the spectral sequence from Corollary 4.4. Then the differential $E_{1d}^1 \rightarrow E_{0d}^1$ is surjective.*

Proof. As notation, for $1 \leq j_1, \dots, j_\ell \leq 2d+1$ let

$$M[j_1, \dots, j_\ell] = H_d(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]}(k); \text{St}(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]})).$$

We have

$$E_{0d}^1 = \bigoplus_{1 \leq j \leq 2d+1} M[j] \quad \text{and} \quad E_{1d}^1 = \bigoplus_{1 \leq j_1 < j_2 \leq 2d+1} M[j_1, j_2].$$

Consider some $1 \leq j \leq 2d+1$. We must prove that when we quotient E_{0d}^1 by the image of the differential $E_{1d}^1 \rightarrow E_{0d}^1$, the summand $M[j]$ is killed. The first step is to show that many $M[j]$ already vanish:

Claim 1. *For $1 \leq j \leq 2d+1$ with j even, we have $M[j] = 0$.*

Proof of claim. Write $j = 2e$, so ${}_k\Phi(\mathbf{L}_{\Delta[2e]}) = \mathbf{A}_{2e-1} \times \mathbf{A}_{2d+1-2e}$. Lemma 9.1 (Levi vanishing and surjectivity) implies that $H_i(\mathbf{L}_{\Delta[2e]}(k); \text{St}(\mathbf{L}_{\Delta[2e]})) = 0$ for

$$i \leq 1 + \lfloor (2e-2)/2 \rfloor + \lfloor (2d-2e)/2 \rfloor = 1 + (e-1) + (d-e) = d.$$

In particular, $M[2e] = H_d(\mathbf{L}_{\Delta[2e]}(k); \text{St}(\mathbf{L}_{\Delta[2e]})) = 0$. □

Now consider $1 \leq j \leq 2d+1$ with j odd. In light of Claim 1, it is enough to prove that $M[j]$ is killed when we quotient E_{0d}^1 by the image of the differential $E_{1d}^1 \rightarrow E_{0d}^1$. Assume first that $j \neq 1$, so

$$(12.3) \quad {}_k\Phi(\mathbf{L}_{\Delta[j]}) = \begin{cases} \mathbf{A}_{j-1} \times \mathbf{A}_{2d-j+1} & \text{if } 1 < j < 2d+1, \\ \mathbf{A}_{2d} & \text{if } j = 2d+1. \end{cases}$$

Since j is odd, in both cases we have

$$(12.4) \quad \mathbf{b}({}_k\Phi(\mathbf{L}_{\Delta[j]})) = d-1.$$

We will show that $M[j]$ is killed by the image of the summand $M[j-1, j]$ of E_{1d}^1 . On the summand $M[j-1, j]$, the differential is the map

$$M[j-1, j] \longrightarrow M[j] \oplus M[j-1] \hookrightarrow E_{0d}^1.$$

Claim 1 says that $M[j-1] = 0$, so to show that this differential kills $M[j]$ it is enough to prove that $M[j-1, j] \rightarrow M[j]$ is surjective. Since the $\mathbf{A}_{j-1}$ in (12.3) has $j-1$ even and positive, this follows from Lemma 9.1 (Levi vanishing and surjectivity). Here we are using the fact that $\mathbf{b}({}_k\Phi(\mathbf{L}_{\Delta[j]})) + 1 = d$; cf. (12.4).

It remains to deal with the case $j = 1$. Note that

$$(12.5) \quad {}_k\Phi(\mathbf{L}_{\Delta[1]}) = \mathbf{A}_{2d},$$

$$(12.6) \quad \mathbf{b}({}_k\Phi(\mathbf{L}_{\Delta[1]})) = \lfloor (2d-1)/2 \rfloor = d-1.$$

In this case, we will use the summand $M[1, 2]$ of E_{1d}^1 . Just like above, on this summand this differential takes the form

$$M[1, 2] \longrightarrow M[2] \oplus M[1] \hookrightarrow E_{0d}^1.$$

Claim 1 says that $M[2] = 0$, and since the $\mathbf{A}_{2d}$ in (12.5) has $2d$ even and positive Lemma 9.1 (Levi vanishing and surjectivity) shows that the map $M[1, 2] \rightarrow M[1]$ is surjective. Here we are using the fact that $\mathbf{b}({}_k\Phi(\mathbf{L}_{\Delta[1]})) + 1 = d$; cf. (12.6). The lemma follows. □

12.3. Differentials, III (type A). Our final differential calculation is:

Lemma 12.3. *Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = \mathbf{A}_{2d}$ for some $d \geq 2$. Assume the $(2d-1)$ -surjectivity and vanishing hypothesis in type A (Definition 8.4). Let E_{pq}^1 be the spectral sequence from Corollary 4.4. Then the differential $E_{2,d-1}^1 \rightarrow E_{1,d-1}^1$ is surjective.*

Proof. As notation, for $1 \leq j_1, \dots, j_\ell \leq 2d$ let

$$M[j_1, \dots, j_\ell] = H_{d-1}(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]}(k); \text{St}(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]})).$$

We have

$$E_{1,d-1}^1 = \bigoplus_{1 \leq j_1 < j_2 \leq 2d} M[j_1, j_2] \quad \text{and} \quad E_{2,d-1}^1 = \bigoplus_{1 \leq j_1 < j_2 < j_3 \leq 2d} M[j_1, j_2, j_3].$$

Consider some $1 \leq j_1 < j_2 \leq 2d$. We must prove that when we quotient $E_{1,d-1}^1$ by the image of the differential $E_{2,d-1}^1 \rightarrow E_{1,d-1}^1$, the summand $M[j_1, j_2]$ is killed. The first step is to show that many $M[j_1, j_2]$ already vanish:

Claim 1. *For $1 \leq j_1 < j_2 \leq 2d$ with either j_1 even or j_2 odd, we have $M[j_1, j_2] = 0$.*

Proof of claim. To simplify our notation, we will let $A_0 = \{0\} \subset \mathbb{R}^0$, regarded as a trivial root system of rank 0. For a product $A_{n_1} \times \dots \times A_{n_m}$ with $n_j \geq 0$, the bound

$$(m-1) + \lfloor (n_1-1)/2 \rfloor + \dots + \lfloor (n_m-1)/2 \rfloor$$

for homology vanishing from Lemma 9.1 (Levi vanishing and surjectivity) is still correct since any n_j with $n_j = 0$ contribute $+1$ to $(m-1)$ and -1 to the rest of the sum.

Now observe that ${}_k\Phi(\mathbf{L}_{\Delta[j_1, j_2]}) = A_{j_1-1} \times A_{j_2-j_1-1} \times A_{2d-j_2}$. By Lemma 9.1 (Levi vanishing and surjectivity), we have $H_i(\mathbf{L}_{\Delta[j_1, j_2]}(k); \text{St}(\mathbf{L}_{\Delta[j_1, j_2]})) = 0$ for

$$\begin{aligned} i &\leq 2 + \lfloor (j_1-2)/2 \rfloor + \lfloor (j_2-j_1-2)/2 \rfloor + \lfloor (2d-j_2-1)/2 \rfloor \\ &= d + \lfloor j_1/2 \rfloor + \lfloor (j_2-j_1)/2 \rfloor + \lfloor (-j_2-1)/2 \rfloor. \end{aligned}$$

It follows that $M[j_1, j_2] = 0$ if the right hand side is at least $d-1$, i.e., if

$$(12.7) \quad \lfloor j_1/2 \rfloor + \lfloor (j_2-j_1)/2 \rfloor + \lfloor (-j_2-1)/2 \rfloor \geq -1.$$

The left hand side of (12.7) is -1 if either j_1 is even or j_2 is odd, and is -2 otherwise. The claim follows. $\square$

Now consider $1 \leq j_1 < j_2 \leq 2d$ with j_1 odd and j_2 even. In light of Claim 1, it is enough to prove that $M[j_1, j_2]$ is killed when we quotient $E_{1,d-1}^1$ by the image of the differential $E_{2,d-1}^1 \rightarrow E_{1,d-1}^1$. Assume first that $j_1 > 1$, so

$$(12.8) \quad {}_k\Phi(\mathbf{L}_{\Delta[j_1, j_2]}) = \begin{cases} A_{j_1-1} \times A_{j_2-j_1-1} \times A_{2d-j_2} & \text{if } j_2 > j_1 + 1, j_2 < 2d, \\ A_{j_1-1} \times A_{2d-j_1-1} & \text{if } j_2 > j_1 + 1, j_2 = 2d, \\ A_{j_1-1} \times A_{2d-j_1-1} & \text{if } j_2 = j_1 + 1, j_2 < 2d, \\ A_{2d-2} & \text{if } j_2 = j_1 + 1, j_2 = 2d. \end{cases}$$

In all four cases, since j_1 is odd and j_2 is even we have

$$(12.9) \quad \mathbf{b}({}_k\Phi(\mathbf{L}_{\Delta[j_1, j_2]})) = d-2.$$

We will show that $M[j_1, j_2]$ is killed by the image of the summand $M[j_1-1, j_1, j_2]$ of $E_{2,d-1}^1$. On the summand $M[j_1-1, j_1, j_2]$, the differential is the map

$$M[j_1-1, j_1, j_2] \longrightarrow M[j_1, j_2] \oplus M[j_1-1, j_2] \oplus M[j_1-1, j_1] \hookrightarrow E_{1,d-1}^1.$$

Since $j_1 - 1$ is even, Claim 1 says that $M[j_1 - 1, j_2] = M[j_1 - 1, j_1] = 0$, so to show that this differential kills $M[j_1, j_2]$ it is enough to prove that $M[j_1 - 1, j_1, j_2] \rightarrow M[j_1, j_2]$ is surjective. Since the first A-term in all four cases of (12.8) has a subscript that is even and positive, this follows from Lemma 9.1 (Levi vanishing and surjectivity). Here we are using the fact that $\mathbf{b}({}_k\Phi(\mathbf{L}_{\Delta_{[j_1, j_2]}))} + 1 = d - 1$; cf. (12.9).

Assume next that $j_2 < 2d$. We remind the reader that j_1 is odd and j_2 is even. We have

$$(12.10) \quad {}_k\Phi(\mathbf{L}_{\Delta_{[j_1, j_2]}}) = \begin{cases} A_{j_1-1} \times A_{j_2-j_1-1} \times A_{2d-j_2} & \text{if } j_1 > 1, j_2 > j_1 + 1, \\ A_{j_2-2} \times A_{2d-j_2} & \text{if } j_1 = 1, j_2 > j_1 + 1, \\ A_{j_2-2} \times A_{2d-j_2} & \text{if } j_1 > 1, j_2 = j_1 + 1, \\ A_{2d-2} & \text{if } j_1 = 1, j_2 = j_1 + 1. \end{cases}$$

In all four cases, since j_1 is odd and j_2 is even we have

$$(12.11) \quad \mathbf{b}({}_k\Phi(\mathbf{L}_{\Delta_{[j_1, j_2]}))} = d - 2.$$

We will show that $M[j_1, j_2]$ is killed by the image of the summand $M[j_1, j_2, j_2 + 1]$ of $E_{2, d-1}^1$. On the summand $M[j_1, j_2, j_2 + 1]$, the differential is the map

$$M[j_1, j_2, j_2 + 1] \longrightarrow M[j_2, j_2 + 1] \oplus M[j_1, j_2 + 1] \oplus M[j_1, j_2] \hookrightarrow E_{1, d-1}^1.$$

Since $j_2 + 1$ is odd, Claim 1 says that $M[j_2, j_2 + 1] = M[j_1, j_2 + 1] = 0$, so to show that this differential kills $M[j_1, j_2]$ it is enough to prove that $M[j_1, j_2, j_2 + 1] \rightarrow M[j_1, j_2]$ is surjective. Since the last A-term in all our cases of (12.10) has a subscript that is even and positive, this follows from Lemma 9.1 (Levi vanishing and surjectivity). Here we are using the fact that $\mathbf{b}({}_k\Phi(\mathbf{L}_{\Delta_{[j_1, j_2]}))} + 1 = d - 1$; cf. (12.11).

The only case not covered by the previous two paragraphs is $j_1 = 1$ and $j_2 = 2d$. We have

$$(12.12) \quad {}_k\Phi(\mathbf{L}_{\Delta_{[1, 2d]}}) = A_{2d-2},$$

$$(12.13) \quad \mathbf{b}({}_k\Phi(\mathbf{L}_{\Delta_{[1, 2d]}})) = \lfloor (2d - 3)/2 \rfloor = d - 2.$$

In this case, we will use the summand $M[1, 2, 2d]$ of $E_{2, d-1}^1$, which makes sense since $d \geq 2$ by assumption. Just like above, on this summand the differential takes the form

$$M[1, 2, 2d] \longrightarrow M[2, 2d] \oplus M[1, 2d] \oplus M[1, 2] \hookrightarrow E_{1, d-1}^1.$$

Claim 1 says that $M[2, 2d] = 0$, and the previous paragraph implies that $M[1, 2]$ dies in the cokernel of the differential. Finally, since the A_{2d-2} in (12.12) has $2d - 2$ even and positive Lemma 9.1 (Levi vanishing and surjectivity) shows that the map $M[1, 2, 2d] \rightarrow M[1, 2d]$ is surjective. Here we are using the fact that $\mathbf{b}({}_k\Phi(\mathbf{L}_{\Delta_{[j_1, j_2]}))} + 1 = d - 1$; cf. (12.13). It follows that $M[1, 2d]$ dies in the cokernel of the differential, as desired. $\square$

Part 3. Vanishing in types B and C and BC (Theorem B.2)

This part of the paper is devoted to Theorem B.2, which is our vanishing result in types B and C and BC. The proofs follow the same outline as those for type A in Part 2.

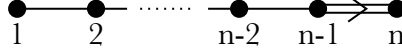
13. VANISHING AND SURJECTIVITY (TYPES B AND C AND BC)

In this section, we first introduce some notation for the standard Levi factors of groups of type B_n and C_n and BC_n . We then state a stronger version of Theorem B.2 and prove it in rank 2.

13.1. Levi factor notation. Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) \in \{B_n, C_n, BC_n\}$ for some $n \geq 2$. We introduce the following convention:

Convention 13.1. Throughout this part of the paper, we will write X_n with X meaning either B or C or BC. These root systems are only defined for $n \geq 2$, but to allow uniform statements we will define $X_1 = A_1$. $\square$

Let $\Delta = {}_k\Delta(\mathbf{G})$ be the set of simple roots of ${}_k\Phi(\mathbf{G})$. Number the elements of Δ from left to right as in the usual Dynkin diagram; for instance, if ${}_k\Phi(\mathbf{G}) = B_n$ we have:



For $1 \leq j_1, \dots, j_\ell \leq n$, let $\Delta[j_1, \dots, j_\ell]$ be the result of removing the simple roots labeled $j_1, \dots, j_\ell$ from Δ . We thus have a standard Levi subgroup $\mathbf{L}_{\Delta[j_1, \dots, j_\ell]}$ of $\mathbf{G}$.

Example 13.2. We have ${}_k\Phi(\mathbf{L}_{\Delta[1]}) = X_{n-1}$ and ${}_k\Phi(\mathbf{L}_{\Delta[n]}) = A_{n-1}$, while for $2 \leq j \leq n-1$ we have ${}_k\Phi(\mathbf{L}_{\Delta[j]}) = A_{j-1} \times X_{n-j}$. In general, for distinct $1 \leq j_1, \dots, j_\ell \leq n$ we have either

$$\begin{aligned} {}_k\Phi(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]}) &= A_{n_1} \times \dots \times A_{n_{m-1}} \times X_{n_m} \quad \text{or} \\ {}_k\Phi(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]}) &= A_{n_1} \times \dots \times A_{n_m} \end{aligned}$$

with $n_1 + \dots + n_m + \ell = n$. $\square$

We have a Reeder map (cf. §2.6) of the form $\text{St}(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]}) \rightarrow \text{St}(\mathbf{G})$, and thus maps $H_i(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]}(k); \text{St}(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]})) \rightarrow H_i(\mathbf{G}(k); \text{St}(\mathbf{G}))$.

13.2. Strong vanishing. The main result we will prove in this part of the paper is:

Theorem B.2'. *Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = X_n$ for some $X \in \{B, C, BC\}$ and $n \geq 2$. Then:*

- $H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ for $i \leq \lfloor (n-2)/2 \rfloor$; and
- letting $\Delta = {}_k\Delta(\mathbf{G})$, the map $H_i(\mathbf{L}_{\Delta[1]}(k); \text{St}(\mathbf{L}_{\Delta[1]})) \rightarrow H_i(\mathbf{G}(k); \text{St}(\mathbf{G}))$ is surjective for $i \leq \lfloor (n-1)/2 \rfloor$.

This strengthens Theorem B.2 by adding the indicated surjectivity statement. To avoid degenerate cases, we prove:

Lemma 13.3. *Theorem B.2' holds for $n = 2$.*

Proof. Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = X_2$ for some $X \in \{B, C, BC\}$. Since a map to 0 is surjective, for $\mathbf{G}$ Theorem B.2' only asserts that $H_0(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$, which follows from Lemma 3.2. $\square$

Because of Lemma 13.3, for the rest of this part of the paper we can assume that $n \geq 3$. We will also assume as an inductive hypothesis that we have already proved Theorem B.2' in smaller ranks. For this, we make the following definition:

Definition 13.4. For $r \geq 2$, the *r-surjectivity and vanishing hypothesis in type BC* is as follows. Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = X_n$ for some $X \in \{B, C, BC\}$ and $2 \leq n \leq r$. Then:

- $H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ for $i \leq \lfloor (n-2)/2 \rfloor$; and
- letting $\Delta = {}_k\Delta(\mathbf{G})$, the map $H_i(\mathbf{L}_{\Delta[1]}(k); \text{St}(\mathbf{L}_{\Delta[1]})) \rightarrow H_i(\mathbf{G}(k); \text{St}(\mathbf{G}))$ is surjective for $i \leq \lfloor (n-1)/2 \rfloor$. $\square$

Remark 13.5. We have already proven Theorem B.1', so we also have available to us vanishing and surjectivity results in type A. $\square$

14. VANISHING AND SURJECTIVITY FOR LEVI SUBGROUPS (TYPES B AND C AND BC)

In this section, we show how to use the r -surjectivity and vanishing hypothesis in type BC to analyze the homology of standard Levi subgroups. Our main result is as follows. Its statement uses the ordering on the simple roots of $A_{n_{j_0}}$ and $X_{n_{j_0}}$ for $X \in \{B, C, BC\}$ discussed in §8.1 and §13.1.

Lemma 14.1 (Levi vanishing and surjectivity). *For some $n \geq 3$, assume the $(n-1)$ -surjectivity and vanishing hypothesis in type BC (Definition 13.4). Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = X_n$ for some $X \in \{B, C, BC\}$. Let $\Delta \subset {}_k\Delta(\mathbf{G})$ be a set of simple roots with $\Delta \neq {}_k\Delta(\mathbf{G})$. Write*

$${}_k\Phi(\mathbf{L}_\Delta) = A_{n_1} \times \cdots \times Z_{n_m} \quad \text{with } Z \in \{A, X\}.$$

We thus have $n_1, \dots, n_{m-1} \geq 1$, and if $Z = X$ then $n_m \geq 2$ while if $Z = A$ then $n_m \geq 1$. Set $b = \mathbf{b}({}_k\Phi(\mathbf{L}_\Delta))$, so

$$b = \begin{cases} (m-1) + \lfloor (n_1-1)/2 \rfloor + \cdots + \lfloor (n_{m-1}-1)/2 \rfloor + \lfloor (n_m-1)/2 \rfloor & \text{if } Z = A, \\ (m-1) + \lfloor (n_1-1)/2 \rfloor + \cdots + \lfloor (n_{m-1}-1)/2 \rfloor + \lfloor (n_m-2)/2 \rfloor & \text{if } Z = X. \end{cases}$$

Then the following hold:

- (i) We have $H_i(\mathbf{L}_\Delta(k); \text{St}(\mathbf{L}_\Delta)) = 0$ for $i \leq b$.
- (ii) For some $1 \leq j_0 \leq n$, assume one of the following:
 - n_{j_0} is even and nonzero, $1 \leq j_0 \leq m-1$, and $\Delta' \subset \Delta$ is the set of simple roots obtained by removing either the first or last simple root from $A_{n_{j_0}}$.
 - n_{j_0} is even and nonzero, $j_0 = m$ and $Z = A$, and $\Delta' \subset \Delta$ is the set of simple roots obtained by removing either the first or last simple root from A_{n_m} .
 - n_{j_0} is odd, $j_0 = m$ and $Z = X$, and $\Delta' \subset \Delta$ is the set of simple roots obtained by removing the first simple root from Z_{n_m} .

Then the map $H_{b+1}(\mathbf{L}_{\Delta'}(k); \text{St}(\mathbf{L}_{\Delta'})) \rightarrow H_{b+1}(\mathbf{L}_\Delta(k); \text{St}(\mathbf{G}))$ is surjective.

Proof. Since $\Delta \neq {}_k\Delta(\mathbf{G})$, we have $n_j \leq n-1$ for $1 \leq j \leq m$. The $(n-1)$ -surjectivity and vanishing hypothesis in type BC thus applies to all reductive groups $\mathbf{H}_m$ with ${}_k\Phi(\mathbf{H}_m) = X_{n_m}$. Theorem B.1' also gives a vanishing and surjectivity result for all reductive groups $\mathbf{H}_j$ with ${}_k\Phi(\mathbf{H}_j) = A_{n_j}$. This gives the hypothesis (♥) in Lemma 6.4 (reducible vanishing). Applying Lemma 6.4, we deduce (i). Similarly, for Δ' as in (ii) it gives the hypotheses (♥) and (♥♥) in Lemma 6.5 (reducible surjectivity). Applying Lemma 6.5, we deduce (ii). $\square$

15. VANISHING REGION (TYPES B AND C AND BC)

Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = X_n$ for some $X \in \{B_n, C_n, BC_n\}$ and $n \geq 3$. Corollary 4.4 gives a spectral sequence E_{pq}^r converging to $H_{p+q}(\mathbf{G}(k); \text{St}(\mathbf{G}))$ with

$$E_{pq}^1 \cong \begin{cases} \bigoplus_{R \in \mathcal{L}_p(\mathbf{G})} H_q(\mathbf{L}_R(k); \text{St}(\mathbf{L}_R)) & \text{if } 0 \leq p \leq n-1 \\ H_q(\mathbf{G}(k); \text{St}(\mathbf{G})^{\otimes 2}) & \text{if } p = n, \\ 0 & \text{otherwise.} \end{cases}$$

The following lemma shows that our inductive hypothesis implies that many terms of this spectral sequence vanish.

Lemma 15.1. *Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = X_n$ for some $X \in \{B_n, C_n, BC_n\}$ and $n \geq 3$. Assume the $(n-1)$ -surjectivity and vanishing hypothesis in type BC (Definition 13.4). Let E_{pq}^1 be the spectral sequence from Corollary 4.4. Then the following hold:*

- For $n = 2d + 2$ with $d \geq 1$, we have $E_{pq}^1 = 0$ for $p + q \leq d$ except for possibly E_{0d}^1 .

- For $n = 2d + 1$ with $d \geq 1$, we have $E_{pq}^1 = 0$ for $p + q \leq d$ except for possibly E_{0d}^1 and $E_{1,d-1}^1$. For $n = 3$ (so $d = 1$), we also have $E_{1,d-1}^1 = 0$.

Proof. This is identical to the proof of Lemma 10.1 in type A. The only difference is that the standard Levi subgroups can be of two types:

- $A_{n_1} \times \cdots \times A_{n_m}$, where the vanishing range for H_i given by Theorem B.1' is

$$i \leq (m-1) + \lfloor (n_1-1)/2 \rfloor + \cdots + \lfloor (n_m-1)/2 \rfloor.$$

Repeatedly applying the inequality $1 + \lfloor a/2 \rfloor + \lfloor b/2 \rfloor \geq \lfloor (a+b+1)/2 \rfloor$ from Lemma 10.2, the right hand side is at least

$$(15.1) \quad \lfloor (n_1 + \cdots + n_m - 1)/2 \rfloor.$$

- $A_{n_1} \times \cdots \times A_{n_{m-1}} \times X_{n_m}$, where the vanishing range for H_i given by Lemma 14.1 (Levi vanishing and surjectivity) is

$$(15.2) \quad i \leq (m-1) + \lfloor (n_1-1)/2 \rfloor + \cdots + \lfloor (n_{m-1}-1)/2 \rfloor + \lfloor (n_m-2)/2 \rfloor.$$

Repeatedly applying the inequality $1 + \lfloor a/2 \rfloor + \lfloor b/2 \rfloor \geq \lfloor (a+b+1)/2 \rfloor$ from Lemma 10.2, the right hand side is at least

$$(15.3) \quad \lfloor (n_1 + \cdots + n_m - 2)/2 \rfloor.$$

The slightly worse vanishing range in (15.3) (as opposed to (15.1), which is the same vanishing range we found in the proof of Lemma 10.1) accounts for the slightly worse range in the statement of the lemma. Finally, the fact that $E_{1,d-1}^1 = 0$ for $d = 1$ follows from the fact that $E_{1,0}^1$ is a direct sum of H_0 -groups, and these all vanish by Lemma 3.2. $\square$

Remark 15.2. For later use, note that we were lucky that there was only one term of the form $\lfloor (n_j - 2)/2 \rfloor$ on the right hand side of (15.2). If there were more than one, then the vanishing range would not be strong enough for the lemma. For instance, the bound

$$i \leq (m-1) + \lfloor (n_1-2)/2 \rfloor + \cdots + \lfloor (n_m-2)/2 \rfloor$$

would not be good enough. Indeed, if all the n_j were odd then the right hand side equals

$$\lfloor (n_1 + \cdots + n_m - m - 1)/2 \rfloor,$$

which is too small to give anything like Lemma 15.1. $\square$

16. REMAINING TASKS (TYPES B AND C AND BC)

Lemma 15.1 implies many cases of Theorem B.2'. To prove the remaining cases, we need to compute some differentials in our spectral sequence. We now explain the structure of the argument, postponing three calculations to the next section. Recall that Theorem B.2' is:

Theorem B.2'. *Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = X_n$ for some $X \in \{B, C, BC\}$ and $n \geq 2$. Then:*

- $H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ for $i \leq \lfloor (n-2)/2 \rfloor$; and
- letting $\Delta = {}_k\Delta(\mathbf{G})$, the map $H_i(\mathbf{L}_{\Delta[1]}(k); \text{St}(\mathbf{L}_{\Delta[1]})) \rightarrow H_i(\mathbf{G}(k); \text{St}(\mathbf{G}))$ is surjective for $i \leq \lfloor (n-1)/2 \rfloor$.

Proof. The proof is by induction on n . We proved the base case $n = 2$ in Lemma 13.3, so we can assume that $n \geq 3$ and that the result is true for smaller ranks, i.e., that the $(n-1)$ -surjectivity and vanishing hypothesis in type BC holds.

