

A NEW APPROACH TO TWISTED HOMOLOGICAL STABILITY, WITH APPLICATIONS TO CONGRUENCE SUBGROUPS

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ABSTRACT. We introduce a new method for proving twisted homological stability, and use it to prove such results for symmetric groups and general linear groups. In addition to sometimes slightly improving the stable range given by the traditional method (due to Dwyer), it is easier to adapt to nonstandard situations. As an illustration of this, we generalize to GL_n of many rings R a theorem of Borel which says that passing from GL_n of a number ring to a finite-index subgroup does not change the rational cohomology. Charney proved this generalization for trivial coefficients, and we extend it to twisted coefficients.

1. INTRODUCTION

A sequence of groups $\{G_n\}_{n=1}^\infty$ exhibits *homological stability* if for each $k \geq 0$, the value $H_k(G_n)$ is independent of n for $n \gg k$. There is a vast literature on this, starting with early work of Nakaoka on symmetric groups [23] and unpublished work of Quillen dealing with $\mathrm{GL}_n(\mathbb{F}_q)$. Many sequences of groups exhibit homological stability: general linear groups over many rings [30], mapping class groups [16], automorphism groups of free groups [18], etc. See [29] for a general framework that encompasses many of these results, as well as a survey of the literature.

1.1. Twisted coefficients. Dwyer [12] showed how to extend this to homology with certain kinds of twisted coefficients. For instance, it follows from his work that if \mathbb{k} is a field, then for all $m \geq 1$ and $k \geq 0$ we have

$$H_k(\mathrm{GL}_n(\mathbb{k}); (\mathbb{k}^n)^{\otimes m}) \cong H_k(\mathrm{GL}_{n+1}(\mathbb{k}); (\mathbb{k}^{n+1})^{\otimes m}) \quad \text{for } n \gg 0.$$

This has been generalized to many different groups and coefficient systems. The most general result we are aware of is in [29], which shows how to prove such a result for “polynomial coefficient systems” on many classes of groups.

1.2. New approach. In this paper, we give an alternate approach to twisted homological stability for polynomial coefficient systems. Our new approach is actually closer to the usual proof for constant coefficients, which uses spectral sequences associated to group actions on carefully chosen simplicial complexes. What we do is encode our twisted coefficient systems using certain kinds of “coefficient systems” on those simplicial complexes. The high connectivity of the simplicial complexes that is used in the constant coefficient proof is replaced by the vanishing of the homology of these simplicial complexes with respect to our “coefficient systems”.

A small advantage of our approach is that it sometimes gives better stability ranges. More importantly, our approach is more flexible than Dwyer’s, which only works well in settings where the traditional proof of homological stability applies in its simplest form. For constant coefficients, the standard proof of homological stability can give useful information even in settings where it does not apply directly. Examples of this from the author’s work include the proofs of [1, Theorem 1.1] and [28, Theorem A]. Our approach to twisted coefficients is

Date: December 5, 2022.

AP was supported in part by NSF grant DMS-1811210.

similarly flexible. We will illustrate this by proving Theorem C below, which shows how the homology of the general linear group changes when you pass to a finite-index congruence subgroup. With twisted coefficients, this seems hard to prove using the traditional approach.

There is a tension between on the one hand developing abstract frameworks like the one in [29] that systematize the homological stability machine, and on the other hand giving flexible tools that can be adapted to nonstandard situations, including ones that are not necessarily about homological stability per se. Both of these goals are important. In writing this paper, we had the latter goal in mind, so our focus will be on basic examples rather than a general abstract result.

Remark 1.1. We wrote this paper in part to provide tools for our paper [25], which applies our machine to study the homology of the moduli space of curves with level structures. The main result of [25] is about untwisted homology, but the proof requires working with twisted homology as well. \square

Remark 1.2. Miller–Patz–Petersen [21] have recently developed an approach to twisted homological stability that shares some ideas with our paper, though the technical details and intended applications are different. Unlike us, they prove a general theorem in the spirit of [29]. Our work was done independently of theirs.¹ \square

1.3. FI-modules. The easiest example to which our results apply is the symmetric group. For these groups we will encode our twisted coefficient systems using Church–Ellenberg–Farb’s theory of FI-modules [10]. Let FI be the category whose objects are finite sets and whose morphisms are injections. For a commutative ring \mathbb{k} , an FI-module over \mathbb{k} is a functor M from FI to the category of \mathbb{k} -modules. For $n \geq 0$, let² $\bar{n} = \{1, \dots, n\}$. Every object of FI is isomorphic to \bar{n} for some $n \geq 0$, so the data of an FI-module M consists of:

- a \mathbb{k} -module $M(\bar{n})$ for each $n \geq 0$, and
- for each injection $f: \bar{n} \rightarrow \bar{m}$, an induced \mathbb{k} -module homomorphism $f_*: M(\bar{n}) \rightarrow M(\bar{m})$.

In particular, the inclusions $\bar{n} \rightarrow \overline{n+1}$ induce a sequence of morphisms

$$(1.1) \quad M(\bar{0}) \rightarrow M(\bar{1}) \rightarrow M(\bar{2}) \rightarrow \dots$$

The group of FI-automorphisms of \bar{n} is the symmetric group \mathfrak{S}_n . This acts on $M(\bar{n})$, making $M(\bar{n})$ into a $\mathbb{k}[\mathfrak{S}_n]$ -module. More generally, for a finite set S the group \mathfrak{S}_S of bijections of S acts on $M(S)$, making $M(S)$ into a $\mathbb{k}[\mathfrak{S}_S]$ -module.

Example 1.3. The group \mathfrak{S}_n acts on \mathbb{k}^n for each $n \geq 0$. We can fit the increasing sequence

$$\mathbb{k}^0 \rightarrow \mathbb{k}^1 \rightarrow \mathbb{k}^2 \rightarrow \dots$$

of \mathfrak{S}_n -representations into an FI-module M over \mathbb{k} by defining

$$M(S) = \mathbb{k}^S \quad \text{for a finite set } S.$$

Here the notation \mathbb{k}^S means the free \mathbb{k} -module with basis S , and an injective map $f: S \rightarrow T$ between finite sets induces a map $f_*: M(S) \rightarrow M(T)$ taking basis elements to basis elements. As a $\mathbb{k}[\mathfrak{S}_n]$ -module, we have $M(\bar{n}) = \mathbb{k}^{\bar{n}} \cong \mathbb{k}^n$. \square

¹See [21, Remark 1.2]. We apologize for taking so long to write our approach up.

²It is more common to write $[n]$ for $\{1, \dots, n\}$, but later on when discussing semisimplicial sets we will need to use the notation $[n]$ for $\{0, \dots, n\}$.

1.4. Polynomial FI-modules. For an FI-module M over \mathbb{k} , the inclusions (1.1) induce maps between homology groups for each k :

$$H_k(\mathfrak{S}_0; M(\bar{0})) \rightarrow H_k(\mathfrak{S}_1; M(\bar{1})) \rightarrow H_k(\mathfrak{S}_2; M(\bar{2})) \rightarrow \cdots .$$

We would like to give conditions under which these stabilize. We start with the following. Fix a functorial coproduct \sqcup on FI, which we think of as “disjoint union”. Letting $*$ be a formal symbol, for a finite set S we then have the finite set $S \sqcup \{*\}$ of cardinality $|S| + 1$.

Definition 1.4. Let \mathbb{k} be a commutative ring and let M be an FI-module over \mathbb{k} .

- The *shifted* FI-module of M , denoted ΣM , is the FI-module over \mathbb{k} defined via the formula $\Sigma M(S) = M(S \sqcup \{*\})$ for a finite set S .
- The *derived* FI-module of M , denoted DM , is the FI-module over \mathbb{k} defined via the formula

$$DM(S) = \frac{M(S \sqcup \{*\})}{\text{Im}(M(S) \rightarrow M(S \sqcup \{*\}))} \quad \text{for a finite set } S. \quad \square$$

Remark 1.5. Morphisms between FI-modules over \mathbb{k} are natural transformations between functors. This makes the collection of FI-modules over \mathbb{k} into an abelian category, where kernels and cokernel are computed pointwise. With these conventions, there is a morphism $M \rightarrow \Sigma M$, and $DM = \text{coker}(M \rightarrow \Sigma M)$. \square

The idea behind the following definition goes back to Dwyer’s work [12], and has been elaborated upon by many people; see [29] for a more complete history.

Definition 1.6. Let \mathbb{k} be a commutative ring and let M be an FI-module over \mathbb{k} . We say that M is *polynomial* of degree $d \geq -1$ starting at $m \in \mathbb{Z}$ if it satisfies the following inductive condition:

- If $d = -1$, then for finite sets S with $|S| \geq m$ we require $M(S) = 0$.
- If $d \geq 0$, then we require the following two conditions:
 - For all injective maps $f: S \rightarrow T$ between finite sets with $|S| \geq m$, the induced map $f_*: M(S) \rightarrow M(T)$ must be an injection.
 - The derived FI-module DM must be polynomial of degree $(d - 1)$ starting at $(m - 1)$. \square

Remark 1.7. If the FI-module M over \mathbb{k} is polynomial of degree d starting at $m \in \mathbb{Z}$, then for all finite sets S with $|S| \geq m$ we have a short exact sequence

$$(1.2) \quad 0 \longrightarrow M(S) \longrightarrow \Sigma M(S) \longrightarrow DM(S) \longrightarrow 0$$

of $\mathbb{k}[\mathfrak{S}_S]$ -modules. \square

Remark 1.8. There is also a notion of a polynomial FI-module being *split*, which can slightly improve the bounds on where stability begins. See [29] for the definition. To avoid complicating our arguments, we will not incorporate this into our results. \square

Example 1.9. An FI-module M over \mathbb{k} is polynomial of degree 0 starting at $m \in \mathbb{Z}$ if it satisfies the following two conditions:

- For all injective maps $f: S \rightarrow T$ between finite sets with $|S| \geq m$, the induced map $f_*: M(S) \rightarrow M(T)$ must be an injection.
- For all finite sets S of with $|S| \geq m - 1$, we must have

$$DM(S) = \frac{M(S \sqcup \{*\})}{\text{Im}(M(S) \rightarrow M(S \sqcup \{*\}))} = 0.$$

In other words, the map $M(S) \rightarrow M(S \sqcup \{*\})$ must be surjective. This implies more generally that for all injective maps $f: S \rightarrow T$ between finite sets with $|S| \geq m - 1$, the induced map $f_*: M(S) \rightarrow M(T)$ must be a surjection.

Combining these two facts, we see that M is polynomial of degree 0 starting at $m \in \mathbb{Z}$ if for all injective maps $f: S \rightarrow T$ between finite sets, the map $f_*: M(S) \rightarrow M(T)$ is an isomorphism if $|S| \geq m$ and a surjection if $|S| = m - 1$. \square

Example 1.10. The FI-module M in Example 1.3 is polynomial of degree 1 starting at 0. More generally, for $d \geq 1$ we can define an FI-module $M^{\otimes d}$ via the formula

$$M^{\otimes d}(S) = (\mathbb{k}^S)^{\otimes d} \quad \text{for a finite set } S.$$

The FI-module $M^{\otimes d}$ is polynomial of degree d starting at 0. \square

Example 1.11. There is a natural notion of an FI-module being generated and related in finite degree (see [10] for the definition). As was observed in [29, Example 4.18], it follows from work of Church–Ellenberg [9] that if M is generated in degree d and related in degree r , then M is polynomial of degree d starting at $r + \min(d, r)$. We remark that though $r + \min(d, r)$ is sharp in general, in practice many such FI-modules are polynomial of degree d starting far earlier than $r + \min(d, r)$, often at 0. \square

1.5. Symmetric groups. Our main theorem about the symmetric group is as follows:

Theorem A. *Let \mathbb{k} be a commutative ring and let M be an FI-module over \mathbb{k} that is polynomial of degree $d \geq -1$ starting at $m \geq 0$. For each $k \geq 0$, the map*

$$H_k(\mathfrak{S}_n; M(\bar{n})) \rightarrow H_k(\mathfrak{S}_{n+1}; M(\overline{n+1}))$$

is an isomorphism for $n \geq 2k + \max(d, m) + 1$ and a surjection for $n = 2k + \max(d, m)$.

In the key case $m = 0$, the stabilization map is thus an isomorphism for $n \geq 2k + d + 1$ and a surjection for $n = 2k + d$. Theorem A is the natural output of the machine we develop, but for $m \gg 0$ the following result sometimes gives better bounds (though often worse ones):

Theorem A'. *Let \mathbb{k} be a commutative ring and let M be an FI-module over \mathbb{k} that is polynomial of degree $d \geq -1$ starting at $m \geq 0$. For each $k \geq 0$, the map*

$$(1.3) \quad H_k(\mathfrak{S}_n; M(\bar{n})) \rightarrow H_k(\mathfrak{S}_{n+1}; M(\overline{n+1}))$$

is an isomorphism for $n \geq \max(m, 2k + 2d + 2)$ and a surjection for $n \geq \max(m, 2k + 2d)$.

Remark 1.12. Theorem A' will be derived from Theorem A using (1.2). \square

The first twisted homological stability result for the symmetric group was proved by Betley [3], who only considered split coefficient systems starting at $m = 0$ (c.f. Remark 1.8) but proved that (1.3) is an isomorphism for $n \geq 2k + d$. Randal-Williams–Wahl [29, Theorem 5.1] showed how to deal with general polynomial coefficients, but in this level of generality could only prove that (1.3) is an isomorphism for $n \geq \max(2m + 1, 2k + 2d + 2)$, though in the split case they could reduce this to $n \geq \max(m + 1, 2k + d + 2)$.

1.6. VIC-modules. We now turn to general linear groups. Let R be a ring, possibly noncommutative. To encode our coefficient systems on $\mathrm{GL}_n(R)$, we will use VIC(R)-modules, which were introduced by the author and Sam [26, 27]. Define VIC(R) to be the following category:

- The objects of VIC(R) are finite-rank free right R -modules.³ For a finite-rank free right R -module A , we will sometimes write $[A]$ for the associated object of VIC(R) to clarify our statements.

³We use right R -modules since this ensures that $\mathrm{GL}_n(R)$ acts on R^n on the left.

- For finite-rank free right R -modules A_1 and A_2 , a $\text{VIC}(R)$ -morphism $[A_1] \rightarrow [A_2]$ is a pair (f, C) , where $f: A_1 \rightarrow A_2$ is an injection and $C \subset A_2$ is a submodule⁴ such that $A_2 = f(A_1) \oplus C$. The composition of $\text{VIC}(R)$ -morphisms $(f_1, C_1): [A_1] \rightarrow [A_2]$ and $(f_2, C_2): [A_2] \rightarrow [A_3]$ is the $\text{VIC}(R)$ -morphism $(f_2 \circ f_1, C_2 \oplus f_1(C_1)): [A_1] \rightarrow [A_3]$.

For a commutative ring \mathbb{k} , a $\text{VIC}(R)$ -module over \mathbb{k} is a functor M from $\text{VIC}(R)$ to the category of \mathbb{k} -modules. Every object of $\text{VIC}(R)$ is isomorphic to R^n for some $n \geq 0$, so the data of a $\text{VIC}(R)$ -module M consists of:

- a \mathbb{k} -module $M(R^n)$ for each $n \geq 0$, and
- for each $\text{VIC}(R)$ -morphism $(f, C): [R^n] \rightarrow [R^m]$, an induced \mathbb{k} -module homomorphism $(f, C)_*: M(R^n) \rightarrow M(R^m)$.

For each n , let $\iota_n: R^n \rightarrow R^{n+1}$ be the inclusion into the first n coordinates. We then have a $\text{VIC}(R)$ -morphism $(\iota_n, 0 \oplus R): [R^n] \rightarrow [R^{n+1}]$. These induce a sequence of morphisms

$$(1.4) \quad M(R^0) \rightarrow M(R^1) \rightarrow M(R^2) \rightarrow \dots$$

The group of $\text{VIC}(R)$ -automorphisms of $[R^n]$ is $\text{GL}_n(R)$. This acts on $M(R^n)$, making $M(R^n)$ into a $\mathbb{k}[\text{GL}_n(R)]$ -module. More generally, for a finite-rank free right R -module A the group $\text{GL}(A)$ acts on $M(A)$.

Example 1.13. We can fit the increasing sequence

$$\mathbb{Z}^0 \rightarrow \mathbb{Z}^1 \rightarrow \mathbb{Z}^2 \rightarrow \dots$$

of $\text{GL}_n(\mathbb{Z})$ -representations into a $\text{VIC}(\mathbb{Z})$ -module M over \mathbb{Z} by defining

$$M(A) = A \quad \text{for a finite-rank free } \mathbb{Z}\text{-module } A.$$

For a $\text{VIC}(\mathbb{Z})$ -morphism $(f, C): [A_1] \rightarrow [A_2]$, the associated map $(f, C)_*: M(A_1) \rightarrow M(A_2)$ is simply f . \square

Example 1.14. Example 1.13 can be generalized as follows. Let R be a ring, let \mathbb{k} be a commutative ring, let V be a \mathbb{k} -module, and let $\lambda: R \rightarrow \text{End}_{\mathbb{k}}(V)$ be a ring homomorphism. Example 1.13 will correspond to $R = \mathbb{k} = V = \mathbb{Z}$ and

$$\lambda(r)(v) = rv \quad \text{for } r \in R = \mathbb{Z} \text{ and } v \in V = \mathbb{Z}.$$

Another example would be $R = \mathbb{k}[G]$ for a group G and V a representation of G over \mathbb{k} . For each $n \geq 0$, the ring homomorphism λ induces a group homomorphism

$$\text{GL}_n(R) \rightarrow \text{GL}_n(\text{End}_{\mathbb{k}}(V)),$$

endowing the \mathbb{k} -module $V^{\oplus n}$ with the structure of a $\mathbb{k}[\text{GL}_n(R)]$ -module. We can fit the increasing sequence

$$V^{\oplus 0} \rightarrow V^{\oplus 1} \rightarrow V^{\oplus 2} \rightarrow \dots$$

of $\text{GL}_n(R)$ -representations into a $\text{VIC}(R)$ -module M by defining

$$M(A) = A \otimes_R V \quad \text{for a finite-rank free right } R\text{-module } A.$$

Here we use λ to regard V as a left R -module. For a $\text{VIC}(R)$ -morphism $(f, C): [A_1] \rightarrow [A_2]$, the induced map $(f, C)_*: M(A_1) \rightarrow M(A_2)$ is $f \otimes \text{id}$. As a $\mathbb{k}[\text{GL}_n(R)]$ -module, we have

$$M(R^n) = R^n \otimes_R V = V^{\oplus n}. \quad \square$$

⁴It would also be reasonable to require C to be free with $\text{rk}(C) = \text{rk}(A_2) - \text{rk}(A_1)$. We will not make this restriction. For the rings we consider, this frequently holds automatically, at least when $\text{rk}(A_2) - \text{rk}(A_1)$ is large enough (see Lemma 8.9).

Example 1.15. In Examples 1.13 and 1.14, the C in a $\text{VIC}(R)$ -morphism played no role. For an easy example of a $\text{VIC}(R)$ -module where it is important, consider the dual M^* of the $\text{VIC}(\mathbb{Z})$ -module M over \mathbb{Z} from Example 1.13:

$$M^*(A) = \text{Hom}(A, \mathbb{Z}) \quad \text{for a finite-rank free } \mathbb{Z}\text{-module } A.$$

For a $\text{VIC}(\mathbb{Z})$ -morphism $(f, C): [A_1] \rightarrow [A_2]$, the associated map $(f, C)_*: M^*(A_1) \rightarrow M^*(A_2)$ takes $\phi \in \text{Hom}(A_1, \mathbb{Z})$ to the composition

$$A_2 \xrightarrow{/C} f(A_1) \xrightarrow{f^{-1}} A_1 \xrightarrow{\phi} \mathbb{Z},$$

where the first map is the projection $A_2 = f(A_1) \oplus C \rightarrow f(A_1)$. \square

1.7. Polynomial VIC-modules. For a $\text{VIC}(R)$ -module M over \mathbb{k} , the inclusions (1.4) induce maps between homology groups for each k :

$$\text{H}_k(\text{GL}_0(R); M(R^0)) \rightarrow \text{H}_k(\text{GL}_1(R); M(R^1)) \rightarrow \text{H}_k(\text{GL}_2(R); M(R^2)) \rightarrow \dots$$

Just like for FI-modules, to make this stabilize we will need to impose a polynomiality condition.⁵ The definitions are similar to those for FI-modules:

Definition 1.16. Let R be a ring, \mathbb{k} be a commutative ring, and M be a $\text{VIC}(R)$ -module over \mathbb{k} .

- The *shifted* $\text{VIC}(R)$ -module of M , denoted ΣM , is the $\text{VIC}(R)$ -module over \mathbb{k} defined via the formula $\Sigma M(A) = M(A \oplus R^1)$ for a finite-rank free right R -module A .
- The *derived* $\text{VIC}(R)$ -module of M , denoted DM , is the $\text{VIC}(R)$ -module over \mathbb{k} defined via the formula

$$DM(A) = \frac{M(A \oplus R^1)}{\text{Im}(M(A) \rightarrow M(A \oplus R^1))} \quad \text{for a finite-rank free right } R\text{-module } A. \quad \square$$

Definition 1.17. Let R be a ring, \mathbb{k} be a commutative ring, and M be a $\text{VIC}(R)$ -module over \mathbb{k} . We say that M is *polynomial* of degree $d \geq -1$ starting at $m \in \mathbb{Z}$ if it satisfies the following inductive condition:

- If $d = -1$, then for all finite-rank free right R -modules A with $\text{rk}(A) \geq m$ we require $M(A) = 0$.
- If $d \geq 0$, then we require the following two conditions:
 - For all $\text{VIC}(R)$ -morphisms $(f, C): [A_1] \rightarrow [A_2]$ with $\text{rk}(A_1) \geq m$, the induced map $(f, C)_*: M(A_1) \rightarrow M(A_2)$ must be an injection.
 - The derived $\text{VIC}(R)$ -module DM must be polynomial of degree $(d - 1)$ starting at $(m - 1)$. \square

Example 1.18. The $\text{VIC}(R)$ -modules in Examples 1.13, 1.14, and 1.15 are all polynomial of degree 1 starting at 0. Letting M be one of these, for $d \geq 0$ we can define another $\text{VIC}(R)$ -module $M^{\otimes d}$ via the formula

$$M^{\otimes d}(A) = (M(A))^{\otimes d} \quad \text{for a finite-rank free right } R\text{-module } A.$$

This is easily seen to be polynomial of degree d starting at 0. \square

⁵For finite rings R , the author and Sam proved in [26, 27] a homological stability result which replaces the polynomiality condition by a much weaker “finite generation” condition. That proof is very different from the one we will give, and cannot possibly work for infinite rings like $R = \mathbb{Z}$.

1.8. General linear groups. We now turn to our stability theorem, which will concern the homology of $\mathrm{GL}_n(R)$. We will need to impose a “stable rank” condition on R called (SR_r) that was introduced by Bass [2]. See §8 below for the definition and a survey. Here we will simply say this condition is satisfied by many rings; in particular, fields satisfy (SR_2) and PIDs satisfy (SR_3) . More generally a ring R that is finitely generated as a module over a Noetherian commutative ring of Krull dimension r satisfies (SR_{r+2}) . Thus for instance if \mathbb{k} is a field and G is a finite group, then the group ring $\mathbb{k}[G]$ satisfies (SR_2) .

Theorem B. *Let R be a ring satisfying (SR_r) , let \mathbb{k} be a commutative ring, and let M be a $\mathrm{VIC}(R)$ -module over \mathbb{k} that is polynomial of degree $d \geq -1$ starting at $m \geq 0$. For each $k \geq 0$, the map*

$$H_k(\mathrm{GL}_n(R); M(R^n)) \rightarrow H_k(\mathrm{GL}_{n+1}(R); M(R^{n+1}))$$

is an isomorphism for $n \geq 2k + \max(2d + r - 1, m, r) + 1$ and a surjection for $n = 2k + \max(2d + r - 1, m, r)$.

In the key case $d \geq 1^6$ and $m = 0$, the stabilization map is thus an isomorphism for $n \geq 2k + 2d + r$ and a surjection for $n = 2k + 2d + r - 1$. Just like for the symmetric group, we will derive from Theorem B the following variant result which for $m \gg 0$ sometimes gives better bounds:

Theorem B'. *Let R be a ring satisfying (SR_r) , let \mathbb{k} be a commutative ring, and let M be a $\mathrm{VIC}(R)$ -module over \mathbb{k} that is polynomial of degree $d \geq -1$ starting at $m \geq 0$. For each $k \geq 0$, the map*

$$(1.5) \quad H_k(\mathrm{GL}_n(R); M(R^n)) \rightarrow H_k(\mathrm{GL}_{n+1}(R); M(R^{n+1}))$$

is an isomorphism for $n \geq \max(m, 2k + 2d + r + 2)$ and a surjection for $n \geq \max(m, 2k + 2d + r)$.

Dwyer [12] proved a version of this for R a PID, though he only worked with split coefficient systems starting at $m = 0$ (c.f. Remark 1.8). He did not identify a stable range. Later van der Kallen [30, Theorem 5.6] extended this to rings satisfying (SR_r) , though again he only worked with split coefficient systems starting at $m = 0$. He proved that (1.5) is an isomorphism for $n \geq 2k + d + r$. Randal-Williams–Wahl [29, Theorem 5.11] then showed how to deal with general polynomial modules, though they only stated a result for ones that started at $m = 0$. They proved that (1.5) is an isomorphism for $n \geq 2k + 2d + r + 1$.

Remark 1.19. The result we will prove is more general than Theorems B and B' and applies to certain subgroups of $\mathrm{GL}_n(R)$ as well.⁷ For instance, if R is commutative then it applies to $\mathrm{SL}_n(R)$. See §9 for the definition of the groups we will consider and Theorems 12.1 and 12.2 for the statement of our theorem. \square

1.9. Congruence subgroups. Our final theorem illustrates how our machinery can be applied to prove a theorem that is not (directly) about homological stability. Borel [5, 6] proved that if Γ is a lattice in $\mathrm{SL}_n(\mathbb{R})$ and V is a rational representation of the algebraic group SL_n , then for $n \gg k$ the homology group $H_k(\Gamma; V)$ depends only on k and V , not on the lattice Γ . In particular, it is unchanged if you pass from Γ to a finite-index subgroup.

We will prove a version of this for $\mathrm{GL}_n(R)$ for rings R satisfying (SR_r) . The basic flavor of the result will be that passing from $\mathrm{GL}_n(R)$ to appropriate finite-index subgroups does not change rational homology, at least in some stable range. The finite-index subgroups we will consider are the finite-index congruence subgroups, which are defined as follows:

⁶That is, for M not constant.

⁷The work of van der Kallen also applies to these subgroups.

Definition 1.20. Let R be a ring and let α be a two sided ideal of R . The *level- α congruence subgroup*⁸ of $\mathrm{GL}_n(R)$, denoted $\mathrm{GL}_n(R, \alpha)$, is the kernel of the natural group homomorphism $\mathrm{GL}_n(R) \rightarrow \mathrm{GL}_n(R/\alpha)$. \square

Our theorem is as follows:

Theorem C. *Let R be a ring satisfying (SR_r) , let \mathbb{k} be a field of characteristic 0, and let M be a $\mathrm{VIC}(R)$ -module over \mathbb{k} that is polynomial of degree $d \geq -1$ starting at $m \geq 0$. Assume furthermore that $M(R^n)$ is a finite-dimensional vector space over \mathbb{k} for all $n \geq 0$. Then for all two sided ideals α of R such that $|R/\alpha| < \infty$, the map*

$$\mathrm{H}_k(\mathrm{GL}_n(R, \alpha); M(R^n)) \rightarrow \mathrm{H}_k(\mathrm{GL}_n(R); M(R^n))$$

is an isomorphism for $n \geq \max(m, 2k + 2d + 2r - 1)$.

We emphasize that Theorem C is *not* a homological stability theorem:⁹ rather than increasing $\mathrm{GL}_n(R)$ to $\mathrm{GL}_{n+1}(R)$, we are decreasing $\mathrm{GL}_n(R)$ by passing to the finite-index subgroup $\mathrm{GL}_n(R, \alpha)$. For untwisted rational coefficients, something like Theorem C is implicit in work of Charney [8]. Though she did not state this, it can easily be derived from her work that for R and \mathbb{k} and α as in Theorem C, the map

$$\mathrm{H}_k(\mathrm{GL}_n(R, \alpha); \mathbb{k}) \rightarrow \mathrm{H}_k(\mathrm{GL}_n(R); \mathbb{k})$$

is an isomorphism for $n \geq 2k + 2r + 4$.¹⁰ Later Cohen [11] proved an analogous result for the symplectic group, and more importantly for us showed how to simplify Charney's argument. Our proof of Theorem C follows the outline of Cohen's proof, but using our new approach to twisted homological stability. It seems very hard to do this using the traditional proof of twisted homological stability.

Remark 1.21. The fact that the field \mathbb{k} in Theorem C has characteristic 0 is essential. This theorem is false over fields of finite characteristic or over more general rings like \mathbb{Z} . It can fail even when M is polynomial of degree 0, i.e., constant. For instance, for $n \geq 3$ we have $\mathrm{H}_1(\mathrm{GL}_n(\mathbb{Z}); \mathbb{Z}) = \mathbb{Z}/2$ with the $\mathbb{Z}/2$ coming from the determinant, but Lee–Szczarba [20] proved that¹¹

$$\mathrm{H}_1(\mathrm{GL}_n(\mathbb{Z}, \ell\mathbb{Z}); \mathbb{Z}) \cong \mathfrak{sl}_n(\mathbb{Z}/\ell) \quad \text{for } n, \ell \geq 3,$$

where $\mathfrak{sl}_n(\mathbb{Z}/\ell)$ is the abelian group of $n \times n$ trace 0 matrices over \mathbb{Z}/ℓ . \square

Remark 1.22. Just like for Theorems B and B', we will actually prove something more general than Theorem C that will apply to congruence subgroups of certain subgroups of $\mathrm{GL}_n(R)$, e.g., to $\mathrm{SL}_n(R)$ if R is commutative. See Theorem 14.3. \square

Remark 1.23. The proof of Theorem C also requires some recent work of Harman [17] classifying certain kinds of finitely generated $\mathrm{VIC}(\mathbb{Z})$ -modules. \square

⁸In many contexts it is common to call these *principal congruence subgroups*, and to define a congruence subgroup as a subgroup that contains a principal congruence subgroup. For general rings R , this more general notion of “congruence subgroups” can have some pathological properties. For number rings R , all nontrivial ideals α of R satisfy $|R/\alpha| < \infty$, so for number rings all principal congruence subgroups are finite-index. This is false for general rings. Even worse, as Tom Church pointed out to me there might exist finite-index subgroups $\Gamma < \mathrm{GL}_n(R)$ that contain principal congruence subgroups, but do not contain finite-index principal congruence subgroups (though I do not know any concrete examples of this).

⁹Though in light of Theorems B and B' it implies that $\mathrm{H}_k(\mathrm{GL}_n(R, \alpha); M(R^n))$ stabilizes.

¹⁰The constant coefficients \mathbb{k} fit into an FI-module of degree 0 starting at 0, so even in this case Theorem C gives the slightly better stable range $n \geq 2k + 2r - 1$.

¹¹Their theorem is actually about congruence subgroups of $\mathrm{SL}_n(\mathbb{Z})$, but for $\ell \geq 3$ all elements of $\mathrm{GL}_n(\mathbb{Z}, \ell\mathbb{Z})$ have determinant 1.

Remark 1.24. If the ring R in Theorem C is a *finite* ring, then we can take $\alpha = \{0\}$, so $\mathrm{GL}_n(R, \alpha)$ is trivial. The case $k = 0$ of the theorem thus implies that under its hypotheses, for R a finite ring the action of $\mathrm{GL}_n(R)$ on $M(R^n)$ is *trivial* for $n \geq \max(m, 2d + 2r)$. It is enlightening to go through our proof and see how it proves this special case. This will also clarify to the reader how the work of Harman discussed in the previous remark is used. \square

Remark 1.25. Though I have not worked out the details, I expect that a version of Theorem C for the symplectic group can also be proved by these methods. Indeed, as I mentioned above our proof is inspired by work of Cohen [11] on the symplectic group. \square

1.10. Outline. We start with three sections of preliminary results in §2 – §4. We then discuss our twisted homological stability machine in §5. To make this useful we also need an accompanying “vanishing theorem”, which is in §6. We then have §7 on symmetric groups, which proves Theorems A and A’. After that, we have four sections of background on rings and general linear groups: §8 discusses the stable rank condition (SR_r), §9 has some K-theoretic background, and §10–§11 introduce some important simplicial complexes associated to $\mathrm{GL}_n(R)$. We then have §12 on general linear groups, which proves Theorems B and B’. We finally turn our attention to congruence subgroups. This requires some preliminary results on unipotent representations that are discussed in §13. We close with §14 on congruence subgroups, which proves Theorem C.

1.11. Acknowledgments. I would like to thank Nate Harman, Jeremy Miller, and Nick Salter for helpful conversations. In particular, I would like to thank Nate Harman for explaining how to prove Lemma 13.2 below. I would also like to thank Tom Church, Benson Farb, Peter Patzt, and Nathalie Wahl for providing comments on a previous draft of this paper. Finally, I would like to thank the referee for an unusually thorough and detailed report.

2. BACKGROUND I: SIMPLICIAL COMPLEXES

This section contains background material on simplicial complexes. Its main purpose is to establish notation. See [15] for more details.

2.1. Basic definitions. A *simplicial complex* X consists of the following data:

- A set $X^{(0)}$ called the *0-simplices* or *vertices*.
- For each $k \geq 1$, a set $X^{(k)}$ of $(k + 1)$ -element subsets of the vertices called the *k-simplices*. These are required to satisfy the following condition:
 - Consider $\sigma \in X^{(k)}$. Then for $\sigma' \subseteq \sigma$ with $|\sigma'| > 0$, we must have $\sigma' \in X^{(|\sigma'|-1)}$. In this case, we call σ' a *face* of σ .

A simplicial complex X has a geometric realization $|X|$ obtained by gluing together geometric k -simplices (one for each k -simplex in $X^{(k)}$) according to the face relation. Whenever we talk about topological properties of X (e.g., being connected), we are referring to its geometric realization.

2.2. Links and Cohen–Macaulay. Let X be a simplicial complex. The *link* of a simplex σ of X , denoted $\mathrm{Link}_X(\sigma)$, is the subcomplex of X consisting of all simplices τ satisfying the following two conditions.

- The simplices τ and σ are disjoint, i.e., have no vertices in common.
- The union $\tau \cup \sigma$ is a simplex.

We say that X is *weakly Cohen–Macaulay* of dimension $n \in \mathbb{Z}$ if it satisfies the following:

- The complex X must be $(n - 1)$ -connected. Here our convention is that a space is (-1) -connected if it is nonempty, and all spaces are k -connected for $k \leq -2$.

- For all k -simplices σ of X , the complex $\text{Link}_X(\sigma)$ must be $(n - k - 2)$ -connected.

Example 2.1. Let X be a PL triangulation of an n -sphere. We claim that X is weakly Cohen–Macaulay of dimension n . There are two things to check:

- The simplicial complex X is $(n - 1)$ -connected, which is clear.
- For a k -simplex σ of X , the complex $\text{Link}_X(\sigma)$ is $(n - k - 2)$ -connected. In fact, since our triangulation is PL the link of σ is a PL triangulation of an $(n - k - 1)$ -sphere. \square

Remark 2.2. The adjective “weak” is here since we do not require X to be n -dimensional, nor do we require links of k -simplices to be $(n - k - 1)$ -dimensional. However, since our conventions imply that the empty set is not (-1) -connected, if X is weakly Cohen–Macaulay of dimension n then X is at least n -dimensional and the links of k -simplices are at least $(n - k - 1)$ -dimensional. \square

One basic property of weakly Cohen–Macaulay complexes is as follows:

Lemma 2.3. *Let X be a simplicial complex that is weakly Cohen–Macaulay of dimension n and let σ be a k -simplex of X . Then $\text{Link}_X(\sigma)$ is weakly Cohen–Macaulay of dimension $(n - k - 1)$.*

Proof. By definition, $\text{Link}_X(\sigma)$ is $(n - k - 2)$ -connected. Also, if τ is an ℓ -simplex of $\text{Link}_X(\sigma)$, then $\tau \cup \sigma$ is a $(k + \ell + 1)$ -simplex of X and $\text{Link}_{\text{Link}_X(\sigma)}(\tau) = \text{Link}_X(\tau \cup \sigma)$. This is $n - (k + \ell + 1) - 2 = n - k - \ell - 3$ connected by assumption. \square

3. BACKGROUND II: SEMISIMPLICIAL SETS AND ORDERED SIMPLICIAL COMPLEXES

The category of simplicial complexes has some undesirable features that make it awkward for homological stability proofs. For instance, if X is a simplicial complex and G is a group acting on X , then one might expect the quotient X/G to be a simplicial complex whose k -simplices are the G -orbits of simplices of X . Unfortunately, this need not hold. In this section, we discuss the category of semisimplicial sets, which does not have such pathologies. See [15]¹² and [13] for more details.

3.1. Semisimplicial sets. Let Δ be the category whose objects are the finite sets $[k] = \{0, \dots, k\}$ with $k \geq 0$ and whose morphisms $[\ell] \rightarrow [k]$ are order-preserving injections. A *semisimplicial set* is a contravariant functor $\mathbb{X}: \Delta \rightarrow \mathbf{Set}$. Unwinding this, \mathbb{X} consists of the following two pieces of data:

- For each $k \geq 0$, a set $\mathbb{X}^k = \mathbb{X}([k])$ called the *k -simplices*.
- For each order-preserving injection $\iota: [\ell] \rightarrow [k]$, a map $\iota^*: \mathbb{X}^k \rightarrow \mathbb{X}^\ell$ called a *face map*. For $\sigma \in \mathbb{X}^k$, the image $\iota^*(\sigma) \in \mathbb{X}^\ell$ is called a *face* of σ .

A semisimplicial set \mathbb{X} has a geometric realization $|\mathbb{X}|$ obtained by gluing geometric n -simplices together (one for each n -simplex) according to the face maps. See [15] for more details. Whenever we talk about topological properties of \mathbb{X} (e.g., being connected), we are referring to its geometric realization.

3.2. Morphisms. Below we will compare semisimplicial sets with simplicial complexes, but first we describe their morphisms. A morphism $f: \mathbb{X} \rightarrow \mathbb{Y}$ between semisimplicial sets \mathbb{X} and \mathbb{Y} is a natural transformation between the functors \mathbb{X} and \mathbb{Y} . In other words, f consists of set maps $f_k: \mathbb{X}^k \rightarrow \mathbb{Y}^k$ for each $k \geq 0$ that commute with the face maps. Such a morphism induces a continuous map $|f|: |\mathbb{X}| \rightarrow |\mathbb{Y}|$. Using this definition, an action of a group G on a semisimplicial set \mathbb{X} consists of actions of G on each set \mathbb{X}^k that commute with the face maps, and \mathbb{X}/G is a semisimplicial set with $(\mathbb{X}/G)^k = \mathbb{X}^k/G$ for each $k \geq 0$.

¹²This reference calls semisimplicial sets Δ -sets

3.3. Semisimplicial sets vs simplicial complexes. Let \mathbb{X} be a semisimplicial set. The *vertices* of \mathbb{X} are the elements of the set \mathbb{X}^0 of 0-simplices. For each k -simplex $\sigma \in \mathbb{X}^k$, we can define an ordered $(k+1)$ -tuple (v_0, \dots, v_k) of vertices via the formula

$$v_i = \iota_i^*(\sigma) \text{ with } \iota_i: [0] \rightarrow [k] \text{ the map } \iota_i(0) = i.$$

We will call v_0, \dots, v_k the vertices of σ . This is similar to a simplicial complex, whose vertices are $(k+1)$ -element sets of vertices. However, there are three essential differences:

- The vertices of a simplex in a semisimplicial set have a natural ordering.
- The vertices of a simplex in a semisimplicial set need not be distinct.
- Simplices in a semisimplicial set are *not* determined by their vertices.

3.4. Ordered simplicial complexes. Ordered simplicial complexes are special kinds of semisimplicial sets where simplices are determined by the vertices, which are also required to be distinct. More precisely, an *ordered simplicial complex* is a semisimplicial set \mathbb{X} of the following form:

- The vertices \mathbb{X}^0 are an arbitrary set V .
- For each $k \geq 1$, the set \mathbb{X}^k of k -simplices is a subset of

$$\{(v_0, \dots, v_k) \mid \text{the } v_i \text{ are distinct elements of } V\} \subset V^k.$$

- For an order-preserving injection $\iota: [\ell] \rightarrow [k]$, the face map $\iota^*: \mathbb{X}^k \rightarrow \mathbb{X}^\ell$ equals¹³

$$\iota^*(v_0, \dots, v_k) = (v_{\iota(0)}, \dots, v_{\iota(\ell)}) \quad \text{for all } (v_0, \dots, v_k) \in \mathbb{X}^k.$$

In particular, for $(v_0, \dots, v_k) \in \mathbb{X}^k$, all tuples obtained by deleting some of the v_i are also simplices.

Remark 3.1. Though many of the concrete spaces we will consider are ordered simplicial complexes, the quotient of an ordered simplicial complex by a group action is not necessarily an ordered simplicial complex. Because of this, we will need general semisimplicial sets for the homological stability machine. \square

3.5. Small ordering. Consider a simplicial complex X . There are two natural ways to associate to X an ordered simplicial complex. First, a *small ordering* of X is an ordered simplicial complex \mathbb{X} with the following properties:

- (a) The vertices of \mathbb{X} are the same as the vertices of X .
- (b) The (unordered) set of vertices of a k -simplex of \mathbb{X} is a k -simplex of X .
- (c) The map $\mathbb{X}^k \rightarrow X^{(k)}$ taking a simplex to its set of vertices is a bijection.

The only difference between \mathbb{X} and X is thus that the vertices of a simplex of \mathbb{X} have a natural ordering, while the vertices of a simplex of X are unordered. It is clear that the geometric realizations $|\mathbb{X}|$ and $|X|$ are homeomorphic.

Small orderings always exist; for instance, if you choose a total ordering on the the vertices of X , then you can construct a small ordering \mathbb{X} of X by letting

$$\mathbb{X}^k = \left\{ (v_0, \dots, v_k) \in \left(X^{(0)} \right)^{k+1} \mid \{v_0, \dots, v_k\} \in X^{(k)} \text{ and } v_0 < \dots < v_k \right\}$$

for all $k \geq 0$. However, since this depends on the total ordering on $X^{(0)}$ it might have fewer symmetries than X . The example we will give of this will be used repeatedly, so we make a formal definition.

¹³Here we write $\iota^*(v_0, \dots, v_k)$ rather than the more precise but ugly $\iota^*((v_0, \dots, v_k))$. We will silently do this kind of thing throughout the paper.

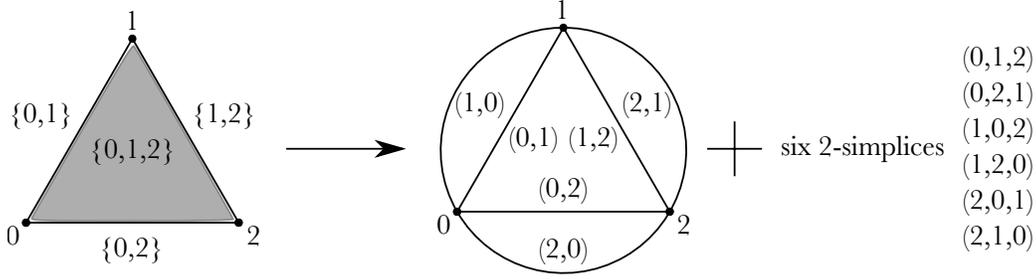


Figure 1. *Left: the 2-simplex Sim_2 . Right: its large ordering Sim_2 . The drawn portion is the 1-skeleton.*

Definition 3.2. For $n \geq 0$, let Sim_n be the n -simplex,¹⁴ i.e., the simplicial complex whose vertex set is $[n] = \{0, \dots, n\}$ and whose k -simplices are all $(k+1)$ -element subsets of $[n]$. The geometric realization $|\text{Sim}_n|$ is the usual geometric n -simplex. \square

Example 3.3. The symmetric group \mathfrak{S}_{n+1} acts on Sim_n via its natural action on $[n] = \{0, \dots, n\}$. However, if we choose a total ordering on $[n]$ and use this to define a small ordering \mathbb{X} of Sim_n as above, then \mathfrak{S}_{n+1} does not act on \mathbb{X} except in the degenerate case $n = 0$. Indeed, it is easy to see that the automorphism group of \mathbb{X} is trivial. \square

Remark 3.4. If a group G acts on a simplicial complex X , then G also acts on the set $\mathcal{O}(X)$ of all small orderings of X . If a small ordering \mathbb{X} of X is a fixed point for the action of G on $\mathcal{O}(X)$, then G acts on \mathbb{X} . \square

Remark 3.5. One way to construct a small ordering of a simplicial complex X is to choose a relation \prec on $X^{(0)}$ such that the restriction of \prec to each simplex of X is a total ordering. Since \prec is only a relation (not even a partial ordering!), not all vertices need to be comparable, and indeed it is natural to only define \prec on vertices v and w such that $\{v, w\}$ is an edge. In many situations there is a “natural” choice of such a \prec that is preserved by the symmetries that are relevant for the problem at hand. For instance, this is how small orderings arise in our followup paper [25]. \square

3.6. Large ordering. If a sufficiently symmetric small ordering of a simplicial complex does not exist, there is another construction that is often useful. Let X be a simplicial complex. The *large ordering* of X , denoted X_{ord} , is the ordered simplicial complex X_{ord} with the same vertex set as X whose k -simplices X_{ord}^k are *all* ordered $(k+1)$ -tuples (v_0, \dots, v_k) of distinct vertices of X such that the unordered set $\{v_0, \dots, v_k\}$ is a k -simplex of X .

The following example of this will be used several times, so we introduce notation for it.

Definition 3.6. For $n \geq 0$, let Sim_n be the large ordering $(\text{Sim}_n)_{\text{ord}}$ of the n -simplex Sim_n . The k -simplices of Sim_n are thus ordered sequences (i_0, \dots, i_k) of distinct elements of $[n]$. \square

Example 3.7. Each Sim_n is contractible. However, none of the Sim_n are contractible except for Sim_0 . For instance, Sim_1 has two 0-cells 0 and 1 and two 1-cells $(0, 1)$ and $(1, 0)$, and its geometric realization is homeomorphic to S^1 . For an even more complicated example, see the picture of Sim_2 in Figure 1. \square

The semisimplicial set X_{ord} is much larger than X ; indeed, each k -simplex of X corresponds to $(k+1)!$ simplices of X_{ord} , one for each total ordering of its vertices. It is clear from its construction that if a group G acts on X , then G also acts on X_{ord} . However, examples

¹⁴It would be natural to instead denote this by Δ^n or Δ_n , but we are already using Δ for the category of finite sets used to define semisimplicial sets.

like Example 3.7 might lead one to think that there is no simple relationship between the topologies of X and X_{ord} . This makes the following theorem of Randal-Williams–Wahl [29, Proposition 2.14] somewhat surprising. See also [19, Proposition 2.10] for an alternate proof.

Theorem 3.8 ([29, Theorem 2.14]). *Let X be a simplicial complex that is weakly Cohen–Macaulay of dimension n . Then X_{ord} is $(n - 1)$ -connected.*

Example 3.9. Since Sim_n is clearly weakly Cohen–Macaulay of dimension n , Theorem 3.8 implies that Sim_n is $(n - 1)$ -connected. The semisimplicial set Sim_n is also called the *complex of injective words* on $(n + 1)$ letters, and the fact that it is $(n - 1)$ -connected was originally proved by Farmer [14]. \square

Remark 3.10. For a simplicial complex X , every small ordering of X appears as a subcomplex of X_{ord} . \square

3.7. Forward link and forward Cohen–Macaulay. Let \mathbb{X} be an ordered simplicial complex. As notation, if $\sigma = (v_0, \dots, v_k)$ and $\tau = (w_0, \dots, w_\ell)$ are ordered sequences of vertices of \mathbb{X} , then we will write $\sigma \cdot \tau$ for $(v_0, \dots, v_k, w_0, \dots, w_\ell)$. Of course, $\sigma \cdot \tau$ need not be a simplex; for instance, its vertices need not be distinct.

Given a simplex σ of \mathbb{X} , the *forward link* of σ , denoted $\overrightarrow{\text{Link}}_{\mathbb{X}}(\sigma)$, is the ordered simplicial complex whose ℓ -simplices are ℓ -simplices τ of \mathbb{X} such that $\sigma \cdot \tau$ is a $(k + \ell + 1)$ -simplex of \mathbb{X} . We will say that \mathbb{X} is *weakly forward Cohen–Macaulay* of dimension n if \mathbb{X} is $(n - 1)$ -connected and for all k -simplices σ of \mathbb{X} , the forward link $\overrightarrow{\text{Link}}_{\mathbb{X}}(\sigma)$ is $(n - k - 2)$ -connected.

One source of examples of this is as follows:

Lemma 3.11. *Let X be a simplicial complex that is weakly Cohen–Macaulay of dimension n . Then its large ordering X_{ord} is weakly forward Cohen–Macaulay of dimension n .*

Proof. Theorem 3.8 implies that X_{ord} is $(n - 1)$ -connected. Let σ be a k -simplex of X_{ord} , and let $\bar{\sigma}$ be the corresponding (unordered) k -simplex of X . Since X is weakly Cohen–Macaulay of dimension n , the complex $\text{Link}_X(\bar{\sigma})$ is weakly Cohen–Macaulay of dimension $(n - k - 1)$. Since

$$\overrightarrow{\text{Link}}_{\mathbb{X}}(\sigma) = (\text{Link}_X(\bar{\sigma}))_{\text{ord}},$$

Theorem 3.8 implies that $\overrightarrow{\text{Link}}_{\mathbb{X}}(\sigma)$ is $(n - k - 2)$ -connected. The lemma follows. \square

4. BACKGROUND III: COEFFICIENT SYSTEMS

In this section, we define coefficient systems on semisimplicial sets. Informally, these are natural assignments of abelian groups to each simplex.

4.1. Simplex category. To formalize this, we introduce the *simplex category* of a semisimplicial set \mathbb{X} , which is the following category¹⁵ $\mathbf{Simp}(\mathbb{X})$:

- The objects of $\mathbf{Simp}(\mathbb{X})$ are the simplices of \mathbb{X} .
- For $\sigma, \sigma' \in \mathbf{Simp}(\mathbb{X})$ with $\sigma \in \mathbb{X}^k$ and $\sigma' \in \mathbb{X}^\ell$, the morphisms from σ to σ' are

$$\text{Mor}(\sigma, \sigma') = \{ \iota: [\ell] \rightarrow [k] \mid \iota \text{ order-preserving injection with } \iota^*(\sigma) = \sigma' \}.$$

This is nonempty precisely when σ' is a face of σ .

The *augmented simplex category* of \mathbb{X} , denoted $\widetilde{\mathbf{Simp}}(\mathbb{X})$, is obtained by adjoining a terminal object $*$ to $\mathbf{Simp}(\mathbb{X})$ that we will call the *(-1) -simplex*.

¹⁵This is related to the poset of simplices of \mathbb{X} , but is *not* always a poset since there can be multiple morphisms between simplices in it.

4.2. Coefficient systems. Let \mathbb{k} be a commutative ring. A *coefficient system* over \mathbb{k} on a semisimplicial set \mathbb{X} is a covariant functor \mathcal{F} from $\mathbf{Simp}(\mathbb{X})$ to the category of \mathbb{k} -modules. Unpacking this, \mathcal{F} consists of the following data:

- For each simplex σ of \mathbb{X} , a \mathbb{k} -module $\mathcal{F}(\sigma)$.
- For $\sigma \in \mathbb{X}^k$ and $\sigma' \in \mathbb{X}^\ell$ and $\iota: [\ell] \rightarrow [k]$ an order-preserving injection with $\iota^*(\sigma) = \sigma'$, a \mathbb{k} -module morphism $\iota^*: \mathcal{F}(\sigma) \rightarrow \mathcal{F}(\sigma')$.

These must satisfy the evident compatibility conditions. Similarly, an *augmented coefficient system* on \mathbb{X} is a covariant functor \mathcal{F} from $\widetilde{\mathbf{Simp}}(\mathbb{X})$ to the category of \mathbb{k} -modules.

Example 4.1. If \mathbb{X} is a semisimplicial set and \mathbb{k} is a commutative ring, then we have the constant coefficient system $\underline{\mathbb{k}}$ on \mathbb{X} with $\underline{\mathbb{k}}(\sigma) = \mathbb{k}$ for all simplices σ . This can be extended to an augmented coefficient system by setting $\underline{\mathbb{k}}(*) = \mathbb{k}$ for the (-1) -simplex $*$. \square

Example 4.2. Recall that \mathbf{Sim}_n is the large ordering of the n -simplex \mathbf{Sim}_n , and that we introduced FI-modules in §1.3. For an FI-module M , we can define a coefficient system $\mathcal{F}_{M,n}$ on \mathbf{Sim}_n via the formula

$$\mathcal{F}_{M,n}(i_0, \dots, i_k) = M([n] \setminus \{i_0, \dots, i_k\}) \quad \text{for a simplex } (i_0, \dots, i_k) \text{ of } \mathbf{Sim}_n.$$

For an order-preserving injective map $\iota: [\ell] \rightarrow [k]$, the induced map

$$\iota^*: \mathcal{F}_{M,n}(i_0, \dots, i_k) \rightarrow \mathcal{F}_{M,n}(i_{\iota(0)}, \dots, i_{\iota(\ell)})$$

is the one induced by the inclusion

$$[n] \setminus \{i_0, \dots, i_k\} \hookrightarrow [n] \setminus \{i_{\iota(0)}, \dots, i_{\iota(\ell)}\}.$$

This can be extended to an augmented coefficient system by setting $\mathcal{F}_{M,n}(*) = M([n])$ for the (-1) -simplex $*$. \square

The collection of coefficient systems (resp. augmented coefficient systems) over \mathbb{k} on \mathbb{X} forms an abelian category.

4.3. Equivariant coefficient systems. Let G be a group, let \mathbb{X} be a semisimplicial set on which G acts, and let \mathbb{k} be a commutative ring. Consider a (possibly augmented) coefficient system \mathcal{F} on \mathbb{X} . We want to equip \mathcal{F} with an “action” of G that is compatible with the G -action on \mathbb{X} . For $g \in G$ and a simplex σ of \mathbb{X} , the data of such an action should include isomorphisms $\mathcal{F}(\sigma) \rightarrow \mathcal{F}(g \cdot \sigma)$, and more generally for $h \in G$ should include isomorphisms $\mathcal{F}(h \cdot \sigma) \rightarrow \mathcal{F}(gh \cdot \sigma)$. Moreover, these isomorphisms should be compatible with \mathcal{F} in an appropriate sense.

We formalize this as follows. For $h \in G$, let \mathcal{F}_h be the coefficient system over \mathbb{k} on \mathbb{X} defined via the formula

$$\mathcal{F}_h(\sigma) = \mathcal{F}(h \cdot \sigma) \quad \text{for a simplex } \sigma \text{ of } \mathbb{X}.$$

We say that \mathcal{F} is a *G -equivariant coefficient system* if for all $g, h \in G$ we are given a natural transformation $\Phi_{g,h}: \mathcal{F}_h \Rightarrow \mathcal{F}_{gh}$. These natural transformations should satisfy the following two properties:

- For all $h \in G$, the natural transformation $\Phi_{1,h}: \mathcal{F}_h \Rightarrow \mathcal{F}_h$ should be the identity natural transformation.
- For all $g_1, g_2 \in G$ and all $h \in H$, we require the two natural transformations

$$\mathcal{F}_h \xrightarrow{\Phi_{g_1 g_2, h}} \mathcal{F}_{g_1 g_2 h} \quad \text{and} \quad \mathcal{F}_h \xrightarrow{\Phi_{g_2, h}} \mathcal{F}_{g_2 h} \xrightarrow{\Phi_{g_1, g_2 h}} \mathcal{F}_{g_1 g_2 h}$$

to be equal.

Let us unpack this a bit. For all simplices σ of \mathbb{X} and all $g, h \in G$, the natural transformation $\Phi_{g,h}$ gives a homomorphism $\mathcal{F}(h \cdot \sigma) \rightarrow \mathcal{F}(gh \cdot \sigma)$. This homomorphism is an isomorphism whose inverse is given by the map $\mathcal{F}(gh \cdot \sigma) \rightarrow \mathcal{F}(h \cdot \sigma)$ induced by $\Phi_{g^{-1},gh}$. Moreover, it must be natural in the sense that if σ' is a face of σ via some face map, then the diagram

$$\begin{array}{ccc} \mathcal{F}(h \cdot \sigma) & \longrightarrow & \mathcal{F}(gh \cdot \sigma) \\ \downarrow & & \downarrow \\ \mathcal{F}(h \cdot \sigma') & \longrightarrow & \mathcal{F}(gh \cdot \sigma') \end{array}$$

must commute. The fact that the natural transformations respect the group law implies that the stabilizer subgroup G_σ acts on $\mathcal{F}(\sigma)$, making it into a $\mathbb{k}[G_\sigma]$ -module. If σ' is a face of σ via some face map, the induced map $\mathcal{F}(\sigma) \rightarrow \mathcal{F}(\sigma')$ is a map of $\mathbb{k}[G_\sigma]$ -modules, where G_σ acts on $\mathcal{F}(\sigma')$ via the inclusion $G_\sigma \hookrightarrow G_{\sigma'}$.