Corollary 4.4 gives a spectral sequence E_{pq}^r converging to $H_{p+q}(\mathbf{G}(k); \text{St}(\mathbf{G}))$, and Lemma 15.1 implies that $E_{pq}^1 = 0$ for $p + q \leq \lfloor (n-1)/2 \rfloor - 1$. This implies that $H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ for $i \leq \lfloor (n-1)/2 \rfloor - 1$. Since our surjectivity claim is trivial when the target is 0, all that remains to prove are the following two claims:

Claim 1. *Assume that $n = 2d + 2$ with $d \geq 1$. Then $H_d(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$.*

In this case, Lemma 15.1 says that the only potentially nonzero term E_{pq}^1 in our spectral sequence with $p + q = d$ is E_{0d}^1 . We will prove in Lemma 17.2 below that the differential $E_{1d}^1 \rightarrow E_{0d}^1$ is surjective, so $E_{0d}^2 = 0$. This implies that $H_d(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$, as desired.

Claim 2. *Assume that $n = 2d + 1$ with $d \geq 1$. Then the map*

$$H_d(\mathbf{L}_{\Delta[1]}(k); \text{St}(\mathbf{L}_{\Delta[1]})) \rightarrow H_d(\mathbf{G}(k); \text{St}(\mathbf{G}))$$

is surjective.

Lemma 15.1 says that the only potentially nonzero terms E_{pq}^1 in our spectral sequence with $p + q = d$ are E_{0d}^1 and $E_{1,d-1}^1$. Lemma 15.1 also says that $E_{1,d-1}^1 = 0$ if $d = 1$, and we will prove in Lemma 17.3 below that the differential $E_{2,d-1}^1 \rightarrow E_{1,d-1}^1$ is surjective for $d \geq 2$. It follows that in all cases $E_{1,d-1}^2 = 0$. We will also prove in Lemma 17.1 below that the summand $H_d(\mathbf{L}_{\Delta[1]}(k); \text{St}(\mathbf{L}_{\Delta[1]}))$ of

$$E_{0d}^1 = \bigoplus_{R \in \mathcal{L}_0(\mathbf{G})} H_d(\mathbf{L}_R(k); \text{St}(\mathbf{L}_R)) = \bigoplus_{j=1}^n H_d(\mathbf{L}_{\Delta[j]}(k); \text{St}(\mathbf{L}_{\Delta[j]}))$$

surjects onto the cokernel of the differential $E_{1d}^1 \rightarrow E_{0d}^1$. It follows that E_{0d}^2 is a quotient of $H_d(\mathbf{L}_{\Delta[1]}(k); \text{St}(\mathbf{L}_{\Delta[1]}))$. Since E_{0d}^2 is the only potentially nonzero term of the form E_{pq}^2 with $p + q = d$, it follows that $H_d(\mathbf{L}_{\Delta[1]}(k); \text{St}(\mathbf{L}_{\Delta[1]}))$ surjects onto $H_d(\mathbf{G}(k); \text{St}(\mathbf{G}))$, as desired. $\square$

17. DIFFERENTIALS (TYPES B AND C AND BC)

This final section of this part of the paper determines the images of three differentials whose calculations were needed in the previous section.

17.1. Differentials, I (types B and C and BC). Our first differential calculation is:

Lemma 17.1. *Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = X_{2d+1}$ for some $X \in \{B_n, C_n, BC_n\}$ and $d \geq 1$. Assume the $2d$ -surjectivity and vanishing hypothesis in type BC (Definition 13.4). Let E_{pq}^1 be the spectral sequence from Corollary 4.4. Then the summand $H_d(\mathbf{L}_{\Delta[1]}(k); \text{St}(\mathbf{L}_{\Delta[1]}))$ of E_{0d}^1 surjects onto the cokernel of the differential $E_{1d}^1 \rightarrow E_{0d}^1$.*

Proof. As notation, for $1 \leq j_1, \dots, j_\ell \leq 2d + 1$ let

$$M[j_1, \dots, j_\ell] = H_d(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]}(k); \text{St}(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]})).$$

We have

$$E_{0d}^1 = \bigoplus_{1 \leq j \leq 2d+1} M[j] \quad \text{and} \quad E_{1d}^1 = \bigoplus_{1 \leq j_1 < j_2 \leq 2d+1} M[j_1, j_2].$$

Consider some $1 < j \leq 2d + 1$. We must prove that when we quotient E_{0d}^1 by the image of the differential $E_{1d}^1 \rightarrow E_{0d}^1$, the summand $M[j]$ of E_{0d}^1 is identified with a subspace of $M[1]$. Note that

$$(17.1) \quad {}_k\Phi(\mathbf{L}_{\Delta[j]}) = \begin{cases} A_{j-1} \times X_{2d+1-j} & \text{if } j \leq 2d-1, \\ A_{2d-1} \times A_1 & \text{if } j = 2d, \\ A_{2d} & \text{if } j = 2d+1, \end{cases}$$

$$(17.2) \quad \mathbf{b}({}_k\Phi(\mathbf{L}_{\Delta[j]})) = \begin{cases} 1 + \lfloor (j-2)/2 \rfloor + \lfloor (2d-j-1)/2 \rfloor = d-1 & \text{if } j \leq 2d-1, \\ 1 + \lfloor (2d-2)/2 \rfloor + \lfloor (1-1)/2 \rfloor = d & \text{if } j = 2d, \\ \lfloor (2d-1)/2 \rfloor = d-1 & \text{if } j = 2d+1. \end{cases}$$

It follows that $M[2d] = 0$, so we do not need to deal with that case. There are three other cases.

The first case is j odd and $j \neq 2d + 1$. Recall that by assumption $j > 1$. The differential $E_{1d}^1 \rightarrow E_{0d}^1$ takes the summand $M[1, j]$ of E_{1d}^1 to E_{0d}^1 via the map

$$M[1, j] \longrightarrow M[j] \oplus M[1] \hookrightarrow E_{0d}^1.$$

Since the A_{j-1} -factor in (17.1) has $j - 1$ even and positive, we can use Lemma 14.1 (Levi vanishing and surjectivity) to see that the map $M[1, j] \rightarrow M[j]$ is surjective. Here we are using the fact that $\mathbf{b}({}_k\Phi(\mathbf{L}_{\Delta[j]})) + 1 = d$; cf. (17.2). Thus when we quotient E_{0d}^1 by the image of the differential, $M[j]$ is identified with a subspace of $M[1]$, as desired.

The second case is $j = 2d + 1$. The differential $E_{1d}^1 \rightarrow E_{0d}^1$ takes the summand $M[1, 2d + 1]$ of E_{1d}^1 to E_{0d}^1 via the map

$$M[1, 2d + 1] \longrightarrow M[2d + 1] \oplus M[1] \hookrightarrow E_{0d}^1.$$

Since the A_{2d} in (17.1) has $2d$ even and positive, we can use Lemma 14.1 (Levi vanishing and surjectivity) to see that the map $M[1, 2d + 1] \rightarrow M[2d + 1]$ is surjective. Here we are using the fact that $\mathbf{b}({}_k\Phi(\mathbf{L}_{\Delta[j]})) + 1 = d$; cf. (17.2). Thus when we quotient E_{0d}^1 by the image of the differential, $M[2d + 1]$ is identified with a subspace of $M[1]$, as desired.

The third case is j even and $j \neq 2d$. The differential $E_{1d}^1 \rightarrow E_{0d}^1$ takes the summand $M[j, j + 1]$ of E_{1d}^1 to E_{0d}^1 via the map

$$M[j, j + 1] \longrightarrow M[j + 1] \oplus M[j] \hookrightarrow E_{0d}^1.$$

Since the X_{2d+1-j} -factor in (17.1) has $2d + 1 - j$ odd, we can use Lemma 14.1 (Levi vanishing and surjectivity) to see that the map $M[j, j + 1] \rightarrow M[j]$ is surjective. Again, we are using the fact that $\mathbf{b}({}_k\Phi(\mathbf{L}_{\Delta[j]})) + 1 = d$; cf. (17.2). Thus when we quotient E_{0d}^1 by the image of the differential, $M[j]$ is identified with a subspace of $M[j + 1]$. Since $j + 1$ is odd, the previous paragraph shows that this quotient identifies $M[j + 1]$ with a subspace of $M[1]$, as desired. $\square$

17.2. Differentials, II (types B and C and BC). Our second differential calculation is:

Lemma 17.2. *Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = X_{2d+2}$ for some $X \in \{B_n, C_n, BC_n\}$ and $d \geq 1$. Assume the $(2d + 1)$ -surjectivity and vanishing hypothesis in type BC (Definition 13.4). Let E_{pq}^1 be the spectral sequence from Corollary 4.4. Then the differential $E_{1d}^1 \rightarrow E_{0d}^1$ is surjective.*

Proof. As notation, for $1 \leq j_1, \dots, j_\ell \leq 2d + 2$ let

$$M[j_1, \dots, j_\ell] = H_d(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]}(k); \text{St}(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]})).$$

We have

$$E_{0d}^1 = \bigoplus_{1 \leq j \leq 2d+2} M[j] \quad \text{and} \quad E_{1d}^1 = \bigoplus_{1 \leq j_1 < j_2 \leq 2d+2} M[j_1, j_2].$$

Consider some $1 \leq j \leq 2d + 2$. We must prove that when we quotient E_{0d}^1 by the image of the differential $E_{1d}^1 \rightarrow E_{0d}^1$, the summand $M[j]$ is killed. The first step is to show that many $M[j]$ already vanish:

Claim 1. *For $1 \leq j \leq 2d + 2$ with j even, we have $M[j] = 0$.*

Proof of claim. Write $j = 2e$, so

$${}_k\Phi(\mathbf{L}_{\Delta[2e]}) = \begin{cases} A_{2e-1} \times X_{2d+2-2e} & \text{if } 2e \neq 2d + 2, \\ A_{2d+1} & \text{if } 2e = 2d + 2. \end{cases}$$

Lemma 14.1 (Levi vanishing and surjectivity) implies that $H_i(\mathbf{L}_{\Delta[2e]}(k); \text{St}(\mathbf{L}_{\Delta[2e]})) = 0$ for

$$i \leq \begin{cases} 1 + \lfloor (2e-2)/2 \rfloor + \lfloor (2d-2e)/2 \rfloor = 1 + (e-1) + (d-e) = d & \text{if } 2e \neq 2d+2, \\ \lfloor 2d/2 \rfloor = d & \text{if } 2e = 2d+2. \end{cases}$$

In particular, $M[2e] = H_d(\mathbf{L}_{\Delta[2e]}(k); \text{St}(\mathbf{L}_{\Delta[2e]})) = 0$. $\square$

Now consider $1 \leq j \leq 2d+2$ with j odd. In light of Claim 1, it is enough to prove that $M[j]$ is killed when we quotient E_{0d}^1 by the image of the differential $E_{1d}^1 \rightarrow E_{0d}^1$. Since j is odd, we have

$$(17.3) \quad {}_k\Phi(\mathbf{L}_{\Delta[j]}) = \begin{cases} A_{j-1} \times X_{2d+2-j} & \text{if } j \neq 1, 2d+1, \\ X_{2d+1} & \text{if } j = 1, \\ A_{2d} \times A_1 & \text{if } j = 2d+1, \end{cases}$$

$$(17.4) \quad \mathbf{b}({}_k\Phi(\mathbf{L}_{\Delta[j]})) = \begin{cases} 1 + \lfloor (j-2)/2 \rfloor + \lfloor (2d-j)/2 \rfloor = d-1 & \text{if } j \neq 1, 2d+1, \\ \lfloor (2d-2)/2 \rfloor = d-1 & \text{if } j = 1, \\ 1 + \lfloor (2d-1)/2 \rfloor + \lfloor (1-1)/2 \rfloor = d & \text{if } j = 2d+1. \end{cases}$$

It follows that $M[2d+1] = 0$, so we do not need to deal with that case. There are two other cases.

The first case is $j \neq 1$. Remember that j is odd. On the summand $M[j-1, j]$, the differential is the map

$$M[j-1, j] \longrightarrow M[j] \oplus M[j-1] \hookrightarrow E_{0d}^1.$$

Claim 1 says that $M[j-1] = 0$, so to show that this differential kills $M[j]$ it is enough to prove that $M[j-1, j] \rightarrow M[j]$ is surjective. Since the A_{j-1} in (17.3) has $j-1$ even and positive, this follows from Lemma 14.1 (Levi vanishing and surjectivity). Here we are using the fact that $\mathbf{b}({}_k\Phi(\mathbf{L}_{\Delta[j]})) + 1 = d$; cf. (17.4).

The second case is $j = 1$. On the summand $M[1, 2]$, the differential is the map

$$M[1, 2] \longrightarrow M[2] \oplus M[1] \hookrightarrow E_{0d}^1.$$

Claim 1 says that $M[2] = 0$, and since the X_{2d+1} in (17.4) has $2d+1$ odd Lemma 14.1 (Levi vanishing and surjectivity) shows that the map $M[1, 2] \rightarrow M[1]$ is surjective. Here we are using the fact that $\mathbf{b}({}_k\Phi(\mathbf{L}_{\Delta[1]})) + 1 = d$; cf. (17.4). The lemma follows. $\square$

17.3. Differentials, III (types B and C and BC). Our final differential calculation is:

Lemma 17.3. *Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = X_{2d+1}$ for some $X \in \{B_n, C_n, BC_n\}$ and $d \geq 2$. Assume the 2d-surjectivity and vanishing hypothesis in type BC (Definition 13.4). Let E_{pq}^1 be the spectral sequence from Corollary 4.4. Then the differential $E_{2,d-1}^1 \rightarrow E_{1,d-1}^1$ is surjective.*

Proof. As notation, for $1 \leq j_1, \dots, j_\ell \leq 2d+1$ let

$$M[j_1, \dots, j_\ell] = H_{d-1}(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]}(k); \text{St}(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]})).$$

We have

$$E_{1,d-1}^1 = \bigoplus_{1 \leq j_1 < j_2 \leq 2d+1} M[j_1, j_2] \quad \text{and} \quad E_{2,d-1}^1 = \bigoplus_{1 \leq j_1 < j_2 < j_3 \leq 2d+1} M[j_1, j_2, j_3].$$

Consider some $1 \leq j_1 < j_2 \leq 2d+1$. We must prove that when we quotient $E_{1,d-1}^1$ by the image of the differential $E_{2,d-1}^1 \rightarrow E_{1,d-1}^1$, the summand $M[j_1, j_2]$ is killed. The first step is to show that many $M[j_1, j_2]$ already vanish:

Claim 1. *If $1 \leq j_1 < j_2 \leq 2d+1$ with either j_1 even or j_2 odd or $j_2 = 2d$, then $M[j_1, j_2] = 0$.*

Proof of claim. To simplify our notation, we will let $A_0 = \{0\} \subset \mathbb{R}^0$, regarded as a trivial root system of rank 0. This does not affect the bounds for homology vanishing in Lemma 14.1 (Levi vanishing and surjectivity). With this convention,

$${}_k\Phi(\mathbf{L}_{\Delta[j_1, j_2]}) = \begin{cases} A_{j_1-1} \times A_{j_2-j_1-1} \times X_{2d+1-j_2} & \text{if } j_2 \neq 2d, 2d+1, \\ A_{j_1-1} \times A_{2d-j_1-1} \times A_1 & \text{if } j_2 = 2d, \\ A_{j_1-1} \times A_{2d-j_1} & \text{if } j_2 = 2d+1. \end{cases}$$

If $j_2 = 2d$ and j_1 is arbitrary, then Lemma 14.1 (Levi vanishing and surjectivity) implies that $H_i(\mathbf{L}_{\Delta[j_1, j_2]}(k); \text{St}(\mathbf{L}_{\Delta[j_1, j_2]})) = 0$ for

$$i \leq 2 + \lfloor (j_1 - 2)/2 \rfloor + \lfloor (2d - j_1 - 2)/2 \rfloor + \lfloor (1 - 1)/2 \rfloor = d + \lfloor j_1/2 \rfloor + \lfloor -j_1/2 \rfloor.$$

The right hand side is at least $d - 1$, so $M[j_1, j_2] = 0$. If $j_2 = 2d + 1$ and j_1 is arbitrary, then Lemma 14.1 (Levi vanishing and surjectivity) implies that $H_i(\mathbf{L}_{\Delta[j_1, j_2]}(k); \text{St}(\mathbf{L}_{\Delta[j_1, j_2]})) = 0$ for

$$i \leq 1 + \lfloor (j_1 - 2)/2 \rfloor + \lfloor (2d - j_1 - 1)/2 \rfloor = d + \lfloor j_1/2 \rfloor + \lfloor (-j_1 - 1)/2 \rfloor = d - 1,$$

so $M[j_1, j_2] = 0$. Finally, if $j_2 \neq 2d, 2d+1$ then Lemma 14.1 (Levi vanishing and surjectivity) implies that $H_i(\mathbf{L}_{\Delta[j_1, j_2]}(k); \text{St}(\mathbf{L}_{\Delta[j_1, j_2]})) = 0$ for

$$\begin{aligned} i &\leq 2 + \lfloor (j_1 - 2)/2 \rfloor + \lfloor (j_2 - j_1 - 2)/2 \rfloor + \lfloor (2d - j_2 - 1)/2 \rfloor \\ &= d + \lfloor j_1/2 \rfloor + \lfloor (j_2 - j_1)/2 \rfloor + \lfloor (-j_2 - 1)/2 \rfloor. \end{aligned}$$

It follows that $M[j_1, j_2] = 0$ if the right hand side is at least $d - 1$, i.e., if

$$(17.5) \quad \lfloor j_1/2 \rfloor + \lfloor (j_2 - j_1)/2 \rfloor + \lfloor (-j_2 - 1)/2 \rfloor \geq -1.$$

The left hand side of (17.5) is -1 if either j_1 is even or j_2 is odd, and is -2 otherwise. The claim follows. $\square$

Now consider $1 \leq j_1 < j_2 \leq 2d - 2$ with j_1 odd and j_2 even. In light of Claim 1, it is enough to prove that $M[j_1, j_2]$ is killed when we quotient $E_{1,d-1}^1$ by the image of the differential $E_{2,d-1}^1 \rightarrow E_{1,d-1}^1$. We have

$${}_k\Phi(\mathbf{L}_{\Delta[j_1, j_2]}) = \begin{cases} A_{j_1-1} \times A_{j_2-j_1-1} \times X_{2d+1-j_2} & \text{if } j_1 > 1 \text{ and } j_2 > j_1 + 1, \\ A_{j_2-2} \times X_{2d+1-j_2} & \text{if } j_1 = 1 \text{ and } j_2 > j_1 + 1, \\ A_{j_1-1} \times X_{2d-j_1} & \text{if } j_1 > 1 \text{ and } j_2 = j_1 + 1, \\ X_{2d-1} & \text{if } j_1 = 1 \text{ and } j_2 = j_1 + 1. \end{cases}$$

In all four cases, since j_1 is odd and j_2 is even we have

$$(17.6) \quad \mathbf{b}({}_k\Phi(\mathbf{L}_{\Delta[j_1, j_2]})) = d - 2.$$

On the summand $M[j_1, j_2, j_2 + 1]$, the differential is the map

$$M[j_1, j_2, j_2 + 1] \longrightarrow M[j_2, j_2 + 1] \oplus M[j_1, j_2 + 1] \oplus M[j_1, j_2] \hookrightarrow E_{1,d-1}^1.$$

Since $j_2 + 1$ is odd, Claim 1 says that $M[j_2, j_2 + 1] = M[j_1, j_2 + 1] = 0$, so to show that this differential kills $M[j_1, j_2]$ it is enough to prove that $M[j_1, j_2, j_2 + 1] \rightarrow M[j_1, j_2]$ is surjective. Since the X-term that appears in all four cases of (12.8) has an odd subscript, this follows from Lemma 14.1 (Levi vanishing and surjectivity). Here we are using the fact that $\mathbf{b}({}_k\Phi(\mathbf{L}_{\Delta[j_1, j_2]})) + 1 = d - 1$; cf. (17.6). $\square$

Part 4. Vanishing in type D (Theorem B.3)

This part of the paper is devoted to Theorem B.3, which is our vanishing result in type D. The proofs follow the same outline as those for type A in Part 2.

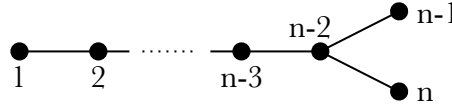
18. VANISHING AND SURJECTIVITY (TYPE D)

In this section, we first introduce some notation for the standard Levi factors of groups of type D_n . We then state a stronger version of Theorem B.3.

18.1. Levi factor notation. Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = D_n$ for some $n \geq 4$. We introduce the following convention:

Convention 18.1. Usually D_n is only defined for $n \geq 4$, but to allow uniform statements we will define $D_3 = A_3$. We do not define D_n for $n \leq 2$. $\square$

Let $\Delta = {}_k\Delta(\mathbf{G})$ be the set of simple roots of ${}_k\Phi(\mathbf{G})$. Number the elements of Δ as follows:



For $1 \leq j_1, \dots, j_\ell \leq n$, let $\Delta[j_1, \dots, j_\ell]$ be the result of removing the simple roots labeled $j_1, \dots, j_\ell$ from Δ . We thus have a standard Levi subgroup $\mathbf{L}_{\Delta[j_1, \dots, j_\ell]}$ of $\mathbf{G}$.

Example 18.2. We have ${}_k\Phi(\mathbf{L}_{\Delta[1]}) = D_{n-1}$ and

$${}_k\Phi(\mathbf{L}_{\Delta[n]}) = {}_k\Phi(\mathbf{L}_{\Delta[n-1]}) = A_{n-1}$$

and ${}_k\Phi(\mathbf{L}_{\Delta[n-2]}) = A_{n-3} \times A_1 \times A_1$, while for $2 \leq j \leq n-3$ we have ${}_k\Phi(\mathbf{L}_{\Delta[j]}) = A_{j-1} \times D_{n-j}$. In general, for distinct $1 \leq j_1, \dots, j_\ell \leq n$ we have either

$$\begin{aligned} {}_k\Phi(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]}) &= A_{n_1} \times \cdots \times A_{n_{m-1}} \times D_{n_m} \quad \text{or} \\ {}_k\Phi(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]}) &= A_{n_1} \times \cdots \times A_{n_m} \end{aligned}$$

with $n_1 + \cdots + n_m + \ell = n$. $\square$

We have a Reeder map (cf. §2.6) of the form $\text{St}(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]}) \rightarrow \text{St}(\mathbf{G})$, and thus maps $H_i(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]}(k); \text{St}(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]})) \rightarrow H_i(\mathbf{G}(k); \text{St}(\mathbf{G}))$.

18.2. Strong vanishing. The main result we will prove in this part of the paper is:

Theorem B.3'. *Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = D_n$ for some $n \geq 4$. Then:*

- $H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ for $i \leq \lfloor (n-3)/2 \rfloor$; and
- letting $\Delta = {}_k\Delta(\mathbf{G})$, the map $H_i(\mathbf{L}_{\Delta[1]}(k); \text{St}(\mathbf{L}_{\Delta[1]})) \rightarrow H_i(\mathbf{G}(k); \text{St}(\mathbf{G}))$ is surjective for $i \leq \lfloor (n-2)/2 \rfloor$.

This strengthens Theorem B.3 by adding the indicated surjectivity statement. Unlike in other types, we do not need to handle the easiest case (that is, $n = 4$) separately. We will therefore continue to assume that $n \geq 4$. We will also assume as an inductive hypothesis that we have already proved Theorem B.3' in smaller ranks. For this, we make the following definition:

Definition 18.3. For $r \geq 3$, the r -surjectivity and vanishing hypothesis in type D is as follows. If $r \leq 3$, then it is trivial.³⁸ Otherwise, if $r \geq 4$ then let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = D_n$ for some $4 \leq n \leq r$. Then:

- $H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ for $i \leq \lfloor (n-3)/2 \rfloor$; and

³⁸Groups of type $D_3 = A_3$ are covered by Theorem B.1', so there is no need to assume anything about them.

- letting $\Delta = {}_k\Delta(\mathbf{G})$, the map $H_i(\mathbf{L}_{\Delta[1]}(k); \text{St}(\mathbf{L}_{\Delta[1]})) \rightarrow H_i(\mathbf{G}(k); \text{St}(\mathbf{G}))$ is surjective for $i \leq \lfloor (n-2)/2 \rfloor$. $\square$

Remark 18.4. We have already proven Theorem B.1', so we also have available to us vanishing and surjectivity results in type A. We have also already proven Theorem B.2' about types B and C and BC, but we will not need this. $\square$

19. VANISHING AND SURJECTIVITY FOR LEVI SUBGROUPS (TYPE D)

In this section, we show how to use the r -surjectivity and vanishing hypothesis in type D to analyze the homology of standard Levi subgroups. Our main result is as follows. Its statement uses the ordering on the simple roots of $A_{n_{j_0}}$ and $D_{n_{j_0}}$ discussed in §8.1 and §18.1.

Lemma 19.1 (Levi vanishing and surjectivity). *For some $n \geq 4$, assume the $(n-1)$ -surjectivity and vanishing hypothesis in type D (Definition 18.3). Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = D_n$. Let $\Delta \subset {}_k\Delta(\mathbf{G})$ be a set of simple roots with $\Delta \neq {}_k\Delta(\mathbf{G})$. Write*

$${}_k\Phi(\mathbf{L}_{\Delta}) = A_{n_1} \times \cdots \times Z_{n_m} \quad \text{with } Z \in \{A, D\}.$$

We thus have $n_1, \dots, n_{m-1} \geq 1$, and if $Z = D$ then $n_m \geq 4$ while if $Z = A$ then $n_m \geq 1$. Set $b = \mathbf{b}({}_k\Phi(\mathbf{L}_{\Delta}))$, so

$$b = \begin{cases} (m-1) + \lfloor (n_1-1)/2 \rfloor + \cdots + \lfloor (n_{m-1}-1)/2 \rfloor + \lfloor (n_m-1)/2 \rfloor & \text{if } Z = A, \\ (m-1) + \lfloor (n_1-1)/2 \rfloor + \cdots + \lfloor (n_{m-1}-1)/2 \rfloor + \lfloor (n_m-3)/2 \rfloor & \text{if } Z = D. \end{cases}$$

Then the following hold:

- (i) *We have $H_i(\mathbf{L}_{\Delta}(k); \text{St}(\mathbf{L}_{\Delta})) = 0$ for $i \leq b$.*
- (ii) *For some $1 \leq j_0 \leq n$, assume one of the following:*
 - n_{j_0} is even and nonzero, $1 \leq j_0 \leq m-1$, and $\Delta' \subset \Delta$ is the set of simple roots obtained by removing either the first or last simple root from $A_{n_{j_0}}$.
 - n_{j_0} is even and nonzero, $j_0 = m$ and $Z = A$, and $\Delta' \subset \Delta$ is the set of simple roots obtained by removing either the first or last simple root from A_{n_m} .
 - n_{j_0} is even and nonzero, $j_0 = m$ and $Z = D$, and $\Delta' \subset \Delta$ is the set of simple roots obtained by removing the first simple root from Z_{n_m} .