Example 4.3. Let M be an FI-module over \mathbb{k} and let $\mathcal{F}_{M,n}$ be the augmented coefficient system on Sim_n from Example 4.2 defined via the formula

$$\mathcal{F}_{M,n}(i_0, \dots, i_k) = M([n] \setminus \{i_0, \dots, i_k\}) \quad \text{for a simplex } (i_0, \dots, i_k) \text{ of } \text{Sim}_n.$$

Recalling that $[n] = \{0, \dots, n\}$, the symmetric group \mathfrak{S}_{n+1} acts on Sim_n . The augmented coefficient system $\mathcal{F}_{M,n}$ can be endowed with the structure of an \mathfrak{S}_{n+1} -equivariant augmented coefficient system in the following way. Consider $g, h \in \mathfrak{S}_{n+1}$. We then define $\Phi_{g,h}$ to be the following natural transformation:

- For a simplex (i_0, \dots, i_k) of Sim_n , let the induced map

$$\mathcal{F}_{M,n}(h(i_0), \dots, h(i_k)) \rightarrow \mathcal{F}_{M,n}(gh(i_0), \dots, gh(i_k))$$

be the map

$$M([n] \setminus \{h(i_0), \dots, h(i_k)\}) \rightarrow M([n] \setminus \{gh(i_0), \dots, gh(i_k)\})$$

induced by the bijection

$$[n] \setminus \{h(i_0), \dots, h(i_k)\} \rightarrow [n] \setminus \{gh(i_0), \dots, gh(i_k)\}$$

obtained by restricting $g \in \mathfrak{S}_{n+1}$ to $[n] \setminus \{h(i_0), \dots, h(i_k)\}$. \square

Another consequence of the fact that the natural transformations respect the group law is that if \mathcal{F} is a G -equivariant coefficient system on \mathbb{X} , then for all $k \geq 0$ the direct sum

$$\bigoplus_{\sigma \in \mathbb{X}^k} \mathcal{F}(\sigma)$$

is a $\mathbb{k}[G]$ -module in a natural way, where the G -action restricts to the G_σ -action on $\mathcal{F}(\sigma)$ for each $\sigma \in \mathbb{X}^k$. This $\mathbb{k}[G]$ -module structure can be described in terms of the quotient \mathbb{X}/G as follows. For each $\tau \in \mathbb{X}^k/G$, fix a lift $\tilde{\tau} \in \mathbb{X}^k$. We then have

$$\bigoplus_{\sigma \in \mathbb{X}^k} \mathcal{F}(\sigma) = \bigoplus_{\tau \in \mathbb{X}^k/G} \left(\bigoplus_{\sigma \in G \cdot \tilde{\tau}} \mathcal{F}(\sigma) \right).$$

Using [7, Corollary III.5.4], we have a $\mathbb{k}[G]$ -module isomorphism

$$\bigoplus_{\sigma \in G \cdot \tilde{\tau}} \mathcal{F}(\sigma) \cong \text{Ind}_{G_{\tilde{\tau}}}^G \mathcal{F}(\tilde{\tau}).$$

We conclude that

$$(4.1) \quad \bigoplus_{\sigma \in \mathbb{X}^k} \mathcal{F}(\sigma) = \bigoplus_{\tau \in \mathbb{X}^k/G} \text{Ind}_{G_{\tilde{\tau}}}^G \mathcal{F}(\tilde{\tau}).$$

4.4. Chain complex and homology. Let \mathbb{X} be a semisimplicial set and let \mathcal{F} be a coefficient system on \mathbb{X} . Define the *simplicial chain complex* of \mathbb{X} with coefficients in \mathcal{F} to be the chain complex $C_\bullet(\mathbb{X}; \mathcal{F})$ defined as follows:

- For $k \geq 0$, we have

$$C_k(\mathbb{X}; \mathcal{F}) = \bigoplus_{\sigma \in \mathbb{X}^k} \mathcal{F}(\sigma).$$

- The boundary map $d: C_k(\mathbb{X}; \mathcal{F}) \rightarrow C_{k-1}(\mathbb{X}; \mathcal{F})$ is $d = \sum_{i=0}^k (-1)^i d_i$, where the map $d_i: C_k(\mathbb{X}; \mathcal{F}) \rightarrow C_{k-1}(\mathbb{X}; \mathcal{F})$ is as follows. Consider $\sigma \in \mathbb{X}^k$. Let $\iota: [k-1] \rightarrow [k]$ be the order-preserving map whose image omits i . Then on the $\mathcal{F}(\sigma)$ factor of $C_n(\mathbb{X}; \mathcal{F})$, the map d_i is

$$\mathcal{F}(\sigma) \xrightarrow{\iota^*} \mathcal{F}(\iota^*(\sigma)) \hookrightarrow \bigoplus_{\sigma' \in \mathbb{X}^{k-1}} \mathcal{F}(\sigma') = C_{k-1}(\mathbb{X}; \mathcal{F}).$$

Define

$$H_k(\mathbb{X}; \mathcal{F}) = H_k(C_\bullet(\mathbb{X}; \mathcal{F})).$$

For an augmented coefficient system \mathcal{F} on \mathbb{X} , define $\tilde{C}_\bullet(\mathbb{X}; \mathcal{F})$ to be the augmented chain complex defined just like we did above but with $\tilde{C}_{-1}(\mathbb{X}; \mathcal{F}) = \mathcal{F}(\ast)$ for the (-1) -simplex \ast and define

$$\tilde{H}_k(\mathbb{X}; \mathcal{F}) = H_k(\tilde{C}_\bullet(\mathbb{X}; \mathcal{F})).$$

Example 4.4. For a semisimplicial set \mathbb{X} and a commutative ring \mathbb{k} , we have

$$H_k(\mathbb{X}; \mathbb{k}) = H_k(|\mathbb{X}|; \mathbb{k}) \quad \text{and} \quad \tilde{H}_k(\mathbb{X}; \mathbb{k}) = \tilde{H}_k(|\mathbb{X}|; \mathbb{k}). \quad \square$$

Remark 4.5. With our definition, $\tilde{H}_{-1}(\mathbb{X}; \mathcal{F})$ is a quotient of $\mathcal{F}(\ast)$. This quotient can sometimes be nonzero. It vanishes precisely when the map

$$\bigoplus_{v \in \mathbb{X}^0} \mathcal{F}(v) \rightarrow \mathcal{F}(\ast)$$

is surjective. □

Note that if a group G acts on \mathbb{X} and \mathcal{F} is a G -equivariant coefficient system, then

$$\cdots \rightarrow \tilde{C}_2(\mathbb{X}; \mathcal{F}) \rightarrow \tilde{C}_1(\mathbb{X}; \mathcal{F}) \rightarrow \tilde{C}_0(\mathbb{X}; \mathcal{F}) \rightarrow \tilde{C}_{-1}(\mathbb{X}; \mathcal{F}) = \mathcal{F}(\ast) \rightarrow 0$$

is a chain complex of $\mathbb{k}[G]$ -modules, and each $\tilde{H}_k(\mathbb{X}; \mathcal{F})$ is a $\mathbb{k}[G]$ -module.

4.5. Long exact sequences. Consider a short exact sequence

$$0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$$

of coefficient systems over \mathbb{k} on \mathbb{X} . For each simplex σ of \mathbb{X} , we thus have a short exact sequence

$$0 \rightarrow \mathcal{F}_1(\sigma) \rightarrow \mathcal{F}_2(\sigma) \rightarrow \mathcal{F}_3(\sigma) \rightarrow 0$$

of \mathbb{k} -modules. These fit together into a short exact sequence

$$0 \rightarrow C_\bullet(\mathbb{X}; \mathcal{F}_1) \rightarrow C_\bullet(\mathbb{X}; \mathcal{F}_2) \rightarrow C_\bullet(\mathbb{X}; \mathcal{F}_3) \rightarrow 0$$

of chain complexes, and thus induce a long exact sequence in homology of the form

$$\cdots \rightarrow H_k(\mathbb{X}; \mathcal{F}_1) \rightarrow H_k(\mathbb{X}; \mathcal{F}_2) \rightarrow H_k(\mathbb{X}; \mathcal{F}_3) \rightarrow H_{k-1}(\mathbb{X}; \mathcal{F}_1) \rightarrow \cdots$$

A similar result holds for augmented coefficient systems and reduced homology.

5. STABILITY I: STABILITY MACHINE

In this section, we describe our machine for proving twisted homological stability.

5.1. Classical homological stability. An *increasing sequence of groups* is an indexed sequence of groups $\{G_n\}_{n=0}^\infty$ such that

$$G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots .$$

For each $k \geq 0$, we get maps

$$H_k(G_0) \rightarrow H_k(G_1) \rightarrow H_k(G_2) \rightarrow \cdots .$$

The classical homological stability machine gives conditions under which these stabilize, i.e., such that the maps $H_k(G_{n-1}) \rightarrow H_k(G_n)$ are isomorphisms for $n \gg k$. One version of it is as follows.

Theorem 5.1 (Classical homological stability). *Let $\{G_n\}_{n=0}^\infty$ be an increasing sequence of groups. For each $n \geq 1$, let \mathbb{X}_n be a semisimplicial set upon which G_n acts. Assume for some $c \geq 2$ that the following hold for all $n \geq 1$:*

- (a) *For all $-1 \leq k \leq \lfloor \frac{n-2}{c} \rfloor$, we have $\tilde{H}_k(\mathbb{X}_n) = 0$.*
- (b) *For all $0 \leq k < n$, the group G_{n-k-1} is the G_n -stabilizer of a k -simplex of \mathbb{X}_n .*
- (c) *For all $0 \leq k < n$, the group G_n acts transitively on the k -simplices of \mathbb{X}_n .*
- (d) *For all $n \geq 2$ and all 1-simplices e of \mathbb{X}_n whose proper faces consist of 0-simplices v and v' , there exists some $\lambda \in G_n$ with $\lambda(v) = v'$ such that λ commutes with all elements of $(G_n)_e$.*

Then for all k the map $H_k(G_{n-1}) \rightarrow H_k(G_n)$ is an isomorphism for $n \geq ck + 2$ and a surjection for $n = ck + 1$.

Proof. This can be proved exactly like [19, Theorem 1.1] – the only major difference between our theorem and [19, Theorem 1.1] is that we assume a weaker connectivity range on the \mathbb{X}_n , which causes stability to happen at a slower rate.¹⁶ We thus omit the details of the proof, but to clarify our indexing conventions we make a few remarks.

The proof is by induction on k . The base case is $k \leq -1$, where the result is trivial since $H_k(G) = 0$ for all groups G when k is negative. We could also start with $k = 0$ since $H_0(G) = \mathbb{Z}$, but later when we work with twisted coefficients even the H_0 statement will be nontrivial. In any case, to go from H_{k-1} to H_k two steps are needed (which is why we require $c \geq 2$).

The first step is to prove that the map $H_k(G_{n-1}) \rightarrow H_k(G_n)$ is surjective for all n sufficiently large, which requires that $\tilde{H}_i(\mathbb{X}_n) = 0$ for $-1 \leq i \leq k-1$. Once this has been done, we then prove that $H_k(G_{n-1}) \rightarrow H_k(G_n)$ is also injective for n one step larger than needed for surjective stability. This requires $\tilde{H}_i(\mathbb{X}_n) = 0$ for $-1 \leq i \leq k$. We remark that condition (d) is used for injective stability but not surjective stability, which is why we only assume it for $n \geq 2$. See Remark 5.3 below for more discussion of condition (d).

This explains our indexing conventions:

- Surjective stability for H_0 starts with $H_0(G_0) \rightarrow H_0(G_1)$, so we need $\tilde{H}_{-1}(\mathbb{X}_1) = 0$.
- Injective stability for H_0 starts with $H_0(G_1) \rightarrow H_0(G_2)$, so we need $\tilde{H}_{-1}(\mathbb{X}_2) = \tilde{H}_0(\mathbb{X}_2) = 0$. \square

5.2. Setup for twisted coefficients. We want to give a version of this with twisted coefficients. Fix a commutative ring \mathbb{k} . An *increasing sequence of groups and modules* is an indexed sequence of pairs $\{(G_n, M_n)\}_{n=0}^\infty$, where the G_n and the M_n are as follows:

- The $\{G_n\}_{n=0}^\infty$ are an increasing sequence of groups.
- Each M_n is a $\mathbb{k}[G_n]$ -module.

¹⁶Another minor difference is that we work with semisimplicial sets rather than simplicial complexes, so unlike [19, Theorem 1.1] we do not need to assume in (b) that the G_n -stabilizer of a simplex σ of \mathbb{X}_n stabilizes σ pointwise. This is automatic for semisimplicial sets.

- As abelian groups, the M_n satisfy

$$M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots .$$

- For each $n \geq 0$, the inclusion $M_n \hookrightarrow M_{n+1}$ is G_n -equivariant, where G_n acts on M_{n+1} via the inclusion $G_n \hookrightarrow G_{n+1}$.

Given an increasing sequence of groups and modules $\{(G_n, M_n)\}_{n=0}^\infty$, for each k we get maps

$$\mathrm{H}_k(G_0; M_0) \rightarrow \mathrm{H}_k(G_1; M_1) \rightarrow \mathrm{H}_k(G_2; M_2) \rightarrow \cdots$$

between the associated twisted homology groups. We want to show that this stabilizes via a machine similar to Theorem 5.1.

We will incorporate the M_n into our machine via a G_n -equivariant augmented coefficient system \mathcal{M}_n on the semisimplicial set \mathbb{X}_n with $\mathcal{M}_n(*) = M_n$ for the (-1) -simplex $*$. Having done this, we will be forced to replace the requirement that $\tilde{\mathrm{H}}_k(\mathbb{X}_n) = 0$ in condition (a) by $\tilde{\mathrm{H}}_k(\mathbb{X}_n; \mathcal{M}_n) = 0$. This is not easy to check, but we will give a useful criterion in §6 below. The twisted analogue of Theorem 5.1 is as follows.

Theorem 5.2 (Twisted homological stability). *Let $\{(G_n, M_n)\}_{n=0}^\infty$ be an increasing sequence of groups and modules. For each $n \geq 1$, let \mathbb{X}_n be a semisimplicial set upon which G_n acts and let \mathcal{M}_n be a G_n -equivariant augmented coefficient system on \mathbb{X}_n . Assume for some $c \geq 2$ that the following hold for all $n \geq 1$:*

- For all $-1 \leq k \leq \lfloor \frac{n-2}{c} \rfloor$, we have $\tilde{\mathrm{H}}_k(\mathbb{X}_n; \mathcal{M}_n) = 0$.
- For all $-1 \leq k < n$, the group G_{n-k-1} is the G_n -stabilizer of a k -simplex σ_k of \mathbb{X}_n with $\mathcal{M}_n(\sigma_k) = M_{n-k-1}$. In particular, $\mathcal{M}_n(*) = M_n$.
- For all $0 \leq k < n$, the group G_n acts transitively on the k -simplices of \mathbb{X}_n .
- For all $n \geq 2$ and all 1-simplices e of \mathbb{X}_n whose proper faces consist of 0-simplices v and v' , there exists some $\lambda \in G_n$ with $\lambda(v) = v'$ such that λ commutes with all elements of $(G_n)_e$ and fixes all elements of $\mathcal{M}_n(e)$.

Then for $k \geq 0$ the map $\mathrm{H}_k(G_{n-1}; M_{n-1}) \rightarrow \mathrm{H}_k(G_n; M_n)$ is an isomorphism for $n \geq ck + 2$ and a surjection for $n = ck + 1$.

The proof of Theorem 5.2 is almost identical to that of Theorem 5.1 (for which we referred to [19, Theorem 1.1]). We will therefore not give full details, but only describe how to construct the key spectral sequence (see Example 5.7 below), which will explain the role played by condition (a).

Remark 5.3. One other potentially confusing thing is how condition (d) is used, and in particular why we have to assume that the λ in it fixes all elements of $\mathcal{M}_n(e)$. As we said above, the corresponding condition (d) in the classical homological stability machine (Theorem 5.1) is used to prove injective stability. It arises in the following way. Let the notation be as in Theorem 5.1. Let e be an edge of \mathbb{X}_n connecting vertices v and v' . The stabilizer $(G_n)_e$ is thus a subgroup of both $(G_n)_v$ and $(G_n)_{v'}$. If $\phi \in G_n$ satisfies $\phi(v) = v'$, then conjugation by ϕ gives an isomorphism $\iota_\phi: (G_n)_v \rightarrow (G_n)_{v'}$. We therefore get two different maps

$$\mathrm{H}_i((G_n)_e) \rightarrow \mathrm{H}_i((G_n)_{v'}) \quad \text{and} \quad \mathrm{H}_i((G_n)_e) \rightarrow \mathrm{H}_i((G_n)_v) \xrightarrow{(\iota_\phi)_*} \mathrm{H}_i((G_n)_{v'}).$$

To prove injective stability, it turns out that you need to ensure that these maps are *equal*. The point here is that their difference equals a differential in a spectral sequence, and one needs to prove that this differential vanishes.

Any choice of $\phi \in G_n$ with $\phi(v) = v'$ will work here. Condition (d) from Theorem 5.1 gives a $\lambda \in G_n$ with $\lambda(v) = v'$ such that λ also commutes with all elements of $(G_n)_e$. This

implies that the maps

$$(G_n)_e \rightarrow (G_n)_{v'} \quad \text{and} \quad (G_n)_e \rightarrow (G_n)_v \xrightarrow{\iota_\lambda} (G_n)_{v'}$$

are actually equal to each other, and therefore induce the same map on H_i . We can thus take $\phi = \lambda$.

In the setting of Theorem 5.2, the G_n -equivariance of the coefficient system \mathcal{M}_n implies that a $\phi \in G_n$ with $\phi(v) = v'$ induces an isomorphism $\phi_*: \mathcal{M}_n(v) \rightarrow \mathcal{M}_n(v')$. The conjugation isomorphism $\iota_\phi: (G_n)_v \rightarrow (G_n)_{v'}$ and $\phi_*: \mathcal{M}_n(v) \rightarrow \mathcal{M}_n(v')$ induce an isomorphism

$$(\iota_\phi, \phi_*)_*: H_i((G_n)_v; \mathcal{M}_n(v)) \rightarrow H_i((G_n)_{v'}; \mathcal{M}_n(v')),$$

so we get two maps

$$H_i((G_n)_e; \mathcal{M}_n(e)) \rightarrow H_i((G_n)_{v'}; \mathcal{M}_n(v'))$$

and

$$H_i((G_n)_e; \mathcal{M}_n(e)) \rightarrow H_i((G_n)_v; \mathcal{M}_n(v)) \xrightarrow{(\iota_\phi, \phi_*)_*} H_i((G_n)_{v'}; \mathcal{M}_n(v')),$$

and again we need these to be equal. This will be ensured by taking $\phi = \lambda$ for the $\lambda \in G_n$ provided by condition (d) in Theorem 5.2, which not only commutes with all elements of $(G_n)_e$ but also fixes all elements of $\mathcal{M}_n(e)$. \square

Remark 5.4. Theorem 5.2 is related to [24, Theorem D]. \square

5.3. Homology of stabilizers. To construct the spectral sequence, we need the following construction. Fix a commutative ring \mathbb{k} . Let G be a group acting on a semisimplicial set \mathbb{X} and let \mathcal{M} be a G -equivariant augmented coefficient system on \mathbb{X} over \mathbb{k} . For a simplex $\tilde{\sigma}$ of \mathbb{X} , the value $\mathcal{M}(\tilde{\sigma})$ is a $\mathbb{k}[G_{\tilde{\sigma}}]$ -module. For $q \geq 0$, define an augmented coefficient system $\mathcal{H}_q(\mathcal{M})$ on \mathbb{X}/G as follows. Consider a simplex σ of \mathbb{X}/G , and let $\tilde{\sigma}$ be a lift of σ to \mathbb{X} . We then define

$$\mathcal{H}_q(\mathcal{M})(\sigma) = H_q(G_{\tilde{\sigma}}; \mathcal{M}(\tilde{\sigma})).$$

To see that this is well-defined, let $\tilde{\sigma}'$ be another lift of σ to \mathbb{X} . There exists some $g \in G$ with $g\tilde{\sigma} = \tilde{\sigma}'$, so $gG_{\tilde{\sigma}}g^{-1} = G_{\tilde{\sigma}'}$ and $g \cdot \mathcal{M}(\tilde{\sigma}) = \mathcal{M}(\tilde{\sigma}')$. Conjugation/multiplication by g thus induces an isomorphism

$$H_q(G_{\tilde{\sigma}}; \mathcal{M}(\tilde{\sigma})) \cong H_q(G_{\tilde{\sigma}'}; \mathcal{M}(\tilde{\sigma}')).$$

What is more, since inner automorphisms induce the identity on homology (even with twisted coefficients; see [7, Proposition III.8.1]), this isomorphism is independent of the choice of g , and thus is completely canonical. That it is a coefficient system follows immediately.

Remark 5.5. If $*$ is the (-1) -simplex of \mathbb{X}/G , then the only possible lift $\tilde{*}$ is the (-1) -simplex of \mathbb{X} , and by definition its stabilizer is the entire group G . It follows that $\mathcal{H}_q(\mathcal{M})(*) = H_q(G; \mathcal{M}(\tilde{*}))$. \square

5.4. Spectral sequence. The spectral sequence that underlies Theorem 5.2 is as follows. In it, our convention is that $\mathcal{H}_q(\mathcal{M}) = \underline{0}$ for $q < 0$.

Theorem 5.6 (Spectral sequence). *Let G be a group acting on a semisimplicial set \mathbb{X} , let \mathbb{k} be a commutative ring, let \mathcal{M} be a G -equivariant augmented coefficient system over \mathbb{k} on \mathbb{X} . Assume that $\tilde{H}_k(\mathbb{X}; \mathcal{M}) = 0$ for $0 \leq k \leq r$. Then there exists a spectral sequence E_{pq}^\bullet with the following properties:*

- (i) *We have $E_{pq}^1 = \tilde{C}_p(\mathbb{X}/G; \mathcal{H}_q(\mathcal{M}))$, and the differential $E_{pq}^1 \rightarrow E_{p-1,q}^1$ is the differential on $\tilde{C}_\bullet(\mathbb{X}/G; \mathcal{H}_q(\mathcal{M}))$. In particular, $E_{pq}^1 = 0$ if $p < -1$ or if $q < 0$.*
- (ii) *For $p + q \leq r$, we have $E_{pq}^\infty = 0$.*

Example 5.7. Let the notation and assumptions be as in Theorem 5.2, and apply Theorem 5.6 to G_n and \mathbb{X}_n and \mathcal{M}_n . Since G_n acts transitively on the k -simplices of \mathbb{X}_n for $0 \leq k < n$, there is a single k -simplex in \mathbb{X}_n/G_n . Since G_{n-k-1} is the G_n stabilizer of a k -simplex σ_k of \mathbb{X}_n with $\mathcal{M}_n(\sigma_k) = M_{n-k-1}$, we conclude that our spectral sequence has

$$E_{pq}^1 \cong \tilde{C}_p(\mathbb{X}_n/G_n; \mathcal{H}_q(\mathcal{M}_n)) = H_q(G_{n-p-1}; M_{n-p-1}) \quad \text{for } -1 \leq p < n. \quad \square$$

Proof of Theorem 5.6. Let

$$\cdots \rightarrow F_2(G) \rightarrow F_1(G) \rightarrow F_0(G) \rightarrow \mathbb{k}$$

be a resolution of the trivial $\mathbb{k}[G]$ -module \mathbb{k} by free $\mathbb{k}[G]$ -modules. The action of G on \mathbb{X} makes $\tilde{C}_\bullet(\mathbb{X}; \mathcal{M})$ into a chain complex of $\mathbb{k}[G]$ -modules, so we can consider the double complex $C_{\bullet, \bullet}$ defined by

$$(5.1) \quad C_{pq} = \tilde{C}_p(\mathbb{X}; \mathcal{M}) \otimes_G F_q(G).$$

The spectral sequence we are looking for converges to the homology of this double complex. In fact, there are two spectral sequences converging to the homology of a double complex, one arising by filtering it “horizontally” and the other by filtering it “vertically” (see, e.g., [7, §VII.3]). We will use the horizontal filtration to show that the homology of (5.1) vanishes up to degree r (conclusion (ii)), and then prove that the vertical filtration gives the spectral sequence described in the theorem.

The spectral sequence arising from the horizontal filtration has

$$E_{pq}^1 \cong H_p\left(\tilde{C}_\bullet(\mathbb{X}; \mathcal{M}) \otimes_G F_q(G)\right).$$

Our assumptions imply that $\tilde{C}_\bullet(\mathbb{X}; \mathcal{M})$ is exact up to degree r , and since $F_q(G)$ is a free $\mathbb{Z}[G]$ -module it follows that $\tilde{C}_\bullet(\mathbb{X}; \mathcal{M}) \otimes_G F_q(G)$ is also exact up to degree r . It follows that $E_{pq}^1 = 0$ for $p \leq r$. We deduce that the homology of the double complex (5.1) vanishes up to degree r .

The spectral sequence arising from the vertical filtration has

$$E_{pq}^1 \cong H_q\left(\tilde{C}_p(\mathbb{X}; \mathcal{M}) \otimes_G F_\bullet(G)\right).$$

This is the spectral sequence that is referred to in the theorem, and we must prove that it satisfies (i). Since $F_\bullet(G)$ is a free resolution of the trivial $\mathbb{k}[G]$ -module \mathbb{Z} , the above expression simplifies to

$$E_{pq}^1 \cong H_q\left(G; \tilde{C}_p(\mathbb{X}; \mathcal{M})\right).$$

For each p -simplex σ of \mathbb{X}/G , fix a lift $\tilde{\sigma}$ to \mathbb{X} . As we discussed in §4.3 (see (4.1)), as a $\mathbb{k}[G]$ -module we have

$$\tilde{C}_p(\mathbb{X}; \mathcal{M}) \cong \bigoplus_{\sigma \in (\mathbb{X}/G)^p} \text{Ind}_{G_{\tilde{\sigma}}}^G \mathcal{M}(\tilde{\sigma}).$$

Plugging this in, we get

$$E_{pq}^1 \cong \bigoplus_{\sigma \in (\mathbb{X}/G)^p} H_q\left(G; \text{Ind}_{G_{\tilde{\sigma}}}^G \mathcal{M}(\tilde{\sigma})\right).$$

Applying Shapiro’s Lemma, the right hand side simplifies to

$$\bigoplus_{\sigma \in (\mathbb{X}/G)^p} H_q\left(G_{\tilde{\sigma}}; \mathcal{M}(\tilde{\sigma})\right) = \tilde{C}_p(\mathbb{X}/G; \mathcal{H}_q(\mathcal{M})).$$

That the differential is as described in (i) is clear. The theorem follows. \square

5.5. Stability machine and finite-index subgroups. We close this section by proving a variant of Theorem 5.2 that we will use to analyze congruence subgroups of $\mathrm{GL}_n(R)$. To motivate its statement, consider a finite-index normal subgroup G' of a group G and a $\mathbb{k}[G]$ -module M over a field \mathbb{k} of characteristic 0. The conjugation action of G on G' induces an action of G on each $\mathrm{H}_k(G'; M)$, and using the transfer map (see [7, Chapter III.9]) one can show that $\mathrm{H}_k(G; M)$ is the G -coinvariants of the action of G on $\mathrm{H}_k(G'; M)$. From this, we see that the map

$$\mathrm{H}_k(G'; M) \rightarrow \mathrm{H}_k(G; M)$$

induced by the inclusion $G' \hookrightarrow G$ is an isomorphism if and only if the action of G on $\mathrm{H}_k(G'; M)$ is trivial. The point of the following is that under the conditions of our stability machine, it is enough to check a weaker version of this triviality (condition (h)).