Then the map $H_{b+1}(\mathbf{L}_{\Delta'}(k); \text{St}(\mathbf{L}_{\Delta'})) \rightarrow H_{b+1}(\mathbf{L}_{\Delta}(k); \text{St}(\mathbf{G}))$ is surjective.

Proof. Since $\Delta \neq {}_k\Delta(\mathbf{G})$, we have $n_j \leq n-1$ for $1 \leq j \leq m$. The $(n-1)$ -surjectivity and vanishing hypothesis in type D thus applies to all reductive groups $\mathbf{H}_m$ with ${}_k\Phi(\mathbf{H}_m) = D_{n_m}$. Theorem B.1' also gives a vanishing and surjectivity result for all reductive groups $\mathbf{H}_j$ with ${}_k\Phi(\mathbf{H}_j) = A_{n_j}$. This gives the hypothesis (♥) in Lemma 6.4 (reducible vanishing). Applying Lemma 6.4, we deduce (i). Similarly, for Δ' as in (ii) it gives the hypotheses (♥) and (♥♥) in Lemma 6.5 (reducible surjectivity). Applying Lemma 6.5, we deduce (ii). $\square$

20. VANISHING REGION (TYPE D)

Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = D_n$ for some $n \geq 4$. Corollary 4.4 gives a spectral sequence E_{pq}^r converging to $H_{p+q}(\mathbf{G}(k); \text{St}(\mathbf{G}))$ with

$$E_{pq}^1 \cong \begin{cases} \bigoplus_{R \in \mathcal{L}_p(\mathbf{G})} H_q(\mathbf{L}_R(k); \text{St}(\mathbf{L}_R)) & \text{if } 0 \leq p \leq n-1 \\ H_q(\mathbf{G}(k); \text{St}(\mathbf{G})^{\otimes 2}) & \text{if } p = n, \\ 0 & \text{otherwise.} \end{cases}$$

The following lemma shows that our inductive hypothesis implies that many terms of this spectral sequence vanish.

Lemma 20.1. *Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = \mathbf{D}_n$ for some $n \geq 4$. Assume the $(n-1)$ -surjectivity and vanishing hypothesis in type D (Definition 18.3). Let E_{pq}^1 be the spectral sequence from Corollary 4.4. Then the following hold:*

- For $n = 2d + 3$ with $d \geq 1$, we have $E_{pq}^1 = 0$ for $p + q \leq d$ except for possibly E_{0d}^1 .
- For $n = 2d + 2$ with $d \geq 1$, we have $E_{pq}^1 = 0$ for $p + q \leq d$ except for possibly E_{0d}^1 and $E_{1,d-1}^1$. For $d = 1$, we also have $E_{1,d-1}^1 = 0$.

Proof. This is identical to the proof of Lemma 10.1 in type A. The only difference is that the standard Levi subgroups can be of two types:

- $A_{n_1} \times \cdots \times A_{n_m}$, where the vanishing range for H_i given by Theorem B.1' is
$$i \leq (m-1) + \lfloor (n_1-1)/2 \rfloor + \cdots + \lfloor (n_m-1)/2 \rfloor.$$
- $A_{n_1} \times \cdots \times A_{n_{m-1}} \times D_{n_m}$, where the vanishing range for H_i given by Lemma 19.1 (Levi vanishing and surjectivity) is
$$i \leq (m-1) + \lfloor (n_1-1)/2 \rfloor + \cdots + \lfloor (n_{m-1}-1)/2 \rfloor + \lfloor (n_m-3)/2 \rfloor.$$

The slightly worse vanishing range in the second case accounts for the slightly worse range in the statement of the lemma. Finally, the fact that $E_{1,d-1}^1 = 0$ for $d = 1$ follows from the fact that $E_{1,0}^1$ is a direct sum of H_0 -groups, and these all vanish by Lemma 3.2. $\square$

21. REMAINING TASKS (TYPE D)

Lemma 20.1 implies many cases of Theorem B.3'. To prove the remaining cases, we need to compute some differentials in our spectral sequence. We now explain the structure of the argument, postponing three calculations to the next section. Recall that Theorem B.3' is:

Theorem B.3'. *Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = \mathbf{D}_n$ for some $n \geq 4$. Then:*

- $H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ for $i \leq \lfloor (n-3)/2 \rfloor$; and
- letting $\Delta = {}_k\Delta(\mathbf{G})$, the map $H_i(\mathbf{L}_{\Delta[1]}(k); \text{St}(\mathbf{L}_{\Delta[1]})) \rightarrow H_i(\mathbf{G}(k); \text{St}(\mathbf{G}))$ is surjective for $i \leq \lfloor (n-2)/2 \rfloor$.

Proof. The proof is by induction on n . We will prove the base case $n = 4$ in exactly the same way we will prove the inductive step, so assume that $n \geq 4$ and that we have proved the result in smaller ranks, i.e., that the $(n-1)$ -surjectivity and vanishing hypothesis in type D holds. This assumption is vacuous if $n = 4$.

Corollary 4.4 gives a spectral sequence E_{pq}^r converging to $H_{p+q}(\mathbf{G}(k); \text{St}(\mathbf{G}))$, and Lemma 20.1 implies that $E_{pq}^1 = 0$ for $p + q \leq \lfloor (n-2)/2 \rfloor - 1$. This implies that $H_i(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$ for $i \leq \lfloor (n-2)/2 \rfloor - 1$. Since our surjectivity claim is trivial when the target is 0, all that remains to prove are the following two claims:

Claim 1. *Assume that $n = 2d + 3$ with $d \geq 1$. Then $H_d(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$.*

In this case, Lemma 20.1 says that the only potentially nonzero term E_{pq}^1 in our spectral sequence with $p + q = d$ is E_{0d}^1 . We will prove in Lemma 22.2 below that the differential $E_{1d}^1 \rightarrow E_{0d}^1$ is surjective, so $E_{0d}^2 = 0$. This implies that $H_d(\mathbf{G}(k); \text{St}(\mathbf{G})) = 0$, as desired.

Claim 2. *Assume that $n = 2d + 2$ with $d \geq 1$. Then the map*

$$H_d(\mathbf{L}_{\Delta[1]}(k); \text{St}(\mathbf{L}_{\Delta[1]})) \rightarrow H_d(\mathbf{G}(k); \text{St}(\mathbf{G}))$$

is surjective.

Lemma 20.1 says that the only potentially nonzero terms E_{pq}^1 in our spectral sequence with $p + q = d$ are E_{0d}^1 and $E_{1,d-1}^1$. Lemma 20.1 also says that $E_{1,d-1}^1 = 0$ if $d = 1$, and we will prove in Lemma 22.3 below that the differential $E_{2,d-1}^1 \rightarrow E_{1,d-1}^1$ is surjective for $d \geq 2$.

It follows that in all cases $E_{1,d-1}^2 = 0$. We will also prove in Lemma 22.1 below that the summand $H_d(\mathbf{L}_{\Delta[1]}(k); \text{St}(\mathbf{L}_{\Delta[1]}))$ of

$$E_{0d}^1 = \bigoplus_{R \in \mathcal{L}_0(\mathbf{G})} H_d(\mathbf{L}_R(k); \text{St}(\mathbf{L}_R)) = \bigoplus_{j=1}^n H_d(\mathbf{L}_{\Delta[j]}(k); \text{St}(\mathbf{L}_{\Delta[j]}))$$

surjects onto the cokernel of the differential $E_{1d}^1 \rightarrow E_{0d}^1$. It follows that E_{0d}^2 is a quotient of $H_d(\mathbf{L}_{\Delta[1]}(k); \text{St}(\mathbf{L}_{\Delta[1]}))$. Since E_{0d}^2 is the only potentially nonzero term of the form E_{pq}^2 with $p + q = d$, it follows that $H_d(\mathbf{L}_{\Delta[1]}(k); \text{St}(\mathbf{L}_{\Delta[1]}))$ surjects onto $H_d(\mathbf{G}(k); \text{St}(\mathbf{G}))$, as desired. $\square$

22. DIFFERENTIALS (TYPE D)

This final section of this part of the paper determines the images of three differentials whose calculations were needed in the previous section.

22.1. Differentials, I (type D). Our first differential calculation is:

Lemma 22.1. *Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = D_{2d+2}$ for some $d \geq 1$. Assume the $(2d+1)$ -surjectivity and vanishing hypothesis in type D (Definition 18.3). Let E_{pq}^1 be the spectral sequence from Corollary 4.4. Then the summand $H_d(\mathbf{L}_{\Delta[1]}(k); \text{St}(\mathbf{L}_{\Delta[1]}))$ of E_{0d}^1 surjects onto the cokernel of the differential $E_{1d}^1 \rightarrow E_{0d}^1$.*

Proof. As notation, for $1 \leq j_1, \dots, j_\ell \leq 2d+2$ let

$$M[j_1, \dots, j_\ell] = H_d(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]}(k); \text{St}(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]})).$$

We have

$$E_{0d}^1 = \bigoplus_{1 \leq j \leq 2d+2} M[j] \quad \text{and} \quad E_{1d}^1 = \bigoplus_{1 \leq j_1 < j_2 \leq 2d+2} M[j_1, j_2].$$

Consider some $1 < j \leq 2d+2$. We must prove that when we quotient E_{0d}^1 by the image of the differential $E_{1d}^1 \rightarrow E_{0d}^1$, the summand $M[j]$ of E_{0d}^1 is identified with a subspace of $M[1]$.

We first prove that $M[j] = 0$ for $2d-1 \leq j \leq 2d+2$. As in Example 18.2, we have

$$\begin{aligned} {}_k\Phi(\mathbf{L}_{\Delta[2d-1]}) &= A_{(2d+2)-4} \times A_3 = A_{2d-2} \times A_3, \\ {}_k\Phi(\mathbf{L}_{\Delta[2d]}) &= A_{(2d+2)-3} \times A_1 \times A_1 = A_{2d-1} \times A_1 \times A_1, \\ {}_k\Phi(\mathbf{L}_{\Delta[2d+1]}) &= {}_k\Phi(\mathbf{L}_{\Delta[2d+2]}) = A_{(2d+2)-1} = A_{2d+1}. \end{aligned}$$

In all of these cases, we can appeal to Theorem B.1' to see that $H_i(\mathbf{L}_{\Delta[j]}(k); \text{St}(\mathbf{L}_{\Delta[j]})) = 0$ for $i \leq d$. Indeed, the bound given by that theorem in the three cases above are

$$\begin{aligned} 1 + \lfloor (2d-3)/2 \rfloor + \lfloor (3-1)/2 \rfloor &= d, \\ 2 + \lfloor (2d-2)/2 \rfloor + \lfloor (1-1)/2 \rfloor + \lfloor (1-1)/2 \rfloor &= d+1, \\ \lfloor 2d/2 \rfloor &= d. \end{aligned}$$

The case $i = d$ shows that $M[j] = 0$, as desired.

It remains to prove that for $1 < j < 2d-2$, the summand $M[j]$ of E_{0d}^1 is identified with a subspace of $M[1]$ when you quotient by the image of the differential $E_{1d}^1 \rightarrow E_{0d}^1$. In these cases, we have

$${}_k\Phi(\mathbf{L}_{\Delta[j]}) = A_{j-1} \times D_{2d+2-j}.$$

The proof for these cases is identical to the proof of Lemma 12.1 in type A. $\square$

22.2. Differentials, II (type D). Our second differential calculation is:

Lemma 22.2. *Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = \mathbf{D}_{2d+3}$ for $d \geq 1$. Assume the $(2d+2)$ -surjectivity and vanishing hypothesis in type D (Definition 18.3). Let E_{pq}^1 be the spectral sequence from Corollary 4.4. Then the differential $E_{1d}^1 \rightarrow E_{0d}^1$ is surjective.*

Proof. As notation, for $1 \leq j_1, \dots, j_\ell \leq 2d+3$ let

$$M[j_1, \dots, j_\ell] = H_d(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]}(k); \text{St}(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]})).$$

We have

$$E_{0d}^1 = \bigoplus_{1 \leq j \leq 2d+3} M[j] \quad \text{and} \quad E_{1d}^1 = \bigoplus_{1 \leq j_1 < j_2 \leq 2d+3} M[j_1, j_2].$$

Consider some $1 \leq j \leq 2d+3$. We must prove that when we quotient E_{0d}^1 by the image of the differential $E_{1d}^1 \rightarrow E_{0d}^1$, the summand $M[j]$ is killed.

Just like in the proof of Lemma 22.1 above, we have $M[j] = 0$ for $2d \leq j \leq 2d+3$. We must therefore only deal with $M[j]$ for $1 \leq j \leq 2d-1$. For these j , we have

$${}_k\Phi(\mathbf{L}_{\Delta[j]}) = A_{j-1} \times D_{2d+3-j}.$$

The proof that these $M[j]$ are killed when we quotient by the image of the differential is identical to the proof of Lemma 12.2 in type A. $\square$

22.3. Differentials, III (type D). Our final differential calculation is:

Lemma 22.3. *Let $\mathbf{G}$ be a reductive group with ${}_k\Phi(\mathbf{G}) = \mathbf{D}_{2d+2}$ for some $d \geq 2$. Assume the $(2d+1)$ -surjectivity and vanishing hypothesis in type D (Definition 18.3). Let E_{pq}^1 be the spectral sequence from Corollary 4.4. Then the differential $E_{2,d-1}^1 \rightarrow E_{1,d-1}^1$ is surjective.*

Proof. As notation, for $1 \leq j_1, \dots, j_\ell \leq 2d+2$ let

$$M[j_1, \dots, j_\ell] = H_{d-1}(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]}(k); \text{St}(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]})).$$

We have

$$E_{1,d-1}^1 = \bigoplus_{1 \leq j_1 < j_2 \leq 2d+2} M[j_1, j_2] \quad \text{and} \quad E_{2,d-1}^1 = \bigoplus_{1 \leq j_1 < j_2 < j_3 \leq 2d+2} M[j_1, j_2, j_3].$$

Consider some $1 \leq j_1 < j_2 \leq 2d+2$. We must prove that when we quotient $E_{1,d-1}^1$ by the image of the differential $E_{2,d-1}^1 \rightarrow E_{1,d-1}^1$, the summand $M[j_1, j_2]$ is killed. We first prove:

Claim 1. *If $1 \leq j_1 < j_2 \leq 2d+2$ are such that $2d-1 \leq j_2 \leq 2d+2$, then $M[j_1, j_2] = 0$.*

Proof of claim. In this case,

$${}_k\Phi(\mathbf{L}_{\Delta[j_1, j_2]}) = A_{n_1} \times \dots \times A_{n_m} \quad \text{with } n_1 + \dots + n_m = 2d.$$

Theorem B.1' therefore says that $H_i(\mathbf{L}_{\Delta[j_1, j_2]}(k); \text{St}(\mathbf{L}_{\Delta[j_1, j_2]})) = 0$ for

$$i \leq (m-1) + \lfloor (n_1-1)/2 \rfloor + \dots + \lfloor (n_m-1)/2 \rfloor.$$

Lemma 10.2 says that for $a, b \in \mathbb{Z}$ we have $1 + \lfloor a/2 \rfloor + \lfloor b/2 \rfloor \geq \lfloor (a+b+1)/2 \rfloor$. Applying this repeatedly, we see that our vanishing range is at least $\lfloor (2d-1)/2 \rfloor = d-1$. We deduce that $M[j_1, j_2] = H_{d-1}(\mathbf{L}_{\Delta[j_1, j_2]}(k); \text{St}(\mathbf{L}_{\Delta[j_1, j_2]})) = 0$. $\square$

In light of Claim 1, we must show that for $1 \leq j_1 < j_2 \leq 2d-2$, the summand $M[j_1, j_2]$ is killed by the differential $E_{2,d-1}^1 \rightarrow E_{1,d-1}^1$. Using the convention that A_0 is the empty root system, for $1 \leq j_1 < j_2 \leq 2d-2$ we have

$${}_k\Phi(\mathbf{L}_{\Delta[j_1, j_2]}) = A_{j_1-1} \times A_{j_2-j_1-1} \times D_{2d+2-j_2}.$$

The proof is identical to the proof of Lemma 12.3 in type A; in fact, it is even easier than Lemma 12.3 since the extremal case $M[1, 2d - 2]$ can be treated exactly the same as the case $M[1, j_2]$ with $1 < j_2 < 2d - 2$ even. $\square$

Part 5. Integral vanishing (Theorem C)

We now generalize our vanishing theorems to the groups³⁹ $\mathrm{SL}_{n+1}(\mathbb{Z})$ and $\mathrm{GL}_{n+1}(\mathbb{Z})$, proving Theorem C. This requires a conjectural partial resolution of their Steinberg representations, which we discuss in §23. The existence of this conjectural resolution is equivalent to the high connectivity of the double Tits building. We discuss some results about reducible Levi subgroups in §24 and state a stronger version of Theorem C in §25. The rest of this part is devoted to its proof, which follows the outline of our proof in type A from Part 2.

Convention 22.4. Throughout this part, SL_{n+1} and GL_{n+1} are always taken to be defined over the field $\mathbb{Q}$. $\square$

23. PARTIAL RESOLUTION OF INTEGRAL STEINBERG REPRESENTATION

The key technical tool underlying our proof of Theorem B was the spectral sequence from Corollary 4.4, which comes from the resolution of the Steinberg representation given by Proposition 4.1. This section explains a conjectural integral refinement of this.

23.1. Notation. We start by introducing some notation we will use throughout this part of the paper. The Tits buildings of SL_{n+1} and GL_{n+1} are isomorphic, and we will denote them by $\mathcal{T}(\mathbb{Q}^{n+1})$. More generally, for a nonzero finite-dimensional $\mathbb{Q}$ -vector space V we will write $\mathcal{T}(V)$ for the complex of flags of nonzero proper subspaces of V .

Let $d = \dim(V) \geq 1$. The Solomon–Tits theorem says that $\mathcal{T}(V)$ is homotopy equivalent to a wedge of $(d - 2)$ -dimensional spheres. For a commutative ring $\mathbb{F}$, we let $\mathrm{St}(V; \mathbb{F}) = \tilde{H}_{d-2}(\mathcal{T}(V); \mathbb{F})$. For $\mathbb{F} = \mathbb{Z}$, we omit $\mathbb{F}$ from our notation and write $\mathrm{St}(V) = \tilde{H}_{d-2}(\mathcal{T}(V))$. The groups $\mathrm{GL}(V)$ and $\mathrm{SL}(V)$ act on $\mathrm{St}(V; \mathbb{F})$. With this notation, $\mathrm{St}(\mathbb{Q}^{n+1}; \mathbb{F})$ is the Steinberg representation of $\mathrm{GL}_{n+1}(\mathbb{Q})$ and $\mathrm{SL}_{n+1}(\mathbb{Q})$.

Now let W be a finite-rank free $\mathbb{Z}$ -module. Write $\mathcal{T}(W)$ for the complex of flags of nonzero proper direct summands of W . Direct summands of W are in bijection with subspaces of $W \otimes \mathbb{Q}$, so $\mathcal{T}(W) = \mathcal{T}(W \otimes \mathbb{Q})$. Letting $r = \mathrm{rank}(W)$, we define $\mathrm{St}(W; \mathbb{F}) = \tilde{H}_{r-2}(\mathcal{T}(W); \mathbb{F})$ and $\mathrm{St}(W) = \tilde{H}_{r-2}(\mathcal{T}(W))$. We thus have $\mathrm{St}(W; \mathbb{F}) \cong \mathrm{St}(W \otimes \mathbb{Q}; \mathbb{F}) \cong \mathrm{St}(\mathbb{Q}^r; \mathbb{F})$.

23.2. Resolution over rationals. Recall from Example 4.2 that the resolution of $\mathrm{St}(\mathbb{Q}^{n+1})$ from Proposition 4.1 takes the form

$$(23.1) \quad 0 \rightarrow (\mathrm{St}(\mathbb{Q}^{n+1}))^{\otimes 2} \rightarrow \mathbf{R}_{n-1} \rightarrow \cdots \rightarrow \mathbf{R}_0 \rightarrow \mathrm{St}(\mathbb{Q}^{n+1}) \rightarrow 0$$

with

$$\mathbf{R}_i = \bigoplus_{V_1 \oplus \cdots \oplus V_{i+2} = \mathbb{Q}^{n+1}} \mathrm{St}(V_1) \otimes \cdots \otimes \mathrm{St}(V_{i+2})$$

for $0 \leq i \leq n - 1$. Here the direct sum is over decompositions $V_1 \oplus \cdots \oplus V_{i+2} = \mathbb{Q}^{n+1}$ with $\dim(V_j) \geq 1$ for $1 \leq j \leq i + 2$.

³⁹We use SL_{n+1} and GL_{n+1} since they have type A_n and we want our numerology to match Part 2.

23.3. Integral refinement. Though it is easy to understand the actions of $\mathrm{GL}_{n+1}(\mathbb{Q})$ and $\mathrm{SL}_{n+1}(\mathbb{Q})$ on decompositions $V_1 \oplus \cdots \oplus V_{i+2} = \mathbb{Q}^{n+1}$, the actions of their subgroups $\mathrm{GL}_{n+1}(\mathbb{Z})$ and $\mathrm{SL}_{n+1}(\mathbb{Z})$ are far more complicated. It would be helpful to restrict to integral decompositions $W_1 \oplus \cdots \oplus W_{i+2} = \mathbb{Z}^{n+1}$. If we do this, we are forced to remove the initial term $(\mathrm{St}(\mathbb{Q}^{n+1}))^{\otimes 2}$. Let

$$(23.2) \quad \begin{array}{ccccccc} \mathbf{S}_{n-1}(\mathbb{Z}^{n+1}) & \rightarrow & \cdots & \rightarrow & \mathbf{S}_0(\mathbb{Z}^{n+1}) & \rightarrow & \mathbf{S}_{-1}(\mathbb{Z}^{n+1}) \rightarrow 0 \\ & & & & & & \parallel \\ & & & & & & \mathrm{St}(\mathbb{Z}^{n+1}) \end{array}$$

be the subcomplex of (23.1) with

$$\mathbf{S}_i(\mathbb{Z}^{n+1}) = \bigoplus_{W_1 \oplus \cdots \oplus W_{i+2} = \mathbb{Z}^{n+1}} \mathrm{St}(W_1) \otimes \cdots \otimes \mathrm{St}(W_{i+2}) \quad \text{for } -1 \leq i \leq n-1.$$

We conjecture that more and more of $\mathbf{S}_\bullet^{n+1}$ becomes exact as n increases. The bound in the following more precise version of this conjecture is exactly what we need to prove our vanishing theorem:

Conjecture 23.1. *For $b \geq 1$, the b -integral resolution conjecture says the following. Consider some $n \geq 3$. Let $b' = \min(b, \lfloor n/2 \rfloor)$. Then the portion*

$$\begin{array}{ccccccc} \mathbf{S}_{b'}(\mathbb{Z}^{n+1}) & \rightarrow & \mathbf{S}_{b'-1}(\mathbb{Z}^{n+1}) & \rightarrow & \cdots & \rightarrow & \mathbf{S}_0(\mathbb{Z}^{n+1}) \rightarrow \mathbf{S}_{-1}(\mathbb{Z}^{n+1}) \rightarrow 0 \\ & & & & & & \parallel \\ & & & & & & \mathrm{St}(\mathbb{Z}^{n+1}) \end{array}$$

of the chain complex (23.2) is exact.

Remark 23.2. The exactness of the entire complex (23.2) for all $n \geq 1$ is equivalent to the Koszulness of the Steinberg monoid of the integers as defined in [19]. By the results of [22], this would imply the Rognes connectivity conjecture [26, Conjecture 12.3]. $\square$

23.4. High connectivity of the double Tits building. We defined the double Tits building $\mathcal{T}^2(\mathbb{Z}^{n+1})$ in §1.7. Since we will not use it directly, we will not recall its definition here. The relevance of $\mathcal{T}^2(\mathbb{Z}^{n+1})$ for us is the following theorem of Miller–Patz–Wilson [22].

Theorem 23.3. *For all $n \geq 1$, we have $H_i(\mathbf{S}_\bullet(\mathbb{Z}^{n+1})) = \tilde{H}_{i+n}(\mathcal{T}^2(\mathbb{Z}^{n+1}))$.*

Proof. Here is how to extract this from [22]. We remark that we will give a self-contained and somewhat different proof in [21]. We will use the notation from [22]. First, $\mathbf{S}_\bullet(\mathbb{Z}^{n+1})$ is the bar resolution from [19] computing $\mathrm{Tor}_i^{\mathrm{St}(\mathbb{Z})}(\mathbb{Z}, \mathbb{Z})_{n+1}$, so

$$(23.3) \quad H_i(\mathbf{S}_\bullet(\mathbb{Z}^{n+1})) = \mathrm{Tor}_{i+2}^{\mathrm{St}(\mathbb{Z})}(\mathbb{Z}, \mathbb{Z})_{n+1}.$$

Here the $i+2$ appears in the Tor term due to a degree shift we are suppressing. Next, [22, Lemma 6.2] shows that

$$(23.4) \quad \mathrm{Tor}_{i+2}^{\mathrm{St}(\mathbb{Z})}(\mathbb{Z}, \mathbb{Z})_{n+1} = \tilde{H}_{i+n+3}(D^{2,0}(\mathbb{Z})),$$

where $D^{2,0}(\mathbb{Z})$ is one of the “higher buildings” defined in [22]. Finally, [22, Lemma 4.22] proves that $D^{2,0}(\mathbb{Z}) \cong \Sigma^3 \mathcal{T}^2(\mathbb{Z}^{n+1})$, so

$$(23.5) \quad \tilde{H}_{i+n+3}(D^{2,0}(\mathbb{Z})) \cong \tilde{H}_{i+n}(\mathcal{T}^2(\mathbb{Z}^{n+1})).$$

The theorem follows from (23.3) and (23.4) and (23.5). $\square$

Theorem 23.3 has the following immediate corollary:

Corollary 23.4. *For $b \geq 1$, the b -integral resolution conjecture holds if and only if the following holds for all $n \geq 3$:*

($\dagger\dagger$) *We have $H_i(\mathcal{T}^2(\mathbb{Z}^{n+1})) = 0$ for $i \leq n - 1 + \min(b, \lfloor n/2 \rfloor)$.*

23.5. Spectral sequence. Our next goal is to extract a spectral sequence from Conjecture 23.1. Letting $\mathbf{G}$ be either GL_{n+1} or SL_{n+1} , for each $R \subset {}_{\mathbb{Q}}\Delta(\mathbf{G})$ there is a standard Levi subgroup $\mathbf{L}_R$. The group $\mathbf{L}_R$ is defined over $\mathbb{Z}$, so $\mathbf{L}_R(\mathbb{Z})$ makes sense. If $\mathbf{G} = \mathrm{GL}_{n+1}$ then

$$\mathbf{L}_R(\mathbb{Z}) = \mathrm{GL}_{n_1+1}(\mathbb{Z}) \times \cdots \times \mathrm{GL}_{n_m+1}(\mathbb{Z})$$

for some $(n_1 + 1) + \cdots + (n_m + 1) = n + 1$, while if $\mathbf{G} = \mathrm{SL}_{n+1}$ then

$$\mathbf{L}_R(\mathbb{Z}) = \ker(\mathrm{GL}_{n_1+1}(\mathbb{Z}) \times \cdots \times \mathrm{GL}_{n_m+1}(\mathbb{Z}) \xrightarrow{\det} \mathbb{Z}^\times).$$

Recall from Corollary 4.4 that the resolution (23.1) yields a spectral sequence E_{pq}^r converging to $H_{p+q}(\mathbf{G}(k); \mathrm{St}(\mathbf{G}))$ with

$$E_{pq}^1 \cong \begin{cases} \bigoplus_{R \in \mathcal{L}_p(\mathbf{G})} H_q(\mathbf{L}_R(k); \mathrm{St}(\mathbf{L}_R)) & \text{if } 0 \leq p \leq n-1 \\ H_q(\mathbf{G}(k); \mathrm{St}(\mathbf{G})^{\otimes 2}) & \text{if } p = n, \\ 0 & \text{otherwise.} \end{cases}$$

The following shows that Conjecture 23.1 yields a similar spectral sequence for $\mathbf{G}(\mathbb{Z})$:

Lemma 23.5. *Assume the b -integral resolution conjecture (Conjecture 23.1) for some $b \geq 1$. Let $\mathbf{G}$ be either GL_{n+1} or SL_{n+1} , let $\mathbb{F}$ be a commutative ring, and let $n \geq 3$. Set $c = \min(b, \lfloor n/2 \rfloor)$. There is then a spectral sequence E_{pq}^r converging to $H_{p+q}(\mathbf{G}(\mathbb{Z}); \mathrm{St}(\mathbb{Z}^{n+1}; \mathbb{F}))$ such that*

$$E_{pq}^1 \cong \bigoplus_{R \in \mathcal{L}_p(\mathbf{G})} H_q(\mathbf{L}_R(\mathbb{Z}); \mathrm{St}(\mathbf{L}_R; \mathbb{F})) \quad \text{if } 0 \leq p \leq c.$$

Proof. Let $\mathbf{S}_\bullet = \mathbf{S}_\bullet(\mathbb{Z}^{n+1})$ be as in Conjecture 23.1. We suppress the $\mathbb{Z}^{n+1}$ to simplify our notation. Let $\mathbf{S}'_{c+1}$ be the kernel of the differential $\mathbf{S}_c \rightarrow \mathbf{S}_{c-1}$. By assumption, we therefore have a resolution

$$0 \rightarrow \mathbf{S}'_{c+1} \rightarrow \mathbf{S}_c \rightarrow \cdots \rightarrow \mathbf{S}_0 \rightarrow \mathrm{St}(\mathbb{Z}^{n+1}) \rightarrow 0.$$

Each term of this is a free abelian group, so by the universal coefficients theorem we can tensor this with $\mathbb{F}$ and get a resolution

$$0 \rightarrow \mathbf{S}'_{c+1} \otimes \mathbb{F} \rightarrow \mathbf{S}_c \otimes \mathbb{F} \rightarrow \cdots \rightarrow \mathbf{S}_0 \otimes \mathbb{F} \rightarrow \mathrm{St}(\mathbb{Z}^{n+1}; \mathbb{F}) \rightarrow 0.$$

As is standard, there is a spectral sequence converging to $H_{p+q}(\mathbf{G}(\mathbb{Z}); \mathrm{St}(\mathbb{Z}^{n+1}; \mathbb{F}))$ with

$$E_{pq}^1 \cong \begin{cases} H_q(\mathbf{G}(\mathbb{Z}); \mathbf{S}_p \otimes \mathbb{F}) & \text{if } 0 \leq p \leq c, \\ H_q(\mathbf{G}(\mathbb{Z}); \mathbf{S}'_{c+1} \otimes \mathbb{F}) & \text{if } p = c+1, \\ 0 & \text{otherwise.} \end{cases}$$

To prove the lemma, it is therefore enough to prove:

Claim. *For $0 \leq p \leq n-1$, we have $H_q(\mathbf{G}(\mathbb{Z}); \mathbf{S}_p \otimes \mathbb{F}) \cong \bigoplus_{R \in \mathcal{L}_p(\mathbf{G})} H_q(\mathbf{L}_R(\mathbb{Z}); \mathrm{St}(\mathbf{L}_R; \mathbb{F}))$.*

We have

$$\begin{aligned} \mathbf{S}_p \otimes \mathbb{F} &= \bigoplus_{W_1 \oplus \cdots \oplus W_{p+2} = \mathbb{Z}^{n+1}} \mathrm{St}(W_1; \mathbb{F}) \otimes \cdots \otimes \mathrm{St}(W_{p+2}; \mathbb{F}) \\ &= \bigoplus_{(n_1+1)+\cdots+(n_{p+2}+1)=n+1} \left(\bigoplus_{\substack{W_1 \oplus \cdots \oplus W_{p+2} = \mathbb{Z}^{n+1} \\ \mathrm{rank}(W_j) = n_j+1}} \mathrm{St}(W_1; \mathbb{F}) \otimes \cdots \otimes \mathrm{St}(W_{p+2}; \mathbb{F}) \right). \end{aligned}$$

For a fixed partition $(n_1 + 1) + \cdots + (n_{p+2} + 1) = n + 1$, the group $\mathbf{G}(\mathbb{Z})$ acts transitively on decompositions $W_1 \oplus \cdots \oplus W_{p+2} = \mathbb{Z}^{n+1}$ with $\text{rank}(W_j) = n_j + 1$ for all $1 \leq j \leq p + 2$. The stabilizer of the standard decomposition $\mathbb{Z}^{n_1+1} \oplus \cdots \oplus \mathbb{Z}^{n_{p+2}+1} = \mathbb{Z}^{n+1}$ is the integer points of one of the standard Levi subgroups $\mathbf{L}_R$ of $\mathbf{G}$: if $\mathbf{G} = \text{GL}_{n+1}$ the stabilizer is

$$\mathbf{L}_R(\mathbb{Z}) = \text{GL}_{n_1+1}(\mathbb{Z}) \times \cdots \times \text{GL}_{n_{p+2}+1}(\mathbb{Z}),$$

while if $\mathbf{G} = \text{SL}_{n+1}$ the stabilizer is

$$\mathbf{L}_R(\mathbb{Z}) = \ker(\text{GL}_{n_1+1}(\mathbb{Z}) \times \cdots \times \text{GL}_{n_{p+2}+1}(\mathbb{Z}) \xrightarrow{\det} \mathbb{Z}^\times).$$

The term $\text{St}(\mathbb{Z}^{n_1+1}; \mathbb{F}) \otimes \cdots \otimes \text{St}(\mathbb{Z}^{n_{p+2}+1}; \mathbb{F})$ is exactly $\text{St}(\mathbf{L}_R; \mathbb{F})$. We deduce that

$$\mathbf{S}_p \otimes \mathbb{F} = \bigoplus_{R \in \mathcal{L}_p(\mathbf{G})} \text{Ind}_{\mathbf{L}_R(\mathbb{Z})}^{\mathbf{G}(\mathbb{Z})} \text{St}(\mathbf{L}_R; \mathbb{F}).$$

The claim now follows from Shapiro's lemma. $\square$

24. REDUCIBLE LEVI SUBGROUPS OVER THE INTEGERS

When we proved Theorem B for groups of type A_n in Part 2, we had to understand their standard Levi subgroups. These subgroups have reducible relative root systems of type $A_{n_1} \times \cdots \times A_{n_m}$, and we analyzed them using general results about reducible root systems from §6. This section discusses the analogue of these results for Levi subgroups of $\text{GL}_{n+1}(\mathbb{Z})$ and $\text{SL}_{n+1}(\mathbb{Z})$.

24.1. Setup. The standard Levi factors of GL_{n+1} and SL_{n+1} are of the forms

$$\text{GL}_{n_1+1} \times \cdots \times \text{GL}_{n_m+1} \quad \text{and}$$

$$\ker(\text{GL}_{n_1+1} \times \cdots \times \text{GL}_{n_m+1} \xrightarrow{\det} \text{GL}_1)$$

with $(n_1 + 1) + \cdots + (n_m + 1) = n + 1$. We will call groups $\mathbf{G}$ of these two forms *standard A-integral Levi subgroups*. Since such $\mathbf{G}$ are defined over $\mathbb{Z}$, the group $\mathbf{G}(\mathbb{Z})$ makes sense. To simplify our notation, we will let $A_0 = \{0\} \subset \mathbb{R}^0$, regarded as a trivial root system of rank 0. With this convention, a standard A-integral Levi subgroup $\mathbf{G}$ is split with root system

$${}_{\mathbb{Q}}\Phi(\mathbf{G}) = A_{n_1} \times \cdots \times A_{n_m}.$$

The A_{n_i} with $n_i = 0$ could be deleted from this.

24.2. Reduction to products. Our first result about these groups is as follows. It will serve as a substitute for Lemmas 6.1 and 6.2 from §6.

Lemma 24.1. *Let $\mathbf{G}$ be a standard A-integral Levi subgroup. Write*

$${}_{\mathbb{Q}}\Phi(\mathbf{G}) = A_{n_1} \times \cdots \times A_{n_m} \quad \text{with } n_1, \dots, n_m \geq 1.$$

Set $\mathbf{H} = \text{SL}_{n_1+1} \times \cdots \times \text{SL}_{n_m+1}$. Let $\mathbb{F}$ be a commutative ring. Then:

- (i) *the group $\mathbf{H}$ is a subgroup of $\mathbf{G}$ and the inclusion $\mathbf{H} \rightarrow \mathbf{G}$ induces a bijection between parabolic subgroups, and thus isomorphisms $\mathcal{T}(\mathbf{H}) \cong \mathcal{T}(\mathbf{G})$ and $\text{St}(\mathbf{H}; \mathbb{F}) \cong \text{Res}_{\mathbf{H}(\mathbb{Z})}^{\mathbf{G}(\mathbb{Z})} \text{St}(\mathbf{G}; \mathbb{F})$.*

If in addition for some $b \geq -1$ we have $H_i(\mathbf{H}(\mathbb{Z}); \text{St}(\mathbf{H})) = 0$ for $i \leq b$, then:

- (ii) *$H_i(\mathbf{G}(\mathbb{Z}); \text{St}(\mathbf{G}; \mathbb{F})) = 0$ for $i \leq b$ and the map*

$$H_{b+1}(\mathbf{H}(\mathbb{Z}); \text{St}(\mathbf{H}; \mathbb{F})) \rightarrow H_{b+1}(\mathbf{G}(\mathbb{Z}); \text{St}(\mathbf{G}; \mathbb{F}))$$

is a surjection.

Proof. Recall that we can add or delete copies of A_0 from our root system without changing it. Since SL_1 is trivial, we can add corresponding SL_{0+1} -factors to $\mathbf{H}$ without changing it. Adding an appropriate number of A_0 -factors to our root system,⁴⁰ we can thus assume that

$$\mathbf{H} = SL_{n_1+1} \times \cdots \times SL_{n_m+1} \subset \mathbf{G} \subset GL_{n_1+1} \times \cdots \times GL_{n_m+1}.$$

This shows that $\mathbf{H}$ is a subgroup of $\mathbf{G}$. The rest of (ii) is obvious.

For (ii), assume that for some $b \geq -1$ we have $H_i(\mathbf{H}(\mathbb{Z}); \text{St}(\mathbf{H})) = 0$ for $i \leq b$. By construction, we have a short exact sequence

$$1 \longrightarrow \mathbf{H}(\mathbb{Z}) \longrightarrow \mathbf{G}(\mathbb{Z}) \longrightarrow A \longrightarrow 1$$

with A abelian. The associated Hochschild–Serre spectral sequence with coefficients in $\text{St}(\mathbf{H}; \mathbb{F}) = \text{St}(\mathbf{G}; \mathbb{F})$ is of the form

$$E_{pq}^2 = H_p(A; H_q(\mathbf{H}(\mathbb{Z}); \text{St}(\mathbf{H}; \mathbb{F}))) \Rightarrow H_{p+q}(\mathbf{G}(\mathbb{Z}); \text{St}(\mathbf{G}; \mathbb{F})).$$

Our vanishing assumption implies that $E_{pq}^2 = 0$ for $q \leq b$, so $H_i(\mathbf{G}(\mathbb{Z}); \text{St}(\mathbf{G}; \mathbb{F})) = 0$ for $i \leq b$ and $H_{b+1}(\mathbf{G}(\mathbb{Z}); \text{St}(\mathbf{G}; \mathbb{F})) = E_{0,b+1}^2$. We deduce that

$$H_{b+1}(\mathbf{G}(\mathbb{Z}); \text{St}(\mathbf{G}; \mathbb{F})) = H_0(A; H_{b+1}(\mathbf{H}(\mathbb{Z}); \text{St}(\mathbf{H}; \mathbb{F}))) = H_{b+1}(\mathbf{H}(\mathbb{Z}); \text{St}(\mathbf{H}; \mathbb{F}))_A,$$

where the subscript indicates that we are taking coinvariants. For any group Γ and any Γ -module M , the map $M \rightarrow M_\Gamma$ is a surjection. In light of the above identity, we deduce that the map $H_{b+1}(\mathbf{H}(\mathbb{Z}); \text{St}(\mathbf{H}; \mathbb{F})) \rightarrow H_{b+1}(\mathbf{G}(\mathbb{Z}); \text{St}(\mathbf{G}; \mathbb{F}))$ is a surjection, as desired. $\square$

Lemma 24.1 has the following corollary:

Corollary 24.2. *Let $n \geq 1$ and let $\mathbb{F}$ be a commutative ring. For some $b \geq -1$, assume that $H_i(SL_{n+1}(\mathbb{Z}); \text{St}(\mathbb{Z}^{n+1}; \mathbb{F})) = 0$ for $i \leq b$. Then $H_i(GL_{n+1}(\mathbb{Z}); \text{St}(\mathbb{Z}^{n+1}; \mathbb{F})) = 0$.*

Proof. Immediate from the case $m = 1$ of Lemma 24.1. $\square$

24.3. Reducible root systems and vanishing. The following result will serve as a substitute for Lemma 6.4 from §6:

Lemma 24.3 (Reducible vanishing). *Let $n_1, \dots, n_m \geq 1$ and let $\mathbb{F}$ be a PID. For each $1 \leq j \leq m$, assume that there is some $b_j \geq -1$ such that the following holds:*

(♥) *We have $H_i(SL_{n_j+1}(\mathbb{Z}); \text{St}(\mathbb{Z}^{n_j+1}; \mathbb{F})) = 0$ for $i \leq b_j$.*

Then for all standard A -integral Levi subgroups $\mathbf{G}$ with ${}_{\mathbb{Q}}\Phi(\mathbf{G}) = A_{n_1} \times \cdots \times A_{n_m}$, we have $H_i(\mathbf{G}(\mathbb{Z}); \text{St}(\mathbf{G}; \mathbb{F})) = 0$ for $i \leq (m-1) + b_1 + \cdots + b_m$.

Proof. By Lemma 24.1, it is enough to prove this vanishing for⁴¹ $\mathbf{G} = SL_{n_1+1} \times \cdots \times SL_{n_m+1}$. The proof will be by induction on m . The base case $m = 1$ follows immediately from (♥), so assume that $m \geq 2$ and that the result is true whenever m is smaller.

As notation, set $\mathbf{A} = SL_{n_1+1} \times \cdots \times SL_{n_{m-1}+1}$ and $\mathbf{B} = SL_{n_m}$, so $\mathbf{G} = \mathbf{A} \times \mathbf{B}$. By Lemma 6.3, we have $\text{St}(\mathbf{A} \times \mathbf{B}; \mathbb{F}) = \text{St}(\mathbf{A}; \mathbb{F}) \otimes \text{St}(\mathbf{B}; \mathbb{F})$. Since $\mathbb{F}$ is a PID and $\text{St}(\mathbf{A}; \mathbb{F})$ and $\text{St}(\mathbf{B}; \mathbb{F})$ are free $\mathbb{F}$ -modules, the Künneth formula applies and shows that $H_i(\mathbf{A}(\mathbb{Z}) \times \mathbf{B}(\mathbb{Z}); \text{St}(\mathbf{A} \times \mathbf{B}; \mathbb{F}))$ fits into a short exact sequence with the following kernel and cokernel:

$$\begin{aligned} & \bigoplus_{i_1+i_2=i} H_{i_1}(\mathbf{A}(\mathbb{Z}); \text{St}(\mathbf{A}; \mathbb{F})) \otimes H_{i_2}(\mathbf{B}(\mathbb{Z}); \text{St}(\mathbf{B}; \mathbb{F})). \\ & \bigoplus_{i_1+i_2=i-1} \text{Tor}(H_{i_1}(\mathbf{A}(\mathbb{Z}); \text{St}(\mathbf{A}; \mathbb{F})), H_{i_2}(\mathbf{B}(\mathbb{Z}); \text{St}(\mathbf{B}; \mathbb{F}))). \end{aligned}$$

By our induction hypothesis and (♥), we have

- $H_{i_1}(\mathbf{A}(\mathbb{Z}); \text{St}(\mathbf{A}; \mathbb{F})) = 0$ for $i_1 \leq (m-2) + b_1 + \cdots + b_{m-1}$; and
- $H_{i_2}(\mathbf{B}(\mathbb{Z}); \text{St}(\mathbf{B}; \mathbb{F})) = 0$ for $i_2 \leq b_m$.

⁴⁰We thus no longer have $n_j \geq 1$ for all j , but this does not affect the truth of the lemma.

⁴¹Note that for $m \geq 2$ this $\mathbf{G}$ is not a standard A -integral Levi subgroup.

Note that if

$$i_1 + i_2 \leq ((m-2) + b_1 + \cdots + b_{m-1}) + b_m + 1 = (m-1) + b_1 + \cdots + b_m,$$

then either

$$i_1 \leq (m-2) + b_1 + \cdots + b_{m-1} \quad \text{or} \quad i_2 \leq b_m.$$

It follows that $H_i(\mathbf{A}(\mathbb{Z}) \times \mathbf{B}(\mathbb{Z}); \text{St}(\mathbf{A} \times \mathbf{B}; \mathbb{F})) = 0$ for $i \leq (m-1) + b_1 + \cdots + b_m$, as desired. $\square$

24.4. Reducible root systems and surjectivity. The following result will serve as a substitute for Lemma 6.5 from §6:

Lemma 24.4 (Reducible surjectivity). *Let $n_1, \dots, n_m \geq 1$ and let $\mathbb{F}$ be a PID. For each $1 \leq j \leq m$, assume that there is some $b_j \geq -1$ such that the following holds:*

(♥) *We have $H_i(\text{SL}_{n_j+1}(\mathbb{Z}); \text{St}(\mathbb{Z}^{n_j+1}; \mathbb{F})) = 0$ for $i \leq b_j$.*

Additionally, for some $1 \leq j_0 \leq m$ assume there exists a set Δ' of simple roots of A_{j_0} such that the following holds:

(♥♥) *Let $\mathbf{L}_{\Delta'}^{j_0}$ be the standard Levi subgroup of $\text{SL}_{n_{j_0}+1}$ corresponding to Δ' . Then the map $H_{b_{j_0}+1}(\mathbf{L}_{\Delta'}^{j_0}(\mathbb{Z}); \text{St}(\mathbf{L}_{\Delta'}^{j_0}; \mathbb{F})) \rightarrow H_{b_{j_0}+1}(\text{SL}_{n_{j_0}+1}(\mathbb{Z}); \text{St}(\mathbb{Z}^{n_{j_0}+1}; \mathbb{F}))$ is surjective.*

Let $\mathbf{G}$ be a standard A -integral Levi subgroup with $_{\mathbb{Q}}\Phi(\mathbf{G}) = A_{n_1} \times \cdots \times A_{n_m}$. Let

$$\overline{\Delta}' = \Delta' \sqcup \bigsqcup_{\substack{1 \leq j \leq m \\ j \neq j_0}} \mathbb{Q}\Delta(\Phi_j),$$

and let $\mathbf{L}_{\overline{\Delta}'}^{\mathbf{G}}$ be the corresponding standard Levi subgroup of $\mathbf{G}$. Then for

$$b = (m-1) + b_1 + \cdots + b_m$$

the map $H_{b+1}(\mathbf{L}_{\overline{\Delta}'}^{\mathbf{G}}(\mathbb{Z}); \text{St}(\mathbf{L}_{\overline{\Delta}'}^{\mathbf{G}}; \mathbb{F})) \rightarrow H_{b+1}(\mathbf{G}(\mathbb{Z}); \text{St}(\mathbf{G}; \mathbb{F}))$ is surjective.

Proof. Let $\mathbf{H} = \text{SL}_{n_1+1} \times \cdots \times \text{SL}_{n_m+1}$. The proof of Lemma 24.3 shows that we have $H_i(\mathbf{H}(\mathbb{Z}); \text{St}(\mathbf{H}; \mathbb{F})) = 0$ for $i \leq b$. Applying Lemma 24.1, we deduce:

- (i) the group $\mathbf{H}$ is a subgroup of $\mathbf{G}$ and the inclusion $\mathbf{H} \rightarrow \mathbf{G}$ induces a bijection between parabolic subgroups, and thus isomorphisms $\mathcal{T}(\mathbf{H}) \cong \mathcal{T}(\mathbf{G})$ and $\text{St}(\mathbf{H}; \mathbb{F}) \cong \text{Res}_{\mathbf{H}(\mathbb{Z})}^{\mathbf{G}(\mathbb{Z})} \text{St}(\mathbf{G}; \mathbb{F})$.
- (ii) $H_i(\mathbf{G}(\mathbb{Z}); \text{St}(\mathbf{G}; \mathbb{F})) = 0$ for $i \leq b$ and the map

$$H_{b+1}(\mathbf{H}(\mathbb{Z}); \text{St}(\mathbf{H}; \mathbb{F})) \rightarrow H_{b+1}(\mathbf{G}(\mathbb{Z}); \text{St}(\mathbf{G}; \mathbb{F}))$$

is a surjection.

Let $\mathbf{L}_{\overline{\Delta}'}^{\mathbf{H}}$ and $\mathbf{L}_{\Delta'}^{j_0}$ be the standard Levi subgroups of $\mathbf{H}$ and $\text{SL}_{n_{j_0}+1}$ corresponding to the simple roots $\overline{\Delta}'$ and Δ' . We therefore have

$$\mathbf{L}_{\overline{\Delta}'}^{\mathbf{H}} = \text{SL}_{n_1+1} \times \cdots \times \mathbf{L}_{\Delta'}^{j_0} \times \cdots \times \text{SL}_{n_m+1}.$$

By construction, $\mathbf{L}_{\overline{\Delta}'}^{\mathbf{H}}$ is a subgroup of $\mathbf{L}_{\overline{\Delta}'}^{\mathbf{G}}$ satisfying $\text{St}(\mathbf{L}_{\overline{\Delta}'}^{\mathbf{H}}; \mathbb{F}) = \text{St}(\mathbf{L}_{\overline{\Delta}'}^{\mathbf{G}}; \mathbb{F})$. We therefore have a commutative diagram

$$\begin{array}{ccc} H_{b+1}(\mathbf{L}_{\overline{\Delta}'}^{\mathbf{H}}(\mathbb{Z}); \text{St}(\mathbf{L}_{\overline{\Delta}'}^{\mathbf{H}}; \mathbb{F})) & \longrightarrow & H_{b+1}(\mathbf{L}_{\overline{\Delta}'}^{\mathbf{G}}(\mathbb{Z}); \text{St}(\mathbf{L}_{\overline{\Delta}'}^{\mathbf{G}}; \mathbb{F})) \\ \downarrow \tilde{f} & & \downarrow f \\ H_{b+1}(\mathbf{H}(\mathbb{Z}); \text{St}(\mathbf{H}; \mathbb{F})) & \longrightarrow & H_{b+1}(\mathbf{G}(\mathbb{Z}); \text{St}(\mathbf{G}; \mathbb{F})) \end{array}$$

From this, we see that to prove that f is surjective, it is enough to prove that $\tilde{f}$ is surjective.

In light of (♥), we can use the Künneth formula just like we did in the proof of Lemma 24.3 (reducible vanishing) to see that

$$H_{b+1}(\mathbf{H}(\mathbb{Z}); \text{St}(\mathbf{H}; \mathbb{F})) = \bigotimes_{j=1}^m H_{b_j+1}(\text{SL}_{n_j+1}(\mathbb{Z}); \text{St}(\mathbb{Z}^{n_j+1}; \mathbb{F})).$$

We thus see that it is enough to prove that the map

$$\begin{aligned} & \left(\bigotimes_{j=1}^{j_0-1} H_{b_j+1}(\text{SL}_{n_j+1}(\mathbb{Z}); \text{St}(\mathbb{Z}^{n_j+1}; \mathbb{F})) \right) \otimes H_{b_{j_0}+1}(\mathbf{L}_{\Delta'}^{j_0}(\mathbb{Z}); \text{St}(\mathbf{L}_{\Delta'}^{j_0}; \mathbb{F})) \\ & \otimes \left(\bigotimes_{j=j_0+1}^m H_{b_j+1}(\text{SL}_{n_j+1}(\mathbb{Z}); \text{St}(\mathbb{Z}^{n_j+1}; \mathbb{F})) \right) \rightarrow \bigotimes_{j=1}^m H_{b_j+1}(\text{SL}_{n_j+1}(\mathbb{Z}); \text{St}(\mathbb{Z}^{n_j+1}; \mathbb{F})) \end{aligned}$$

is surjective. Since the maps on all but one tensor factor are the identity, this is equivalent to the surjectivity of the map

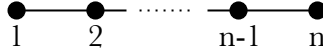
$$H_{b_{j_0}+1}(\mathbf{L}_{\Delta'}^{j_0}(\mathbb{Z}); \text{St}(\mathbf{L}_{\Delta'}^{j_0}; \mathbb{F})) \rightarrow H_{b_{j_0}+1}(\text{SL}_{j_0+1}(\mathbb{Z}); \text{St}(\mathbb{Z}^{j_0+1}; \mathbb{F}))$$

on the remaining tensor factor, which is exactly (♥♥). $\square$

25. VANISHING AND SURJECTIVITY (INTEGRAL)

In this section, we first recall our notation for the standard Levi factors of groups of type A_n . We then state our conditional vanishing theorems and prove them for ranks at most 2. We remark that Theorem C is proved immediately after the statement of Theorem C'.

25.1. Levi factor notation. Let $\mathbf{G}$ be either GL_{n+1} or SL_{n+1} , so ${}_{\mathbb{Q}}\Phi(\mathbf{G}) = A_n$. We recall the notation from Part 2 for its Levi factors. Let $\Delta = {}_{\mathbb{Q}}\Delta(\mathbf{G})$ be the set of simple roots of ${}_{\mathbb{Q}}\Phi(\mathbf{G}) = A_n$. Number the elements of Δ from left to right as in the usual Dynkin diagram:



For $1 \leq j_1, \dots, j_\ell \leq n$, let $\Delta[j_1, \dots, j_\ell]$ be the result of removing the simple roots labeled $j_1, \dots, j_\ell$ from Δ . We have a standard Levi subgroup $\mathbf{L}_{\Delta[j_1, \dots, j_\ell]}$ of $\mathbf{G}$.