Theorem 5.8. *Let $\{(G_n, M_n)\}_{n=0}^\infty$ be an increasing sequence of groups and modules. For each $n \geq 1$, let \mathbb{X}_n be a semisimplicial set upon which G_n acts and let \mathcal{M}_n be a G_n -equivariant augmented coefficient system on \mathbb{X}_n . Assume for some $c \geq 2$ that conditions (a)-(d) of Theorem 5.2 hold, so by that theorem $\mathrm{H}_k(G_n; \mathcal{M}_n)$ stabilizes. Furthermore, assume that $\{G'_n\}_{n=0}^\infty$ is an increasing sequence of groups such that each G'_n is a finite-index normal subgroup of G_n , and that the following hold:*

- (e) Each M_n is a vector space over a field \mathbb{k} of characteristic 0.
- (f) For the k -simplex σ_k of \mathbb{X}_n from condition (b) whose G_n -stabilizer is G_{n-k-1} , the G'_n -stabilizer of σ_k is G'_{n-k-1} .
- (g) The quotient \mathbb{X}_n/G'_n is $\lfloor \frac{n-2}{c} \rfloor$ -connected.
- (h) For $k \geq 0$ and $n \geq ck+2$, the action of G_n on $\mathrm{H}_k(G'_n; M_n)$ induced by the conjugation action of G_n on G'_n fixes pointwise the image of the stabilization map

$$(5.2) \quad \mathrm{H}_k(G'_{n-1}; M_{n-1}) \rightarrow \mathrm{H}_k(G'_n; M_n).$$

Then for $n \geq ck+2$ the map $\mathrm{H}_k(G'_n; M_n) \rightarrow \mathrm{H}_k(G_n; M_n)$ induced by the inclusion $G'_n \hookrightarrow G_n$ is an isomorphism.

Proof. The proof will be by induction on k . The base case $k \leq -1$ is trivial, so assume that $k \geq 0$ and that the result is true for all smaller k . Consider some $n \geq ck+2$. As we discussed before the theorem, because of (e) to prove that $\mathrm{H}_k(G'_n; M_n) \cong \mathrm{H}_k(G_n; M_n)$ it is enough to prove that G_n acts trivially on $\mathrm{H}_k(G'_n; M_n)$. Condition (h) implies that to do this, it is enough to prove that $\mathrm{H}_k(G'_n; M_n)$ is the span of the G_n -orbit of the image of the stabilization map (5.2).

Condition (a) says that $\tilde{\mathrm{H}}_i(\mathbb{X}_n; \mathcal{M}_n) = 0$ for $-1 \leq i \leq \lfloor \frac{n-2}{c} \rfloor$. Since $n \geq ck+2$, this vanishing holds for $-1 \leq i \leq k$. Applying Theorem 5.6, we obtain a spectral sequence E_{pq}^r with the following properties:

- (i) We have $E_{pq}^1 = \tilde{\mathrm{C}}_p(\mathbb{X}_n/G'_n; \mathcal{H}_q(\mathcal{M}_n))$, and the differential $E_{pq}^1 \rightarrow E_{p-1,q}^1$ is the differential on $\tilde{\mathrm{C}}_\bullet(\mathbb{X}_n/G'_n; \mathcal{H}_q(\mathcal{M}_n))$. In particular, $E_{pq}^1 = 0$ if $p < -1$ or if $q < 0$.
- (ii) For $p+q \leq k$, we have $E_{pq}^\infty = 0$.

Unwinding the definition of $\mathcal{H}_q(\mathcal{M}_n)$, we see that for $*$ the (-1) -simplex of \mathbb{X}_n we have

$$E_{-1,k}^1 = \tilde{\mathrm{C}}_{-1}(\mathbb{X}_n/G'_n; \mathcal{H}_k(\mathcal{M}_n)) = \mathrm{H}_k((G'_n)_*; \mathcal{M}_n(*)) = \mathrm{H}_k(G'_n; M_n).$$

For each simplex τ of \mathbb{X}_n/G'_n , fix a lift $\tilde{\tau}$ to \mathbb{X}_n . We then have

$$(5.3) \quad E_{0,k}^1 = \tilde{\mathrm{C}}_0(\mathbb{X}_n/G'_n; \mathcal{H}_k(\mathcal{M}_n)) = \bigoplus_{\tau \in (\mathbb{X}_n/G'_n)^0} \mathrm{H}_k((G'_n)_{\tilde{\tau}}; \mathcal{M}_n(\tilde{\tau})).$$

By conditions (b) and (f), we have

$$\mathrm{H}_k((G'_n)_{\sigma_0}; \mathcal{M}_n(\sigma_0)) = \mathrm{H}_k(G'_{n-1}; M_{n-1}).$$

Condition (c) says that G_n acts transitively on the 0-simplices of \mathbb{X}_n . For each $\tau \in (\mathbb{X}_n/G'_n)^0$, there thus exists some $g \in G_n$ such that $g \cdot \sigma_0 = \tilde{\tau}$. We thus have

$$gG'_{n-1}g^{-1} = (G'_n)_{\tilde{\tau}} \quad \text{and} \quad g \cdot M_{n-1} = \mathcal{M}_n(\tilde{\tau}),$$

where the second equality uses the G_n -equivariant structure on \mathcal{M}_n .

From these observations, we see that the image of the differential

$$E_{0,k}^1 = \bigoplus_{\tau \in (\mathbb{X}_n/G'_n)^0} \mathrm{H}_k((G'_n)_{\tilde{\tau}}; \mathcal{M}_n(\tilde{\tau})) \rightarrow \mathrm{H}_k(G'_n; M_n) = E_{-1,k}^1$$

is the span of the G_n -orbit of the image of the stabilization map

$$\mathrm{H}_k(G'_{n-1}; M_{n-1}) \rightarrow \mathrm{H}_k(G'_n; M_n).$$

To see that $\mathrm{H}_k(G'_n; M_n)$ is the span of the G_n -orbit of the image of this stabilization map, it is therefore enough to prove that the differential $E_{0,k}^1 \rightarrow E_{-1,k}^1$ is surjective.

The only nonzero differentials that can possibly hit $E_{-1,k}^r$ for some $r \geq 1$ are

$$\begin{aligned} E_{0,k}^1 &\rightarrow E_{-1,k}^1 \\ E_{1,k-1}^2 &\rightarrow E_{-1,k}^2 \\ &\vdots \\ E_{k,0}^{k+1} &\rightarrow E_{-1,k}^{k+1}. \end{aligned}$$

By (ii) above we have $E_{-1,k}^\infty = 0$, so to prove that the differential $E_{0,k}^1 \rightarrow E_{-1,k}^1$ is surjective it is enough to prove that $E_{i,k-i}^{i+1} = 0$ for $1 \leq i \leq k$. We will actually prove the following more general result:

Claim. *Fix some $0 \leq q \leq k-1$. Then $E_{pq}^2 = 0$ for $-1 \leq p \leq k-q$.*

The terms E_{pq}^1 with $-1 \leq p \leq k-q+1$ along with the relevant E^1 -differentials are

$$\tilde{\mathcal{C}}_{-1}(\mathbb{X}_n/G'_n; \mathcal{H}_q(\mathcal{M}_n)) \leftarrow \tilde{\mathcal{C}}_0(\mathbb{X}_n/G'_n; \mathcal{H}_q(\mathcal{M}_n)) \leftarrow \cdots \leftarrow \tilde{\mathcal{C}}_{k-q+1}(\mathbb{X}_n/G'_n; \mathcal{H}_q(\mathcal{M}_n)).$$

To prove that the E_{pq}^2 with $-1 \leq p \leq k-q$ are all zero, we must prove that this chain complex is exact except possibly at its rightmost term $\tilde{\mathcal{C}}_{k-q+1}(\mathbb{X}_n/G'_n; \mathcal{H}_q(\mathcal{M}_n))$.

For a p -simplex τ of \mathbb{X}_n/G'_n , we have by definition

$$\mathcal{H}_q(\mathcal{M}_n)(\tau) = \mathrm{H}_q((G'_n)_{\tilde{\tau}}; \mathcal{M}_n(\tilde{\tau})).$$

Here $\tilde{\tau}$ is our chosen lift of τ to \mathbb{X}_n . Setting $V = \mathrm{H}_q(G_n; M_n)$, the inclusion $(G'_n)_{\tilde{\tau}} \hookrightarrow G_n$ along with the map

$$\mathcal{M}_n(\tilde{\tau}) \rightarrow \mathcal{M}_n(*) = M_n \quad \text{with } * \text{ the } (-1)\text{-simplex of } \mathbb{X}_n$$

coming from our coefficient system induce a map

$$(5.4) \quad \mathcal{H}_q(\mathcal{M}_n)(\tau) = \mathrm{H}_q((G'_n)_{\tilde{\tau}}; \mathcal{M}_n(\tilde{\tau})) \rightarrow \mathrm{H}_q(G_n; M_n) = V.$$

Letting \underline{V} be the constant coefficient system on \mathbb{X}_n/G'_n with value V , the maps (5.4) assemble to a map of coefficient systems $\mathcal{H}_q(\mathcal{M}_n) \rightarrow \underline{V}$.

Letting $f_p: \tilde{\mathcal{C}}_p(\mathbb{X}_n/G'_n; \mathcal{H}_q(\mathcal{M}_n)) \rightarrow \tilde{\mathcal{C}}_p(\mathbb{X}_n/G'_n; \underline{V})$ be the induced map on reduced chain complexes, we have a commutative diagram

$$\begin{array}{ccccccc} \tilde{\mathcal{C}}_{-1}(\mathbb{X}_n/G'_n; \mathcal{H}_q(\mathcal{M}_n)) & \leftarrow & \tilde{\mathcal{C}}_0(\mathbb{X}_n/G'_n; \mathcal{H}_q(\mathcal{M}_n)) & \leftarrow & \cdots & \leftarrow & \tilde{\mathcal{C}}_{k-q+1}(\mathbb{X}_n/G'_n; \mathcal{H}_q(\mathcal{M}_n)) \\ \downarrow f_{-1} & & \downarrow f_0 & & & & \downarrow f_{k-q+1} \\ \tilde{\mathcal{C}}_{-1}(\mathbb{X}_n/G'_n; \underline{V}) & \leftarrow & \tilde{\mathcal{C}}_0(\mathbb{X}_n/G'_n; \underline{V}) & \leftarrow & \cdots & \leftarrow & \tilde{\mathcal{C}}_{k-q+1}(\mathbb{X}_n/G'_n; \underline{V}). \end{array}$$

Condition (g) says that \mathbb{X}_n/G'_n is $\lfloor \frac{n-2}{c} \rfloor$ -connected. Since $n \geq ck + 2$, this means that \mathbb{X}_n/G'_n is k -connected. Since $0 \leq q \leq k - 1$, we deduce that the bottom chain complex of this diagram is exact except possibly at its rightmost term. To prove that the top chain complex is exact except possibly at its rightmost term, it is thus enough to prove that f_p is an isomorphism for $-1 \leq p \leq k - q$ and a surjection for $p = k - q + 1$.

Letting τ be a p -simplex of \mathbb{X}_n/G'_n , the map $\mathcal{H}_q(\mathcal{M}_n)(\tau) \rightarrow \underline{V}(\tau)$ is the map

$$(5.5) \quad \mathbb{H}_q((G'_n)_{\tilde{\tau}}; \mathcal{M}_n(\tilde{\tau})) \rightarrow \mathbb{H}_q(G_n; M_n)$$

We must prove that this is an isomorphism for $-1 \leq p \leq k - q$ and a surjection for $p = k - q + 1$. For $-1 \leq p \leq k - q + 1$, the fact that $q \geq 0$ and $c \geq 2$ imply that

$$p \leq k - q + 1 \leq k + 1 \leq ck + 2 \leq n.$$

Condition (c) thus says that for $-1 \leq p \leq k - q + 1$, the group G_n acts transitively on the p -simplices of \mathbb{X}_n . In particular, if p is in this range then the p -simplex $\tilde{\tau}$ is in the same G_n -orbit as the p -simplex σ_p from conditions (b) and (f), which satisfies

$$\mathbb{H}_q((G'_n)_{\sigma_p}; \mathcal{M}_n(\sigma_p)) = \mathbb{H}_q(G'_{n-p-1}; M_{n-p-1}).$$

Whether or not (5.5) is an isomorphism/surjection is invariant under the G_n -action, so we deduce that it is enough to prove that

$$\mathbb{H}_q(G'_{n-p-1}; M_{n-p-1}) \rightarrow \mathbb{H}_q(G_n; M_n)$$

is an isomorphism for $-1 \leq p \leq k - q$ and a surjection for $p = k - q + 1$. Factor this as

$$\mathbb{H}_q(G'_{n-p-1}; M_{n-p-1}) \rightarrow \mathbb{H}_q(G_{n-p-1}; M_{n-p-1}) \rightarrow \mathbb{H}_q(G_n; M_n).$$

Condition (e) says that M_{n-p-1} is a vector space over a field of characteristic 0. Since G'_{n-p-1} is a finite-index subgroup of G_{n-p-1} , the transfer map (see [7, Chapter III.9]) therefore implies that the first map is always a surjection, and our inductive hypothesis says that it is an isomorphism if $n - p - 1 \geq cq + 2$. Also, Theorem 5.2 says that the second map is an isomorphism if $n - p \geq cq + 2$ and a surjection if $n - p = cq + 1$. To prove the theorem, it is thus enough to prove that $n - p - 1 \geq cq + 2$ if $-1 \leq p \leq k - q$ and that $n - p \geq cq + 1$ if $p = k - q + 1$.

We check this as follows. If $-1 \leq p \leq k - q$, then since $n \geq ck + 2$ and $q \leq k - 1$ and $c \geq 2$ we have

$$\begin{aligned} n - p - 1 &\geq (ck + 2) - (k - q) - 1 = (c - 1)k + q + 1 \\ &\geq (c - 1)(q + 1) + q + 1 = cq + c \\ &\geq cq + 2. \end{aligned}$$

If instead $p = k - q + 1$, then we have

$$\begin{aligned} n - p &\geq (ck + 2) - (k - q + 1) = (c - 1)k + q + 1 \\ &\geq (c - 1)(q + 1) + q + 1 = cq + c \\ &\geq cq + 1. \end{aligned} \quad \square$$

6. STABILITY II: THE VANISHING THEOREM

Let \mathbb{X} be an ordered simplicial complex.¹⁷ To use Theorem 5.2, we need a way to prove that $\tilde{\mathbb{H}}_k(\mathbb{X}; \mathcal{F}) = 0$ in a range for an augmented coefficient system \mathcal{F} . This section contains a useful criterion for this that applies in many situations. Since we will be talking about

¹⁷The results in this section work naturally in this level of generality rather than for general semisimplicial sets.

augmented coefficient systems, we will need to discuss the (-1) -simplex, which corresponds to the *empty* ordered tuple of vertices, so we will write it as \emptyset .

6.1. Polynomiality. Our criterion applies to augmented coefficient systems \mathcal{F} that are *polynomial* of degree $d \geq -1$ up to dimension $e \geq -1$. This condition is inspired by the notion of polynomial FI-modules (Definition 1.6). It is defined inductively in d as follows:

- A coefficient system \mathcal{F} is polynomial of degree -1 up to dimension e if for all simplices¹⁸ σ of dimension at most e , we have $\mathcal{F}(\sigma) = 0$. In particular, $\mathcal{F}(\emptyset) = 0$.
- A coefficient system \mathcal{F} is polynomial of degree $d \geq 0$ up to dimension e if it satisfies the following two conditions:
 - If σ is a simplex of dimension at most e , then the map $\mathcal{F}(\sigma) \rightarrow \mathcal{F}(\emptyset)$ is injective.¹⁹
 - Let w be a vertex of \mathbb{X} . Let $D_w\mathcal{F}$ be the coefficient system on the forward link $\overrightarrow{\text{Link}}_{\mathbb{X}}(w)$ defined by the formula

$$D_w\mathcal{F}(\sigma) = \frac{\mathcal{F}(\sigma)}{\text{Im}(\mathcal{F}(w \cdot \sigma) \rightarrow \mathcal{F}(\sigma))} \quad \text{for a simplex } \sigma \text{ of } \overrightarrow{\text{Link}}_{\mathbb{X}}(w).$$

Then $D_w\mathcal{F}$ must be polynomial of degree $d - 1$ up to dimension e .

Remark 6.1. To give a sense as to what this definition means, consider the edge case where \mathcal{F} is polynomial of degree $d \geq 0$ up to dimension (-1) . Unwinding the above definition, we see that it is equivalent to requiring that for all d -simplices (v_0, \dots, v_d) , the map

$$\bigoplus_{i=0}^d \mathcal{F}(v_i) \rightarrow \mathcal{F}(\emptyset)$$

is surjective. As long as \mathbb{X} is at least d -dimensional, this implies in particular that $\tilde{H}_{-1}(\mathbb{X}; \mathcal{F}) = 0$. Our vanishing theorem below will generalize this. \square

6.2. Key example. The following lemma provides motivation for this definition.

Lemma 6.2. *Let M be an FI-module that is polynomial of degree $d \geq -1$ starting at $m \geq 0$ (see Definition 1.6). Fix²⁰ some $n \geq \max(m - 1, 0)$, and let $\mathcal{F}_{M,n}$ be the coefficient system on Sim_n discussed in Example 4.2, so*

$$\mathcal{F}_{M,n}(i_0, \dots, i_k) = M([n] \setminus \{i_0, \dots, i_k\}) \quad \text{for all simplices } (i_0, \dots, i_k) \text{ of } \text{Sim}_n.$$

Then $\mathcal{F}_{M,n}$ is polynomial of degree d up to dimension $n - m$.

Proof. The proof will be by induction on d . If $d = -1$, then consider a simplex $\sigma = (i_0, \dots, i_k)$ with $k \leq n - m$. We allow $k = -1$, in which case $\sigma = \emptyset$. We then have

$$\mathcal{F}_{M,n}(\sigma) = M([n] \setminus \{i_0, \dots, i_k\}).$$

Since

$$(6.1) \quad |[n] \setminus \{i_0, \dots, i_k\}| = (n + 1) - (k + 1) = n - k \geq m,$$

it follows from the fact that M is polynomial of degree -1 starting at m that $\mathcal{F}_{M,n}(\sigma) = 0$, as desired.

¹⁸Including the (-1) -simplex, so this has content even if $e = -1$.

¹⁹This implies that if σ is a simplex of dimension at most e and σ' is a face of σ , then the map $\mathcal{F}(\sigma) \rightarrow \mathcal{F}(\sigma')$ is injective.

²⁰The purpose of the condition $n \geq m - 1$ is simply to ensure that $n - m \geq -1$. Here $n - m$ is the dimension we are claiming that $\mathcal{F}_{M,n}$ is polynomial of degree d up to, and for this to make sense we need it to be at least -1 .

Now assume that $d \geq 0$. There are two things to check. For the first, let $\sigma = (i_0, \dots, i_k)$ be a simplex with $k \leq n - m$. We must prove that the map $\mathcal{F}_{M,n}(\sigma) \rightarrow \mathcal{F}_{M,n}(\emptyset)$ is injective, i.e., that the map

$$M([n] \setminus \{i_0, \dots, i_k\}) \rightarrow M([n])$$

is injective. The calculation (6.1) shows that this injectivity follows from the fact that M is polynomial of degree d starting at m .

For the second, let w be any vertex. Let $D_w \mathcal{F}_{M,n}$ be the coefficient system on $\overrightarrow{\text{Link}}_{\text{Sim}_n}(w)$ defined by the formula

$$D_w \mathcal{F}_{M,n}(\sigma) = \frac{\mathcal{F}_{M,n}(\sigma)}{\text{Im}(\mathcal{F}_{M,n}(w \cdot \sigma) \rightarrow \mathcal{F}_{M,n}(\sigma))} \quad \text{for a simplex } \sigma \text{ of } \overrightarrow{\text{Link}}_{\text{Sim}_n}(w).$$

We must prove that $D_w \mathcal{F}_{M,n}$ is polynomial of degree $d - 1$ up to dimension $n - m$. We have $w \in [n] = \{0, \dots, n\}$, and what we have to prove is invariant under the action of the symmetric group on $[n]$. Applying an appropriate element of this symmetric group, we can assume without loss of generality that $w = n$, so $\overrightarrow{\text{Link}}_{\text{Sim}_n}(w) = \text{Sim}_{n-1}$.

Recall that we defined the derived FI-module DM in Definition 1.4. By construction, there is an isomorphism between the coefficient systems $D_w \mathcal{F}_{M,n}$ and $\mathcal{F}_{DM,n-1}$ on Sim_{n-1} . Since M is polynomial of degree d starting at m , the FI-module DM is polynomial of degree $d - 1$ starting at $m - 1$. By induction, $D_w \mathcal{F}_{M,n}$ is polynomial of degree $d - 1$ up to dimension $(n - 1) - (m - 1) = n - m$, as desired. \square

6.3. Subcomplexes. A subcomplex of an ordered simplicial complex is also an ordered simplicial complex. Our definition behaves well under restriction:

Lemma 6.3. *Let \mathbb{X} be an ordered simplicial complex and let \mathcal{F} be an augmented coefficient system on \mathbb{X} that is polynomial of degree d up to dimension e . Let \mathbb{X}' be a subcomplex of \mathbb{X} and let \mathcal{F}' be the restriction of \mathcal{F} to \mathbb{X}' . Then \mathcal{F}' is polynomial of degree d up to dimension e .*

Proof. The proof is by induction on d . The base case $d = -1$ is trivial: if σ is a simplex of \mathbb{X}' of dimension at most e , then $\mathcal{F}'(\sigma) = \mathcal{F}(\sigma) = 0$. Assume now that $d \geq 0$ and that the result is true for all smaller degrees. We must check two conditions.

For the first, consider a simplex σ of \mathbb{X}' of dimension at most e . We must prove that the map $\mathcal{F}'(\sigma) \rightarrow \mathcal{F}'(\emptyset)$ is injective. This is immediate from the corresponding property for the map $\mathcal{F}(\sigma) \rightarrow \mathcal{F}(\emptyset)$.

For the second, let w be a vertex of \mathbb{X}' . Let $D_w \mathcal{F}'$ be the coefficient system on $\overrightarrow{\text{Link}}_{\mathbb{X}'}(w)$ defined by the formula

$$D_w \mathcal{F}'(\sigma) = \frac{\mathcal{F}'(\sigma)}{\text{Im}(\mathcal{F}'(w \cdot \sigma) \rightarrow \mathcal{F}'(\sigma))} \quad \text{for a simplex } \sigma \text{ of } \overrightarrow{\text{Link}}_{\mathbb{X}'}(w).$$

We must prove that $D_w \mathcal{F}'$ is polynomial of degree $d - 1$ up to dimension e . Let $D_w \mathcal{F}$ be the coefficient system on $\overrightarrow{\text{Link}}_{\mathbb{X}}(w)$ defined by the formula

$$D_w \mathcal{F}(\sigma) = \frac{\mathcal{F}(\sigma)}{\text{Im}(\mathcal{F}(w \cdot \sigma) \rightarrow \mathcal{F}(\sigma))} \quad \text{for a simplex } \sigma \text{ of } \overrightarrow{\text{Link}}_{\mathbb{X}}(w).$$

By definition, $D_w \mathcal{F}$ is polynomial of degree $d - 1$ up to dimension e . Since $\overrightarrow{\text{Link}}_{\mathbb{X}'}(w)$ is a subcomplex of $\overrightarrow{\text{Link}}_{\mathbb{X}}(w)$ and $D_w \mathcal{F}'$ is the restriction of $D_w \mathcal{F}$ to $\overrightarrow{\text{Link}}_{\mathbb{X}'}(w)$, the desired result follows from our inductive hypothesis. \square

6.4. Vanishing theorem. Our vanishing theorem is as follows:²¹

Theorem 6.4. *For some $N \geq -1$ and $d \geq 0$, let \mathbb{X} be an ordered simplicial complex that is weakly forward Cohen–Macaulay of dimension $N + d + 1$ and let \mathcal{F} be an augmented coefficient system on \mathbb{X} that is polynomial of degree d up to dimension N . Then $\tilde{H}_k(\mathbb{X}; \mathcal{F}) = 0$ for $-1 \leq k \leq N$.*

Proof. The proof will be by induction on d . For the base case $d = -1$, by definition the coefficient system \mathcal{F} equals the constant coefficient system $\underline{0}$ on the N -skeleton of \mathbb{X} . This trivially implies that for $-1 \leq k \leq N$ we have $\tilde{H}_k(\mathbb{X}; \mathcal{F}) = 0$.

Assume now that $d \geq 0$ and that the theorem is true for smaller d . We divide the rest of the proof into four steps. The descriptions of the steps describe the objects that are introduced during that step and what the step reduces the theorem to, with the fourth step proving the result. The inductive hypothesis and the polynomiality assumption are only invoked in the fourth step.

Step 1. *For $n \geq -1$, we introduce the shifted coefficient systems \mathcal{F}_n and reduce the theorem to proving that $\tilde{H}_k(\mathbb{X}; \mathcal{F}_{n-1}) \cong \tilde{H}_k(\mathbb{X}; \mathcal{F}_n)$ for $-1 \leq k \leq N$ and $0 \leq n \leq N + 1$.*

For a simplex $\sigma = (v_0, \dots, v_k)$ of \mathbb{X} and $n \geq -1$, define

$$\sigma_{>n} = \begin{cases} (v_{n+1}, \dots, v_k) & \text{if } k > n \\ \emptyset & \text{if } k \leq n. \end{cases}$$

Observe that if σ' is a face of σ , then $\sigma'_{>n}$ is a face of $\sigma_{>n}$. We can thus define an augmented coefficient system \mathcal{F}_n on \mathbb{X} via the formula

$$\mathcal{F}_n(\sigma) = \mathcal{F}(\sigma_{>n}) \quad \text{for a simplex } \sigma \text{ of } \mathbb{X}.$$

The maps $\mathcal{F}_n(\sigma) \rightarrow \mathcal{F}_n(\sigma')$ when σ' is a face of σ are the ones induced by \mathcal{F} . The augmented coefficient system \mathcal{F}_{N+1} equals the constant coefficient system $\underline{\mathcal{F}(\emptyset)}$ on all simplices of dimension at most $(N + 1)$. The fact that \mathbb{X} is weakly forward Cohen–Macaulay of dimension $(N + d + 1)$ implies that \mathbb{X} is $N + d \geq N$ connected and hence

$$\tilde{H}_k(\mathbb{X}; \mathcal{F}_{N+1}) = \tilde{H}_k(\mathbb{X}; \underline{\mathcal{F}(\emptyset)}) = 0 \quad \text{for } -1 \leq k \leq N.$$

We want to prove a similar equality with \mathcal{F}_{N+1} replaced with \mathcal{F} . We can fit \mathcal{F} and \mathcal{F}_{N+1} into a chain of maps

$$\mathcal{F} = \mathcal{F}_{-1} \rightarrow \mathcal{F}_0 \rightarrow \mathcal{F}_1 \rightarrow \dots \rightarrow \mathcal{F}_{N+1}.$$

To prove the theorem, it is enough to prove that the map $\mathcal{F}_{n-1} \rightarrow \mathcal{F}_n$ induces an isomorphism $\tilde{H}_k(\mathbb{X}; \mathcal{F}_{n-1}) \cong \tilde{H}_k(\mathbb{X}; \mathcal{F}_n)$ for $0 \leq n \leq N + 1$ and $-1 \leq k \leq N$.

Step 2. *Fix some $0 \leq n \leq N + 1$. We introduce the quotiented shifted coefficient systems $\overline{\mathcal{F}}_n$ and reduce the theorem to proving that $\tilde{H}_k(\mathbb{X}; \overline{\mathcal{F}}_n) = 0$ for $n \leq k \leq N + 1$.*

Define

$$\begin{aligned} \overline{\mathcal{F}}_n &= \text{coker}(\mathcal{F}_{n-1} \rightarrow \mathcal{F}_n), \\ \mathcal{F}'_{n-1} &= \text{Im}(\mathcal{F}_{n-1} \rightarrow \mathcal{F}_n), \\ \mathcal{F}''_{n-1} &= \text{ker}(\mathcal{F}_{n-1} \rightarrow \mathcal{F}_n). \end{aligned}$$

²¹To connect this to Remark 6.1, note that this remark deals with the case $N = -1$. In that case, Theorem 6.4 requires \mathbb{X} to be weakly forward Cohen–Macaulay of dimension d , and asserts that then $\tilde{H}_{-1}(\mathbb{X}; \mathcal{F}) = 0$ if \mathcal{F} is polynomial of degree d up to dimension (-1) . Using our convention that a space is (-1) -connected precisely when it is nonempty, the assumption that \mathbb{X} is weakly forward Cohen–Macaulay of dimension d implies that \mathbb{X} is at least d -dimensional (c.f. Remark 2.2), as required in Remark 6.1 for $\tilde{H}_{-1}(\mathbb{X}; \mathcal{F}) = 0$.