25.2. Strong vanishing. The central results in this part of the paper are as follows.

Theorem C'.1 (2 and 3 invertible). *Assume the b-integral resolution conjecture (Conjecture 23.1) for some $b \geq 1$. Let $\mathbb{F} = \mathbb{Z}[1/2, 1/3]$. Then:*

- $H_i(\text{SL}_{n+1}(\mathbb{Z}); \text{St}(\mathbb{Z}^{n+1}; \mathbb{F})) = 0$ for $i \leq \min(b, \lfloor (n-1)/2 \rfloor)$; and
- letting $\Delta = {}_{\mathbb{Q}}\Delta(\text{SL}_{n+1})$, the maps

$$H_i(\mathbf{L}_{\Delta[1]}(\mathbb{Z}); \text{St}(\mathbf{L}_{\Delta[1]}; \mathbb{F})) \rightarrow H_i(\text{SL}_{n+1}(\mathbb{Z}); \text{St}(\mathbb{Z}^{n+1}; \mathbb{F})) \quad \text{and}$$

$$H_i(\mathbf{L}_{\Delta[n]}(\mathbb{Z}); \text{St}(\mathbf{L}_{\Delta[n]}; \mathbb{F})) \rightarrow H_i(\text{SL}_{n+1}(\mathbb{Z}); \text{St}(\mathbb{Z}^{n+1}; \mathbb{F}))$$

are both surjective for $i \leq \min(b, \lfloor n/2 \rfloor)$.

Theorem C'.2 (Integral). *Assume the b-integral resolution conjecture (Conjecture 23.1) for some $b \geq 1$. Then:*

- $H_i(\text{SL}_{n+1}(\mathbb{Z}); \text{St}(\mathbb{Z}^{n+1})) = 0$ for $i \leq \min(b, \lfloor (n-1)/3 \rfloor)$; and
- letting $\Delta = {}_{\mathbb{Q}}\Delta(\text{SL}_{n+1})$, the maps

$$H_i(\mathbf{L}_{\Delta[1]}(\mathbb{Z}); \text{St}(\mathbf{L}_{\Delta[1]})) \rightarrow H_i(\text{SL}_{n+1}(\mathbb{Z}); \text{St}(\mathbb{Z}^{n+1})) \quad \text{and}$$

$$H_i(\mathbf{L}_{\Delta[n]}(\mathbb{Z}); \text{St}(\mathbf{L}_{\Delta[n]})) \rightarrow H_i(\text{SL}_{n+1}(\mathbb{Z}); \text{St}(\mathbb{Z}^{n+1}))$$

are both surjective for $i \leq \min(b, \lfloor n/3 \rfloor)$.

We will discuss the proofs of these theorems soon, but first we show they imply:

Theorem C'. *Assume the b -integral resolution conjecture (Conjecture 23.1) for some $b \geq 1$. Then $H_i(\mathrm{GL}_{n+1}(\mathbb{Z}); \mathrm{St}(\mathbb{Z}^{n+1}; \mathbb{F})) = H_i(\mathrm{SL}_{n+1}(\mathbb{Z}); \mathrm{St}(\mathbb{Z}^{n+1}; \mathbb{F})) = 0$ for all commutative rings $\mathbb{F}$ and all $n, i \geq 0$ such that:*

- $i \leq \min(b, \lfloor (n-1)/2 \rfloor)$ if 2 and 3 are invertible in $\mathbb{F}$; and
- $i \leq \min(b, \lfloor (n-1)/3 \rfloor)$ in general.

Before proving Theorem C', note that the only difference between it and Theorem C is that Theorem C' assumes the b -integral resolution conjecture, while Theorem C assumes:

(†) The space $\mathcal{T}^2(\mathbb{Z}^n)$ is $n - 2 + \min(b, \lfloor (n-1)/2 \rfloor)$ -connected.

Corollary 23.4 says that (†) implies the b -integral resolution conjecture, so Theorem C' implies Theorem C.

Proof of Theorem C', assuming Theorems C'.1 and C'.2. By Corollary 24.2, it is enough to prove Theorem C' for $\mathrm{SL}_{n+1}(\mathbb{Z})$. For $\mathrm{SL}_{n+1}(\mathbb{Z})$, the only difference between the vanishing result in Theorem C' and those of Theorems C'.1 and C'.2 are the coefficients $\mathbb{F}$, which are more general in Theorem C'. Lemma 25.1 below (which generalizes Lemma 3.1) shows that Theorems C'.1 and C'.2 imply the slightly more general vanishing in Theorem C'. $\square$

Lemma 25.1. *Let $\mathbf{G}$ be a standard A -integral Levi subgroup, let $\mathcal{P}$ be a set of primes in $\mathbb{Z}$, and let $\mathbb{Z}_{\mathcal{P}} = \mathbb{Z}[1/p \mid p \in \mathcal{P}]$. Let $b \geq 0$ be such that $H_i(\mathbf{G}(\mathbb{Z}); \mathrm{St}(\mathbf{G}; \mathbb{Z}_{\mathcal{P}})) = 0$ for $i \leq b$. Then for all commutative rings $\mathbb{F}$ such that each $p \in \mathcal{P}$ is invertible in $\mathbb{F}$ we have $H_i(\mathbf{G}(\mathbb{Z}); \mathrm{St}(\mathbf{G}; \mathbb{F})) = 0$ for $i \leq b$.*

Proof. Letting $F_{\bullet} \rightarrow \mathbb{Z}$ be a free resolution of $\mathbf{G}(\mathbb{Z})$ -modules, the homology groups of $F_{\bullet} \otimes_{\mathbf{G}(\mathbb{Z})} \mathrm{St}(\mathbf{G}; \mathbb{Z}_{\mathcal{P}})$ are $H_i(\mathbf{G}(\mathbb{Z}); \mathrm{St}(\mathbf{G}; \mathbb{Z}_{\mathcal{P}}))$. Since $\mathrm{St}(\mathbf{G}; \mathbb{Z}_{\mathcal{P}})$ is a free $\mathbb{Z}_{\mathcal{P}}$ -module, each $F_i \otimes_{\mathbf{G}(\mathbb{Z})} \mathrm{St}(\mathbf{G}; \mathbb{Z}_{\mathcal{P}})$ is also a free $\mathbb{Z}_{\mathcal{P}}$ -module. Since $\mathbb{Z}_{\mathcal{P}}$ is a PID, we can apply the universal coefficients theorem to

$$F_{\bullet} \otimes_{\mathbf{G}(\mathbb{Z})} \mathrm{St}(\mathbf{G}; \mathbb{Z}_{\mathcal{P}}) \otimes_{\mathbb{Z}_{\mathcal{P}}} \mathbb{F} = F_{\bullet} \otimes_{\mathbf{G}(\mathbb{Z})} \mathrm{St}(\mathbf{G}; \mathbb{F}),$$

which computes $H_i(\mathbf{G}(\mathbb{Z}); \mathrm{St}(\mathbf{G}; \mathbb{F}))$. The result is a short exact sequence

$$0 \rightarrow H_i(\mathbf{G}(\mathbb{Z}); \mathrm{St}(\mathbf{G}; \mathbb{Z}_{\mathcal{P}})) \otimes \mathbb{F} \rightarrow H_i(\mathbf{G}(\mathbb{Z}); \mathrm{St}(\mathbf{G}; \mathbb{F})) \rightarrow \mathrm{Tor}(H_{i-1}(\mathbf{G}(\mathbb{Z}); \mathrm{St}(\mathbf{G})), \mathbb{Z}_{\mathcal{P}}) \rightarrow 0.$$

The lemma follows. $\square$

25.3. Initial results. Before discussing the proofs of Theorems C'.1 and C'.2, we prove two preliminary results.

Lemma 25.2. *Let $\mathbf{G} = \mathrm{SL}_{n+1}$ for some $n \geq 1$. For all commutative rings $\mathbb{F}$, the images of the maps*

$$\begin{aligned} H_i(\mathbf{L}_{\Delta[1]}(\mathbb{Z}); \mathrm{St}(\mathbf{L}_{\Delta[1]}; \mathbb{F})) &\rightarrow H_i(\mathbf{G}(\mathbb{Z}); \mathrm{St}(\mathbf{G}; \mathbb{F})) \quad \text{and} \\ H_i(\mathbf{L}_{\Delta[n]}(\mathbb{Z}); \mathrm{St}(\mathbf{L}_{\Delta[n]}; \mathbb{F})) &\rightarrow H_i(\mathbf{G}(\mathbb{Z}); \mathrm{St}(\mathbf{G}; \mathbb{F})) \end{aligned}$$

are the same.

Proof. We have

$$\begin{aligned} \mathbf{L}_{\Delta[1]}(\mathbb{Z}) &= \ker(\mathrm{GL}_1(\mathbb{Z}) \times \mathrm{GL}_n(\mathbb{Z}) \xrightarrow{\det} \mathbb{Z}^{\times}), \\ \mathbf{L}_{\Delta[n]}(\mathbb{Z}) &= \ker(\mathrm{GL}_n(\mathbb{Z}) \times \mathrm{GL}_1(\mathbb{Z}) \xrightarrow{\det} \mathbb{Z}^{\times}). \end{aligned}$$

These are conjugate subgroups of $\mathrm{SL}_{n+1}(\mathbb{Z})$, with the conjugating matrix an appropriate signed permutation matrix. The lemma now follows from the fact that inner automorphisms act trivially on group homology. $\square$

Lemma 25.3. *Theorems C'.1 and C'.2 hold for $n \leq 2$.*

Proof. For $n = 0$, these two theorems assert nothing. We therefore only need to handle SL_2 and SL_3 . Examining the conclusions of Theorems C'.1 and C'.2 for these groups, it is enough to prove the following two things:

Claim 1. $H_0(\mathrm{SL}_{n+1}(\mathbb{Z}); \mathrm{St}(\mathbb{Z}^{n+1}; \mathbb{F})) = 0$ for all commutative rings $\mathbb{F}$ and all $n \geq 1$.

This was proved by Lee–Szczarba [18, Theorem 1.3].

Claim 2. For all commutative rings $\mathbb{F}$ such that 2 and 3 are invertible in $\mathbb{F}$, we have $H_1(\mathrm{SL}_3(\mathbb{Z}); \mathrm{St}(\mathbb{Z}^3; \mathbb{F})) = 0$. We remark that this makes the surjectivity statement from Theorem C'.1 trivially true for $\mathrm{SL}_3(\mathbb{Z})$.

For a field $\mathbb{F}$ of characteristic 0, the paper [15] proves that $H_1(\mathrm{SL}_n(\mathbb{Z}); \mathrm{St}(\mathbb{Z}^n; \mathbb{F})) = 0$ for $n \geq 3$. In fact, as was pointed out in [27] the proof in [15] shows more generally that this holds if $\mathbb{F}$ is a commutative ring in which p is invertible for all primes $p \leq n$, giving the claim. Since [27] does not explain the argument, we give the details. We remark that [19, Theorem 1.10] proves that for an arbitrary commutative ring $\mathbb{F}$ we have $H_1(\mathrm{SL}_n(\mathbb{Z}); \mathrm{St}(\mathbb{Z}^n; \mathbb{F})) = 0$ for $n \geq 6$, but our main interest is in the case $n = 3$, so this result does not suffice for us.

The main theorem of [15] gives a presentation

$$\mathbf{X}_1 \rightarrow \mathbf{X}_0 \rightarrow \mathrm{St}(\mathbb{Z}^n) \rightarrow 0.$$

See §31 for more about this presentation. All the terms here are free abelian groups, so we can tensor this with $\mathbb{F}$ to get a presentation

$$\mathbf{Y}_1 \rightarrow \mathbf{Y}_0 \rightarrow \mathrm{St}(\mathbb{Z}^n; \mathbb{F}) \rightarrow 0.$$

Let $\mathbf{Y}_2$ be the kernel of the map $\mathbf{Y}_1 \rightarrow \mathbf{Y}_0$, so we have a resolution

$$0 \rightarrow \mathbf{Y}_2 \rightarrow \mathbf{Y}_1 \rightarrow \mathbf{Y}_0 \rightarrow \mathrm{St}(\mathbb{Z}^n; \mathbb{F}) \rightarrow 0.$$

This gives a spectral sequence of the form

$$E_{pq}^1 \cong H_q(\mathrm{SL}_n(\mathbb{Z}); \mathbf{Y}_p) \Rightarrow H_{p+q}(\mathrm{SL}_n(\mathbb{Z}); \mathrm{St}(\mathbb{Z}^n; \mathbb{F})).$$

To prove that $H_1(\mathrm{SL}_n(\mathbb{Z}); \mathrm{St}(\mathbb{Z}^n; \mathbb{F})) = 0$, it is enough to prove that $E_{10}^1 = E_{01}^1 = 0$.

The term $E_{10}^1 = H_0(\mathrm{SL}_n(\mathbb{Z}); \mathbf{Y}_1)$ equals the coinvariants of the action of $\mathrm{SL}_n(\mathbb{Z})$ on $\mathbf{Y}_1$. In [15], the authors prove that $\mathbf{X}_1$ is generated by elements σ such that there exists $f \in \mathrm{SL}_n(\mathbb{Z})$ with $f(\sigma) = -\sigma$. In the coinvariants, we thus have $2\sigma = 0$. Since 2 is invertible in $\mathbb{F}$, this implies that the image of σ in $(\mathbf{Y}_1)_{\mathrm{SL}_n(\mathbb{Z})} = (\mathbf{X}_1 \otimes \mathbb{F})_{\mathrm{SL}_n(\mathbb{Z})}$ is 0. Thus $E_{10}^1 = 0$, as desired.

For $E_{01}^1 = H_1(\mathrm{SL}_n(\mathbb{Z}); \mathbf{Y}_0)$, in [15] the authors describe $\mathbf{X}_0$ as an induced representation, which we now describe. Let Λ_n be the subgroup of $\mathrm{SL}_n(\mathbb{Z})$ consisting of signed permutation matrices with determinant 1. Let $\mathbb{Z}_{\mathrm{sign}}$ and $\mathbb{F}_{\mathrm{sign}}$ be $\mathbb{Z}$ and $\mathbb{F}$ with the Λ_n -actions given by multiplication by the sign of the permutation.⁴² The paper [15] then proves that

$$\mathbf{X}_0 = \mathrm{Ind}_{\Lambda_n}^{\mathrm{SL}_n(\mathbb{Z})} \mathbb{Z}_{\mathrm{sign}}, \quad \text{so } \mathbf{Y}_0 = \mathrm{Ind}_{\Lambda_n}^{\mathrm{SL}_n(\mathbb{Z})} \mathbb{F}_{\mathrm{sign}}.$$

By Shapiro's lemma, we have

$$E_{01}^1 = H_1(\mathrm{SL}_n(\mathbb{Z}); \mathbf{Y}_0) \cong H_1(\Lambda_n; \mathbb{F}_{\mathrm{sign}}).$$

The group $\Lambda_n < \mathrm{SL}_n(\mathbb{Z})$ is a subgroup of the whole signed permutation group, and thus has order dividing $2^n n!$. In particular, since all primes $p \leq n$ are invertible in $\mathbb{F}$ it follows that multiplication by $|\Lambda_n|$ is invertible in $\mathbb{F}$. This implies that $E_{01}^1 = 0$, as desired. $\square$

⁴²This is not the determinant, which is always 1. The signs in the signed permutation matrices play no role in it.

Remark 25.4. For $\mathrm{SL}_3(\mathbb{Z})$ we could only verify surjectivity⁴³ directly after inverting 2 and 3. Integrally, we will be forced to start by proving surjectivity for $\mathrm{SL}_4(\mathbb{Z})$. This is the ultimate source of the $1/3$ slope in Theorem C'. One might hope that we could prove a weaker theorem with a $1/2$ slope; for instance, that $H_i(\mathrm{SL}_n(\mathbb{Z}); \mathrm{St}(\mathrm{SL}_n)) = 0$ for $i \leq \lfloor (n-2)/2 \rfloor$ and that surjectivity holds for $i = \lfloor (n-1)/2 \rfloor$. However, these ranges would not give a sufficient vanishing range in our spectral sequence argument; cf. Remark 15.2. $\square$

25.4. Proof when 2 and 3 are invertible. We now turn to Theorems C'.1 and C'.2. We will give full details for Theorem C'.2 below, and to get a sense as to how the argument works a reader might want to read that first. For Theorem C'.1, the ranges in it for vanishing and surjectivity are exactly the same as those we proved in type A_n in Part 2. The proof of Theorem C'.1 follows Part 2 closely, with the following changes:

- Throughout, work with $\mathbb{F} = \mathbb{Z}[1/2, 1/3]$ instead of $\mathbb{F} = \mathbb{Z}$. Since $\mathbb{Z}[1/2, 1/3]$ is a PID, this causes no problems in the proof (e.g., in the uses of the Künneth formula).
- The spectral sequence from Lemma 23.5 replaces the spectral sequence for general reductive groups from Corollary 4.4. The existence of this spectral sequence is where we use the b -integral resolution conjecture (Conjecture 23.1).
- To handle reducible root systems, substitute the results about reducible Levi subgroups from §24 for the corresponding results from §6.

Because the arguments are so similar, we will not give the full details of the proof.

25.5. Proof over the integers. The rest of this part of the paper is devoted to the proof of Theorem C'.2. For the proof, we will assume that we have already proved Theorem C'.2 in smaller ranks. For this, we make the following definition:

Definition 25.5. For $b \geq 1$ and $r \geq 0$, the *integral (b, r) -surjectivity and vanishing hypothesis* is as follows:

- The b -integral resolution (Conjecture 23.1) holds.
- For all $n \leq r$, we have $H_i(\mathrm{SL}_{n+1}(\mathbb{Z}); \mathrm{St}(\mathbb{Z}^{n+1})) = 0$ for $i \leq \min(b, \lfloor (n-1)/3 \rfloor)$.
- For all $n \leq r$, letting $\Delta = {}_{\mathbb{Q}}\Delta(\mathrm{SL}_{n+1})$ the maps

$$\begin{aligned} H_i(\mathbf{L}_{\Delta[1]}(\mathbb{Z}); \mathrm{St}(\mathbf{L}_{\Delta[1]})) &\rightarrow H_i(\mathrm{SL}_{n+1}(\mathbb{Z}); \mathrm{St}(\mathbb{Z}^{n+1})) \quad \text{and} \\ H_i(\mathbf{L}_{\Delta[n]}(\mathbb{Z}); \mathrm{St}(\mathbf{L}_{\Delta[n]})) &\rightarrow H_i(\mathrm{SL}_{n+1}(\mathbb{Z}); \mathrm{St}(\mathbb{Z}^{n+1})) \end{aligned}$$

are both surjective for $i \leq \min(b, \lfloor n/3 \rfloor)$. $\square$

26. VANISHING AND SURJECTIVITY FOR LEVI SUBGROUPS (INTEGRAL)

In this section, we show how to use the integral (b, r) -surjectivity and vanishing hypothesis to analyze the homology of standard Levi subgroups. As notation, for $\Phi = A_{n_1} \times \cdots \times A_{n_m}$ define

$$\widehat{\mathbf{b}}(\Phi) = (m-1) + \lfloor (n_1-1)/3 \rfloor + \cdots + \lfloor (n_m-1)/3 \rfloor.$$

Our main result is as follows. Its statement uses the ordering on the simple roots of $A_{n_{j_0}}$ discussed in §25.1.

Lemma 26.1 (Levi vanishing and surjectivity). *Assume the integral $(b, n-1)$ -surjectivity and vanishing hypothesis (Definition 25.5). Let $\Delta \subset {}_{\mathbb{Q}}\Delta(\mathrm{SL}_{n+1})$ be a set of simple roots with $\Delta \neq {}_{\mathbb{Q}}\Delta(\mathrm{SL}_{n+1})$. Write*

$${}_{\mathbb{Q}}\Phi(\mathbf{L}_{\Delta}) = A_{n_1} \times \cdots \times A_{n_m}.$$

⁴³We actually proved vanishing for $\mathrm{SL}_3(\mathbb{Z})$. Knowing this does not seem to improve our ultimate result.

Set⁴⁴

$$c = \widehat{\mathbf{b}}(\mathbb{Q}\Phi(\mathbf{L}_\Delta)) = (m-1) + \lfloor (n_1-1)/3 \rfloor + \cdots + \lfloor (n_m-1)/3 \rfloor.$$

Then the following hold:

- (i) We have $H_i(\mathbf{L}_\Delta(\mathbb{Z}); \text{St}(\mathbf{L}_\Delta)) = 0$ for $i \leq \min(b, c)$.
- (ii) For some $1 \leq j_0 \leq n$, assume that n_{j_0} is divisible by 3. Let $\Delta' \subset \Delta$ be the set of simple roots obtained by removing either the first or last simple root from $A_{n_{j_0}}$, so

$$\mathbb{Q}\Phi(\mathbf{L}_{\Delta'}) = A_{n_1} \times \cdots \times A_{n_{j_0}-1} \times \cdots \times A_{n_m}.$$

Finally, assume that $c+1 \leq b$. Then the map

$$H_{c+1}(\mathbf{L}_{\Delta'}(\mathbb{Z}); \text{St}(\mathbf{L}_{\Delta'})) \rightarrow H_{c+1}(\mathbf{L}_\Delta(\mathbb{Z}); \text{St}(\mathbf{G}))$$

is surjective.

Proof. Since $\Delta \neq \mathbb{Q}\Delta(\mathbf{G})$, we have $n_j \leq n-1$ for $1 \leq j \leq m$. The integral $(b, n-1)$ -surjectivity and vanishing hypothesis thus applies to all the groups SL_{n_j+1} . This gives the hypothesis (♥) in Lemma 24.3 (reducible vanishing). Applying Lemma 24.3, we deduce (i). Similarly, for Δ' as in (ii) it gives the hypotheses (♥) and (♥♥) in Lemma 24.4 (reducible surjectivity). Applying Lemma 24.4, we deduce (ii). $\square$

27. VANISHING REGION (INTEGRAL)

If the b -integral resolution conjecture (Conjecture 23.1) holds, then for $n \geq 3$ Lemma 23.5 gives a spectral sequence E_{pq}^r converging to $H_{p+q}(\text{SL}_{n+1}(\mathbb{Z}); \text{St}(\mathbb{Z}^{n+1}))$ with⁴⁵

$$E_{pq}^1 \cong \bigoplus_{R \in \mathcal{L}_p(\mathbf{G})} H_q(\mathbf{L}_R(\mathbb{Z}); \text{St}(\mathbf{L}_R)) \quad \text{if } 0 \leq p \leq \min(b, \lfloor n/2 \rfloor)$$

The following lemma shows that our inductive hypothesis implies that many terms of this spectral sequence vanish.

Lemma 27.1. *Assume the integral $(b, n-1)$ -surjectivity and vanishing hypothesis (Definition 25.5) for some $b \geq 1$ and $n \geq 3$. Let E_{pq}^1 be the spectral sequence converging to $H_{p+q}(\text{SL}_{n+1}(\mathbb{Z}); \text{St}(\mathbb{Z}^{n+1}))$ from Lemma 23.5. Then the following hold:*

- Let $d = \lfloor n/3 \rfloor$. Then $E_{pq}^1 = 0$ for $p+q \leq \min(b, d)$, except for possibly E_{0d}^1 when n is congruent to 0 or 1 modulo 3. There are thus no exceptions if $b < d$ or n is congruent to 2 modulo 3.

Proof. Our goal is to prove a vanishing result for E_{pq}^1 . The terms in question all have $p \leq \min(b, d)$, so they all satisfy

$$E_{pq}^1 \cong \bigoplus_{R \in \mathcal{L}_p(\mathbf{G})} H_q(\mathbf{L}_R(\mathbb{Z}); \text{St}(\mathbf{L}_R)).$$

Consider some $R \in \mathcal{L}_p(\mathbf{G})$. We will prove that the integral $(b, n-1)$ -surjectivity and vanishing hypothesis implies that $H_q(\mathbf{L}_R(\mathbb{Z}); \text{St}(\mathbf{L}_R)) = 0$ whenever $q \leq b$ and $p+q \leq d$, with the possible exception of $(p, q) = (0, d)$ if n is congruent to 0 or 1 modulo 3. This will give the lemma.

Since R is obtained by deleting $p+1$ simple roots from the Dynkin diagram of A_n , we have

$$\mathbb{Q}\Phi(\mathbf{L}_R) = A_{n_1} \times \cdots \times A_{n_m} \quad \text{with } n_1 + \cdots + n_m + p + 1 = n.$$

⁴⁴In other versions of this lemma we used b for this bound, but we change to c since b is being used for something else.

⁴⁵Here we switched from Δ to R to make our notation match that of the corresponding sections of Part 2.

Lemma 26.1 (Levi vanishing and surjectivity) implies that $H_q(\mathbf{L}_R(\mathbb{Z}); \text{St}(\mathbf{L}_R)) = 0$ for q such that $q \leq b$ and

$$(27.1) \quad q \leq \widehat{\mathbf{b}}_{(\mathbb{Q}} \Phi(\mathbf{L}_R)) = (m-1) + \lfloor (n_1-1)/3 \rfloor + \cdots + \lfloor (n_m-1)/3 \rfloor.$$

For $a_1, a_2 \in \mathbb{Z}$, Lemma 10.2 implies that $1 + \lfloor a_1/3 \rfloor + \lfloor a_2/3 \rfloor \geq \lfloor (a_1 + a_2 + 1)/3 \rfloor$. Applying this repeatedly, we deduce that the right hand side of (27.1) is at least

$$\lfloor (n_1 + \cdots + n_m - 1)/3 \rfloor = \lfloor (n - p - 2)/3 \rfloor.$$

It follows that $H_q(\mathbf{L}_R(\mathbb{Z}); \text{St}(\mathbf{L}_R)) = 0$ for q such that $q \leq b$ and $q \leq \lfloor (n - p - 2)/3 \rfloor$, or equivalently

$$(27.2) \quad p + q \leq p + \lfloor (n - p - 2)/3 \rfloor = \lfloor (n + 2p - 2)/3 \rfloor.$$

There are now three cases:

- If $p \geq 1$, then the right hand side of (27.2) is at least $\lfloor n/3 \rfloor = d$.
- If $p = 0$ and n is congruent to 2 modulo 3, then the right hand side of (27.2) is $\lfloor (n-2)/3 \rfloor = \lfloor n/3 \rfloor = d$.
- If $p = 0$ and n is congruent to 0 or 1 modulo 3, then the right hand side of (27.2) is $\lfloor (n-2)/3 \rfloor = \lfloor n/3 \rfloor - 1 = d - 1$.

All of this implies what we are trying to show: that $H_q(\mathbf{L}_R(\mathbb{Z}); \text{St}(\mathbf{L}_R)) = 0$ for q such that $q \leq b$ and $p + q \leq \lfloor n/3 \rfloor$, with the possible exception of $(p, q) = (0, d)$ if n is congruent to 0 or 1 modulo 3. $\square$

28. REMAINING TASKS (INTEGRAL)

Lemma 27.1 implies many cases of Theorem C'.2. To prove the remaining cases, we need to compute some differentials in our spectral sequence. We now explain the structure of the argument, postponing three calculations to the next section. Recall that Theorem C'.2 is:

Theorem C'.2. *Assume the b -integral resolution conjecture (Conjecture 23.1) for some $b \geq 1$. Then:*

- $H_i(\text{SL}_{n+1}(\mathbb{Z}); \text{St}(\mathbb{Z}^{n+1})) = 0$ for $i \leq \min(b, \lfloor (n-1)/3 \rfloor)$; and
- letting $\Delta = {}_{\mathbb{Q}}\Delta(\text{SL}_{n+1})$, the maps

$$\begin{aligned} H_i(\mathbf{L}_{\Delta[1]}(\mathbb{Z}); \text{St}(\mathbf{L}_{\Delta[1]})) &\rightarrow H_i(\text{SL}_{n+1}(\mathbb{Z}); \text{St}(\mathbb{Z}^{n+1})) \quad \text{and} \\ H_i(\mathbf{L}_{\Delta[n]}(\mathbb{Z}); \text{St}(\mathbf{L}_{\Delta[n]})) &\rightarrow H_i(\text{SL}_{n+1}(\mathbb{Z}); \text{St}(\mathbb{Z}^{n+1})) \end{aligned}$$

are both surjective for $i \leq \min(b, \lfloor n/3 \rfloor)$.

Proof. The proof is by induction on n . We proved the base cases $n \leq 2$ in Lemma 25.3, so we can assume that $n \geq 3$ and that the result is true for smaller ranks, i.e., that the integral $(b, n-1)$ -surjectivity and vanishing hypothesis holds.

Lemma 23.5 gives a spectral sequence E_{pq}^r converging to $H_{p+q}(\text{SL}_{n+1}(\mathbb{Z}); \text{St}(\mathbb{Z}^{n+1}))$. Letting $d = \lfloor n/3 \rfloor$, Lemma 27.1 implies that $E_{pq}^1 = 0$ for $p + q \leq \min(b, d)$ except for possibly E_{0d}^1 when n is congruent to 0 or 1 modulo 3. This implies that $H_i(\text{SL}_{n+1}(\mathbb{Z}); \text{St}(\mathbb{Z}^{n+1})) = 0$ for

$$i \leq \begin{cases} b & \text{if } b < d, \\ d & \text{if } b \geq d \text{ and } n \text{ is congruent to 2 modulo 3,} \\ d-1 & \text{if } b \geq d \text{ and } n \text{ is congruent to 0 or 1 modulo 3.} \end{cases}$$

Since our surjectivity claim is trivial when the target is 0, all that remains to prove are the following two claims:

Claim 1. *Assume that $b \geq d$ and that n is congruent to 1 modulo 3, so $d = \lfloor (n-1)/3 \rfloor$. Then $H_d(\text{SL}_{n+1}(\mathbb{Z}); \text{St}(\mathbb{Z}^{n+1})) = 0$.*

In this case, Lemma 27.1 says that the only potentially nonzero term E_{pq}^1 in our spectral sequence with $p + q = d$ is E_{0d}^1 . We will prove in Lemma 29.2 below that the differential $E_{1d}^1 \rightarrow E_{0d}^1$ is surjective, so $E_{0d}^2 = 0$. This implies that $H_d(\mathrm{SL}_{n+1}(\mathbb{Z}); \mathrm{St}(\mathbb{Z}^{n+1})) = 0$, as desired.

Claim 2. *Assume that $b \geq d$ and that n is congruent to 0 modulo 3, so $d - 1 = \lfloor (n - 1)/3 \rfloor$. Then the maps*

$$\begin{aligned} H_d(\mathbf{L}_{\Delta[1]}(\mathbb{Z}); \mathrm{St}(\mathbf{L}_{\Delta[1]})) &\rightarrow H_d(\mathrm{SL}_{n+1}(\mathbb{Z}); \mathrm{St}(\mathbb{Z}^{n+1})) \quad \text{and} \\ H_d(\mathbf{L}_{\Delta[n]}(\mathbb{Z}); \mathrm{St}(\mathbf{L}_{\Delta[n]})) &\rightarrow H_d(\mathrm{SL}_{n+1}(\mathbb{Z}); \mathrm{St}(\mathbb{Z}^{n+1})) \end{aligned}$$

are both surjective.

Lemma 25.2 says that these maps have the same image, so it is enough to prove that

$$(28.1) \quad H_d(\mathbf{L}_{\Delta[1]}(\mathbb{Z}); \mathrm{St}(\mathbf{L}_{\Delta[1]})) \oplus H_d(\mathbf{L}_{\Delta[n]}(\mathbb{Z}); \mathrm{St}(\mathbf{L}_{\Delta[n]}))$$

surjects onto $H_d(\mathrm{SL}_{n+1}(\mathbb{Z}); \mathrm{St}(\mathbb{Z}^{n+1}))$. Lemma 27.1 says that the only potentially nonzero term E_{pq}^1 in our spectral sequence with $p + q = d$ is E_{0d}^1 . We will prove in Lemma 29.1 below that the summand (28.1) of

$$E_{0d}^1 = \bigoplus_{R \in \mathcal{L}_0(\mathbf{G})} H_d(\mathbf{L}_R(\mathbb{Z}); \mathrm{St}(\mathbf{L}_R)) = \bigoplus_{j=1}^n H_d(\mathbf{L}_{\Delta[j]}(\mathbb{Z}); \mathrm{St}(\mathbf{L}_{\Delta[j]}))$$

surjects onto the cokernel of the differential $E_{1d}^1 \rightarrow E_{0d}^1$. It follows that E_{0d}^2 is a quotient of (28.1). Since E_{0d}^2 is the only potentially nonzero term of the form E_{pq}^2 with $p + q = d$, it follows that (28.1) surjects onto $H_d(\mathrm{SL}_{n+1}(\mathbb{Z}); \mathrm{St}(\mathbb{Z}^{n+1}))$, as desired. $\square$

29. DIFFERENTIALS (INTEGRAL)

This final section of this part of the paper determines the images of two differentials whose calculations were needed in the previous section.

29.1. Differentials, I (integral). Our first differential calculation is:

Lemma 29.1. *Consider SL_{n+1} with $n = 3d$ for some $d \geq 1$. Assume the integral $(b, n - 1)$ -surjectivity and vanishing hypothesis (Definition 25.5) for some $b \geq d$. Let E_{pq}^1 be the spectral sequence from Lemma 23.5. Then the summand*

$$H_d(\mathbf{L}_{\Delta[1]}(\mathbb{Z}); \mathrm{St}(\mathbf{L}_{\Delta[1]})) \oplus H_d(\mathbf{L}_{\Delta[2d]}(\mathbb{Z}); \mathrm{St}(\mathbf{L}_{\Delta[2d]}))$$

of E_{0d}^1 surjects onto the cokernel of the differential $E_{1d}^1 \rightarrow E_{0d}^1$.

Proof. As notation, for $1 \leq j_1, \dots, j_\ell \leq 3d$ let

$$M[j_1, \dots, j_\ell] = H_d(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]}(\mathbb{Z}); \mathrm{St}(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]})).$$

We have

$$E_{0d}^1 = \bigoplus_{1 \leq j \leq 3d} M[j] \quad \text{and} \quad E_{1d}^1 = \bigoplus_{1 \leq j_1 < j_2 \leq 3d} M[j_1, j_2].$$

Consider some $1 < j < 3d$. We must prove that when we quotient E_{0d}^1 by the image of the differential $E_{1d}^1 \rightarrow E_{0d}^1$, the summand $M[j]$ of E_{0d}^1 is identified with a subspace of $M[1] \oplus M[3d]$. The first step is to show that many $M[j]$ already vanish:

Claim 1. *For $1 < j < 3d$ with $j = 3e + 2$ for some $0 \leq e < d$, we have $M[j] = 0$.*

Proof of claim. Since ${}_{\mathbb{Q}}\Phi(\mathbf{L}_{\Delta[3e+2]}) = A_{3e+1} \times A_{3d-3e-2}$, Lemma 26.1 (Levi vanishing and surjectivity) implies that $H_i(\mathbf{L}_{\Delta[3e+2]}(\mathbb{Z}); \text{St}(\mathbf{L}_{\Delta[3e+2]})) = 0$ for

$$i \leq \widehat{\mathbf{b}}({}_{\mathbb{Q}}\Phi(\mathbf{L}_{\Delta[3e+2]})) = 1 + \lfloor 3e/3 \rfloor + \lfloor (3d - 3e - 3)/3 \rfloor = 1 + e + (d - e - 1) = d.$$

In particular, $M[3e + 2] = H_d(\mathbf{L}_{\Delta[3e+2]}(\mathbb{Z}); \text{St}(\mathbf{L}_{\Delta[3e+2]})) = 0$. $\square$

In light of Claim 1, there are two remaining cases. The first is that $j = 3e + 1$ for some $1 \leq e < d$, so

$$(29.1) \quad {}_{\mathbb{Q}}\Phi(\mathbf{L}_{\Delta[j]}) = A_{3e} \times A_{3d-3e-1},$$

$$(29.2) \quad \begin{aligned} \widehat{\mathbf{b}}({}_{\mathbb{Q}}\Phi(\mathbf{L}_{\Delta[j]})) &= 1 + \lfloor (3e - 1)/3 \rfloor + \lfloor (3d - 3e - 2)/3 \rfloor \\ &= 1 + (e - 1) + (d - e - 1) = d - 1. \end{aligned}$$

The differential $E_{1d}^1 \rightarrow E_{0d}^1$ takes the summand $M[1, j]$ of E_{1d}^1 to E_{0d}^1 via the map

$$M[1, j] \xrightarrow{f \oplus (-g)} M[j] \oplus M[1] \hookrightarrow E_{0d}^1.$$

To prove that this differential identifies $M[j]$ with a subspace of $M[1]$, we must prove that $f: M[1, j] \rightarrow M[j]$ is surjective. Since the A_{3e} in (29.1) has $3e$ a positive multiple of 3, this follows from Lemma 26.1 (Levi vanishing and surjectivity). Here we are using the fact that $\widehat{\mathbf{b}}({}_{\mathbb{Q}}\Phi(\mathbf{L}_{\Delta[j]})) + 1 = d$; cf. (29.2).

The second case is that $j = 3e$ for some $1 \leq e < d$, so

$$(29.3) \quad {}_{\mathbb{Q}}\Phi(\mathbf{L}_{\Delta[j]}) = A_{3e-1} \times A_{3d-3e},$$

$$(29.4) \quad \begin{aligned} \widehat{\mathbf{b}}({}_{\mathbb{Q}}\Phi(\mathbf{L}_{\Delta[j]})) &= 1 + \lfloor (3e - 2)/3 \rfloor + \lfloor (3d - 3e - 1)/3 \rfloor \\ &= 1 + (e - 1) + (d - e - 1) = d - 1. \end{aligned}$$

The differential $E_{1d}^1 \rightarrow E_{0d}^1$ takes the summand $M[j, 3d]$ of E_{1d}^1 to E_{0d}^1 via the map

$$M[j, 3d] \xrightarrow{f \oplus (-g)} M[3d] \oplus M[j] \hookrightarrow E_{0d}^1.$$

To prove that this differential identifies $M[j]$ with a subspace of $M[3d]$, we must prove that $f: M[j, 3d] \rightarrow M[j]$ is surjective. Since the A_{3d-3e} in (29.3) has $3d - 3e$ a positive multiple of 3, this follows from Lemma 26.1 (Levi vanishing and surjectivity). Here we are using the fact that $\widehat{\mathbf{b}}({}_{\mathbb{Q}}\Phi(\mathbf{L}_{\Delta[j]})) + 1 = d$; cf. (29.4). $\square$

29.2. Differentials, II (integral). Our second and final differential calculation is:

Lemma 29.2. *Consider SL_{n+1} with $n = 3d + 1$ for some $d \geq 1$. Assume the integral $(b, n - 1)$ -surjectivity and vanishing hypothesis (Definition 25.5) for some $b \geq d$. Let E_{pq}^1 be the spectral sequence from Lemma 23.5. Then the differential $E_{1d}^1 \rightarrow E_{0d}^1$ is surjective.*

Proof. As notation, for $1 \leq j_1, \dots, j_\ell \leq 3d + 1$ let

$$M[j_1, \dots, j_\ell] = H_d(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]}(\mathbb{Z}); \text{St}(\mathbf{L}_{\Delta[j_1, \dots, j_\ell]})).$$

We have

$$E_{0d}^1 = \bigoplus_{1 \leq j \leq 3d+1} M[j] \quad \text{and} \quad E_{1d}^1 = \bigoplus_{1 \leq j_1 < j_2 \leq 3d+1} M[j_1, j_2].$$

Consider some $1 \leq j \leq 3d + 1$. We must prove that when we quotient E_{0d}^1 by the image of the differential $E_{1d}^1 \rightarrow E_{0d}^1$, the summand $M[j]$ is killed. The first step is to show that many $M[j]$ already vanish:

Claim 1. *For $1 \leq j \leq 3d + 1$ with j equal to either $3e + 2$ with $0 \leq e < d$ or $3e$ with $1 \leq e \leq d$, we have $M[j] = 0$.*

Proof of claim. Since ${}_{\mathbb{Q}}\Phi(\mathbf{L}_{\Delta[j]}) = A_{j-1} \times A_{3d+1-j}$, Lemma 26.1 (Levi vanishing and surjectivity) implies that $H_i(\mathbf{L}_{\Delta[j]}(\mathbb{Z}); \text{St}(\mathbf{L}_{\Delta[j]})) = 0$ for

$$i \leq \widehat{\mathbf{b}}({}_{\mathbb{Q}}\Phi(\mathbf{L}_{\Delta[j]})) = 1 + \lfloor (j-2)/3 \rfloor + \lfloor (3d-j+1)/3 \rfloor.$$

We must prove that the right hand side of this is d for j of the indicated forms:

- If $j = 3e + 2$ with $0 \leq e < d$, then the right hand side is

$$1 + \lfloor 3e/3 \rfloor + \lfloor 3d - 3e - 1 \rfloor = 1 + e + (d - e - 1) = d.$$

- If $j = 3e$ with $1 \leq e \leq d$, then the right hand side is

$$1 + \lfloor (3e-2) \rfloor + \lfloor (3d - 3e + 1)/3 \rfloor = 1 + (e-1) + (d-e) = d. \quad \square$$

Now consider $1 \leq j \leq 3d+1$ with $j = 3e+1$ for some $0 \leq e \leq d$. In light of Claim 1, it is enough to prove that $M[j]$ is killed when we quotient E_{0d}^1 by the image of the differential $E_{1d}^1 \rightarrow E_{0d}^1$. Assume first that $j \neq 1$, so

$$(29.5) \quad {}_{\mathbb{Q}}\Phi(\mathbf{L}_{\Delta[j]}) = \begin{cases} A_{3e} \times A_{3d-3e} & \text{if } e \neq d, \\ A_{3e} & \text{if } e = d. \end{cases}$$

In both cases, we have

$$(29.6) \quad \widehat{\mathbf{b}}({}_{\mathbb{Q}}\Phi(\mathbf{L}_{\Delta[j]})) = d - 1.$$

We will show that $M[j]$ is killed by the image of the summand $M[j-1, j]$ of E_{1d}^1 . On the summand $M[j-1, j]$, the differential is the map

$$M[j-1, j] \xrightarrow{f \oplus (-g)} M[j] \oplus M[j-1] \hookrightarrow E_{0d}^1.$$

Claim 1 says that $M[j-1] = 0$, so to show that this differential kills $M[j]$ it is enough to prove that $f: M[j-1, j] \rightarrow M[j]$ is surjective. Since the A_{3e} in both cases of (29.5) has $3e$ a positive multiple of 3, this follows from Lemma 26.1 (Levi vanishing and surjectivity). Here we are using the fact that $\widehat{\mathbf{b}}({}_{\mathbb{Q}}\Phi(\mathbf{L}_{\Delta[j]}) + 1 = d$; cf. (29.6).

It remains to deal with the case $j = 1$, so

$$(29.7) \quad {}_{\mathbb{Q}}\Phi(\mathbf{L}_{\Delta[j]}) = A_{3d},$$

$$(29.8) \quad \widehat{\mathbf{b}}({}_{\mathbb{Q}}\Phi(\mathbf{L}_{\Delta[j]})) = \lfloor (3d-1)/3 \rfloor = d - 1.$$

In this case, we will use the summand $M[1, 2]$ of E_{1d}^1 . Just like above, on this summand this differential takes the form

$$M[1, 2] \longrightarrow M[2] \oplus M[1] \hookrightarrow E_{0d}^1.$$

Claim 1 says that $M[2] = 0$, and since the A_{3d} in (29.7) has $3d$ a positive multiple of 3 Lemma 26.1 (Levi vanishing and surjectivity) implies that the map $M[1, 2] \rightarrow M[1]$ is surjective. Here we are using the fact that $\widehat{\mathbf{b}}({}_{\mathbb{Q}}\Phi(\mathbf{L}_{\Delta[j]}) + 1 = d$; cf. (29.8). The lemma follows. $\square$

Part 6. The double Tits building (Theorem D)

We defined the double Tits building $\mathcal{T}^2(\mathbb{Z}^n)$ in §1.7. In this final part of the paper, we prove Theorem D, which says that $\mathcal{T}^2(\mathbb{Z}^n)$ is n -connected for $n \geq 4$. See the introductory §30 for an outline.

30. OUTLINE OF CONNECTIVITY PROOF

In this introductory section, we first introduce some notation, then translate Theorem D into a homological statement, and finally outline our proof of it.

30.1. Steinberg module and lines. Let V be a rank- r free $\mathbb{Z}$ -module with $r \geq 1$. Recall that $\mathcal{T}(V) \cong \mathcal{T}(\mathrm{GL}_r)$ is the complex of flags of nonzero proper direct summands of V and $\mathrm{St}(V) = \tilde{H}_{r-2}(\mathcal{T}(V)) \cong \mathrm{St}(\mathrm{GL}_r)$. A *line* in V is a one-dimensional direct summand. Each line L can be written as $L = \langle x \rangle$ for an $x \in V$ that is *primitive*, that is, only divisible by ± 1 . The x in $L = \langle x \rangle$ is unique up to multiplication by ± 1 .

30.2. Apartments. For lines $L_1, \dots, L_r \subset V$, the *apartment class* $\mathbb{A}[[L_1, \dots, L_r]] \in \mathrm{St}(V)$ is as follows. Let X_r be the simplicial complex with vertices $[r] = \{1, \dots, r\}$ and simplices nonempty proper subsets of $[r]$. The complex X_r is isomorphic to the barycentric subdivision of the boundary of an $(r-1)$ -simplex. Let $f: X_r \rightarrow \mathcal{T}(V)$ be the simplicial map taking the simplex corresponding to $I \subsetneq [r]$ to $\langle L_i \mid i \in I \rangle$. The ordering on $[r]$ gives $X_r \cong \mathbb{S}^{r-2}$ an orientation. Let $[X_r] \in \tilde{H}_{r-2}(X_r) \cong \mathbb{Z}$ be the fundamental class. Then

$$\mathbb{A}[[L_1, \dots, L_r]] = f_*([X_r]) \in \tilde{H}_{r-2}(\mathcal{T}(V)) = \mathrm{St}(V).$$

Permuting the L_i changes the orientation, so for σ in the symmetric group $\mathfrak{S}_r$ we have

$$\mathbb{A}[[L_{\sigma(1)}, \dots, L_{\sigma(r)}]] = (-1)^{|\sigma|} \mathbb{A}[[L_1, \dots, L_r]].$$

We have $\mathbb{A}[[L_1, \dots, L_r]] = 0$ if the L_i are linearly dependent, i.e., if they do not span $V \otimes \mathbb{Q}$. The Solomon–Tits theorem says that $\mathrm{St}(V)$ is spanned by apartment classes. In fact, Ash–Rudolph [3] proved that $\mathrm{St}(V)$ is spanned by *integral apartment classes*, i.e., apartment classes $\mathbb{A}[[L_1, \dots, L_r]]$ such that the lines L_i satisfy $V = L_1 \oplus \dots \oplus L_r$.

Remark 30.1. Assume that the L_i are linearly independent. Identifying V with $\mathbb{Z}^r$ and $\mathrm{St}(V)$ with $\mathrm{St}(\mathrm{GL}_r)$, in the notation of §2.4 the apartment $\mathbb{A}[[L_1, \dots, L_r]]$ is the apartment $\mathbb{A}_g$ with $g \in \mathrm{GL}_r(\mathbb{Q})$ the matrix whose columns are $v_i \in \mathbb{Z}^r$ with $L_i = \langle v_i \rangle$. $\square$

30.3. Steinberg multiplication. Let V and W be $\mathbb{Z}$ -modules with $V \cong \mathbb{Z}^r$ and $W \cong \mathbb{Z}^s$. There is a natural bilinear multiplication $\mathrm{St}(V) \otimes \mathrm{St}(W) \rightarrow \mathrm{St}(V \oplus W)$. For apartment classes $\mathbb{A}[[L_1, \dots, L_r]] \in \mathrm{St}(V)$ and $\mathbb{A}[[L'_1, \dots, L'_s]] \in \mathrm{St}(W)$, their product is

$$\mathbb{A}[[L_1, \dots, L_r]] \cdot \mathbb{A}[[L'_1, \dots, L'_s]] = \mathbb{A}[[L_1, \dots, L_r, L'_1, \dots, L'_s]].$$

Here we are identifying V and W with the corresponding subspaces of $V \oplus W$. This product is classical. One place where it appears explicitly is [14]. See [1] for recent work on it. This product appeared earlier in our paper in a different guise. Indeed, identify V with $\mathbb{Z}^r$ and W with $\mathbb{Z}^s$. We can then identify our product with a map

$$\mathrm{St}(\mathrm{GL}_r \times \mathrm{GL}_s) = \mathrm{St}(\mathrm{GL}_r) \otimes \mathrm{St}(\mathrm{GL}_s) = \mathrm{St}(\mathbb{Z}^r) \otimes \mathrm{St}(\mathbb{Z}^s) \longrightarrow \mathrm{St}(\mathbb{Z}^{r+s}) = \mathrm{St}(\mathrm{GL}_{r+s}).$$

This product map $\mathrm{St}(\mathrm{GL}_r \times \mathrm{GL}_s) \rightarrow \mathrm{St}(\mathrm{GL}_{r+s})$ is exactly the Reeder map described in §2.6.

30.4. Chain complex for Steinberg. For $i \geq -1$, recall from §23.3 that

$$\mathbf{S}_i(\mathbb{Z}^n) = \bigoplus_{V_1 \oplus \dots \oplus V_{i+2} = \mathbb{Z}^n} \mathrm{St}(V_1) \otimes \dots \otimes \mathrm{St}(V_{i+2}).$$

Here and throughout the rest of the paper the direct sum is over all decompositions $V_1 \oplus \dots \oplus V_{i+1} = \mathbb{Z}^n$ with $V_j \neq 0$ for all $1 \leq j \leq i+2$. As a special case, we have

$\mathbf{S}_{-1}(\mathbb{Z}^n) = \text{St}(\mathbb{Z}^n)$. As we discussed in §23.3, these fit into a chain complex

$$0 \rightarrow \mathbf{S}_{n-2}(\mathbb{Z}^n) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathbf{S}_0(\mathbb{Z}^n) \xrightarrow{\partial} \mathbf{S}_{-1}(\mathbb{Z}^n) \rightarrow 0.$$

$$\parallel$$

$$\text{St}(\mathbb{Z}^n)$$

For a decomposition $V_1 \oplus \cdots \oplus V_{i+2} = \mathbb{Z}^n$ and $\theta_j \in \text{St}(V_j)$ for $1 \leq j \leq i+2$, denote $\theta_1 \otimes \cdots \otimes \theta_{i+2} \in \mathbf{S}_i(\mathbb{Z}^n)$ by $[\theta_1 | \cdots | \theta_{i+2}]$. The boundary map then has the familiar form

$$\partial[\theta_1 | \cdots | \theta_{i+2}] = \sum_{j=1}^{i+1} (-1)^{j-1} [\theta_1 | \cdots | \theta_j \cdot \theta_{j+1} | \cdots | \theta_{i+2}] \in \mathbf{S}_{i-1}(\mathbb{Z}^n).$$

30.5. Main theorem. The main theorem we will prove in this part of the paper is:

Theorem D'. *For $n \geq 4$, the chain complex*

$$\mathbf{S}_2(\mathbb{Z}^n) \xrightarrow{\partial} \mathbf{S}_1(\mathbb{Z}^n) \xrightarrow{\partial} \mathbf{S}_0(\mathbb{Z}^n) \xrightarrow{\partial} \text{St}(\mathbb{Z}^n) \longrightarrow 0$$

is exact.

Before outlining the proof of Theorem D', we explain why it implies Theorem D:

Theorem D. *The complex $\mathcal{T}^2(\mathbb{Z}^n)$ is n -connected for $n \geq 4$.*

Proof, assuming Theorem D'. Let $n \geq 4$. Theorem 23.3 says that

$$\tilde{H}_{i+n-1}(\mathcal{T}^2(\mathbb{Z}^n)) = H_i(\mathbf{S}_\bullet(\mathbb{Z}^n)) \quad \text{for all } i.$$

Theorem D' implies that $H_i(\mathbf{S}_\bullet(\mathbb{Z}^n)) = 0$ for $i \leq 1$, so $\tilde{H}_i(\mathcal{T}^2(\mathbb{Z}^n)) = 0$ for $i \leq n$. To complete the proof, it is enough to prove that the fundamental group of $\mathcal{T}^2(\mathbb{Z}^n)$ is trivial. Let $f: \mathbb{S}^1 \rightarrow \mathcal{T}^2(\mathbb{Z}^n)$ be a map. By definition, $\mathcal{T}^2(\mathbb{Z}^n)$ contains a copy of $\mathcal{T}(\mathbb{Z}^n)$. It follows from the proof of [22, Proposition 5.7] that $f: \mathbb{S}^1 \rightarrow \mathcal{T}^2(\mathbb{Z}^n)$ can be homotoped such that its image lies in $\mathcal{T}(\mathbb{Z}^n)$. The Solomon–Tits theorem says that $\mathcal{T}(\mathbb{Z}^n)$ is $(n-3)$ -connected. Since $n \geq 4$, it is in particular 1-connected, so f is homotopic to a constant map, as desired. $\square$

30.6. Outline of rest of paper. The rest of this paper is devoted to proving Theorem D'. Our proof has three steps. In §31 we recall a three-step resolution of $\text{St}(\mathbb{Z}^r)$ that was proved by Brück–Miller–Patz–Sroka–Wilson [9], generalizing classical work on modular symbols and [12, 15]. Next, in §32 we construct a double complex out of this 3-step resolution and prove that its homology agrees with that of $\mathbf{S}_\bullet(\mathbb{Z}^n)$ in a range of degrees. Finally, in §33 we use this double complex to prove Theorem D'.