We thus have short exact sequences of coefficient systems

$$(6.2) \quad 0 \longrightarrow \mathcal{F}'_{n-1} \longrightarrow \mathcal{F}_n \longrightarrow \overline{\mathcal{F}}_n \longrightarrow 0$$

and

$$(6.3) \quad 0 \longrightarrow \mathcal{F}''_{n-1} \longrightarrow \mathcal{F}_{n-1} \longrightarrow \mathcal{F}'_{n-1} \longrightarrow 0.$$

Both of these induce long exact sequences in homology. Let us focus first on the one associated to (6.3), which contains segments of the form

$$(6.4) \quad \tilde{\mathbb{H}}_k(\mathbb{X}; \mathcal{F}''_{n-1}) \longrightarrow \tilde{\mathbb{H}}_k(\mathbb{X}; \mathcal{F}_{n-1}) \longrightarrow \tilde{\mathbb{H}}_k(\mathbb{X}; \mathcal{F}'_{n-1}) \longrightarrow \tilde{\mathbb{H}}_{k-1}(\mathbb{X}; \mathcal{F}''_{n-1}).$$

Since \mathcal{F} is polynomial of degree d up to dimension N , for all simplices σ of dimension at most N and all faces σ' of σ the map $\mathcal{F}(\sigma) \rightarrow \mathcal{F}(\sigma')$ is injective. This implies that the map $\mathcal{F}_{n-1}(\sigma) \rightarrow \mathcal{F}_n(\sigma)$ is injective as long as σ has dimension at most N , and thus that $\mathcal{F}''_{n-1}(\sigma) = 0$. It follows that $\tilde{\mathbb{H}}_k(\mathbb{X}; \mathcal{F}''_{n-1}) = 0$ for $k \leq N$. Combining this with (6.4), we see that

$$(6.5) \quad \tilde{\mathbb{H}}_k(\mathbb{X}; \mathcal{F}_{n-1}) \cong \tilde{\mathbb{H}}_k(\mathbb{X}; \mathcal{F}'_{n-1}) \quad \text{for } -1 \leq k \leq N.$$

We now turn to the long exact sequence associated to (6.2), which contains the segment

$$\tilde{\mathbb{H}}_{k+1}(\mathbb{X}; \overline{\mathcal{F}}_n) \longrightarrow \tilde{\mathbb{H}}_k(\mathbb{X}; \mathcal{F}'_{n-1}) \longrightarrow \tilde{\mathbb{H}}_k(\mathbb{X}; \mathcal{F}_n) \longrightarrow \tilde{\mathbb{H}}_k(\mathbb{X}; \overline{\mathcal{F}}_n).$$

In light of (6.5), this implies that to prove that the map $\mathcal{F}_{n-1} \rightarrow \mathcal{F}_n$ induces an isomorphism $\tilde{\mathbb{H}}_k(\mathbb{X}; \mathcal{F}_{n-1}) \cong \tilde{\mathbb{H}}_k(\mathbb{X}; \mathcal{F}_n)$ for $-1 \leq k \leq N$, it is enough to prove that

$$\tilde{\mathbb{H}}_k(\mathbb{X}; \overline{\mathcal{F}}_n) = 0 \quad \text{for } -1 \leq k \leq N+1.$$

This can be simplified a little further: if σ is a k -simplex with $k \leq n-1$, then $\mathcal{F}_{n-1}(\sigma) = \mathcal{F}_n(\sigma) = \mathcal{F}(\emptyset)$, so $\overline{\mathcal{F}}_n(\sigma) = 0$. The group $\tilde{\mathbb{H}}_k(\mathbb{X}; \overline{\mathcal{F}}_n)$ is thus automatically 0 for $k \leq n-1$, so it is actually enough to prove that

$$\tilde{\mathbb{H}}_k(\mathbb{X}; \overline{\mathcal{F}}_n) = 0 \quad \text{for } n \leq k \leq N+1.$$

Step 3. Fix some $0 \leq n \leq N+1$. For each n -simplex τ , we construct a coefficient system $\overline{\mathcal{F}}_{n,\tau}$ and prove that $\overline{\mathcal{F}}_n$ is the direct sum of the $\overline{\mathcal{F}}_{n,\tau}$ as τ ranges over the n -simplices, reducing the theorem to proving that $\tilde{\mathbb{H}}_k(\mathbb{X}; \overline{\mathcal{F}}_{n,\tau}) = 0$ for $n \leq k \leq N+1$.

Consider a k -simplex σ of \mathbb{X} with $k \geq n$. We claim that the following holds:

- Let σ' be a face of σ . Then the map $\overline{\mathcal{F}}_n(\sigma) \rightarrow \overline{\mathcal{F}}_n(\sigma')$ is the zero map unless σ' also has dimension at least n and the first $(n+1)$ vertices of σ and σ' are equal.

To prove this, it is enough to prove the following special case:

- Let $(w_0, \dots, w_n, v_0, \dots, v_m)$ be a simplex of \mathbb{X} and let $0 \leq i \leq n$. Then the map $\overline{\mathcal{F}}_n(w_0, \dots, w_n, v_0, \dots, v_m) \rightarrow \overline{\mathcal{F}}_n(w_0, \dots, \widehat{w}_i, \dots, w_n, v_0, \dots, v_m)$ is the zero map.

For this, observe that

$$\overline{\mathcal{F}}_n(w_0, \dots, w_n, v_0, \dots, v_m) = \frac{\mathcal{F}(v_0, \dots, v_m)}{\text{Im}(\mathcal{F}(w_n, v_0, \dots, v_m) \rightarrow \mathcal{F}(v_0, \dots, v_m))}$$

and

$$\overline{\mathcal{F}}_n(w_0, \dots, \widehat{w}_i, \dots, w_n, v_0, \dots, v_m) = \frac{\mathcal{F}(v_1, \dots, v_m)}{\text{Im}(\mathcal{F}(v_0, \dots, v_m) \rightarrow \mathcal{F}(v_1, \dots, v_m))}.$$

The map between these is visibly the zero map.

For each n -simplex τ of \mathbb{X} , this suggests defining a coefficient system $\overline{\mathcal{F}}_{n,\tau}$ on \mathbb{X} via the formula

$$\overline{\mathcal{F}}_{n,\tau}(\sigma) = \begin{cases} \overline{\mathcal{F}}_n(\sigma) & \text{if } \sigma \text{ starts with } \tau \\ 0 & \text{otherwise} \end{cases} \quad \text{for a simplex } \sigma \text{ of } \mathbb{X}.$$

By the above, we have a decomposition

$$\overline{\mathcal{F}}_n = \bigoplus_{\tau \in \mathbb{X}^n} \overline{\mathcal{F}}_{n,\tau}$$

of coefficient systems. To prove that

$$\tilde{\mathbb{H}}_k(\mathbb{X}; \overline{\mathcal{F}}_n) = 0 \quad \text{for } n \leq k \leq N + 1,$$

it is thus enough to prove that for all n -simplices τ we have

$$\tilde{\mathbb{H}}_k(\mathbb{X}; \overline{\mathcal{F}}_{n,\tau}) = 0 \quad \text{for } n \leq k \leq N + 1.$$

Step 4. Fix an n -simplex τ with $0 \leq n \leq N + 1$. We prove that $\tilde{\mathbb{H}}_k(\mathbb{X}; \overline{\mathcal{F}}_{n,\tau}) = 0$ for $n \leq k \leq N + 1$.

Define an augmented coefficient system \mathcal{H} on $\overrightarrow{\text{Link}}_{\mathbb{X}}(\tau)$ via the formula

$$\mathcal{H}(\sigma) = \overline{\mathcal{F}}_{n,\tau}(\tau \cdot \sigma) \quad \text{for a simplex } \sigma \text{ of } \overrightarrow{\text{Link}}_{\mathbb{X}}(\tau).$$

On the level of reduced chain complexes, up to multiplying the differentials by $(-1)^{n+1}$ we have

$$\tilde{\mathbb{C}}_{\bullet}(\overrightarrow{\text{Link}}_{\mathbb{X}}(\tau); \mathcal{H}) \cong \tilde{\mathbb{C}}_{\bullet+n+1}(\mathbb{X}; \overline{\mathcal{F}}_{n,\tau}),$$

so $\tilde{\mathbb{H}}_k(\overrightarrow{\text{Link}}_{\mathbb{X}}(\tau); \mathcal{H}) \cong \tilde{\mathbb{H}}_{k+n+1}(\mathbb{X}; \overline{\mathcal{F}}_{n,\tau})$. It is thus enough to prove that $\tilde{\mathbb{H}}_k(\overrightarrow{\text{Link}}_{\mathbb{X}}(\tau); \mathcal{H}) = 0$ for $-1 \leq k \leq N - n$.

Write $\tau = (w_0, \dots, w_n)$. For a simplex σ of $\overrightarrow{\text{Link}}_{\mathbb{X}}(\tau)$, we have

$$\mathcal{H}(\sigma) = \overline{\mathcal{F}}_{n,\tau}((w_0, \dots, w_n) \cdot \sigma) = \frac{\mathcal{F}(\sigma)}{\text{Im}(\mathcal{F}(w_n \cdot \sigma) \rightarrow \mathcal{F}(\sigma))}.$$

Let $D_{w_n}\mathcal{F}$ be the coefficient system on $\overrightarrow{\text{Link}}_{\mathbb{X}}(w_n)$ defined via the formula

$$D_{w_n}\mathcal{F}(\sigma) = \frac{\mathcal{F}(\sigma)}{\text{Im}(\mathcal{F}(w_n \cdot \sigma) \rightarrow \mathcal{F}(\sigma))} \quad \text{for a simplex } \sigma \text{ of } \overrightarrow{\text{Link}}_{\mathbb{X}}(w_n).$$

By definition, $D_{w_n}\mathcal{F}$ is polynomial of degree $(d-1)$ up to dimension N . Since \mathcal{H} is the restriction of $D_{w_n}\mathcal{F}$ to $\overrightarrow{\text{Link}}_{\mathbb{X}}(\tau)$, Lemma 6.3 implies that \mathcal{H} is also polynomial of degree $(d-1)$ up to dimension N . In fact, all we will need is the weaker statement that it is polynomial of degree $(d-1)$ up to dimension $N - n$.

Since \mathbb{X} is weakly forward Cohen–Macaulay of dimension $N + d + 1$, the forward link $\overrightarrow{\text{Link}}_{\mathbb{X}}(\tau)$ of the n -simplex τ is weakly forward Cohen–Macaulay of dimension²² $N + d - n$. Since

$$N + d - n = (N - n) + (d - 1) + 1,$$

we can apply our inductive hypothesis to conclude that $\tilde{\mathbb{H}}_k(\overrightarrow{\text{Link}}_{\mathbb{X}}(\tau); \mathcal{H}) = 0$ for $-1 \leq k \leq N - n$, as desired. \square

²²This uses the observation that if τ is a simplex of \mathbb{X} and τ' is a simplex of $\overrightarrow{\text{Link}}_{\mathbb{X}}(\tau)$, then $\overrightarrow{\text{Link}}_{\overrightarrow{\text{Link}}_{\mathbb{X}}(\tau)}(\tau') = \overrightarrow{\text{Link}}_{\mathbb{X}}(\tau \cdot \tau')$.

7. STABILITY FOR SYMMETRIC GROUPS

We now turn to applications of our machinery, starting with Theorems A and A'.

Proof of Theorem A. We first recall the statement. Let \mathbb{k} be a commutative ring and let M be an FI-module over \mathbb{k} that is polynomial of degree $d \geq -1$ starting at $m \geq 0$. For each $k \geq 0$, we must prove that the map

$$H_k(\mathfrak{S}_n; M(\bar{n})) \rightarrow H_k(\mathfrak{S}_{n+1}; M(\overline{n+1}))$$

is an isomorphism for $n \geq 2k + \max(d, m) + 1$ and a surjection for $n = 2k + \max(d, m)$. This is trivial for $d = -1$ since in that case these homology groups are 0 in the indicated range, so we can assume that $d \geq 0$.

Recall that $\bar{n} = \{1, \dots, n\}$ and $[n-1] = \{0, \dots, n-1\}$. The group \mathfrak{S}_n acts on both $M(\bar{n})$ and $M([n-1])$, and there is a $\mathbb{k}[\mathfrak{S}_n]$ -module isomorphism $M(\bar{n}) \cong M([n-1])$. In light of this, it is enough to deal with the map

$$H_k(\mathfrak{S}_n; M([n-1])) \rightarrow H_k(\mathfrak{S}_{n+1}; M([n])),$$

which will fit into our framework a little better.

The group \mathfrak{S}_n acts on Sim_{n-1} . Let $\mathcal{F}_{M,n-1}$ be the \mathfrak{S}_n -equivariant augmented system of coefficients on Sim_{n-1} from Example 4.3, so

$$\mathcal{F}_{M,n-1}(i_0, \dots, i_k) = M([n-1] \setminus \{i_0, \dots, i_k\}) \quad \text{for a simplex } (i_0, \dots, i_k) \text{ of } \text{Sim}_{n-1}.$$

The following claim will be used to show that with an appropriate degree shift, this all satisfies the hypotheses of Theorem 5.2.

Claim. *Assume that $n \geq \max(m, 1)$, so we can apply Lemma 6.2 to the coefficient system $\mathcal{F}_{M,n-1}$ on Sim_{n-1} . The following then hold:*

- (A) *For all $-1 \leq k \leq n - \max(d, m - 1) - 2$, we have $\tilde{H}_k(\text{Sim}_{n-1}; \mathcal{F}_{M,n-1}) = 0$.*
- (B) *For all $-1 \leq k \leq n - 1$, the group \mathfrak{S}_{n-k-1} is the \mathfrak{S}_n -stabilizer of a k -simplex σ_k of Sim_{n-1} with $\mathcal{F}_{M,n-1}(\sigma_k) = M([n-k-2])$.*
- (C) *For all $0 \leq k \leq n - 1$, the group \mathfrak{S}_n acts transitively on the k -simplices of Sim_{n-1} .*
- (D) *For all $n \geq 2$ and all 1-simplices e of Sim_{n-1} of the form $e = (i_0, i_1)$, there exists some $\lambda \in \mathfrak{S}_n$ with $\lambda(i_0) = i_1$ such that λ commutes with all elements of $(\mathfrak{S}_n)_e$ and fixes all elements of $\mathcal{F}_{M,n-1}(e)$.*

Proof of claim. For (A), Lemma 6.2 says that $\mathcal{F}_{M,n-1}$ is a polynomial coefficient system of degree d up to dimension $n - m - 1$. Also, Sim_{n-1} is the large ordering of the $(n-1)$ -simplex Sim_{n-1} and Sim_{n-1} is weakly Cohen–Macaulay of dimension $(n-1)$, so by Lemma 3.11 the ordered simplicial complex Sim_{n-1} is weakly forward Cohen–Macaulay of dimension $(n-1)$. Letting

$$N = \min(n - m - 1, n - d - 2) = n - \max(d, m - 1) - 2,$$

the above shows that Sim_{n-1} is weakly forward Cohen–Macaulay of dimension $N + d + 1$ and $\mathcal{F}_{M,n-1}$ is a polynomial coefficient system of degree d up to dimension N . Theorem 6.4 thus implies that $\tilde{H}_k(\text{Sim}_{n-1}; \mathcal{F}_{M,n-1}) = 0$ for $-1 \leq k \leq N$.

For (B), the group \mathfrak{S}_{n-k-1} is the \mathfrak{S}_n -stabilizer of the k -simplex

$$\sigma_k = \begin{cases} (\emptyset) & \text{if } k = -1, \\ (n-k-1, n-k+1, \dots, n-1) & \text{if } 0 \leq k \leq n-1 \end{cases}$$

of Sim_{n-1} , and by definition

$$\mathcal{F}_{M,n-1}(\sigma_k) = M([n-1] \setminus \{n-k-1, n-k+1, \dots, n-1\}) = M([n-k-2]).$$

Condition (C) is obvious.

For (D), simply let λ be the transposition (i_0, i_1) , which acts trivially on

$$\mathcal{F}_{M,n-1}(e) = M([n-1] \setminus \{i_0, i_1\})$$

by basic properties of FI-modules. \square

Let $e = \max(d, m)$, so since $m \geq 0$ we have $e \geq 0$. For $n \geq 0$ define

$$G_n = \mathfrak{S}_{n+e} \quad \text{and} \quad M_n = M([n+e-1])$$

and for $n \geq 1$ define

$$\mathbb{X}_n = \text{Sim}_{n+e-1} \quad \text{and} \quad \mathcal{M}_n = \mathcal{F}_{M,n+e-1}.$$

This makes sense since $n+e \geq n$.²³ We then have

$$G_0 \subset G_1 \subset G_2 \subset \cdots,$$

and since M is a polynomial FI-module of degree d starting at $m \geq 0$ the maps $M_n \rightarrow M_{n+1}$ are injective for $n \geq 0$,²⁴ so we have an increasing sequence

$$M_0 \subset M_1 \subset M_2 \subset \cdots.$$

In other words, the pair $\{(G_n, M_n)\}_{n=0}^\infty$ is an increasing sequence of groups and modules in the sense of §5.2.

The above claim verifies the conditions of Theorem 5.2 with $c = 2$. The shift by e is needed for condition (a) of Theorem 5.2, which requires that $\tilde{H}_k(\mathbb{X}_n; \mathcal{M}_n) = 0$ for all $n \geq 1$ and $-1 \leq k \leq \lfloor \frac{n-2}{2} \rfloor$. Conclusion (A) of the Claim says that $\tilde{H}_k(\text{Sim}_{n+e-1}; \mathcal{F}_{M,n+e-1}) = 0$ for $n+e \geq \max(m, 1)$ ²⁵ and $-1 \leq k \leq (n+e) - \max(d, m-1) - 2$, which implies the desired range of vanishing for $\tilde{H}_k(\mathbb{X}_n; \mathcal{M}_n)$ since

$$(n+e) - \max(d, m-1) - 2 = n + \max(d, m) - \max(d, m-1) - 2 \geq n - 2 \geq \left\lfloor \frac{n-2}{2} \right\rfloor$$

for $n \geq 1$. Applying Theorem 5.2, we deduce that the map

$$\mathrm{H}_k(\mathfrak{S}_{n+e-1}; M([n+e-2])) \rightarrow \mathrm{H}_k(\mathfrak{S}_{n+e}; M([n+e-1]))$$

is an isomorphism for $n \geq 2k+2$ and a surjection for $n = 2k+1$, which implies that

$$\mathrm{H}_k(\mathfrak{S}_n; M([n-1])) \rightarrow \mathrm{H}_k(\mathfrak{S}_{n+1}; M([n]))$$

is an isomorphism for $n \geq 2k+e+1$ and a surjection for $n = 2k+e$. \square

Proof of Theorem A'. We first recall the statement. Let \mathbb{k} be a commutative ring and let M be an FI-module over \mathbb{k} that is polynomial of degree $d \geq -1$ starting at $m \geq 0$. For each $k \geq 0$, we must prove that the map

$$(7.1) \quad \mathrm{H}_k(\mathfrak{S}_n; M(\bar{n})) \rightarrow \mathrm{H}_k(\mathfrak{S}_{n+1}; M(\overline{n+1}))$$

is an isomorphism for $n \geq \max(m, 2k+2d+2)$ and a surjection for $n \geq \max(m, 2k+2d)$.

The proof will be by double induction on d and m . There are two base cases:

- The first is where $m = 0$ and $d \geq 0$. Theorem A says in this case that (7.1) is an isomorphism for

$$n \geq 2k + \max(d, m) + 1 = 2k + \max(d, 0) + 1 = 2k + d + 1$$

and a surjection for

$$n = 2k + \max(d, m) = 2k + \max(d, 0) = 2k + d.$$

²³Note that if $n+e = 0$ then $[n+e-1] = [-1] = \emptyset$, which can be plugged into the FI-module M .

²⁴Remember, $[n+e-1]$ has $n+e$ elements, and for $n \geq 0$ we have $n+e \geq e = \max(d, m) \geq m$.

²⁵Which holds for $n \geq 1$ since then $n+e = n + \max(d, m) \geq 1 + m \geq \max(m, 1)$. Here we are using the fact that $m \geq 0$.

Since $d \geq 0$, these bounds are even stronger than our purported bounds of

$$n \geq \max(m, 2k + 2d + 2) = \max(0, 2k + 2d + 2) = 2k + 2d + 2$$

for (7.1) to be an isomorphism and

$$n \geq \max(m, 2k + 2d) = \max(0, 2k + 2d) = 2k + 2d$$

for (7.1) to be a surjection.

- The second is where $m \geq 0$ and $d = -1$. In this case, by the definition of an FI-module being polynomial of degree -1 starting at m we have for $n \geq m$ that $M(\bar{n}) = 0$ and hence $H_k(\mathfrak{S}_n; M(\bar{n})) = 0$. In other words, for $n \geq m$ the domain and codomain of (7.1) are both 0, so it is trivially an isomorphism.

Assume now that $m \geq 1$ and $d \geq 0$, and that the theorem is true for all such (m', d') with $m' \leq m$ and $d' \leq d$ such that either $m' < m$ or $d' < d$ (or both). As in Definition 1.4, let ΣM be the shifted FI-module and DM be the derived FI-module. For $n \geq m$, we have a short exact sequence

$$(7.2) \quad 0 \longrightarrow M(\bar{n}) \longrightarrow \Sigma M(\bar{n}) \longrightarrow DM(\bar{n}) \longrightarrow 0$$

of $\mathbb{k}[\mathfrak{S}_n]$ -modules (see Remark 1.7). The FI-module ΣM is polynomial of degree d starting at $(m - 1)$, and the FI-module DM is polynomial of degree $(d - 1)$ starting at $(m - 1)$.

To simplify our notation, for all $r \geq 0$ and all $\mathbb{k}[\mathfrak{S}_r]$ -modules N , we will denote $H_k(\mathfrak{S}_r; N)$ by $H_k(N)$. The long exact sequence in \mathfrak{S}_n -homology associated to (7.2) maps to the one in \mathfrak{S}_{n+1} -homology, so for $n \geq m$ and all k we have a commutative diagram

$$\begin{array}{ccccccccc} H_{k+1}(\Sigma M(\bar{n})) & \rightarrow & H_{k+1}(DM(\bar{n})) & \rightarrow & H_k(M(\bar{n})) & \rightarrow & H_k(\Sigma M(\bar{n})) & \rightarrow & H_k(DM(\bar{n})) \\ \downarrow g_1 & & \downarrow g_2 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 \\ H_{k+1}(\Sigma M(\overline{n+1})) & \rightarrow & H_{k+1}(DM(\overline{n+1})) & \rightarrow & H_k(M(\overline{n+1})) & \rightarrow & H_k(\Sigma M(\overline{n+1})) & \rightarrow & H_k(DM(\overline{n+1})) \end{array}$$

with exact rows. Our inductive hypothesis says the following about the g_i and f_i :

- Since ΣM is polynomial of degree d starting at $(m - 1)$, the map f_2 is an isomorphism for $n \geq \max(m - 1, 2k + 2d + 2)$ and a surjection for $n \geq \max(m - 1, 2k + 2d)$. Also, the map g_1 is an isomorphism for

$$n \geq \max(m - 1, 2(k + 1) + 2d + 2) = \max(m - 1, 2k + 2d + 4)$$

and a surjection for $n \geq \max(m - 1, 2k + 2d + 2)$.

- Since DM is polynomial of degree $(d - 1)$ starting at $(m - 1)$, the map f_3 is an isomorphism for

$$n \geq \max(m - 1, 2k + 2(d - 1) + 2) = \max(m - 1, 2k + 2d)$$

and a surjection for $n \geq \max(m - 1, 2k + 2d - 2)$. Also, the map g_2 is an isomorphism for

$$n \geq \max(m - 1, 2(k + 1) + 2(d - 1) + 2) = \max(m - 1, 2k + 2d + 2)$$

and a surjection for $n \geq \max(m - 1, 2k + 2d)$.

For $n \geq \max(m, 2k + 2d + 2)$, the maps g_2 and f_2 and f_3 are isomorphisms and the map g_1 is a surjection, so by the five-lemma the map f_1 is an isomorphism. For $n \geq \max(m, 2k + 2d)$, the maps g_2 and f_2 are surjections and the map f_3 is an isomorphism, so by the five-lemma²⁶ the map f_1 is a surjection. The claim follows. \square

²⁶Or, more precisely, one of the four-lemmas.

8. THE STABLE RANK OF RINGS

The rest of the paper is devoted to the general linear group and its congruence subgroups. We begin with a discussion of a ring-theoretic condition called the stable rank that was introduced by Bass [2]. To make this paper a bit more self-contained,²⁷ we will include the proofs of the results we need (most of which are due to Bass) if they are short.

8.1. Unimodular vectors. The starting point is the following.

Definition 8.1. Let R be a ring. A vector $v \in R^n$ is *unimodular* if there is a homomorphism $\phi: R^n \rightarrow R$ of right R -modules such that $\phi(v) = 1$. \square

Example 8.2. The columns of matrices in $\mathrm{GL}_n(R)$ are unimodular. \square

The following lemma might clarify this definition.

Lemma 8.3. Let R be a ring and let $v = (c_1, \dots, c_n) \in R^n$. Then v is unimodular if and only if there exist $a_1, \dots, a_n \in R$ such that $a_1c_1 + \dots + a_nc_n = 1$.

Proof. Immediate from the fact that all right R -module morphisms $\phi: R^n \rightarrow R$ are of the form $\phi(x_1, \dots, x_n) = a_1x_1 + \dots + a_nx_n$ for some $a_1, \dots, a_n \in R$. \square

8.2. Stable rank. In light of Example 8.2, one might hope that all unimodular vectors in R^n can appear as columns of matrices in $\mathrm{GL}_n(R)$. This holds if R is a field or if $R = \mathbb{Z}$, but does not hold in general. To clarify this, we make the following definition.

Definition 8.4. A ring R satisfies *Bass's stable rank condition* (SR_r) if the following holds for all $n \geq r$. Let $(c_1, \dots, c_n) \in R^n$ be a unimodular vector. Then there exist $b_1, \dots, b_{n-1} \in R$ such that $(c_1 + b_1c_n, \dots, c_{n-1} + b_{n-1}c_n) \in R^{n-1}$ is unimodular. \square

Example 8.5. This condition only makes sense for $r \geq 2$. It is easy to see that fields satisfy (SR_2) and that PIDs satisfy (SR_3) . More generally, if a ring R is finitely generated as a module over a Noetherian commutative ring S of Krull dimension r , then R satisfies (SR_{r+2}) (see [2, Theorem V.3.5]). \square

8.3. Elementary matrices and their action on unimodular vectors. Recall that an *elementary matrix* in $\mathrm{GL}_n(R)$ is a matrix that differs from the identity matrix at exactly one off-diagonal position. Let $\mathrm{EL}_n(R)$ be the subgroup of $\mathrm{GL}_n(R)$ generated by elementary matrices. One of the key properties of rings satisfying Bass's stable rank condition is the following lemma, which implies in particular that under its hypotheses all unimodular vectors appear as columns of matrices in $\mathrm{GL}_n(R)$. In fact, they even appear as columns of matrices in $\mathrm{EL}_n(R)$.

Lemma 8.6. Let R be a ring satisfying (SR_r) and let $n \geq r$. Then $\mathrm{EL}_n(R)$ acts transitively on the set of unimodular vectors in R^n .

Proof. Let $v \in R^n$ be a unimodular vector. It is enough to find some $M \in \mathrm{EL}_n(R)$ such that $M \cdot v = (0, \dots, 0, 1)$. Write $v = (c_1, \dots, c_n)$. Using elementary matrices, for any distinct $1 \leq i, j \leq n$ and any $\lambda \in R$ we can add λ times the i^{th} entry to the j^{th} one, changing the entry c_j to $c_j + \lambda c_i$. We will use a series of these moves to transform v into $(0, \dots, 0, 1)$.

Applying (SR_r) , we can add multiples of the last entry to the other ones to ensure that the first $(n-1)$ entries of v form a unimodular vector in R^{n-1} . Using Lemma 8.3, we can find $a_1, \dots, a_{n-1} \in R$ such that $a_1c_1 + \dots + a_{n-1}c_{n-1} = 1$. For each $1 \leq i \leq n-1$, add $(1 - c_n)a_i$ times the i^{th} entry to the n^{th} one. This transforms v into $(c_1, \dots, c_{n-1}, 1)$. Finally, add appropriate multiples of the last entry of v to the other ones to transform it into $(0, \dots, 0, 1)$. \square

²⁷And to avoid depending on [2], which is out of print.