31. THREE-STEP PARTIAL RESOLUTION OF STEINBERG FROM MODULAR SYMBOLS

In this section, we describe a three-step resolution of the Steinberg representation that is different from the one in Theorem D'. Let V be a free $\mathbb{Z}$ -module of rank $r \geq 1$.

31.1. Frames and augmentations. A *frame* of V is an ordered collection $(L_1, \dots, L_r)$ of lines in V such that $V = L_1 \oplus \cdots \oplus L_r$. Associated to this frame is an integral apartment class $\mathbb{A}[[L_1, \dots, L_r]] \in \text{St}(V)$. A *partial frame* of V is an ordered collection of lines in V that can be extended to a frame. Fix a partial frame $F = (L_1, \dots, L_s)$ for V . For each $1 \leq j \leq s$, write $L_j = \langle v_j \rangle$ for some $v_j \in V$. Let $[s] = \{1, \dots, s\}$. An *augmentation* of F is a set $F' = \{L'_1, \dots, L'_{s'}\}$ of lines in V constructed as follows:

- Choose disjoint subsets $I_1, \dots, I_m \subset [s]$ such that each I_j has either 2 or 3 elements.
- For each 2-element set $I_j = \{p_1, p_2\}$, pick signs $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ and add the line $\langle \epsilon_1 v_{p_1} + \epsilon_2 v_{p_2} \rangle$ to F' .

- For each 3-element set $I_j = \{p_1, p_2, p_3\}$, pick signs $\epsilon_1, \epsilon_2, \epsilon_3 \in \{\pm 1\}$ and add to F' either:
 - the line $\langle \epsilon_1 v_{p_1} + \epsilon_2 v_{p_2} + \epsilon_3 v_{p_3} \rangle$; or
 - the two lines $\langle \epsilon_1 v_{p_1} + \epsilon_2 v_{p_2} \rangle$ and $\langle \epsilon_1 v_{p_1} + \epsilon_2 v_{p_2} + \epsilon_3 v_{p_3} \rangle$.

An *augmented partial frame* is an ordered collection of lines that as an unordered set is the union of a partial frame F and some augmentation F' of F . An *unordered augmented partial frame* is the unordered set underlying an augmented partial frame. Here is an example:

Example 31.1. Assume that V has rank 15, and let $\{v_1, \dots, v_{15}\}$ be a basis of V . The following is then an augmented partial frame of V :

$$\begin{aligned} &(\langle v_1 \rangle, \langle v_2 \rangle, \langle v_1 - v_2 \rangle, \langle v_3 \rangle, \langle v_4 \rangle, \langle v_5 \rangle, \langle -v_3 + v_4 - v_5 \rangle, \\ &\langle v_6 \rangle, \langle v_7 \rangle, \langle v_8 \rangle, \langle -v_6 + v_7 \rangle, \langle -v_6 + v_7 + v_8 \rangle, \langle v_9 \rangle, \langle v_{10} \rangle, \langle v_9 + v_{10} \rangle, \langle v_{11} \rangle). \end{aligned} \quad \square$$

By construction, the collection of augmented partial frames of V is closed under re-ordering and passing to subsets. Also, the span of the lines in an augmented partial frame of V is a direct summand of V .

31.2. Chain complex. Recall that V is a free $\mathbb{Z}$ -module of rank $r \geq 1$. For $i \geq 0$, define $\mathbf{X}_i(V)$ to be the free abelian group given by generators and relations as follows. The generators of $\mathbf{X}_i(V)$ are formal symbols $\llbracket L_1, \dots, L_{r+i} \rrbracket$, where $(L_1, \dots, L_{r+i})$ is an $(r+i)$ -element augmented partial frame for V . The relations of $\mathbf{X}_i(V)$ are:

- For a generator $\llbracket L_1, \dots, L_{r+i} \rrbracket$ such that the L_j do not span⁴⁶ $V \otimes \mathbb{Q}$, the relation $\llbracket L_1, \dots, L_{r+i} \rrbracket = 0$.
- For a generator $\llbracket L_1, \dots, L_{r+i} \rrbracket$ and $\sigma \in \mathfrak{S}_{r+i}$, the relation $\llbracket L_{\sigma(1)}, \dots, L_{\sigma(r+i)} \rrbracket = (-1)^{|\sigma|} \llbracket L_1, \dots, L_{r+i} \rrbracket$.

There is a differential $\delta: \mathbf{X}_i(V) \rightarrow \mathbf{X}_{i-1}(V)$ defined in the usual way:⁴⁷

$$\delta \llbracket L_1, \dots, L_{r+i} \rrbracket = \sum_{j=1}^{r+i} (-1)^{j-1} \llbracket L_1, \dots, \widehat{L_j}, \dots, L_{r+i} \rrbracket.$$

This makes sense since as is easily checked δ takes relations to relations. We therefore have a chain complex $(\mathbf{X}_\bullet(V), \delta)$. The low-degree terms of this chain complex have the following descriptions:

Example 31.2. The nonzero generators of $\mathbf{X}_0(V)$ are exactly the $\llbracket \langle v_1 \rangle, \dots, \langle v_r \rangle \rrbracket$ such that $\{v_1, \dots, v_r\}$ is a basis for $V \cong \mathbb{Z}^r$. $\square$

Example 31.3. If $r = 1$, then $\mathbf{X}_1(V) = 0$. If $r \geq 1$, then the nonzero generators of $\mathbf{X}_1(V)$ are exactly those that after reordering their entries (and thus introducing a sign) are of one of the following two forms for some basis $\{v_1, \dots, v_r\}$ of V :

- $\llbracket \langle v_1 \rangle, \dots, \langle v_r \rangle, \langle \epsilon_1 v_1 + \epsilon_2 v_2 \rangle \rrbracket$ for signs $\epsilon_1, \epsilon_2 \in \{\pm 1\}$.
- $\llbracket \langle v_1 \rangle, \dots, \langle v_r \rangle, \langle \epsilon_1 v_1 + \epsilon_2 v_2 + \epsilon_3 v_3 \rangle \rrbracket$ for signs $\epsilon_1, \epsilon_2, \epsilon_3 \in \{\pm 1\}$. This second kind of generator only appears when $r \geq 3$. $\square$

Example 31.4. If $r \leq 2$, then $\mathbf{X}_2(V) = 0$. If $r \geq 3$, then the nonzero generators of $\mathbf{X}_2(V)$ are exactly those that after reordering their entries (and thus introducing a sign) are of one of the following four forms for some basis $\{v_1, \dots, v_r\}$ of V :

- $\llbracket \langle v_1 \rangle, \dots, \langle v_r \rangle, \langle \epsilon_1 v_1 + \epsilon_2 v_2 \rangle, \langle \epsilon_1 v_1 + \epsilon_2 v_2 + \epsilon_3 v_3 \rangle \rrbracket$ for signs $\epsilon_1, \epsilon_2, \epsilon_3 \in \{\pm 1\}$.

⁴⁶Since the span of the L_j is a direct summand of V , it would be equivalent to require that the L_j do not span V .

⁴⁷Here we are using the fact that the set of augmented partial frames is closed under reordering and passing to subsets, which we observed at the end of §31.1.

- $\llbracket \langle v_1 \rangle, \dots, \langle v_r \rangle, \langle \epsilon_1 v_1 + \epsilon_2 v_2 \rangle, \langle \epsilon_3 v_3 + \epsilon_4 v_4 \rangle \rrbracket$ for signs $\epsilon_1, \dots, \epsilon_4 \in \{\pm 1\}$. This second kind of generator only appears when $r \geq 4$.
- $\llbracket \langle v_1 \rangle, \dots, \langle v_r \rangle, \langle \epsilon_1 v_1 + \epsilon_2 v_2 + \epsilon_3 v_3 \rangle, \langle \epsilon_4 v_4 + \epsilon_5 v_5 \rangle \rrbracket$ for signs $\epsilon_1, \dots, \epsilon_5 \in \{\pm 1\}$. This third kind of generator only appears when $r \geq 5$.
- $\llbracket \langle v_1 \rangle, \dots, \langle v_r \rangle, \langle \epsilon_1 v_1 + \epsilon_2 v_2 + \epsilon_3 v_3 \rangle, \langle \epsilon_4 v_4 + \epsilon_5 v_5 + \epsilon_6 v_6 \rangle \rrbracket$ for signs $\epsilon_1, \dots, \epsilon_6 \in \{\pm 1\}$. This fourth kind of generator only appears when $r \geq 6$. $\square$

31.3. Resolution of Steinberg. There is a map $\psi: \mathbf{X}_0(V) \rightarrow \text{St}(V)$ taking a generator $\llbracket L_1, \dots, L_r \rrbracket$ to the corresponding apartment class $\mathbb{A}[\llbracket L_1, \dots, L_r \rrbracket]$. This map makes sense since the two relations in $\mathbf{X}_0(V)$ go to relations between apartment classes in $\text{St}(V)$; cf. the relations in §30.2. Generalizing classical work of Manin about modular symbols, Bykovskii [12] proved⁴⁸ that the image of the differential $\delta: \mathbf{X}_1(V) \rightarrow \mathbf{X}_0(V)$ is contained in $\ker(\psi)$ and that the resulting chain complex

$$(31.1) \quad \mathbf{X}_1(V) \xrightarrow{\delta} \mathbf{X}_0(V) \xrightarrow{\psi} \text{St}(V) \longrightarrow 0$$

is exact. In other words, (31.1) is a presentation of $\text{St}(V)$. See [13, 15] for alternate proofs. This was generalized by Brück–Miller–Patz–Sroka–Wilson [9]. The following is a small variant on their result:

Theorem 31.5. *For a free $\mathbb{Z}$ -module V of rank $r \geq 1$, we have an exact sequence*

$$\mathbf{X}_2(V) \xrightarrow{\delta} \mathbf{X}_1(V) \xrightarrow{\delta} \mathbf{X}_0(V) \xrightarrow{\psi} \text{St}(V) \longrightarrow 0.$$

Proof. In light of Examples (31.2) – (31.4), the indicated resolution has the same generators $\mathbf{X}_0(V)$ and relations $\mathbf{X}_1(V)$ as the resolution from [9], but $\mathbf{X}_2(V)$ is larger than the corresponding term from [9]. However, since we are not claiming anything about the kernel of $\delta: \mathbf{X}_2(V) \rightarrow \mathbf{X}_1(V)$, this does not affect the result we are trying to prove. $\square$

Remark 31.6. The elements of $\mathbf{X}_2(V)$ that do not appear in [9] are there to allow the multiplication we define in §32.1 below. $\square$

32. DOUBLE COMPLEX OF RESOLUTIONS

We now assemble the $\mathbf{X}_\bullet(V)$ into a double complex by imitating the construction of the bar resolution $\mathbf{S}_\bullet(\mathbb{Z}^n)$ of $\text{St}(\mathbb{Z}^n)$ discussed in §30.4.

32.1. Complex multiplication. Let V and W be $\mathbb{Z}$ -modules with $V \cong \mathbb{Z}^r$ and $W \cong \mathbb{Z}^s$. Just like for the Steinberg representation (cf. §30.3), there is a natural bilinear multiplication $\mathbf{X}_i(V) \otimes \mathbf{X}_j(W) \rightarrow \mathbf{X}_{i+j}(V \oplus W)$ given as follows. Consider generators $\llbracket L_1, \dots, L_{r+i} \rrbracket$ and $\llbracket L'_1, \dots, L'_{s+j} \rrbracket$ of $\mathbf{X}_i(V)$ and $\mathbf{X}_j(W)$. We then define

$$\llbracket L_1, \dots, L_{r+i} \rrbracket \cdot \llbracket L'_1, \dots, L'_{s+j} \rrbracket = \llbracket L_1, \dots, L_{r+i}, L'_1, \dots, L'_{s+j} \rrbracket \in \mathbf{X}_{i+j}(V \oplus W).$$

Here we are identifying V and W with the corresponding subspaces of $V \oplus W$. We remark that this product is why we defined augmented partial frames like we did instead of just including the elements needed by [9]. This product is compatible with the differentials δ on our chain complexes in the sense that

$$(32.1) \quad \delta(\kappa \cdot \kappa') = \delta(\kappa) \cdot \kappa' + (-1)^{r+i} \kappa \cdot \delta(\kappa') \quad \text{for } \kappa \in \mathbf{X}_i(V) \text{ and } \kappa' \in \mathbf{X}_j(W).$$

⁴⁸Actually, Bykovskii had fewer relations than we do, but adding more relations is no problem for a presentation.

32.2. Double complex. For $p \geq -1$ and $q \geq 0$, define

$$\mathbf{X}_{pq}(\mathbb{Z}^n) = \bigoplus_{V_1 \oplus \cdots \oplus V_{p+2} = \mathbb{Z}^n} \left(\bigoplus_{i_1 + \cdots + i_{p+2} = q} \mathbf{X}_{i_1}(V_1) \otimes \cdots \otimes \mathbf{X}_{i_{p+2}}(V_{p+2}) \right).$$

Here the direct sums are over decompositions $V_1 \oplus \cdots \oplus V_{p+2} = \mathbb{Z}^n$ and $i_1 + \cdots + i_{p+2} = q$ such that $V_j \neq 0$ and $i_j \geq 0$ for all $1 \leq j \leq p+2$. Fixing such decompositions $V_1 \oplus \cdots \oplus V_{p+2} = \mathbb{Z}^n$ and $i_1 + \cdots + i_{p+2} = q$, consider elements $\kappa_j \in \mathbf{X}_{i_j}(V_j)$ for $1 \leq j \leq p+2$. We will denote the element of $\mathbf{X}_{pq}(\mathbb{Z}^n)$ corresponding to $\kappa_1 \otimes \cdots \otimes \kappa_{p+2}$ by $[\kappa_1 | \dots | \kappa_{p+2}]$. When we want to write specific elements of $\mathbf{X}_{pq}(\mathbb{Z}^n)$, we will use the following convention:

Convention 32.1. For generators κ_j of $\mathbf{X}_{i_j}(V_j)$, we will omit the inner brackets when writing $[\kappa_1 | \dots | \kappa_{p+2}]$. For instance, for $p = 1$ if $\kappa_1 = [L_1, L_2]$ and $\kappa_2 = [L_3, L_4, L_5]$ and $\kappa_3 = [L_6]$, then we will write $[L_1, L_2 | L_3, L_4, L_5 | L_6]$ instead of $[[L_1, L_2] | [L_3, L_4, L_5] | [L_6]]$. $\square$

Let $r_j = \text{rank}(V_j)$. Define two differentials $\partial: \mathbf{X}_{pq}(\mathbb{Z}^n) \rightarrow \mathbf{X}_{p-1,q}(\mathbb{Z}^n)$ and $\delta: \mathbf{X}_{pq}(\mathbb{Z}^n) \rightarrow \mathbf{X}_{p,q-1}(\mathbb{Z}^n)$ via the formulas

$$\begin{aligned} \partial[\kappa_1 | \dots | \kappa_{p+2}] &= \sum_{j=1}^{p+1} (-1)^{j-1} [\kappa_1 | \dots | \kappa_j \cdot \kappa_{j+1} | \dots | \kappa_{p+2}] \\ \delta[\kappa_1 | \dots | \kappa_{p+2}] &= \sum_{j=1}^{p+2} (-1)^{r_1 + i_1 + \cdots + r_{j-1} + i_{j-1}} [\kappa_1 | \dots | \delta(\kappa_j) | \dots | \kappa_{p+2}]. \end{aligned}$$

Using (32.1), these differentials make $\mathbf{X}_{\bullet,\bullet}(\mathbb{Z}^n)$ into a double complex.

32.3. Double complex and Steinberg. We now return to the chain complex

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathbf{S}_{n-2}(\mathbb{Z}^n) & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & \mathbf{S}_0(\mathbb{Z}^n) & \xrightarrow{\partial} & \mathbf{S}_{-1}(\mathbb{Z}^n) & \rightarrow & 0. \\ & & & & & & & & \parallel & & \\ & & & & & & & & \text{St}(\mathbb{Z}^n) & & \end{array}$$

Our main result in this section is:

Lemma 32.2. *For all $n \geq 1$, we have $H_i(\mathbf{X}_{\bullet,\bullet}(\mathbb{Z}^n)) \cong H_i(\mathbf{S}_{\bullet}(\mathbb{Z}^n))$ for $i \leq 0$. Moreover, $H_1(\mathbf{X}_{\bullet,\bullet}(\mathbb{Z}^n))$ surjects onto $H_1(\mathbf{S}_{\bullet}(\mathbb{Z}^n))$.*

Proof. Consider the spectral sequence computing $H_i(\mathbf{X}_{\bullet,\bullet}(\mathbb{Z}^n))$ where we take homology with respect to the differential $\delta: \mathbf{X}_{pq}(\mathbb{Z}^n) \rightarrow \mathbf{X}_{p,q-1}(\mathbb{Z}^n)$ first, so

$$E_{pq}^1 = H_q(\mathbf{X}_{p,\bullet}(\mathbb{Z}^n)) \Rightarrow H_{p+q}(\mathbf{X}_{\bullet,\bullet}(\mathbb{Z}^n)).$$

To prove the lemma, it is enough to prove that the bottom rows of the E^1 -page of this spectral sequence are as follows:

$$\begin{array}{c|ccccccc} q = 2 & * & \leftarrow & * & \leftarrow & * & \leftarrow & * & \leftarrow & \cdots \\ q = 1 & 0 & \leftarrow & * & \leftarrow & * & \leftarrow & * & \leftarrow & \cdots \\ q = 0 & \mathbf{S}_{-1}(\mathbb{Z}^n) & \xleftarrow{\partial} & \mathbf{S}_0(\mathbb{Z}^n) & \xleftarrow{\partial} & \mathbf{S}_1(\mathbb{Z}^n) & \xleftarrow{\partial} & \mathbf{S}_2(\mathbb{Z}^n) & \xleftarrow{\partial} & \cdots \\ & p = -1 & & p = 0 & & p = 1 & & p = 2 & & \end{array}$$

The $*$ entries are unspecified. To do this, it is enough to prove the following claim. This claim will show that the entries have the indicated form (and in fact all the entries on the $q = 1$ line vanish). That the differentials are correct will be immediate from the definitions.

Claim. Fix $p \geq -1$. For $q \geq 0$, we have $H_q(\mathbf{X}_{p,\bullet}(\mathbb{Z}^n)) \cong \begin{cases} \mathbf{S}_p(\mathbb{Z}^n) & \text{if } q = 0, \\ 0 & \text{if } q = 1. \end{cases}$

For a decomposition $V_1 \oplus \cdots \oplus V_{p+2} = \mathbb{Z}^n$, let

$$\mathbf{X}_{pq}(V_1, \dots, V_{p+2}) = \bigoplus_{i_1 + \cdots + i_{p+2} = q} \mathbf{X}_{i_1}(V_1) \otimes \cdots \otimes \mathbf{X}_{i_{p+2}}(V_{p+2}).$$

This forms a subcomplex of the chain complex $\mathbf{X}_{p,\bullet}(\mathbb{Z}^n)$, and

$$\mathbf{X}_{p,\bullet}(\mathbb{Z}^n) = \bigoplus_{V_1 \oplus \cdots \oplus V_{p+2} = \mathbb{Z}^n} \mathbf{X}_{p,\bullet}(V_1, \dots, V_{p+2}).$$

Since

$$\mathbf{S}_p(\mathbb{Z}^n) = \bigoplus_{V_1 \oplus \cdots \oplus V_{p+2} = \mathbb{Z}^n} \text{St}(V_1) \otimes \cdots \otimes \text{St}(V_{p+2}),$$

it is enough to prove that for a fixed decomposition $V_1 \oplus \cdots \oplus V_{p+2} = \mathbb{Z}^n$, we have

$$(32.2) \quad H_q(\mathbf{X}_{p,\bullet}(V_1, \dots, V_{p+2})) \cong \begin{cases} \text{St}(V_1) \otimes \cdots \otimes \text{St}(V_{p+2}) & \text{if } q = 0, \\ 0 & \text{if } q = 1. \end{cases}$$

For each $1 \leq j \leq p+2$, we have a chain complex $\mathbf{X}_\bullet(V_j)$, and Theorem 31.5 says that

$$(32.3) \quad H_q(\mathbf{X}_\bullet(V_j)) \cong \begin{cases} \text{St}(V_j) & \text{if } q = 0, \\ 0 & \text{if } q = 1. \end{cases}$$

It is almost the case that

$$(32.4) \quad \mathbf{X}_{p,\bullet}(V_1, \dots, V_{p+2}) = \mathbf{X}_\bullet(V_1) \otimes \cdots \otimes \mathbf{X}_\bullet(V_{p+2}).$$

The only issue is that while this tensor product has the same terms as $\mathbf{X}_{p,\bullet}(V_1, \dots, V_{p+2})$, its differentials are not right. To fix this, let $r_j = \text{rank}(V_j)$ and for each $2 \leq j \leq p+2$ change $\mathbf{X}_\bullet(V_j)$ by multiplying its differential by $(-1)^{r_1 + \cdots + r_{j-1}}$. This does not affect (32.3), and now (32.4) holds. Since each $\mathbf{X}_q(V_j)$ is a free $\mathbb{Z}$ -module, the desired result (32.2) now follows from (32.3) and (32.4) along with the Künneth formula. $\square$

33. THREE-STEP BAR RESOLUTION OF STEINBERG

In this final section we first establish some notation and then prove Theorem D'.

33.1. Restricted bar resolution of a set. Let S be a nonempty finite set. Fix a set $\mathfrak{R}$ of disjoint nonempty subsets of S . An *ordered subset* of S is a subset of S equipped with a linear order. We will talk about ordered subsets of S using ordinary set notation and terminology; for instance, if A is an ordered subset of S and $R \in \mathfrak{R}$ then we will say that $R \subset A$ if each $r \in R$ appears in A . For disjoint ordered subsets A and A' of S , write $A \cdot A'$ for the ordered subset of S obtained by concatenating A and A' .

For $i \geq -1$, define $\mathbf{Z}_i(S, \mathfrak{R})$ to be the free abelian group given by generators and relations as follows. The generators of $\mathbf{Z}_i(S, \mathfrak{R})$ are formal symbols $[A_1 | \dots | A_{i+2}]$ where:

- the A_j are disjoint nonempty ordered subsets of S ; and
- each $s \in S$ appears exactly once in $A_1 \cdot \dots \cdot A_{i+2}$; and
- for each $R \in \mathfrak{R}$ there exists some $1 \leq j \leq i+2$ such that $R \subset A_j$.

The relations of $\mathbf{Z}_i(S, \mathfrak{R})$ are:

- Let $[A_1 | \dots | A_{i+2}]$ be a generator of $\mathbf{Z}_i(S, \mathfrak{R})$. For each $1 \leq j \leq i+2$, let A'_j be an ordered set obtained by permuting the ordered set A_j and let $\lambda_j \in \{\pm 1\}$ be the sign of this permutation. Then

$$[A'_1 | \dots | A'_{i+2}] = (-1)^{\lambda_1 + \cdots + \lambda_{i+2}} [A_1 | \dots | A_{i+2}].$$

There is a differential $\partial: \mathbf{Z}_i(S, \mathfrak{R}) \rightarrow \mathbf{Z}_{i-1}(S, \mathfrak{R})$ defined in the usual way:

$$\partial[A_1 | \dots | A_{i+2}] = \sum_{j=1}^{i+1} (-1)^{j-1} [A_1 | \dots | A_j \cdot A_{j+1} | \dots | A_{i+2}].$$

This makes $\mathbf{Z}_\bullet(S, \mathfrak{R})$ into a chain complex. We have:

Lemma 33.1. *Let S be a nonempty finite set and $\mathfrak{R}$ be a set of disjoint nonempty subsets of S . Set*

$$d = |S \setminus \bigsqcup_{R \in \mathfrak{R}} R| + |\mathfrak{R}|.$$

Then

$$H_i(\mathbf{Z}_\bullet(S, \mathfrak{R})) = \begin{cases} \mathbb{Z} & \text{if } i = d - 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. To simplify our notation, for each $s \in S$ with $s \notin \bigsqcup_{R \in \mathfrak{R}} R$ add the one-element set $\{s\}$ to $\mathfrak{R}$. This does not change $\mathbf{Z}_\bullet(S, \mathfrak{R})$ or d , and ensures that $S = \bigsqcup_{R \in \mathfrak{R}} R$ and $d = |\mathfrak{R}|$.

Let $\mathbf{W}(\mathfrak{R})$ be the poset of nonempty proper subsets of $\mathfrak{R}$. Viewed as a semisimplicial set, $\mathbf{W}(\mathfrak{R})$ is isomorphic to the barycentric subdivision of the boundary of the simplex with $d = |\mathfrak{R}|$ vertices. This implies that its geometric realization $|\mathbf{W}(\mathfrak{R})|$ is homeomorphic to the sphere $\mathbb{S}^{d-2}$. Letting $\tilde{\mathbf{C}}_\bullet(\mathbf{W}(\mathfrak{R}))$ be the reduced chain complex of the semisimplicial set $\mathbf{W}(\mathfrak{R})$, it follows that

$$H_i(\tilde{\mathbf{C}}_\bullet(\mathbf{W}(\mathfrak{R}))) = \begin{cases} \mathbb{Z} & \text{if } i = d - 2, \\ 0 & \text{otherwise.} \end{cases}$$

It is therefore enough to prove that $\mathbf{Z}_\bullet(S, \mathfrak{R})$ is isomorphic to $\tilde{\mathbf{C}}_\bullet(\mathbf{W}(\mathfrak{R}))$.

Fix a total ordering on S . The desired isomorphism $\phi: \mathbf{Z}_\bullet(S, \mathfrak{R}) \rightarrow \tilde{\mathbf{C}}_\bullet(\mathbf{W}(\mathfrak{R}))$ is as follows. Consider a generator $[A_1 | \dots | A_{i+2}]$ of $\mathbf{Z}_i(S, \mathfrak{R})$. The ordered set $A_1 \cdot \dots \cdot A_{i+2}$ is a permutation of the ordered set S . Let $\epsilon \in \{\pm 1\}$ be the sign of this permutation. Also, for $1 \leq j \leq i+2$ set $B_j = \{R \in \mathfrak{R} \mid R \subset A_j\}$. Let σ be the following i -simplex of $\mathbf{W}(\mathfrak{R})$:

$$B_1 \subsetneq (B_1 \sqcup B_2) \subsetneq \dots \subsetneq (B_1 \sqcup \dots \sqcup B_{i+1}).$$

Note that if $i = -1$ then σ is the empty simplex, which we view as the unique (-1) -simplex used to form the reduced chain complex $\tilde{\mathbf{C}}_\bullet(\mathbf{W}(\mathfrak{R}))$. We then define $\phi([A_1 | \dots | A_{i+2}]) = \epsilon \sigma \in \tilde{\mathbf{C}}_i(\mathbf{W}(\mathfrak{R}))$. It is clear that this is a well-defined isomorphism of chain complexes. $\square$

33.2. Proof of main theorem. We close this paper by proving Theorem D':

Theorem D'. *For $n \geq 4$, the chain complex*

$$\mathbf{S}_2(\mathbb{Z}^n) \xrightarrow{\partial} \mathbf{S}_1(\mathbb{Z}^n) \xrightarrow{\partial} \mathbf{S}_0(\mathbb{Z}^n) \xrightarrow{\partial} \text{St}(\mathbb{Z}^n) \longrightarrow 0$$

is exact.