8.4. Generation by elementary matrices. If \mathbb{k} is a field then $\mathrm{EL}_n(\mathbb{k}) = \mathrm{SL}_n(\mathbb{k})$, so $\mathrm{GL}_n(\mathbb{k})$ is generated by $\mathrm{EL}_n(\mathbb{k})$ and $\mathrm{GL}_1(\mathbb{k})$, which is embedded in $\mathrm{GL}_n(\mathbb{k})$ via the upper left hand corner matrix embedding. The following lemma shows that the stable rank condition implies a similar type of result:

Lemma 8.7. *Let R be a ring satisfying (SR_r) and let $n \geq r - 1$. Then $\mathrm{GL}_n(R)$ is generated by $\mathrm{EL}_n(R)$ and $\mathrm{GL}_{r-1}(R) \subset \mathrm{GL}_n(R)$.*

Proof. The proof will be by induction on n . The base case $n = r - 1$ is trivial, so assume that $n \geq r$ and that the lemma is true for smaller n . Let $\{v_1, \dots, v_n\}$ be the standard basis for the right R -module R^n and let $C = \bigoplus_{i=1}^{n-1} v_i \cdot R$. Consider some $M \in \mathrm{GL}_n(R)$. Applying Lemma 8.8 below, we can find some $N \in \mathrm{EL}_n(R)$ such that $N \cdot (M \cdot v_n) = v_n$ and $N \cdot (M \cdot C) = C$. It follows that $NM \in \mathrm{GL}_{n-1}(R)$, so by induction NM lies in the subgroup generated by elementary matrices and $\mathrm{GL}_{r-1}(R)$. We conclude that M does as well. \square

The above proof used the following lemma, which refines Lemma 8.6.

Lemma 8.8. *Let R be a ring satisfying (SR_r) and let $n \geq r$. Let $x, y \in R^n$ be unimodular vectors and $C, D \subset R^n$ be R -submodules such that $R^n = C \oplus x \cdot R$ and $R^n = D \oplus y \cdot R$. Then there exists some $M \in \mathrm{EL}_n(R)$ such that $M \cdot x = y$ and $M \cdot C = D$.*

Proof. Let $\{v_1, \dots, v_n\}$ be the standard basis for the right R -module R^n . It is enough to deal with the case where $x = v_n$ and $C = \bigoplus_{i=1}^{n-1} v_i \cdot R$. What is more, Lemma 8.6 says that $\mathrm{EL}_n(R)$ acts transitively on unimodular vectors in R^n , so we can assume without loss of generality that $y = v_n$ as well. Let $\rho: R^n \rightarrow R$ be the projection with $D = \ker(\rho)$ and $\rho(v_n) = 1$. For $1 \leq i \leq n - 1$, let $\lambda_i = \rho(v_i)$. Define $M \in \mathrm{GL}_n(R)$ via the formula

$$M \cdot v_i = v_i - v_n \cdot \lambda_i \quad \text{for } 1 \leq i \leq n - 1, \text{ and } M \cdot v_n = v_n.$$

It is easy to see that M can be written as a product of $(n - 1)$ elementary matrices, so $M \in \mathrm{EL}_n(R)$. For $1 \leq i \leq n - 1$, we have

$$\rho(M \cdot v_i) = \rho(v_i) - \rho(v_n)\lambda_i = \lambda_i - \lambda_i = 0,$$

so $M \cdot v_i \in D$. We conclude that $M \cdot C = D$. \square

8.5. Stable freeness. For a general ring R , there can exist non-free R -modules C that are stably free in the sense that $C \oplus R^k \cong R^n$ for some $k \geq 1$. The stable rank condition prevents this, at least if $n - k$ (which should be thought of as the ‘‘rank’’ of C) is large enough:

Lemma 8.9. *Let R be a ring satisfying (SR_r) . Let C be a right R -module such that $C \oplus R^k \cong R^n$. Assume that $n - k \geq r - 1$. Then $C \cong R^{n-k}$.*

Proof. It is enough to deal with the case where $k = 1$, so $n \geq r$. Identify C with a submodule of R^n and let $x \in R^n$ be a unimodular vector with $R^n = C \oplus x \cdot R$. Letting $\{v_1, \dots, v_n\}$ be the standard basis for R^n , Lemma 8.8 implies that there exists some $M \in \mathrm{GL}_n(R)$ with $M \cdot x = v_n$ and $M \cdot C = \bigoplus_{i=1}^{n-1} v_i \cdot R$. In particular, $M \cdot C \cong R^{n-1}$, so $C \cong R^{n-1}$ as well. \square

8.6. Quotients of rings. The following lemma shows that the stable rank condition is preserved by quotients:

Lemma 8.10. *Let R be a ring satisfying (SR_r) and let \mathfrak{q} be a two-sided ideal in R . Then R/\mathfrak{q} satisfies (SR_r) .*

Proof. Let $n \geq r$ and let $\bar{v} \in (R/\mathfrak{q})^n$ be a unimodular vector. It is enough to prove that \bar{v} can be lifted to a unimodular vector in R^n . Let $(c_1, \dots, c_n) \in R^n$ be any vector projecting to \bar{v} . Since \bar{v} is unimodular, Lemma 8.3 implies that there exist $a_1, \dots, a_n \in R$ and $q \in \mathfrak{q}$ such

that $a_1c_1 + \cdots + a_nc_n = 1 + q$. It follows that $(c_1, \dots, c_n, q) \in R^{n+1}$ is unimodular, so by (SR_r) we can find $b_1, \dots, b_n \in R$ such that $v = (c_1 + b_1q, \dots, c_n + b_nq) \in R^n$ is unimodular. The vector v projects to \bar{v} . \square

8.7. Stable rank modulo an ideal. The following lemma gives a variant of the stable rank condition that takes into account an ideal.

Lemma 8.11. *Let R be a ring satisfying (SR_r) and let \mathfrak{q} be a two-sided ideal of R . For some $n \geq r$, let $(c_1, \dots, c_n) \in R^n$ be a unimodular vector with $c_n \in \mathfrak{q}$. Then there exists $b_1, \dots, b_{n-1} \in \mathfrak{q}$ such that $(c_1 + b_1c_n, \dots, c_{n-1} + b_{n-1}c_n) \in R^{n-1}$ is unimodular.*

Proof. Using Lemma 8.3, we can find $a_1, \dots, a_n \in R$ with $a_1c_1 + \cdots + a_nc_n = 1$. We claim that $(c_1, \dots, c_{n-1}, c_na_nc_n) \in R^n$ is unimodular. Indeed, we have

$$\begin{aligned} & (a_1 + a_nc_na_1)c_1 + \cdots + (a_{n-1} + a_nc_na_{n-1})c_{n-1} + (a_n)c_na_nc_n \\ &= (a_1c_1 + \cdots + a_{n-1}c_{n-1}) + a_nc_n(a_1c_1 + \cdots + a_nc_n) \\ &= (a_1c_1 + \cdots + a_{n-1}c_{n-1}) + a_nc_n = 1, \end{aligned}$$

as desired. Applying (SR_r) , we can find $b'_1, \dots, b'_{n-1} \in R$ such that $(c_1 + b'_1c_na_nc_n, \dots, c_{n-1} + b'_{n-1}c_na_nc_n) \in R^{n-1}$ is unimodular. Since $c_n \in \mathfrak{q}$, so is $b_i = b'_i c_n a_n$. The lemma follows. \square

8.8. Elementary congruence subgroups and their action on unimodular vectors. Recall that if \mathfrak{q} is an ideal in R , then $\text{GL}_n(R, \mathfrak{q})$ is the level- \mathfrak{q} congruence subgroup of $\text{GL}_n(R)$, i.e., the kernel of the map $\text{GL}_n(R) \rightarrow \text{GL}_n(R/\mathfrak{q})$. An elementary matrix lies in $\text{GL}_n(R, \mathfrak{q})$ precisely when its single off-diagonal entry lies in \mathfrak{q} . Let $\text{EL}_n(R, \mathfrak{q})$ be the subgroup of $\text{EL}_n(R)$ that is **normally** generated by elementary matrices lying in $\text{GL}_n(R, \mathfrak{q})$.

Lemma 8.6 says that the stable rank condition implies that $\text{EL}_n(R)$ acts transitively on the set of unimodular vectors when n is sufficiently large. The following lemma strengthens this:

Lemma 8.12. *Let R be a ring satisfying (SR_r) and let \mathfrak{q} be an ideal in R . For some $n \geq r$, let $v, v' \in R^n$ be unimodular vectors that map to the same vector in $(R/\mathfrak{q})^n$. Then there exists some $M \in \text{EL}_n(R, \mathfrak{q})$ with $M \cdot v = v'$.*

Proof. We will prove this in two steps.

Step 1. *This is true if $v' = (1, 0, \dots, 0)$.*

Since v and $(1, 0, \dots, 0)$ map to the same vector in $(R/\mathfrak{q})^n$, we can write $v = (1 + q_1, q_2, \dots, q_n)$ with $q_1, \dots, q_n \in \mathfrak{q}$. Recall that an elementary matrix lies in $\text{GL}_n(R, \mathfrak{q})$ if its single off-diagonal entry lies in \mathfrak{q} . Just like in the proof of Lemma 8.6, using such elementary matrices, for any distinct $1 \leq i, j \leq n$ and any $\lambda \in \mathfrak{q}$ we can add λ times the i^{th} entry to the j^{th} one. We will use a series of these moves (plus one extra trick) to transform v into $(1, 0, \dots, 0)$.

Applying the relative version of (SR_r) given by Lemma 8.11, we can add \mathfrak{q} -multiples of the last entry to the other ones to ensure that the first $(n-1)$ entries of v form a unimodular vector in R^{n-1} . Using Lemma 8.3, we can thus find $a_1, \dots, a_{n-1} \in R$ such that $a_1(1 + q_1) + a_2q_2 + \cdots + a_{n-1}q_{n-1} = 1$. For each $1 \leq i \leq n-1$, add $(q_1 - q_n)a_i \in \mathfrak{q}$ times the i^{th} entry to the n^{th} one. This transforms v into $(1 + q_1, q_2, \dots, q_{n-1}, q_1)$.

At this point, we will do something slightly tricky. Let $E \in \text{EL}_n(R)$ be the elementary matrix that subtracts the n^{th} row from the first one (notice that E does not lie in $\text{GL}_n(R, \mathfrak{q})$). We thus have

$$E \cdot v = E \cdot (1 + q_1, q_2, \dots, q_{n-1}, q_1) = (1, q_2, \dots, q_{n-1}, q_1).$$

We can then find a product $F \in \text{EL}_n(R, \mathfrak{q})$ of elementary matrices such that

$$F \cdot (1, q_2, \dots, q_{n-1}, q_1) = (1, 0, \dots, 0).$$

Indeed, F first subtracts $q_2 \in \mathfrak{q}$ times the first entry from the 2nd, then subtracts $q_3 \in \mathfrak{q}$ times the first entry from the 3rd, etc., and finishes by subtracting q_1 times the first entry from the n^{th} . Since E fixes $(1, 0, \dots, 0)$, it follows that

$$E^{-1}FE \cdot v = (1, 0, \dots, 0).$$

Since $\text{EL}_n(R, \mathfrak{q})$ is a normal subgroup of $\text{EL}_n(R)$, we have $M = E^{-1}FE \in \text{EL}_n(R, \mathfrak{q})$, as desired.

Step 2. *This is true in general.*

Set $w = (1, 0, \dots, 0)$. Lemma 8.6 says that there exists some $A \in \text{EL}_n(R)$ with $A \cdot v' = w$. Applying the previous step to $A \cdot v$, we can find some $B \in \text{EL}_n(R, \mathfrak{q})$ with $B \cdot A \cdot v = w$, so $A^{-1}BA \cdot v = A^{-1} \cdot w = v'$. Since $\text{EL}_n(R, \mathfrak{q})$ is a normal subgroup of $\text{EL}_n(R)$, it follows that $M = A^{-1}BA$ lies in $\text{EL}_n(R, \mathfrak{q})$. \square

8.9. Action of congruence subgroups on split unimodular vectors. The following lemma refines Lemma 8.12 in the same way that Lemma 8.8 refines Lemma 8.6:

Lemma 8.13. *Let R be a ring satisfying (SR_r) , let \mathfrak{q} be an ideal in R , and let $n \geq r$. Let $x, y \in R^n$ be unimodular vectors and $C, D \subset R^n$ be R -submodules such that $R^n = C \oplus x \cdot R$ and $R^n = D \oplus y \cdot R$. Assume that x and y (resp. C and D) map to the same vector (resp. submodule) in $(R/\mathfrak{q})^n$. Then there exists some $M \in \text{EL}_n(R, \mathfrak{q})$ with $M \cdot x = y$ and $M \cdot C = D$.*

Proof. Using Lemma 8.12, we can assume without loss of generality that $x = y$. Let $\rho: R^n \rightarrow R$ be the projection with $D = \ker(\rho)$ and $\rho(x) = 1$. Lemma 8.9 implies that C and D are both free of rank $(n-1)$. Let $\{v_1, \dots, v_{n-1}\}$ be a free basis for C and let $v_n = x = y$, so $\{v_1, \dots, v_n\}$ is a free basis for R^n . For $1 \leq i \leq n-1$, let $\lambda_i = \rho(v_i)$. Since C and D map to the same submodule of $(R/\mathfrak{q})^n$, we have $\lambda_i \in \mathfrak{q}$. Define $M \in \text{GL}_n(R, \mathfrak{q})$ via the formula

$$M \cdot v_i = v_i - v_n \cdot \lambda_i \quad \text{for } 1 \leq i \leq n-1, \text{ and } M \cdot v_n = v_n.$$

It is easy to write M as a product of elementary matrices lying in $\text{GL}_n(R, \mathfrak{q})$, so $M \in \text{EL}_n(R, \mathfrak{q})$. For $1 \leq i \leq n-1$, we have

$$\rho(M \cdot v_i) = \rho(v_i) - \rho(v_n)\lambda_i = \lambda_i - \lambda_i = 0,$$

so $M \cdot v_i \in D$. We conclude that $M \cdot C = D$, as desired. \square

9. SUBGROUPS FROM K-THEORY

Fix a ring R . This section discusses subgroups of $\text{GL}_n(R)$ arising from algebraic K-theory. One can view them as generalizations of the special linear group.

9.1. Absolute K-groups. Define

$$\text{GL}(R) = \bigcup_{n=1}^{\infty} \text{GL}_n(R) \quad \text{and} \quad \text{EL}(R) = \bigcup_{n=1}^{\infty} \text{EL}_n(R).$$

It turns out that $\text{EL}(R) = [\text{GL}(R), \text{GL}(R)]$ (this is the ‘‘Whitehead Lemma’’; see [22, Lemma 3.1]). By definition, the first algebraic K-theory group of R is the abelian group

$$\text{K}_1(R) = \text{H}_1(\text{GL}(R)) = \text{GL}(R)/\text{EL}(R).$$

For a subgroup $\mathcal{K} \subset \text{K}_1(R)$, we define $\text{GL}_n^{\mathcal{K}}(R)$ to be the preimage of \mathcal{K} under the projection

$$\text{GL}_n(R) \hookrightarrow \text{GL}(R) \longrightarrow \text{K}_1(R).$$

It follows that $\text{EL}_n(R) \subset \text{GL}_n^{\mathcal{K}}(R) \subset \text{GL}_n(R)$.

Example 9.1. Let R be a ring. For $\mathcal{K} = K_1(R)$, we have $GL_n^{\mathcal{K}}(R) = GL_n(R)$. \square

Example 9.2. Let R be a commutative ring and let $\det: GL(R) \rightarrow R^\times$ be the determinant map. Since the determinant of an elementary matrix is 1, the map \det induces a map

$$\mathfrak{d}: K_1(R) = GL(R)/EL(R) \rightarrow R^\times.$$

Letting $\mathcal{K} = \ker(\mathfrak{d})$, we then have $GL_n^{\mathcal{K}}(R) = SL_n(R)$. More generally, if $\mathcal{D} \subset R^\times$ is a subgroup and $\mathcal{K} = \mathfrak{d}^{-1}(\mathcal{D})$, then $GL_n^{\mathcal{K}}(R)$ is the subgroup of $GL_n(R)$ consisting of matrices M with $\det(M) \in \mathcal{D}$. \square

9.2. Relative K-groups. Let α be a two-sided ideal of R . Define

$$GL(R, \alpha) = \bigcup_{n=1}^{\infty} GL_n(R, \alpha) \quad \text{and} \quad EL(R, \alpha) = \bigcup_{n=1}^{\infty} EL_n(R, \alpha).$$

Just like in the absolute case, $EL(R, \alpha)$ is a normal subgroup of $GL(R, \alpha)$ with abelian quotient, and this abelian quotient is the first relative K-theory group of (R, α) :

$$K_1(R, \alpha) = GL(R, \alpha)/EL(R, \alpha).$$

See [22, Lemma 4.2]. There is also a relative version of the Whitehead lemma discussed above that says that

$$EL(R, \alpha) = [GL(R, \alpha), GL(R)].$$

See [22, Lemma 4.3]. In particular, $EL(R, \alpha)$ is generally larger than $[GL(R, \alpha), GL(R, \alpha)]$.

For a subgroup $\mathcal{K}' \subset K_1(R, \alpha)$, we define $GL_n^{\mathcal{K}'}(R, \alpha)$ to be the preimage of \mathcal{K}' under the composition

$$GL_n(R, \alpha) \hookrightarrow GL(R, \alpha) \longrightarrow K_1(R, \alpha).$$

Letting $\mathcal{K} \subset K_1(R)$ be the image of \mathcal{K} under the map $K_1(R, \alpha) \rightarrow K_1(R)$, we have $GL_n^{\mathcal{K}'}(R, \alpha) \subset GL_n^{\mathcal{K}}(R)$. Just like in the above examples, by choosing \mathcal{K}' appropriately we can have $GL_n^{\mathcal{K}'}(R, \alpha) = GL_n(R, \alpha)$ or if R is commutative $GL_n^{\mathcal{K}'}(R, \alpha) = SL_n(R, \alpha)$.

9.3. Injective stability. To clarify the nature of these groups, we need the following deep theorem of Vaserstein (“injective stability for K_1 ”):

Theorem 9.3 (Vaserstein, [31]). *Let R be a ring satisfying (SR_r) and let $n \geq r$. Then $EL_n(R)$ is a normal subgroup of $GL_n(R)$, and the composition*

$$GL_n(R)/EL_n(R) \longrightarrow GL(R)/EL(R) = K_1(R)$$

is an isomorphism. Moreover, if α is a two-sided ideal of R , then $EL_n(R, \alpha)$ is a normal subgroup of $GL_n(R, \alpha)$, and the composition

$$GL_n(R, \alpha)/EL_n(R, \alpha) \longrightarrow GL(R, \alpha)/EL(R, \alpha) = K_1(R, \alpha)$$

is an isomorphism.

This allows us to deal with another example:

Example 9.4. Let R be a ring satisfying (SR_r) and let $n \geq r$. Set $\mathcal{K} = 0$. Then by Theorem 9.3 we have $GL_n^{\mathcal{K}}(R) = EL_n(R)$, and for a two-sided ideal α of R we have $GL_n^{\mathcal{K}}(R, \alpha) = EL_n(R, \alpha)$. \square

It also allows us to prove the following useful result:

Lemma 9.5. *Let R be a ring satisfying (SR_r) and let $n \geq r$. For all subgroups $\mathcal{K} \subset K_1(R)$, we have*

$$\mathrm{GL}_{n+m}^{\mathcal{K}}(R) \cap \mathrm{GL}_n(R) = \mathrm{GL}_n^{\mathcal{K}}(R) \quad \text{for all } m \geq 1.$$

Similarly, for all two-sided ideals α of R and for all subgroups $\mathcal{K}' \subset K_1(R, \alpha)$, we have

$$\mathrm{GL}_{n+m}^{\mathcal{K}'}(R, \alpha) \cap \mathrm{GL}_n(R, \alpha) = \mathrm{GL}_n^{\mathcal{K}'}(R, \alpha) \quad \text{for all } m \geq 1.$$

Proof. It is enough to deal with the case $m = 1$. By Theorem 9.3, the maps

$$\mathrm{GL}_n(R)/\mathrm{EL}_n(R) \rightarrow \mathrm{GL}_{n+1}(R)/\mathrm{EL}_{n+1}(R) \rightarrow K_1(R)$$

are both isomorphisms. This implies that

$$\mathrm{GL}_{n+1}^{\mathcal{K}}(R) \cap \mathrm{GL}_n(R) = \mathrm{GL}_n^{\mathcal{K}}(R).$$

The relative statement is proved similarly. \square

9.4. Elementary vs relative elementary subgroups. Let α be a two-sided ideal of R and let $\mathcal{K}' \subset K_1(R, \alpha)$ be a subgroup. Define

$$\overline{\mathrm{EL}}_n^{\mathcal{K}'}(R, \alpha) = \mathrm{GL}_n^{\mathcal{K}'}(R, \alpha) \cap \mathrm{EL}_n(R).$$

We thus have $\mathrm{EL}_n(R, \alpha) \subset \overline{\mathrm{EL}}_n^{\mathcal{K}'}(R, \alpha)$. This need not be an equality:

Lemma 9.6. *Let R be a ring satisfying (SR_r) , let $n \geq r$, let α be a two-sided ideal of R , and let $\mathcal{K}' \subset K_1(R, \alpha)$ be a subgroup. Let $\mathcal{L} \subset K_1(R, \alpha)$ be the intersection of \mathcal{K}' with the kernel of the map $K_1(R, \alpha) \rightarrow K_1(R)$. We then have a short exact sequence*

$$1 \longrightarrow \mathrm{EL}_n(R, \alpha) \longrightarrow \overline{\mathrm{EL}}_n^{\mathcal{K}'}(R, \alpha) \longrightarrow \mathcal{L} \longrightarrow 0.$$

Proof. By Theorem 9.3, the group $\overline{\mathrm{EL}}_n^{\mathcal{K}'}(R, \alpha)$ is the kernel of the composition

$$\mathrm{GL}_n^{\mathcal{K}'}(R, \alpha) \hookrightarrow \mathrm{GL}_n(R) \rightarrow K_1(R).$$

Theorem 9.3 also implies that this composition can be factored as

$$\mathrm{GL}_n^{\mathcal{K}'}(R, \alpha) \rightarrow \mathrm{GL}_n^{\mathcal{K}'}(R, \alpha)/\mathrm{EL}_n(R, \alpha) \cong \mathcal{K}' \hookrightarrow K_1(R, \alpha) \rightarrow K_1(R).$$

Combining these two facts, we deduce that

$$\overline{\mathrm{EL}}_n^{\mathcal{K}'}(R, \alpha)/\mathrm{EL}_n(R, \alpha) \cong \ker(\mathcal{K}' \hookrightarrow K_1(R, \alpha) \rightarrow K_1(R)) = \mathcal{L},$$

which is equivalent to the claimed short exact sequence. \square

The point of introducing the group $\overline{\mathrm{EL}}_n^{\mathcal{K}'}(R, \alpha)$ is the following lemma:

Lemma 9.7. *Let R be a ring satisfying (SR_r) , let $n \geq r$, and let α be a two-sided ideal of R . Let $\mathcal{K} \subset K_1(R)$ and $\mathcal{K}' \subset K_1(R, \alpha)$ be subgroups such that \mathcal{K}' maps to a subgroup $\overline{\mathcal{K}}'$ of \mathcal{K} under the map $K_1(R, \alpha) \rightarrow K_1(R)$. We then have a commutative diagram with exact rows and injective columns*

$$\begin{array}{ccccccc} 1 & \longrightarrow & \overline{\mathrm{EL}}_n^{\mathcal{K}'}(R, \alpha) & \longrightarrow & \mathrm{GL}_n^{\mathcal{K}'}(R, \alpha) & \longrightarrow & \overline{\mathcal{K}}' \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathrm{EL}_n(R) & \longrightarrow & \mathrm{GL}_n^{\mathcal{K}}(R) & \longrightarrow & \mathcal{K} \longrightarrow 1 \end{array}$$

Proof. Immediate from Theorem 9.3 and the definitions. \square

9.5. Finite-index subgroups. Our next goal is to clarify when $\mathrm{GL}_n^{\mathcal{K}'}(R, \alpha)$ is a finite-index subgroup of $\mathrm{GL}_n^{\mathcal{K}}(R)$. This requires the following lemma:

Lemma 9.8. *Let R be a ring and let α be a two-sided ideal of R with $|R/\alpha| < \infty$. Then the map $\mathrm{K}_1(R, \alpha) \rightarrow \mathrm{K}_1(R)$ has finite kernel and cokernel.*

Proof. Immediate from the following exact sequence [22, Theorem 6.2]:

$$\mathrm{K}_2(R/\alpha) \rightarrow \mathrm{K}_1(R, \alpha) \rightarrow \mathrm{K}_1(R) \rightarrow \mathrm{K}_1(R/\alpha). \quad \square$$

Our result is as follows:

Lemma 9.9. *Let R be a ring satisfying (SR_r) , let $n \geq r$, and let α be a two-sided ideal of R with $|R/\alpha| < \infty$. The following then hold:*

- (i) $\mathrm{EL}_n(R, \alpha)$ is a finite-index subgroup of $\mathrm{EL}_n(R)$.
- (ii) Let $\mathcal{K} \subset \mathrm{K}_1(R)$ and $\mathcal{K}' \subset \mathrm{K}_1(R, \alpha)$ be subgroups such that \mathcal{K}' maps to a finite-index subgroup of \mathcal{K} under the map $\mathrm{K}_1(R, \alpha) \rightarrow \mathrm{K}_1(R)$. Then $\mathrm{GL}_n^{\mathcal{K}'}(R, \alpha)$ is a finite-index subgroup of $\mathrm{GL}_n^{\mathcal{K}}(R)$.

Proof. We start with (i). Since $\mathrm{GL}_n(R, \alpha)$ is the kernel of the map $\mathrm{GL}_n(R) \rightarrow \mathrm{GL}_n(R/\alpha)$ and $|R/\alpha| < \infty$, the group $\mathrm{GL}_n(R, \alpha)$ is a finite-index subgroup of $\mathrm{GL}_n(R)$. Letting

$$\overline{\mathrm{EL}}_n(R, \alpha) = \overline{\mathrm{EL}}_n^{\mathrm{K}_1(R, \alpha)}(R, \alpha),$$

by definition $\overline{\mathrm{EL}}_n(R, \alpha)$ is the intersection of $\mathrm{EL}_n(R) \subset \mathrm{GL}_n(R)$ with the finite-index subgroup $\mathrm{GL}_n(R, \alpha)$ of $\mathrm{GL}_n(R)$. It follows that $\overline{\mathrm{EL}}_n(R, \alpha)$ is a finite-index subgroup of $\mathrm{EL}_n(R)$. Finally, by Lemmas 9.6 and 9.8, the group $\mathrm{EL}_n(R, \alpha)$ is a finite-index subgroup of $\overline{\mathrm{EL}}_n(R, \alpha)$. Summarizing this all, both inclusions

$$\mathrm{EL}_n(R, \alpha) \subset \overline{\mathrm{EL}}_n(R, \alpha) \subset \mathrm{EL}_n(R)$$

are inclusions of finite-index subgroups, so $\mathrm{EL}_n(R, \alpha)$ is a finite-index subgroup of $\mathrm{EL}_n(R)$, as desired.

We next prove (ii). Let $\overline{\mathcal{K}'}$ be the image of \mathcal{K}' in \mathcal{K} . Lemma 9.7 gives a commutative diagram with exact rows and injective columns

$$\begin{array}{ccccccc} 1 & \longrightarrow & \overline{\mathrm{EL}}_n^{\mathcal{K}'}(R, \alpha) & \longrightarrow & \mathrm{GL}_n^{\mathcal{K}'}(R, \alpha) & \longrightarrow & \overline{\mathcal{K}'} \longrightarrow 1 \\ & & \downarrow i_1 & & \downarrow i_2 & & \downarrow i_3 \\ 1 & \longrightarrow & \mathrm{EL}_n(R) & \longrightarrow & \mathrm{GL}_n^{\mathcal{K}}(R) & \longrightarrow & \mathcal{K} \longrightarrow 1 \end{array}$$

The vertical maps i_1, i_2, i_3 are inclusions, and one of our assumptions is that $\overline{\mathcal{K}'}$ is a finite-index subgroup of \mathcal{K} . Since $\mathrm{EL}_n(R, \alpha)$ is a finite-index subgroup of $\mathrm{EL}_n(R)$ and

$$\mathrm{EL}_n(R, \alpha) \subset \overline{\mathrm{EL}}_n^{\mathcal{K}'}(R, \alpha) \subset \mathrm{EL}_n(R),$$

the group $\overline{\mathrm{EL}}_n^{\mathcal{K}'}(R, \alpha)$ is a finite-index subgroup of $\mathrm{EL}_n(R)$. Since both i_1 and i_3 are inclusions of finite-index subgroups, we conclude that i_2 is also an inclusion of a finite-index subgroup, as desired. \square

10. COMPLEX OF SPLIT PARTIAL BASES

We now introduce a simplicial complex called the complex of split partial bases, which is a tiny variant on one introduced by Charney [8].

Remark 10.1. The lemmas we prove in this section are all essentially due to Charney [8]. Many of them are also proved in [29, §5.3]. We include proofs because they are short, and also because it is annoyingly nontrivial to match up our notation and indexing conventions with those papers. \square

10.1. Definition. Let R be a ring and let M be a right R -module. We say that elements $y_1, \dots, y_m \in M$ are *linearly independent* if the map $R^m \rightarrow M$ taking $(\lambda_1, \dots, \lambda_m) \in R^m$ to $y_1 \cdot \lambda_1 + \dots + y_m \cdot \lambda_m$ is injective. The *complex of split partial bases* for M , denoted $\mathcal{B}(M)$, is the following simplicial complex:

- The vertices are pairs $(x; C)$ with $x \in M$ a linearly independent element²⁸ and $C \subseteq M$ a submodule such that $M = C \oplus x \cdot R$.
- A collection $\{(x_0; C_0), \dots, (x_k; C_k)\}$ of vertices forms a k -simplex if $x_i \in C_j$ for all distinct $0 \leq i, j \leq k$.

The group $\text{Aut}_R(M)$ acts on $\mathcal{B}(M)$ on the left. In particular, $\text{GL}_n(R)$ acts on $\mathcal{B}(R^n)$.

10.2. Splitting from a simplex. The following lemma might clarify the definition of a simplex in $\mathcal{B}(M)$.

Lemma 10.2. *Let R be a ring, let M be a right R -module, and let $\{(x_0; C_0), \dots, (x_k; C_k)\}$ be a k -simplex of $\mathcal{B}(M)$. Setting $C = C_0 \cap \dots \cap C_k$, we then have*

$$(10.1) \quad M = C \oplus \bigoplus_{i=0}^k x_i \cdot R.$$

In particular, the elements x_0, \dots, x_k are linearly independent.

Proof. The proof is by induction on k . The base case $k = 0$ is trivial, so assume that $k > 0$ and that the lemma is true for all smaller k . For $0 \leq i \leq k-1$, let $C'_i = C_i \cap C_k$. For these i , we claim that $C_k = C'_i \oplus x_i \cdot R$. Since $x_i \in C_k$ and $M = C_i \oplus x_i \cdot R$, it is enough to show that for all $z \in C_k$, we can write $z = z' + x_i \cdot a$ for some $z' \in C'_i$ and $a \in R$. Since $M = C_i \oplus x_i \cdot R$, we can write $z = z' + x_i \cdot a$ for some $z' \in C_i$ and $a \in R$, and what we must show is that $z' \in C'_i$. But since $z, x_i \in C_k$ we have $z' = z - x_i \cdot a \in C_k$, so $z' \in C'_i$, as desired.

It follows that $\{(x_0; C'_0), \dots, (x_{k-1}; C'_{k-1})\}$ is a $(k-1)$ -simplex of $\mathcal{B}(C_k)$. Since

$$C = C_0 \cap \dots \cap C_k = C'_0 \cap \dots \cap C'_{k-1},$$

our inductive hypothesis implies that

$$C_k = C \oplus \bigoplus_{i=0}^{k-1} x_i \cdot R.$$

Since $M = C_k \oplus x_k \cdot R$, equation (10.1) follows. \square

This has the following useful consequence:

Lemma 10.3. *Let R be a ring and let M be a right R -module. The k -simplices of $\mathcal{B}(M)$ are in bijection with tuples $(\{x_0, \dots, x_k\}; C)$, where $\{x_0, \dots, x_k\}$ is an unordered collection of $(k+1)$ linearly independent elements of M and $C \subset M$ is a submodule such that*

$$M = C \oplus \bigoplus_{i=0}^k x_i \cdot R.$$

²⁸This means that $x \cdot \lambda \neq 0$ for all nonzero $\lambda \in R$.

Proof. For a k -simplex $\{(x_0; C_0), \dots, (x_k, C_k)\}$ of $\mathcal{B}(M)$, by Lemma 10.2 we can associate the tuple

$$(\{x_0, \dots, x_k\}; \bigcap_{i=0}^k C_i).$$

Conversely, consider such a tuple $(\{x_0, \dots, x_k\}; C)$. For $0 \leq j \leq k$, define

$$C_j = C \oplus \bigoplus_{\substack{0 \leq i \leq k \\ i \neq j}} x_i \cdot R.$$

Then $\{(x_0; C_0), \dots, (x_k; C_k)\}$ is a k -simplex of $\mathcal{B}(M)$. It is clear that these two operations are inverses to one another. \square

Subsequently, we will freely move between the notations $\{(x_0; C_0), \dots, (x_k; C_k)\}$ and $(\{x_0, \dots, x_k\}; C)$ for the k -simplices of $\mathcal{B}(M)$.

10.3. Split partial bases and unimodular vectors. We now focus our attention on $\mathcal{B}(R^n)$. The first observation about this complex is as follows:

Lemma 10.4. *Let R be a ring and let $x \in R^n$. Then x is unimodular if and only if there exists some submodule $C \subset R^n$ such that $(x; C)$ is a vertex of $\mathcal{B}(R^n)$.*

Proof. If x is unimodular, then by definition we can find a homomorphism $\phi: R^n \rightarrow R$ of right R -modules such that $\phi(x) = 1$. This implies that x is linearly independent, and letting $C = \ker(\phi)$ the pair $(x; C)$ is a vertex of $\mathcal{B}(R^n)$. Conversely, if $(x; C)$ is a vertex of $\mathcal{B}(R^n)$, then we can define $\phi: R^n \rightarrow R$ to be the composition of the projection

$$R^n = C \oplus x \cdot R \xrightarrow{\text{proj}} x \cdot R$$

with the map $x \cdot R \rightarrow R$ taking x to 1. \square

10.4. Links in complex of split partial bases. If R satisfies a stable rank condition, then the links in $\mathcal{B}(R^n)$ are well-behaved:

Lemma 10.5. *Let R be a ring satisfying (SR_r) . For some n and k with $k \leq n - r$, let σ be a k -simplex of $\mathcal{B}(R^n)$. We then have $\text{Link}_{\mathcal{B}(R^n)}(\sigma) \cong \mathcal{B}(R^{n-k-1})$.*

Proof. Let $\sigma = (\{x_0, \dots, x_k\}; C)$. Since

$$R^n = C \oplus \bigoplus_{i=0}^k x_i \cdot R,$$

Lemma 8.9 implies that $C \cong R^{n-k-1}$, so it is enough to prove that $\text{Link}_{\mathcal{B}(R^n)}(\sigma)$ and $\mathcal{B}(C)$ are isomorphic. For this, note that the maps $\phi: \text{Link}_{\mathcal{B}(R^n)}(\sigma) \rightarrow \mathcal{B}(C)$ and $\psi: \mathcal{B}(C) \rightarrow \text{Link}_{\mathcal{B}(R^n)}(\sigma)$ defined on ℓ -simplices via the formulas

$$\phi(\{y_0, \dots, y_\ell\}; D) = (\{y_0, \dots, y_\ell\}; D \cap C)$$

and

$$\psi(\{z_0, \dots, z_\ell\}; E) = (\{z_0, \dots, z_\ell\}; E \oplus \bigoplus_{i=0}^k x_i \cdot R)$$

are inverse isomorphisms. \square

10.5. General linear group action. We now turn to the action of $\mathrm{GL}_n(R)$ and its elementary subgroup $\mathrm{EL}_n(R)$ on $\mathcal{B}(R^n)$. Our main result will be as follows:

Lemma 10.6. *Let R be a ring satisfying (SR_r) . Let*

$$\{(x_0; C_0), \dots, (x_k; C_k)\} \quad \text{and} \quad \{(y_0; D_0), \dots, (y_k; D_k)\}$$

be k -simplices of $\mathcal{B}(R^n)$. Assume that $k \leq n - r$. The following then hold:

- (a) *There exists $M \in \mathrm{EL}_n(R)$ such that $M \cdot (x_i; C_i) = (y_i; D_i)$ for $0 \leq i \leq k$.*
- (b) *Let \mathfrak{q} be a two-sided ideal of R and let $\pi: R^n \rightarrow (R/\mathfrak{q})^n$ be the projection. Assume that $\pi(x_i) = \pi(y_i)$ and $\pi(C_i) = \pi(D_i)$ for $0 \leq i \leq k$. Then the M in part (a) can be chosen to lie in $\mathrm{EL}_n(R, \mathfrak{q})$.*

Proof. Lemma 10.4 implies that the x_i and y_j are unimodular. For $k = 0$, the lemma is now immediate from Lemmas 8.8 and 8.13. The general case can be deduced from this by induction using Lemma 10.5. \square

10.6. Topology. Recall that we defined what it means for a simplicial complex to be weakly Cohen-Macaulay in §2.2. The complex of split partial bases has this property:

Theorem 10.7. *Let R be a ring satisfying (SR_r) and let $n \geq r - 1$. Then $\mathcal{B}(R^n)$ is weakly Cohen-Macaulay of dimension $\lfloor \frac{n-r+1}{2} \rfloor$.*

The key to the proof of Theorem 10.7 is the following theorem. It is essentially due to Charney [8, Theorem 3.5], though she works with a slightly different complex, so one needs to adapt her proof. For a complete proof of exactly the statement below, see [29, Lemma 5.10] (or rather its proof – the lemma in the reference deals with the large ordering of $\mathcal{B}(R^n)$, but proves this theorem along the way):

Theorem 10.8. *Let R be a ring satisfying (SR_r) . Then for all $n \geq 1$, the complex $\mathcal{B}(R^n)$ is $\lfloor \frac{n-r-1}{2} \rfloor$ -connected.*

Remark 10.9. Here is a guide for connecting our notation and indexing conventions to those of [29, Lemma 5.10]. Let R be a ring satisfying (SR_r) . The paper [29] defines

$$s = \mathrm{sr}(R) = \min \{k \mid R \text{ satisfies } (\mathrm{SR}_{k+1})\} = r - 1.$$

Our complex $\mathcal{B}(R^n)$ is the complex $S_{n-s}(R^s, R)$ of [29]. The proof of [29, Lemma 5.10] shows that $S_{n-s}(R^s, R)$ is

$$\left\lfloor \frac{(n-s)-2}{2} \right\rfloor = \left\lfloor \frac{n-(r-1)-2}{2} \right\rfloor = \left\lfloor \frac{n-r-1}{2} \right\rfloor$$

connected. The complex $W_{n-s}(R^s, R)$ in the statement of [29, Lemma 5.10] is the large ordering of $S_{n-s}(R^s, R)$. \square

Proof of Theorem 10.7. Theorem 10.8 says that $\mathcal{B}(R^n)$ is

$$\left\lfloor \frac{n-r+1}{2} \right\rfloor - 1 = \left\lfloor \frac{n-r-1}{2} \right\rfloor$$

connected, so what we must prove is that for all k -simplices σ of $\mathcal{B}(R^n)$, the complex $\mathrm{Link}_{\mathcal{B}(R^n)}(\sigma)$ is $(\lfloor \frac{n-r-1}{2} \rfloor - k - 1)$ -connected. This is a non-tautological condition precisely when

$$(10.2) \quad \frac{n-r-1}{2} - k - 1 \geq -1, \quad \text{i.e., when } k \leq \frac{n-r-1}{2}.$$

The connectivity condition we want will follow from Theorem 10.8 if $\mathrm{Link}_{\mathcal{B}(R^n)}(\sigma) \cong \mathcal{B}(R^{n-k-1})$, which by Lemma 10.5 holds if

$$(10.3) \quad k \leq n - r.$$

So we must prove that (10.2) implies (10.3), i.e., that

$$\frac{n-r-1}{2} \leq n-r.$$

This holds precisely when $n-r \geq -1$, which is exactly our assumption $n \geq r-1$. \square

11. THE LARGE ORDERING OF THE COMPLEX OF SPLIT PARTIAL BASES

Let R be a ring. Recall that we introduced $\text{VIC}(R)$ -modules in §1.6. We now discuss the large ordering of $\mathcal{B}(R^n)$ and show how to use a $\text{VIC}(R)$ -module M to define a $\text{GL}_n(R)$ -equivariant coefficient system on it (or, rather, on an appropriate subcomplex).

11.1. Large ordering and shifted large ordering. For $n \geq 0$, define $\mathbb{B}(R^n)$ to be the large ordering of $\mathcal{B}(R^n)$. As we discussed in Lemma 10.3, the k -simplices of $\mathcal{B}(R^n)$ can be identified with tuples $(\{x_0, \dots, x_k\}; C)$, where $x_0, \dots, x_k \in R^n$ are linearly independent elements and $C \subset R^n$ is a submodule such that

$$R^n = C \oplus \bigoplus_{i=0}^k x_i \cdot R.$$

The k -simplices of $\mathbb{B}(R^n)$ are obtained by imposing an ordering on the x_i , so can be identified with ordered tuples $(x_0, \dots, x_k; C)$ where the x_i and C are as above. It will be technically annoying that sometimes C is not a free module. To fix this, assume that R satisfies (SR_r) , and define $\mathbb{B}(R^{n,r})$ to be the semisimplicial subset of $\mathbb{B}(R^{n+r})$ consisting of simplices of dimension at most n . The group $\text{GL}_{n+r}(R)$ acts on $\mathbb{B}(R^{n,r})$, and the following lemma shows that $\mathbb{B}(R^{n,r})$ avoids some of the annoying features of $\mathbb{B}(R^n)$. Recall that we defined groups $\text{GL}_n^{\mathcal{K}}(R)$ in §9.

Lemma 11.1. *Let R be a ring satisfying (SR_r) and let $n \geq 0$. The following hold:*

- (i) *For each k -simplex $(x_0, \dots, x_k; C)$ of $\mathbb{B}(R^{n,r})$, we have $C \cong R^{n+r-k-1}$.*
- (ii) *For all k and all subgroups $\mathcal{K} \subset \text{K}_1(R)$, the group $\text{GL}_{n+r}^{\mathcal{K}}(R)$ acts transitively on the k -simplices of $\mathbb{B}(R^{n,r})$.*
- (iii) *For all two-sided ideals α of R , we have*

$$\mathbb{B}(R^{n,r}) / \text{EL}_{n+r}(R, \alpha) = \mathbb{B}((R/\alpha)^{n,r}).$$

- (iv) *The semisimplicial set $\mathbb{B}(R^{n,r})$ is the large ordering of a weakly Cohen–Macaulay complex of dimension $\lfloor \frac{n+1}{2} \rfloor$.*

Proof. Conclusions (i) and (ii) follow from Lemmas 8.9 and 10.6 along with the fact that we have restricted the simplices of $\mathbb{B}(R^{n,r})$ to have dimension at most n . Here for (ii) we are using the fact that $\text{EL}_{n+r}(R) \subset \text{GL}_{n+r}^{\mathcal{K}}(R)$.

For (iii), the projection

$$\pi: \mathbb{B}(R^{n,r}) \longrightarrow \mathbb{B}((R/\alpha)^{n,r})$$

is $\text{GL}_{n+r}(R)$ -equivariant, where the group $\text{GL}_{n+r}(R)$ acts on $\mathbb{B}((R/\alpha)^{n,r})$ via the projection $\text{GL}_{n+r}(R) \rightarrow \text{GL}_{n+r}(R/\alpha)$. Lemma 8.10 says that R/α satisfies (SR_r) , so by Lemma 10.6 the group $\text{EL}_{n+r}(R/\alpha)$ acts transitively on the k -simplices of $\mathbb{B}((R/\alpha)^{n,r})$ for all k . Since $\text{EL}_{n+r}(R)$ maps surjectively onto $\text{EL}_{n+r}(R/\alpha)$ and π is $\text{EL}_{n+r}(R)$ -equivariant, it follows that π is surjective. Since $\text{GL}_{n+1}(R, \alpha)$ is the kernel of the map $\text{GL}_{n+r}(R) \rightarrow \text{GL}_{n+r}(R/\alpha)$, it acts trivially on $\mathbb{B}((R/\alpha)^{n,r})$. It follows that its subgroup $\text{EL}_{n+r}(R, \alpha)$ also acts trivially on $\mathbb{B}((R/\alpha)^{n,r})$, so the map π descends to a map

$$\bar{\pi}: \mathbb{B}(R^{n,r}) / \text{EL}_{n+r}(R, \alpha) \rightarrow \mathbb{B}((R/\alpha)^{n,r}).$$

The map $\bar{\pi}$ is surjective since π is. Lemma 10.6 implies that two simplices of $\mathbb{B}(R^{n,r})$ that map to the same simplex of $\mathbb{B}((R/\alpha)^{n,r})$ differ by an element of $\text{EL}_{n+r}(R, \alpha)$, and thus are

identified with the same simplex of $\mathbb{B}(R^{n,r})/\mathrm{EL}_{n+r}(R, \alpha)$. It follows that $\bar{\pi}$ is injective, and hence an isomorphism, as desired.

For (iv), Theorem 10.7 says that $\mathcal{B}(R^{n+r})$ is weakly Cohen–Macaulay of dimension $\lfloor \frac{n+1}{2} \rfloor$. Since $n \geq 0$, we have $n \geq \lfloor \frac{n+1}{2} \rfloor$, so the n -skeleton of $\mathcal{B}(R^{n+r})$ is also weakly Cohen–Macaulay of dimension $\lfloor \frac{n+1}{2} \rfloor$, as desired. \square

11.2. Coefficient system. Let R be a ring satisfying (SR_r) , let \mathbb{k} be a commutative ring, and let M be a $\mathrm{VIC}(R)$ -module over \mathbb{k} . For each $n \geq 0$, let $\mathcal{G}_{M,n,r}$ be the augmented coefficient system on $\mathbb{B}(R^{n,r})$ defined via the formula

$$\mathcal{G}_{M,n,r}(x_0, \dots, x_k; C) = M(C) \quad \text{for a simplex } (x_0, \dots, x_k; C) \text{ of } \mathbb{B}(R^{n,r}),$$

where our convention is that the -1 -simplex of $\mathbb{B}(R^{n,r})$ is $(; R^{n+r})$. For this to make sense, we need C to be a free R -module, which is ensured by Lemma 11.1.(i). To see how $\mathcal{G}_{M,n,r}$ behaves under face maps, note that for a simplex $\sigma = (x_0, \dots, x_k; C)$ of $\mathbb{B}(R^{n,r})$, the faces of σ are of the form

$$\sigma' = (x_{i_0}, \dots, x_{i_\ell}; C') \quad \text{with} \quad C' = C \oplus \bigoplus_{j \notin \{i_0, \dots, i_\ell\}} x_j \cdot R$$

for increasing sequences $0 \leq i_0 < \dots < i_\ell \leq k$. Letting $\iota: C \rightarrow C'$ be the natural inclusion and letting

$$D = \bigoplus_{j \notin \{i_0, \dots, i_\ell\}} x_j \cdot R,$$

the induced map $\mathcal{G}_{M,n,r}(\sigma) \rightarrow \mathcal{G}_{M,n,r}(\sigma')$ is the one induced by the $\mathrm{VIC}(R)$ -structure on M as follows:

$$\mathcal{G}_{M,n,r}(\sigma) = M(C) \xrightarrow{(\iota, D)^*} M(C') = \mathcal{G}_{M,n,r}(\sigma').$$

Just like in Example 4.3, the augmented coefficient system $\mathcal{G}_{M,n,r}$ is $\mathrm{GL}_{n+r}(R)$ -equivariant. It satisfies the following lemma:

Lemma 11.2 (c.f. Lemma 6.2). *Let R be a ring satisfying (SR_r) , let \mathbb{k} be a commutative ring, and let M be a $\mathrm{VIC}(R)$ -module over \mathbb{k} that is polynomial of degree $d \geq -1$ starting at $m \geq 0$ (see Definition 1.17). Fix²⁹ some $n \geq \max(m - r, 0)$, and let $\mathcal{G}_{M,n,r}$ be the coefficient system on $\mathbb{B}(R^{n,r})$ discussed above. Then $\mathcal{G}_{M,n,r}$ is polynomial of degree d up to dimension $n + r - m - 1$.*

Proof. The proof will be by induction on d . If $d = -1$, then consider a simplex $\sigma = (x_0, \dots, x_k; C)$ with $k \leq n + r - m - 1$. Lemma 11.1.(i) says that $C \cong R^{n+r-k-1}$, and since

$$(11.1) \quad n + r - k - 1 \geq n + r - (n + r - m - 1) - 1 = m$$

the fact that M is polynomial of degree -1 starting at m implies that $\mathcal{G}_{M,n,r}(\sigma) = M(C) = 0$, as desired.

Now assume that $d \geq 0$. There are two things to check. For the first, let $\sigma = (x_0, \dots, x_k; C)$ be a simplex with $k \leq n + r - m - 1$. We must prove that the map $\mathcal{G}_{M,n,r}(\sigma) \rightarrow \mathcal{G}_{M,n,r}(\emptyset)$ is injective, i.e., that the map

$$M(C) \rightarrow M(R^{n+r})$$

is injective. The calculation (11.1) shows that this injectivity follows from the fact that M is polynomial of degree d starting at m .

²⁹The purpose of the condition $n \geq m - r$ is simply to ensure that $n + r - m - 1 \geq -1$. Here $n + r - m - 1$ is the dimension we are claiming that $\mathcal{G}_{M,n,r}$ is polynomial of degree d up to, and for this to make sense we need it to be at least -1 .

For the second, let w be any vertex. Let $D_w \mathcal{G}_{M,b,r}$ be the coefficient system on the forward link $\overrightarrow{\text{Link}}_{\mathbb{B}(R^{n,r})}(w)$ defined by the formula

$$D_w \mathcal{G}_{M,b,r}(\sigma) = \frac{\mathcal{G}_{M,n,r}(\sigma)}{\text{Im}(\mathcal{G}_{M,n,r}(w \cdot \sigma) \rightarrow \mathcal{G}_{M,n,r}(\sigma))} \quad \text{for a simplex } \sigma \text{ of } \overrightarrow{\text{Link}}_{\mathbb{B}(R^{n,r})}(w).$$

We must prove that $D_w \mathcal{G}_{M,b,r}$ is polynomial of degree $d - 1$ up to dimension $n + r - m - 1$. This condition is invariant under the action of $\text{GL}_{n+r}(R)$. Letting $\{v_1, \dots, v_{n+r}\}$ be the standard basis for R^{n+r} , Lemma 11.1.(ii) says that by applying an appropriate element of $\text{GL}_{n+r}(R)$ to w we can assume without loss of generality that $w = (v_{n+r}; R^{n+r-1})$. This implies that

$$\overrightarrow{\text{Link}}_{\mathbb{B}(R^{n,r})}(w) = \mathbb{B}(R^{n-1,r}).$$

Recall that we defined the derived $\text{VIC}(R)$ -module DM in Definition 1.16. By construction, there is an isomorphism between the coefficient systems $D_w \mathcal{G}_{M,b,r}$ and $\mathcal{G}_{DM,n-1,r}$ on $\mathbb{B}(R^{n-1,r})$. Since M is polynomial of degree d starting at m , the $\text{VIC}(R)$ -module DM is polynomial of degree $d - 1$ starting at $m - 1$. By induction, $D_w \mathcal{G}_{M,b,r}$ is polynomial of degree $d - 1$ starting at

$$(n - 1 + r) - (m - 1) - 1 = n + r - m - 1,$$

as desired. \square

12. STABILITY FOR GENERAL LINEAR GROUPS

We now are in a position to prove Theorems B and B'. In fact, we will prove more general results that also deal with the groups $\text{GL}_n^{\mathcal{K}}(R)$ discussed in §9. The following generalizes Theorem B.

Theorem 12.1. *Let R be a ring satisfying (SR_r) , let $\mathcal{K} \subset \text{K}_1(R)$ be a subgroup, let \mathbb{k} be a commutative ring, and let M be a $\text{VIC}(R)$ -module over \mathbb{k} that is polynomial of degree $d \geq -1$ starting at $m \geq 0$. For each $k \geq 0$, the map*

$$\text{H}_k(\text{GL}_n^{\mathcal{K}}(R); M(R^n)) \rightarrow \text{H}_k(\text{GL}_{n+1}^{\mathcal{K}}(R); M(R^{n+1}))$$

is an isomorphism for $n \geq 2k + \max(2d + r - 1, m, r) + 1$ and a surjection for $n = 2k + \max(2d + r - 1, m, r)$.

Proof. The group $\text{GL}_{n+r}^{\mathcal{K}}(R)$ acts on $\mathbb{B}(R^{n,r})$. Let $\mathcal{G}_{M,n,r}$ be the $\text{GL}_{n+r}^{\mathcal{K}}(R)$ -equivariant augmented system of coefficients on $\mathbb{B}(R^{n,r})$ discussed in §11.2, so

$$\mathcal{G}_{M,n,r}(x_0, \dots, x_k; C) = M(C) \quad \text{for a simplex } (x_0, \dots, x_k; C) \text{ of } \mathbb{B}(R^{n,r}).$$

The following claim will be used to show that with an appropriate degree shift, this all satisfies the hypotheses of Theorem 5.2.

Claim. *Assume that $n \geq \max(m - r, 0)$, so we can apply Lemma 11.2. The following then hold:*

- (A) *For all $-1 \leq k \leq \min(\lfloor \frac{n-2d-1}{2} \rfloor, n + r - m - 1)$, we have $\tilde{\text{H}}_k(\mathbb{B}(R^{n,r}); \mathcal{G}_{M,n,r}) = 0$.*
- (B) *For all $-1 \leq k < n$, the group $\text{GL}_{n+r-k-1}^{\mathcal{K}}(R)$ is the $\text{GL}_{n+r}^{\mathcal{K}}(R)$ -stabilizer of a k -simplex σ_k of $\mathbb{B}(R^{n,r})$ with $\mathcal{G}_{M,n,r}(\sigma_k) = M(R^{n+r-k-1})$.*
- (C) *For all $0 \leq k \leq n$, the group $\text{GL}_{n+r}^{\mathcal{K}}(R)$ acts transitively on the k -simplices of $\mathbb{B}(R^{n,r})$.*
- (D) *For all $n \geq 2$ and all 1-simplices e of $\mathbb{B}(R^{n,r})$ of the form $e = ((x_0; C_0), (x_1; C_1))$, there exists some $\lambda \in \text{GL}_{n+r}^{\mathcal{K}}(R)$ with $\lambda(x_0; C_0) = (x_1; C_1)$ such that λ commutes with all elements of $(\text{GL}_{n+r}^{\mathcal{K}}(R))_e$ and fixes all elements of $\mathcal{G}_{M,n,r}(e)$.*

Proof of claim. For (A), Lemma 11.2 says that $\mathcal{G}_{M,n,r}$ is a polynomial coefficient system of degree d up to dimension $n+r-m-1$. Also, by Lemma 11.1.(iv) the semisimplicial set $\mathbb{B}(R^{n,r})$ is the large ordering of a simplicial complex that is weakly Cohen–Macaulay of dimension $\lfloor \frac{n+1}{2} \rfloor$. Letting

$$N = \min\left(\left\lfloor \frac{n+1}{2} \right\rfloor - d - 1, n+r-m-1\right) = \min\left(\left\lfloor \frac{n-2d-1}{2} \right\rfloor, n+r-m-1\right),$$

the complex $\mathbb{B}(R^{n,r})$ is the large ordering of a simplicial complex that is weakly Cohen–Macaulay of dimension $N+d+1$ and $\mathcal{G}_{M,n,r}$ is a polynomial coefficient system of degree d up to dimension N . Theorem 6.4 thus implies that $\tilde{H}_k(\mathbb{B}(R^{n,r}); \mathcal{G}_{M,n,r}) = 0$ for $-1 \leq k \leq N$.

For (B), let $\{v_1, \dots, v_{n+r}\}$ be the standard basis for R^{n+r} . By Lemma 9.5, the group $\mathrm{GL}_{n+r-k-1}^{\mathcal{K}}(R)$ is the $\mathrm{GL}_{n+r}^{\mathcal{K}}(R)$ -stabilizer of the k -simplex

$$\sigma_k = \begin{cases} (R^{n+r}) & \text{if } k = -1, \\ (v_{n+r-k}, v_{n+r-k+1}, \dots, v_{n+r}; R^{n+r-k-1}) & \text{if } 0 \leq k < n \end{cases}$$

of $\mathbb{B}(R^{n,r})$, and by definition

$$\mathcal{G}_{M,n,r}(\sigma_k) = M(R^{n+r-k-1}).$$

Condition (C) is Lemma 11.1.(ii).