Proof. This theorem asserts that $H_i(\mathbf{S}_\bullet(\mathbb{Z}^n)) = 0$ for $i \leq 1$. By Lemma 32.2, it is enough to prove that $H_i(\mathbf{X}_{\bullet, \bullet}(\mathbb{Z}^n)) = 0$ for $i \leq 1$. Consider the spectral sequence computing $H_i(\mathbf{X}_{\bullet, \bullet}(\mathbb{Z}^n))$ where we take homology with respect to the differential $\partial: \mathbf{X}_{pq}(\mathbb{Z}^n) \rightarrow \mathbf{X}_{p-1, q}(\mathbb{Z}^n)$ first, so

$$E_{qp}^1 = H_p(\mathbf{X}_{\bullet, q}(\mathbb{Z}^n)) \Rightarrow H_{p+q}(\mathbf{X}_{\bullet, \bullet}(\mathbb{Z}^n)).$$

We will prove the bottom rows of the E^1 -page of this spectral sequence are as follows:

$$\begin{array}{c|ccccccc}
p = 1 & 0 & \leftarrow & * & \leftarrow & * & \leftarrow & \cdots \\
p = 0 & 0 & \leftarrow & E_{10}^1 & \xleftarrow{\delta} & E_{20}^1 & \leftarrow & \cdots \\
p = -1 & 0 & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & \cdots \\
\hline
& q = 0 & & q = 1 & & q = 2 & &
\end{array}$$

Here the $*$ entries are unspecified and the entry E_{10}^1 is 0 for $n \geq 5$. This will prove the theorem for $n \geq 5$. For $n = 4$, we will also prove that the indicated differential $\delta: E_{20}^1 \rightarrow E_{10}^1$ is surjective, so $E_{10}^2 = 0$. This will prove the theorem for $n = 4$.

For an unordered⁴⁹ augmented partial frame $S = \{L_1, \dots, L_{n+q}\}$ of $\mathbb{Z}^n$ that spans $\mathbb{Z}^n$, denote by $\mathbf{X}_{pq}[S]$ the subspace of $\mathbf{X}_{pq}(\mathbb{Z}^n)$ spanned by nonzero generators κ such that you get $\pm[L_1, \dots, L_{n+q}]$ if you delete the $(p+1)$ occurrences of the bar symbol $|$ from κ . Note that in such κ the L_j might appear in a different order than in $[L_1, \dots, L_{n+q}]$. These form a chain subcomplex $\mathbf{X}_{\bullet,q}[S]$ of $\mathbf{X}_{\bullet,q}(\mathbb{Z}^n)$, and $\mathbf{X}_{\bullet,q}(\mathbb{Z}^n)$ is the direct sum of its subcomplexes $\mathbf{X}_{\bullet,q}[S]$. We must prove the following four claims:

Claim 0. *Let $S = \{L_1, \dots, L_n\}$ be an unordered augmented partial frame of $\mathbb{Z}^n$ that spans $\mathbb{Z}^n$. Then $H_i(\mathbf{X}_{\bullet,0}[S]) = 0$ for $i \leq 1$ and $n \geq 4$.*

By definition, we have $\mathbf{X}_{\bullet,0}[S] \cong \mathbf{Z}_{\bullet}(S, \emptyset)$. It therefore follows from Lemma 33.1 that $H_i(\mathbf{X}_{\bullet,0}[S]) = 0$ for $i \leq n-3$. For $n \geq 4$ this vanishing holds for $i \leq 1$, as desired.

Claim 1. *Let $S = \{L_1, \dots, L_{n+1}\}$ be an unordered augmented partial frame of $\mathbb{Z}^n$ that spans $\mathbb{Z}^n$. Then $H_i(\mathbf{X}_{\bullet,1}[S]) = 0$ in the following two situations:*

- if $i \leq 0$ and $n \geq 5$; or
- if $i \leq -1$ and $n = 4$.

As we described in Example 31.3, there exists a basis $\{v_1, \dots, v_n\}$ for $\mathbb{Z}^n$ such that S is of one of the following two forms:

- (i) we have $S = \{\langle v_1 \rangle, \dots, \langle v_n \rangle, \langle \epsilon_1 v_1 + \epsilon_2 v_2 \rangle\}$ with $\epsilon_1, \epsilon_2 \in \{\pm 1\}$.
- (ii) we have $S = \{\langle v_1 \rangle, \dots, \langle v_n \rangle, \langle \epsilon_1 v_1 + \epsilon_2 v_2 + \epsilon_3 v_3 \rangle\}$ with $\epsilon_1, \epsilon_2, \epsilon_3 \in \{\pm 1\}$.

Assume first that we are in case (i), so $S = \{\langle v_1 \rangle, \dots, \langle v_n \rangle, \langle \epsilon_1 v_1 + \epsilon_2 v_2 \rangle\}$ with $\epsilon_1, \epsilon_2 \in \{\pm 1\}$. In the generators of $\mathbf{X}_{\bullet,1}[S]$, the bar symbols $|$ cannot separate the three terms in $R = \{\langle v_1 \rangle, \langle v_2 \rangle, \langle \epsilon_1 v_1 + \epsilon_2 v_2 \rangle\}$. It follows that $\mathbf{X}_{\bullet,1}[S] \cong \mathbf{Z}_{\bullet}(S, \{R\})$. Lemma 33.1 thus implies that $H_i(\mathbf{X}_{\bullet,1}[S]) = 0$ for

$$i \leq d-3 \quad \text{for } d = (n+1-3) + 1 = n-1.$$

For $n \geq 4$ this vanishing holds for $i \leq 0$, giving the claim.

Assume next that we are in case (ii), so $S = \{\langle v_1 \rangle, \dots, \langle v_n \rangle, \langle \epsilon_1 v_1 + \epsilon_2 v_2 + \epsilon_3 v_3 \rangle\}$ with $\epsilon_1, \epsilon_2, \epsilon_3 \in \{\pm 1\}$. Letting

$$R' = \{\langle v_1 \rangle, \langle v_2 \rangle, \langle v_3 \rangle, \langle \epsilon_1 v_1 + \epsilon_2 v_2 + \epsilon_3 v_3 \rangle\},$$

just like in the previous paragraph we have $\mathbf{X}_{\bullet,1}[S] \cong \mathbf{Z}_{\bullet}(S, \{R'\})$. It therefore follows from Lemma 33.1 that $H_i(\mathbf{X}_{\bullet,1}[S]) = 0$ for

$$i \leq d-3 \quad \text{for } d = (n+1-4) + 1 = n-2.$$

For $n = 4$ this vanishing holds for $i = -1$, while for $n \geq 5$ this vanishing holds for $i \leq 0$.

Assume now that $n = 4$. For later use, note that Lemma 33.1 also says that

$$(33.1) \quad H_0(\mathbf{X}_{\bullet,1}[S]) \cong \mathbb{Z}.$$

⁴⁹Since S is not ordered, we write it like a set rather than an ordered set.

We can write down a generator for $H_0(\mathbf{X}_{\bullet,1}[S])$ as follows. For $1 \leq j \leq 4$, let $L_j = \langle v_j \rangle$. Define $L_{123} = \langle \epsilon_1 v_1 + \epsilon_2 v_2 + \epsilon_3 v_3 \rangle$. Using Convention §32.1, let

$$\kappa = \llbracket L_4 | L_1, L_2, L_3, L_{123} \rrbracket - \llbracket L_1, L_2, L_3, L_{123} | L_4 \rrbracket \in \mathbf{X}_{01}(\mathbb{Z}^4).$$

We have

$$\begin{aligned} \partial \kappa &= \llbracket L_4, L_1, L_2, L_3, L_{123} \rrbracket - \llbracket L_1, L_2, L_3, L_{123}, L_4 \rrbracket \\ &= \llbracket L_1, L_2, L_3, L_{123}, L_4 \rrbracket - \llbracket L_1, L_2, L_3, L_{123}, L_4 \rrbracket = 0, \end{aligned}$$

so we have $[\kappa] \in H_0(\mathbf{X}_{\bullet,1}(\mathbb{Z}^4))$. It follows from the proof of Lemma 33.1 that $[\kappa]$ generates $H_0(\mathbf{X}_{\bullet,1}[S]) \cong \mathbb{Z}$.

Claim 2. *Let $S = \{L_1, \dots, L_{n+2}\}$ be an augmented partial frame of $\mathbb{Z}^n$ that spans $\mathbb{Z}^n$. Then $H_{-1}(\mathbf{X}_{\bullet,2}[S]) = 0$ for $n \geq 4$.*

Assume that $n \geq 4$. As we described in Example 31.4, there exists a basis $\{v_1, \dots, v_n\}$ for $\mathbb{Z}^n$ such that the unordered set S is of one of the following four forms:

- (i) we have $S = \{\langle v_1 \rangle, \dots, \langle v_n \rangle, \langle \epsilon_1 v_1 + \epsilon_2 v_2 \rangle, \langle \epsilon_1 v_1 + \epsilon_2 v_2 + \epsilon_3 v_3 \rangle\}$ with $\epsilon_1, \epsilon_2, \epsilon_3 \in \{\pm 1\}$.
- (ii) we have $S = \{\langle v_1 \rangle, \dots, \langle v_n \rangle, \langle \epsilon_1 v_1 + \epsilon_2 v_2 \rangle, \langle \epsilon_3 v_3 + \epsilon_4 v_4 \rangle\}$ with $\epsilon_1, \dots, \epsilon_4 \in \{\pm 1\}$.
- (iii) we have $S = \{\langle v_1 \rangle, \dots, \langle v_n \rangle, \langle \epsilon_1 v_1 + \epsilon_2 v_2 + \epsilon_3 v_3 \rangle, \langle \epsilon_4 v_4 + \epsilon_5 v_5 \rangle\}$ with $\epsilon_1, \dots, \epsilon_5 \in \{\pm 1\}$.

This third case only appears when $n \geq 5$.

- (iv) we have $S = \{\langle v_1 \rangle, \dots, \langle v_n \rangle, \langle \epsilon_1 v_1 + \epsilon_2 v_2 + \epsilon_3 v_3 \rangle, \langle \epsilon_4 v_4 + \epsilon_5 v_5 + \epsilon_6 v_6 \rangle\}$ with $\epsilon_1, \dots, \epsilon_6 \in \{\pm 1\}$. This fourth case only appears when $n \geq 6$.

The analysis of each of these cases uses Lemma 33.1 just like in Claim 1. In each case, there is some $\mathfrak{R}$ such that

$$\mathbf{X}_{\bullet,2}[S] \cong \mathbf{Z}_{\bullet}(S, \mathfrak{R}).$$

We summarize the $\mathfrak{R}$ and the resulting vanishing range

$$i \leq ((n+2) - \left| \bigsqcup_{R \in \mathfrak{R}} R \right|) + |\mathfrak{R}| - 3$$

for $H_i(\mathbf{X}_{\bullet,2}[S]) \cong H_i(\mathbf{Z}_{\bullet}(S, \mathfrak{R}))$ in the following table:

	$\mathfrak{R} = \{\mathbf{R}_1, \dots, \mathbf{R}_k\}$ with	vanishing range for H_i
(i)	$R_1 = \{\langle v_1 \rangle, \langle v_2 \rangle, \langle v_3 \rangle, \langle \epsilon_1 v_1 + \epsilon_2 v_2 \rangle, \langle \epsilon_1 v_1 + \epsilon_2 v_2 + \epsilon_3 v_3 \rangle\}$	$((n+2) - 5) + 1 - 3$ $= n - 5$
(ii)	$R_1 = \{\langle v_1 \rangle, \langle v_2 \rangle, \langle \epsilon_1 v_1 + \epsilon_2 v_2 \rangle\}$ $R_2 = \{\langle v_3 \rangle, \langle v_4 \rangle, \langle \epsilon_3 v_3 + \epsilon_4 v_4 \rangle\}$	$((n+2) - 6) + 2 - 3$ $= n - 5$
(iii)	$R_1 = \{\langle v_1 \rangle, \langle v_2 \rangle, \langle v_3 \rangle, \langle \epsilon_1 v_1 + \epsilon_2 v_2 + \epsilon_3 v_3 \rangle\}$ $R_2 = \{\langle v_4 \rangle, \langle v_5 \rangle, \langle \epsilon_4 v_4 + \epsilon_5 v_5 \rangle\}$	$((n+2) - 7) + 2 - 3$ $= n - 6$
(iv)	$R_1 = \{\langle v_1 \rangle, \langle v_2 \rangle, \langle v_3 \rangle, \langle \epsilon_1 v_1 + \epsilon_2 v_2 + \epsilon_3 v_3 \rangle\}$ $R_2 = \{\langle v_4 \rangle, \langle v_5 \rangle, \langle v_6 \rangle, \langle \epsilon_4 v_4 + \epsilon_5 v_5 + \epsilon_6 v_6 \rangle\}$	$((n+2) - 8) + 2 - 3$ $= n - 7$

Since $n \geq 4$ in all cases by assumption and additionally $n \geq 5$ in case (iii) and $n \geq 6$ in case (iv), we conclude that in all cases $H_{-1}(\mathbf{X}_{\bullet,2}[S]) = 0$.

Claim (surjectivity). *For $n = 4$, the differential $\delta: E_{20}^1 \rightarrow E_{10}^1$ is surjective.*

As we observed at the end of the proof Claim 1 (see the paragraph after (33.1)), the term E_{10}^1 is generated by elements $[\kappa]$ constructed as follows. Let $v_1, \dots, v_4$ be a basis for $\mathbb{Z}^4$. For

$1 \leq j \leq 4$, set $L_j = \langle v_j \rangle$. Next, let $\epsilon_1, \epsilon_2, \epsilon_3 \in \{\pm 1\}$ and set $L_{123} = \langle \epsilon_1 v_1 + \epsilon_2 v_2 + \epsilon_3 v_3 \rangle$. Define

$$\kappa = \llbracket L_4 | L_1, L_2, L_3, L_{123} \rrbracket - \llbracket L_1, L_2, L_3, L_{123} | L_4 \rrbracket \in \mathbf{X}_{01}(\mathbb{Z}^4).$$

We then have $\partial\kappa = 0$, so there is an element $[\kappa] \in H_0(\mathbf{X}_{\bullet,0}(\mathbb{Z}^4)) = E_{10}^1$.

We must prove that $[\kappa]$ is in the image of the differential $\delta: E_{20}^1 \rightarrow E_{10}^1$. Set $L_{12} = \langle \epsilon_1 v_1 + \epsilon_2 v_2 \rangle$. Define

$$\eta = \llbracket L_{12}, L_1, L_2, L_3, L_{123} | L_4 \rrbracket + \llbracket L_4 | L_{12}, L_1, L_2, L_3, L_{123} \rrbracket \in \mathbf{X}_{02}(\mathbb{Z}^4).$$

We have

$$\begin{aligned} \partial\eta &= \llbracket L_{12}, L_1, L_2, L_3, L_{123}, L_4 \rrbracket + \llbracket L_4, L_{12}, L_1, L_2, L_3, L_{123} \rrbracket \\ &= \llbracket L_{12}, L_1, L_2, L_3, L_{123}, L_4 \rrbracket - \llbracket L_{12}, L_1, L_2, L_3, L_{123}, L_4 \rrbracket = 0. \end{aligned}$$

It follows that we have $[\eta] \in H_0(\mathbf{X}_{\bullet,2}(\mathbb{Z}^4)) = E_{20}^1$. The differential $\delta: E_{20}^1 \rightarrow E_{10}^1$ is induced by the differential $\delta: \mathbf{X}_{02}(\mathbb{Z}^4) \rightarrow \mathbf{X}_{01}(\mathbb{Z}^4)$ discussed in §32.2, which in turn is induced by the differential discussed in §31.2. To prove the claim, it is enough to prove that $[\delta\eta] = [\kappa]$.

The element $\delta\eta$ is an alternating sum of the result of deleting terms from η . If we delete L_4 then the result is 0. From this, we see that $\delta\eta = \kappa + \kappa_2 + \cdots + \kappa_5$ with

$$\begin{aligned} \kappa &= \llbracket L_1, L_2, L_3, L_{123} | L_4 \rrbracket - \llbracket L_4 | L_1, L_2, L_3, L_{123} \rrbracket \\ \kappa_2 &= -\llbracket L_{12}, L_2, L_3, L_{123} | L_4 \rrbracket + \llbracket L_4 | L_{12}, L_2, L_3, L_{123} \rrbracket \\ \kappa_3 &= \llbracket L_{12}, L_1, L_3, L_{123} | L_4 \rrbracket - \llbracket L_4 | L_{12}, L_1, L_3, L_{123} \rrbracket \\ \kappa_4 &= -\llbracket L_{12}, L_1, L_2, L_{123} | L_4 \rrbracket + \llbracket L_4 | L_{12}, L_1, L_2, L_{123} \rrbracket \\ \kappa_5 &= \llbracket L_{12}, L_1, L_2, L_3 | L_4 \rrbracket - \llbracket L_4 | L_{12}, L_1, L_2, L_3 \rrbracket \end{aligned}$$

Each κ_j satisfies $\partial\kappa_j = 0$. For instance,

$$\begin{aligned} \partial\kappa_2 &= -\llbracket L_{12}, L_2, L_3, L_{123}, L_4 \rrbracket + \llbracket L_4, L_{12}, L_2, L_3, L_{123} \rrbracket \\ &= -\llbracket L_{12}, L_2, L_3, L_{123}, L_4 \rrbracket + \llbracket L_{12}, L_2, L_3, L_{123}, L_4 \rrbracket = 0. \end{aligned}$$

We therefore have elements $[\kappa_j] \in H_0(\mathbf{X}_{\bullet,0}(\mathbb{Z}^4)) = E_{10}^1$, and can write

$$[\delta\eta] = [\kappa] + [\kappa_2] + \cdots + [\kappa_5].$$

To prove that $[\delta\eta] = [\kappa]$, it is enough to prove that all the κ_j satisfy $[\kappa_j] = 0$. In fact, all of them are terms that we proved are zero during the proof of Claim 1. For example, to see that $[\kappa_2] = 0$, let $\{v'_1, \dots, v'_4\}$ be the basis

$$(v'_1, v'_2, v'_3, v'_4) = (\epsilon_1 v_1 + \epsilon_2 v_2, \epsilon_3 v_3, v_2, v_4).$$

The collection of lines $S = \{L_{12}, L_2, L_3, L_{123}, L_4\}$ used to form κ_2 are then of the form

$$S = \{\langle v'_1 \rangle, \langle v'_2 \rangle, \langle v'_3 \rangle, \langle v'_4 \rangle, \langle v'_1 + v'_2 \rangle\}.$$

This makes it clear that κ_1 is one of the terms proved to vanish in case (i) of the proof of Claim 1. $\square$

REFERENCES

- [1] A. Ash, J. Miller, & P. Patzt, Hopf algebras, Steinberg modules, and the unstable cohomology of $\mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{GL}_n(\mathbb{Z})$, preprint 2024. [arXiv:2404.13776](https://arxiv.org/abs/2404.13776) (Cited on page 63.)
- [2] A. Ash, A. Putman, & S. Sam, Homological vanishing for the Steinberg representation, *Compos. Math.* 154 (2018), no. 6, 1111–1130. (Cited on pages 2, 5, and 6.)
- [3] A. Ash & L. Rudolph, The modular symbol and continued fractions in higher dimensions, *Invent. Math.* 55 (1979), no. 3, 241–250. (Cited on page 63.)
- [4] A. Borel, *Linear algebraic groups*, second edition, Graduate Texts in Mathematics, 126, Springer-Verlag, New York, 1991. (Cited on pages 2, 7, 22, and 23.)

- [5] A. Borel & J.-P. Serre, Corners and arithmetic groups, *Comment. Math. Helv.* 48 (1973), 436–491. (Cited on page 5.)
- [6] A. Borel & J. Tits, Groupes réductifs, *Publ. Math. I.H.E.S.* 27 (1965), 55–152. (Cited on page 7.)
- [7] A. Borel & J. Tits, Compléments à l'article: “Groupes réductifs”, *Inst. Hautes Études Sci. Publ. Math.* No. 41 (1972), 253–276. (Cited on page 7.)
- [8] K. S. Brown, *Buildings*, Springer, New York, 1989. (Cited on pages 3, 8, 9, and 16.)
- [9] B. Brück, J. Miller, P. Patzt, R. Sroka, & J. Wilson, On the codimension-two cohomology of $\mathrm{SL}_n(\mathbb{Z})$, *Adv. Math.* 451 (2024), 109795. (Cited on pages 5, 7, 64, and 66.)
- [10] B. Brück, P. Patzt, & R. Sroka, A presentation of symplectic Steinberg modules and cohomology of $\mathrm{Sp}_{2g}(\mathbb{Z})$, preprint 2023. [arXiv:2306.03180](#) (Cited on page 5.)
- [11] B. Brück, Y. Santos Rego, & R. Sroka, On the top-dimensional cohomology of arithmetic Chevalley groups, *Proc. Amer. Math. Soc.* 152 (2024), 4131–4139. (Cited on page 5.)
- [12] V.A. Bykovskii, Generating elements of the annihilating ideal for modular symbols, *Funktsional. Anal. i Prilozhen.* 37 (2003), no. 4, 27–38, 95; translation in *Funct. Anal. Appl.* 37 (2003), no. 4, 263–272. (Cited on pages 64 and 66.)
- [13] S. Charlton, D. Radchenko, & D. Rudenko, Multiple polylogarithms and the Steinberg module, preprint 2025. [arXiv:2505.02202](#) (Cited on pages 18 and 66.)
- [14] T. Church, B. Farb, & A. Putman, A stability conjecture for the unstable cohomology of $\mathrm{SL}_n(\mathbb{Z})$, mapping class groups, and $\mathrm{Aut}(F_n)$, in “Algebraic Topology: Applications and New Directions”, 55–70, *Contemp. Math.* 620 (2014), Amer. Math. Soc., Providence, RI. (Cited on pages 2, 5, and 63.)
- [15] T. Church & A. Putman, The codimension-one cohomology of $\mathrm{SL}_n \mathbb{Z}$, *Geom. Topol.* 21 (2017), no. 2, 999–1032. (Cited on pages 5, 7, 56, 64, and 66.)
- [16] C. W. Curtis, The Steinberg character of a finite group with a (B, N) -pair, *J. Algebra* 4 (1966), 433–441. (Cited on page 3.)
- [17] J. E. Humphreys, The Steinberg representation, *Bull. Amer. Math. Soc. (N.S.)* 16 (1987), no. 2, 247–263. (Cited on page 3.)
- [18] R. Lee & R.H. Szczarba, On the homology and cohomology of congruence subgroups, *Invent. Math.* 33 (1976) 1, 15–53. (Cited on pages 5, 7, and 56.)
- [19] J. Miller, R. Nagpal, & P. Patzt, Stability in the high-dimensional cohomology of congruence subgroups, *Compos. Math.* 156 (2020), no. 4, 822–861. (Cited on pages 6, 7, 18, 49, and 56.)
- [20] J. Miller, R. Nagpal, & P. Patzt, Corrigendum to “Stability in the high-dimensional cohomology of congruence subgroups” [*Compos. Math.* 156 (2020), no. 4, 822–861] [arXiv:2508.14945](#) (Cited on page 18.)
- [21] J. Miller, P. Patzt, & A. Putman, The double Tits building and the bar resolution of the Steinberg representation, in preparation. (Cited on page 49.)
- [22] J. Miller, P. Patzt, & J. Wilson, On rank filtrations of algebraic K-theory and Steinberg modules, to appear in *J. Eur. Math. Soc.* [arXiv:2303.00245](#). (Cited on pages 2, 5, 6, 18, 49, and 64.)
- [23] J. S. Milne, *Algebraic groups*, Cambridge Studies in Advanced Mathematics, 170, Cambridge Univ. Press, Cambridge, 2017. (Cited on pages 2, 7, and 22.)
- [24] A. Putman & A. Snowden, The Steinberg representation is irreducible, *Duke Math. J.* 172 (2023), no. 4, 775–808. (Cited on page 3.)
- [25] M. Reeder, The Steinberg module and the cohomology of arithmetic groups, *J. Algebra* 141 (1991), no. 2, 287–315. (Cited on page 10.)
- [26] J. Rognes, A spectrum level rank filtration in algebraic K-theory, *Topology* 31 (1992), no. 4, 813–845. (Cited on pages 5, 6, and 49.)
- [27] M. Sikirić, P. Elbaz-Vincent, A. Kupers, & J. Martinet, Voronoi complexes in higher dimensions, cohomology of $\mathrm{GL}_N(\mathbb{Z})$ for $N \geq 8$, and the triviality of $K_8(\mathbb{Z})$, preprint 2019. [arXiv:1910.11598](#) (Cited on page 56.)
- [28] L. Solomon, The Steinberg character of a finite group with BN -pair, in *Theory of Finite Groups (Symposium, Harvard Univ., Cambridge, Mass., 1968)*, 213–221, Benjamin, New York. (Cited on page 3.)
- [29] R. Steinberg, A geometric approach to the representations of the full linear group over a Galois field, *Trans. Amer. Math. Soc.* 71 (1951), 274–282. (Cited on page 3.)
- [30] R. Steinberg, Prime power representations of finite linear groups, *Canadian J. Math.* 8 (1956), 580–591. (Cited on page 3.)
- [31] R. Steinberg, Prime power representations of finite linear groups. II, *Canadian J. Math.* 9 (1957), 347–351. (Cited on page 3.)
- [32] R. Steinberg, Comments on the Papers, in Robert Steinberg, *Collected Works*, 7, American Mathematical Society, Providence, RI, 1997. (Cited on page 3.)

- [33] J. Tits, Classification of algebraic semisimple groups, in *Algebraic Groups and Discontinuous Subgroups* (Proc. Sympos. Pure Math., Boulder, Colo., 1965), pp. 33–62 American Mathematical Society, Providence, RI, 1966. (Cited on page [11](#).)
- [34] J. Tits, Buildings of spherical type and finite BN-pairs, Lecture Notes in Mathematics, Vol. 386, Springer-Verlag, Berlin, 1974. (Cited on pages [3](#) and [8](#).)

DEPT OF MATHEMATICS; PURDUE UNIVERSITY; 150 N. UNIVERSITY; WEST LAFAYETTE, IN 47907
Email address: `jeremykmiller@purdue.edu`

DEPT OF MATHEMATICS; UNIVERSITY OF OKLAHOMA; 601 ELM AVE; NORMAN, OK 73019
Email address: `ppatzt@ou.edu`

DEPT OF MATHEMATICS; UNIVERSITY OF NOTRE DAME; 255 HURLEY HALL; NOTRE DAME, IN 46556
Email address: `andyp@nd.edu`