For (D), define $\lambda: R^{n+r} \rightarrow R^{n+r}$ to be the R -module homomorphism defined via the formulas

$$\lambda(x_0) = x_1 \quad \text{and} \quad \lambda(x_1) = -x_0 \quad \text{and} \quad \lambda|_{C_0 \cap C_1} = \mathrm{id}.$$

This lies in $\mathrm{EL}_{n+r}(R) \subset \mathrm{GL}_{n+r}^{\mathcal{K}}(R)$; indeed, the group $\mathrm{SL}_2(\mathbb{Z})$ is generated by elementary matrices and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

so λ can be written as a product of elementary matrices. Here we are using the fact that $n+r \geq r$, so by Theorem 9.3 the group $\mathrm{EL}_{n+r}(R)$ is a normal subgroup of $\mathrm{GL}_{n+r}(R)$. In particular, the fact that something can be written as a product of elementary matrices is independent of the choice of basis. The map λ acts trivially on

$$\mathcal{G}_{M,n,r}(e) = M(C_0 \cap C_1)$$

by basic properties of $\mathrm{VIC}(R)$ -modules. □

Let $e = \max(2d-1, m-r, 0) \geq 0$. For $n \geq 0$ define

$$G_n = \mathrm{GL}_{n+e+r}^{\mathcal{K}}(R) \quad \text{and} \quad M_n = M(R^{n+e+r})$$

and for $n \geq 1$ define

$$\mathbb{X}_n = \mathbb{B}(R^{n+e,r}) \quad \text{and} \quad \mathcal{M}_n = \mathcal{G}_{M,n+e,r}.$$

This makes sense since $n+e \geq n \geq 0$ for $n \geq 0$. We then have

$$G_0 \subset G_1 \subset G_2 \subset \dots,$$

and since M is a polynomial $\mathrm{VIC}(R)$ -module of degree d starting at $m \geq 0$ the maps $M_n \rightarrow M_{n+1}$ are injective for $n \geq 0$,³⁰ so we have an increasing sequence

$$M_0 \subset M_1 \subset M_2 \subset \dots.$$

In other words, the pair $\{(G_n, M_n)\}_{n=0}^{\infty}$ is an increasing sequence of groups and modules in the sense of §5.2.

The above claim verifies the conditions of Theorem 5.2 with $c = 2$. The shift by $e+r$ is needed for condition (a) of Theorem 5.2, which requires that $\tilde{H}_k(\mathbb{X}_n; \mathcal{M}_n) = 0$ for all $n \geq 1$

³⁰The point here is that for $n \geq 0$ we have $n+e+r \geq n + \max(2d-1, m-r, 0) + r \geq n+m \geq m$.

and $-1 \leq k \leq \lfloor \frac{n-2}{2} \rfloor$. Conclusion (A) of the Claim says that $\widetilde{H}_k(\mathbb{B}(R^{n+e,r}); \mathcal{G}_{M,n+e,r}) = 0$ for $n+e \geq \max(m-r, 0)$ ³¹ and $-1 \leq k \leq \min(\lfloor \frac{(n+e)-2d-1}{2} \rfloor, (n+e)+r-m-1)$, which implies the desired range of vanishing for $\widetilde{H}_k(\mathbb{X}_n; \mathcal{M}_n)$ since

$$\left\lfloor \frac{(n+e)-2d-1}{2} \right\rfloor \geq \left\lfloor \frac{(n+2d-1)-2d-1}{2} \right\rfloor = \left\lfloor \frac{n-2}{2} \right\rfloor \quad \text{for all } n$$

and

$$(n+e)+r-m-1 \geq n+(m-r)+r-m-1 = n-1 \geq \left\lfloor \frac{n-2}{2} \right\rfloor \quad \text{for all } n \geq 0,$$

so

$$\left\lfloor \frac{n-2}{2} \right\rfloor \leq \min\left(\left\lfloor \frac{(n+e)-2d-1}{2} \right\rfloor, (n+e)+r-m-1\right) \quad \text{for all } n \geq 0.$$

Applying Theorem 5.2, we deduce that the map

$$H_k(\mathrm{GL}_{n+e+r-1}^{\mathcal{K}}(R); M(R^{n+e+r-1})) \rightarrow H_k(\mathrm{GL}_{n+e+r}^{\mathcal{K}}(R); M(R^{n+e+r}))$$

is an isomorphism for $n \geq 2k+2$ and a surjection for $n = 2k+1$, which implies that

$$H_k(\mathrm{GL}_n^{\mathcal{K}}(R); M(R^n)) \rightarrow H_k(\mathrm{GL}_{n+1}^{\mathcal{K}}(R); M(R^{n+1}))$$

is an isomorphism for

$$n \geq 2k+(e+r-1)+2 = 2k+\max(2d-1, m-r, 0)+r+1 = 2k+\max(2d+r-1, m, r)+1$$

and a surjection for

$$n = 2k+\max(2d+r-1, m, r). \quad \square$$

The following theorem generalizes Theorem B':

Theorem 12.2. *Let R be a ring satisfying (SR_r) , let $\mathcal{K} \subset \mathrm{K}_1(R)$ be a subgroup, let \mathbb{k} be a commutative ring, and let M be a $\mathrm{VIC}(R)$ -module over \mathbb{k} that is polynomial of degree $d \geq -1$ starting at $m \geq 0$. For each $k \geq 0$, the map*

$$(12.1) \quad H_k(\mathrm{GL}_n^{\mathcal{K}}(R); M(R^n)) \rightarrow H_k(\mathrm{GL}_{n+1}^{\mathcal{K}}(R); M(R^{n+1}))$$

is an isomorphism for $n \geq \max(m, 2k+2d+r+2)$ and a surjection for $n \geq \max(m, 2k+2d+r)$.

Proof. The proof will be by double induction on d and m . There are three base cases:

- The first is where $m = 0$ and $d = 0$. Theorem 12.1 says in this case that (12.1) is an isomorphism for

$$n \geq 2k+\max(2d+r-1, m, r)+1 = 2k+\max(r-1, 0, r)+1 = 2k+r+1$$

and a surjection for

$$n = 2k+\max(2d+r-1, m, r) = 2k+\max(r-1, 0, r) = 2k+r.$$

These bounds are even stronger than our purported bounds of

$$n \geq \max(m, 2k+2d+r+2) = \max(0, 2k+r+2) = 2k+r+2$$

for (12.1) to be an isomorphism and

$$n = \max(m, 2k+2d+r) = \max(0, 2k+r) = 2k+r$$

for (12.1) to be a surjection.

³¹Which holds for $n \geq 0$ since then $n+e = n+\max(2d-1, m-r, 0) \geq \max(m-r, 0)$.

- The second is where $m = 0$ and $d \geq 1$. Theorem 12.1 says in this case that (12.1) is an isomorphism for

$$n \geq 2k + \max(2d + r - 1, m, r) + 1 = 2k + \max(2d + r - 1, 0, r) + 1 = 2k + 2d + r$$

and a surjection for

$$n = 2k + \max(2d + r - 1, m, r) = 2k + \max(2d + r - 1, 0, r) = 2k + 2d + r - 1.$$

These bounds are even stronger than our purported bounds of

$$n \geq \max(m, 2k + 2d + r + 2) = \max(0, 2k + 2d + r + 2) = 2k + 2d + r + 2$$

for (12.1) to be an isomorphism and

$$n = \max(m, 2k + 2d + r) = \max(0, 2k + 2d + r) = 2k + 2d + r$$

for (12.1) to be a surjection.

- The third is where $m \geq 0$ and $d = -1$. In this case, by the definition of a $\text{VIC}(R)$ -module being polynomial of degree -1 starting at m we have for $n \geq m$ that $M(R^n) = 0$ and hence $H_k(\text{GL}_n^{\mathcal{K}}(R); M(R^n)) = 0$. In other words, for $n \geq m$ the domain and codomain of (12.1) are both 0, so it is trivially an isomorphism.

Assume now that $m \geq 1$ and $d \geq 0$, and that the theorem is true for all such (m', d') with $m' \leq m$ and $d' \leq d$ such that either $m' < m$ or $d' < d$ (or both). As in Definition 1.16, let ΣM be the shifted $\text{VIC}(R)$ -module and DM be the derived $\text{VIC}(R)$ -module. For $n \geq m$, we have a short exact sequence

$$(12.2) \quad 0 \longrightarrow M(R^n) \longrightarrow \Sigma M(R^n) \longrightarrow DM(R^n) \longrightarrow 0$$

of $\mathbb{k}[\text{GL}_n^{\mathcal{K}}(R)]$ -modules. The $\text{VIC}(R)$ -module ΣM is polynomial of degree d starting at $(m - 1)$, and the $\text{VIC}(R)$ -module DM is polynomial of degree $(d - 1)$ starting at $(m - 1)$.

To simplify our notation, for all $s \geq 0$ and all $\mathbb{k}[\text{GL}_s^{\mathcal{K}}(R)]$ -modules N , we will denote $H_k(\text{GL}_s^{\mathcal{K}}(R); N)$ by $H_k(N)$. The long exact sequence in $\text{GL}_n^{\mathcal{K}}(R)$ -homology associated to (12.2) maps to the one in $\text{GL}_{n+1}^{\mathcal{K}}(R)$ -homology, so for $n \geq m$ and all k we have a commutative diagram

$$\begin{array}{ccccccccc} H_{k+1}(\Sigma M(R^n)) & \rightarrow & H_{k+1}(DM(R^n)) & \rightarrow & H_k(M(R^n)) & \rightarrow & H_k(\Sigma M(R^n)) & \rightarrow & H_k(DM(R^n)) \\ \downarrow g_1 & & \downarrow g_2 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 \\ H_{k+1}(\Sigma M(R^{n+1})) & \rightarrow & H_{k+1}(DM(R^{n+1})) & \rightarrow & H_k(M(R^{n+1})) & \rightarrow & H_k(\Sigma M(R^{n+1})) & \rightarrow & H_k(DM(R^{n+1})) \end{array}$$

with exact rows. Our inductive hypothesis says the following about the g_i and f_i :

- Since ΣM is polynomial of degree d starting at $(m - 1)$, the map f_2 is an isomorphism for $n \geq \max(m - 1, 2k + 2d + r + 2)$ and a surjection for $n \geq \max(m - 1, 2k + 2d + r)$. Also, the map g_1 is an isomorphism for

$$n \geq \max(m - 1, 2(k + 1) + 2d + r + 2) = \max(m - 1, 2k + 2d + r + 4)$$

and a surjection for $n \geq \max(m - 1, 2k + 2d + r + 2)$.

- Since DM is polynomial of degree $(d - 1)$ starting at $(m - 1)$, the map f_3 is an isomorphism for

$$n \geq \max(m - 1, 2k + 2(d - 1) + r + 2) = \max(m - 1, 2k + 2d + r)$$

and a surjection for $n \geq \max(m - 1, 2k + 2d + r - 2)$. Also, the map g_2 is an isomorphism for

$$n \geq \max(m - 1, 2(k + 1) + 2(d - 1) + r + 2) = \max(m - 1, 2k + 2d + r + 2)$$

and a surjection for $n \geq \max(m - 1, 2k + 2d + r)$.

For $n \geq \max(m, 2k+2d+r+2)$, the maps g_2 and f_2 and f_3 are isomorphisms and the map g_1 is a surjection, so by the five-lemma the map f_1 is an isomorphism. For $n \geq \max(m, 2k+2d+r)$, the maps g_2 and f_2 are surjections and the map f_3 is an isomorphism, so by the five-lemma³² the map f_1 is a surjection. The claim follows. \square

13. UNIPOTENCE AND ITS CONSEQUENCES

We now turn our attention to congruence subgroups. Before we can prove Theorem C, we need some preliminary results about unipotent representations.

13.1. Unipotent representations. Let \mathbb{k} be a field and let V be a vector space over \mathbb{k} . A *unipotent operator* on V is a linear map $f: V \rightarrow V$ that can be written as $f = \text{id}_V + \phi$ where $\phi: V \rightarrow V$ is nilpotent, i.e., there exists some $k \geq 1$ such that $\phi^k = 0$. If G is a group and V is a $\mathbb{k}[G]$ -module, then we say that V is a *unipotent representation* of G if all elements of G act on V via unipotent operators. We will mostly be interested in abelian G , where this can be checked on generators:

Lemma 13.1. *Let G be an abelian group generated by a set S , let \mathbb{k} be a field, and let V be a $\mathbb{k}[G]$ -module. Assume that each $s \in S$ acts on V via a unipotent operator. Then V is a unipotent representation of G .*

Proof. It is enough to prove that if $g_1, g_2 \in G$ are elements that both act on V via unipotent operators, then $g_1 g_2$ also acts via a unipotent operator. Let g_i act on V via the linear map $f_i: V \rightarrow V$, and write $f_i = \text{id}_V + \phi_i$ with ϕ_i nilpotent. We thus have

$$f_1 f_2 = \text{id}_V + \phi_1 + \phi_2 + \phi_1 \phi_2.$$

Since the g_i commute, the ϕ_i also commute. This implies that $\phi_1 + \phi_2 + \phi_1 \phi_2$ is nilpotent, so $f_1 f_2$ is a unipotent operator. \square

13.2. Unipotence and VIC(R)-modules. The following shows how to find many unipotent operators within a $\text{VIC}(R)$ -module. We thank Harman for explaining its proof to us.

Lemma 13.2. *Let R be a ring, let \mathbb{k} be a field of characteristic 0, and let M be a $\text{VIC}(R)$ -module over \mathbb{k} that is polynomial of degree d starting at m . Assume that $M(R^n)$ is a finite-dimensional vector space over \mathbb{k} for all n . Then there exists some $u \geq 0$ such that for $n \geq u$, all elementary matrices in $\text{GL}_n(R)$ act on $M(R^n)$ via unipotent operators.*

Proof. Whether or not an operator is unipotent is unchanged by field extensions, so without loss of generality we can assume that \mathbb{k} is algebraically closed. Via the ring homomorphism $\mathbb{Z} \rightarrow R$, regard M as a $\text{VIC}(\mathbb{Z})$ -module. This does not change the fact that it is polynomial of degree d starting at m . We can thus appeal to a theorem of Harman [17, Proposition 4.4]³³ saying that there exists some $u \geq 0$ such that for all $n \geq u$, the action of $\text{SL}_n(\mathbb{Z})$ on $M(\mathbb{Z}^n) = M(R^n)$ extends to a rational representation of the algebraic group SL_n .

³²Or, more precisely, one of the four-lemmas.

³³The statement of Harman's theorem requires $\mathbb{k} = \mathbb{C}$, but the proof just uses the fact that \mathbb{k} is algebraically closed and has characteristic 0. Harman's theorem also requires M to be a finitely generated $\text{VIC}(\mathbb{Z})$ -module. This turns out to be a consequence of the fact that M is polynomial of degree d and each $M(\mathbb{Z}^n)$ is finite-dimensional. Indeed, since each $M(\mathbb{Z}^n)$ is assumed to be finite-dimensional it is enough to prove that there exists some $N \geq 0$ such that M is generated in degree N , i.e., for all $n \geq N$, the map

$$\bigoplus_{\substack{(f,C): [\mathbb{Z}^n] \rightarrow [\mathbb{Z}^{n+1}] \\ \text{VIC}(\mathbb{Z})\text{-morphism}}} M(\mathbb{Z}^n) \rightarrow M(\mathbb{Z}^{n+1})$$

is surjective. This can be proved by induction on the polynomial degree d . The base case $d = -1$ is trivial since in that case $M(\mathbb{Z}^n) = 0$ for $n \gg 0$, so assume that $d \geq 0$ and that the result is true for smaller degrees. The derived $\text{VIC}(\mathbb{Z})$ -module DM is polynomial of degree $d - 1$. By induction, we can thus choose $N \geq 0$

Increasing u if necessary, we can assume that $u \geq 3$. Consider some $n \geq u$. For distinct $1 \leq i, j \leq n$ and $r \in R$, let $e_{ij}^r \in \mathrm{GL}_n(R)$ be the elementary matrix obtained from the identity by putting r at position (i, j) . We must check that each e_{ij}^r acts on $M(R^n)$ as a unipotent operator.

Since the action of $\mathrm{SL}_n(\mathbb{Z})$ on $M(R^n)$ extends to a rational representation of SL_n , each elementary matrix in $\mathrm{SL}_n(\mathbb{Z})$ acts as a unipotent operator [4, Theorem I.4.4]. For distinct $1 \leq i, j, k \leq n$, we have the Steinberg relation

$$e_{ij}^r = [e_{ik}^1, e_{kj}^r] = e_{ik}^1 e_{kj}^r (e_{ik}^1)^{-1} (e_{kj}^r)^{-1}.$$

Manipulating this, we get

$$(e_{ik}^1)^{-1} e_{ij}^r = e_{kj}^r (e_{ik}^1)^{-1} (e_{kj}^r)^{-1}.$$

Since e_{ik}^1 acts on $M(R^n)$ as a unipotent operator and the class of unipotent operators is closed under conjugation and inversion, the right hand side of this expression acts on $M(R^n)$ as a unipotent operator. The matrices

$$e_{ik}^1 \quad \text{and} \quad (e_{ik}^1)^{-1} e_{ij}^r$$

commute and act on $M(R^n)$ as unipotent operators, so by Lemma 13.1 their product e_{ij}^r acts on $M(R^n)$ as a unipotent operator, as desired. \square

13.3. Stability of invariants. If V is a module over a commutative ring \mathbb{k} and $f: V \rightarrow V$ is a module homomorphism, then let

$$V^f = \{v \in V \mid f(v) = v\}$$

denote the submodule of invariants. One basic property of unipotent operators on vector spaces of characteristic 0 is as follows:

Lemma 13.3. *Let V be a finite-dimensional vector space over a field \mathbb{k} of characteristic 0 and let $f: V \rightarrow V$ be a unipotent operator. Then for all $n \geq 1$ we have $V^f = V^{f^n}$.*

Proof. Since $V^f \subset V^{f^n}$, it is enough to prove that $\dim(V^f) = \dim(V^{f^n})$. Write $f = \mathrm{id}_V + \phi$ with ϕ nilpotent. Set $c_i = \binom{n}{i}$, so

$$f^n = \mathrm{id}_V + \phi' \quad \text{with} \quad \phi' = \sum_{i=1}^n c_i \phi^i.$$

We have

$$V^f = \ker(\phi) \quad \text{and} \quad V^{f^n} = \ker(\phi'),$$

such that DM is generated in degree N , so for $n \geq N$ the map

$$\bigoplus_{\substack{(f, C): [\mathbb{Z}^n] \rightarrow [\mathbb{Z}^{n+1}] \\ \text{VIC}(\mathbb{Z})\text{-morphism}}} DM(\mathbb{Z}^n) \rightarrow DM(\mathbb{Z}^{n+1})$$

is surjective. Since

$$DM(\mathbb{Z}^m) = \frac{M(\mathbb{Z}^m \oplus \mathbb{Z})}{\mathrm{Im}(M(\mathbb{Z}^m) \rightarrow M(\mathbb{Z}^m \oplus \mathbb{Z}))} \quad \text{for all } m \geq 0,$$

we deduce that for $n \geq N$ the map

$$M(\mathbb{Z}^{n+1}) \oplus \left(\bigoplus_{\substack{(f, C): [\mathbb{Z}^n] \rightarrow [\mathbb{Z}^{n+1}] \\ \text{VIC}(\mathbb{Z})\text{-morphism}}} M(\mathbb{Z}^n \oplus \mathbb{Z}) \right) \rightarrow M(\mathbb{Z}^{n+1} \oplus \mathbb{Z}) = M(\mathbb{Z}^{n+2})$$

is surjective, so M is generated in degree $N + 1$.

so our goal is to prove that $\ker(\phi)$ and $\ker(\phi')$ have the same dimension. Since \mathbb{k} has characteristic 0, we have $c_1 = n \neq 0$. This allows us to write

$$\phi' = c_1 \phi \circ \left(\text{id}_V + \sum_{i=1}^{n-1} \frac{c_{i+1}}{c_1} \phi^i \right) = c_1 \phi \circ (\text{id}_V + \phi'') \quad \text{with} \quad \phi'' = \sum_{i=1}^{n-1} \frac{c_{i+1}}{c_1} \phi^i.$$

The linear map ϕ'' is nilpotent, so $\text{id}_V + \phi''$ is invertible. It follows that

$$\dim \ker(\phi') = \dim \ker(c_1 \phi \circ (\text{id}_V + \phi'')) = \dim \ker(c_1 \phi) = \dim \ker(\phi),$$

as desired. \square

13.4. Stability of homology. Lemma 13.3 is the key input to the following result.

Lemma 13.4. *Let G be an abelian group and let V be a finite-dimensional unipotent representation of G over a field \mathbb{k} of characteristic 0. Then for all finite-index subgroups $G' < G$, the inclusion map $G' \hookrightarrow G$ induces an isomorphism $H_k(G'; V) \cong H_k(G; V)$ for all $k \geq 0$.*

Proof. We divide the proof into several steps.

Step 1. *This holds if G is an infinite cyclic group.*

We thus have $G = \mathbb{Z}$ and $G' = n\mathbb{Z}$ for some $n \geq 1$. Recall that the zeroth homology group is the coinvariants, which is isomorphic to the invariants of the dual. The dual representation V^* is also a unipotent representation of G , so by Lemma 13.3 we have

$$H_0(G; V) = (V^*)^G = (V^*)^{G'} = H_0(G'; V).$$

For the first homology group, since the circle is a classifying space for $G = \mathbb{Z}$, we can apply Poincaré duality and use Lemma 13.3 to see that

$$H_1(G; V) \cong H^0(G; V) = V^G = V^{G'} = H^0(G'; V) \cong H_1(G'; V).$$

Finally, for $k \geq 2$, we have $H_k(G; V) = H_k(G'; V) = 0$.

Step 2. *This holds if G is a finite abelian group.*

In this case, we claim that V is a trivial representation of G . Indeed, since G is finite it is enough to prove that every nontrivial unipotent operator $f: V \rightarrow V$ has infinite order. Write $f = \text{id}_V + \phi$ with $\phi \neq 0$ a nilpotent operator. For all $n \geq 1$, we then have

$$f^n = \text{id}_V + \sum_{i=1}^n \binom{n}{i} \phi^i.$$

Since \mathbb{k} has characteristic 0, we have $\binom{n}{1} = n \neq 0$, so the coefficient of ϕ in this expression is nonzero. Letting r be the order of ϕ , the operators $\{\text{id}_V, \phi, \phi^2, \dots, \phi^{r-1}\}$ are linearly independent in the vector space of linear operators, so $f^n \neq \text{id}_V$, as claimed.

From this, we see that

$$H_k(G'; V) = H_k(G; V) = \begin{cases} V & \text{if } k = 0, \\ 0 & \text{if } k \geq 1. \end{cases}$$

Step 3. *This holds if G is a finitely generated abelian group.*

We can find a chain of subgroups

$$G = G_1 \supset G_2 \supset \dots \supset G_n = G'$$

where for each $1 \leq i < n$ the group G_{i+1} is an index- p_i subgroup of G_i for some prime p_i . From this, we see that we can assume without loss of generality that G' is an index- p subgroup of G for some prime p .

Since G' is an index- p subgroup of the finitely generated abelian group G , we can write $G = C \oplus G''$ and $G' = C' \oplus G''$ with C a cyclic subgroup of G and C' an index- p subgroup of C . We thus have a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C' & \longrightarrow & G' & \longrightarrow & G'' & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 1 & \longrightarrow & C & \longrightarrow & G & \longrightarrow & G'' & \longrightarrow & 1. \end{array}$$

This induces a map between the Hochschild–Serre spectral sequences computing $H_\bullet(G'; V)$ and $H_\bullet(G; V)$. On the E^2 -page, this morphism takes the form

$$(13.1) \quad H_p(G''; H_q(C'; V)) \rightarrow H_p(G''; H_q(C; V)).$$

The previous two steps imply that the inclusion $C' \hookrightarrow C$ induces isomorphisms $H_q(C'; V) \cong H_q(C; V)$ for all q , so the map (13.1) is an isomorphism for all p and q . The spectral sequence comparison theorem now implies that $H_k(G'; V) \cong H_k(G; V)$ for all k , as desired.

Step 4. *This holds if G is an arbitrary abelian group.*

Let \mathfrak{F} be the set of finitely generated subgroups of G . We thus have

$$G = \varinjlim_{H \in \mathfrak{F}} H \quad \text{and} \quad G' = \varinjlim_{H \in \mathfrak{F}} H \cap G'.$$

Since homology commutes with direct limits, this reduces us to the previous case. \square

14. STABILITY FOR CONGRUENCE SUBGROUPS

We now turn to the proof of Theorem C. We will actually prove three increasingly stronger results, with the third a generalization of Theorem C.

14.1. Elementary matrices and unipotence. The first is as follows, which we regard as the heart of the whole proof. The differences between this result and Theorem C are as follows:

- It assumes that elementary matrices act unipotently on $M(R^n)$.
- It concerns the elementary congruence subgroup $\text{EL}_n(R, \alpha)$ rather than $\text{GL}_n(R, \alpha)$.
- Finally, it has a better range of stability.

Theorem 14.1. *Let R be a ring satisfying (SR_r) , let \mathbb{k} be a field of characteristic 0, and let M be a $\text{VIC}(R)$ -module over \mathbb{k} that is polynomial of degree $d \geq -1$ starting at $m \geq 0$. For each $n \geq 0$, assume that $M(R^n)$ is a finite-dimensional vector space over \mathbb{k} and that each elementary matrix in $\text{GL}_n(R)$ acts unipotently on $M(R^n)$. Then for all two sided ideals α of R such that $|R/\alpha| < \infty$, the map*

$$H_k(\text{EL}_n(R, \alpha); M(R^n)) \rightarrow H_k(\text{EL}_n(R); M(R^n))$$

is an isomorphism for $n \geq 2k + \max(2d + r - 1, m, r) + 1$.

Proof. Letting $\mathcal{K} = 0$, Example 9.4 says that for $n \geq r$ we have $\text{GL}_n^{\mathcal{K}}(R) = \text{EL}_n(R)$. In light of this, Theorem 12.1 implies that the stabilization map

$$H_k(\text{EL}_n(R); M(R^n)) \rightarrow H_k(\text{EL}_{n+1}(R); M(R^{n+1}))$$

is an isomorphism for $n \geq 2k + \max(2d + r - 1, m, r) + 1$. We proved this by verifying the conditions of our stability machine (Theorem 5.2) with the parameter $c = 2$ (corresponding to the multiple 2 in front of k in $n \geq 2k + \max(2d + r - 1, m, r) + 1$). Letting

$$e = \max(2d - 1, m - r, 0) \geq 0,$$

the inputs to Theorem 5.2 were

$$G_n = \text{EL}_{n+e+r}(R) \quad \text{and} \quad M_n = M(R^{n+e+r}) \quad \text{for } n \geq 0$$

and

$$\mathbb{X}_n = \mathbb{B}(R^{n+e,r}) \quad \text{and} \quad \mathcal{M}_n = \mathcal{G}_{M,n+e,r} \quad \text{for } n \geq 1.$$

For $n \geq r$, Lemma 9.9 implies that $\text{EL}_n(R, \alpha)$ is a finite-index normal subgroup of $\text{EL}_n(R)$. To prove that the map

$$\text{H}_k(\text{EL}_n(R, \alpha); M(R^n)) \rightarrow \text{H}_k(\text{EL}_n(R); M(R^n))$$

is an isomorphism for $n \geq 2k + \max(2d + r - 1, m, r) + 1$, it is enough to verify the additional hypotheses of Theorem 5.8 for

$$G'_n = \text{EL}_{n+e+r}(R, \alpha).$$

These additional hypotheses are numbered (e)-(h), and we verify each of them in turn.

Condition (e) says that each M_n is a vector space over a field \mathbb{k} of characteristic 0, which is one of our hypotheses.

Condition (f) says that for the k -simplex σ_k of \mathbb{X}_n from condition (b) of Theorem 5.2 whose G_n -stabilizer is G_{n-k-1} , the G'_n -stabilizer of σ_k is G'_{n-k-1} . Looking back at our proof of Theorem 12.1, the simplex σ_k is as follows. Let $\{v_1, \dots, v_{n+e+r}\}$ be the standard basis for R^{n+e+r} . We then have

$$\sigma_k = \begin{cases} (R^{n+e+r}) & \text{if } k = -1, \\ (v_{n+e+r-k}, v_{n+e+r-k+1}, \dots, v_{n+e+r}; R^{n+e+r-k-1}) & \text{if } 0 \leq k < n. \end{cases}$$

Thus the $\text{GL}_{n+e+r}(R)$ -stabilizer of σ_k is $\text{GL}_{n+e+r-k-1}(R)$, and by Lemma 9.5 the $G_n = \text{EL}_{n+e+r}(R)$ stabilizer of σ_k is $G_{n-k-1} = \text{EL}_{n+e+r-k-1}(R)$. Another application of Lemma 9.5 says that the $G'_n = \text{EL}_{n+e+r}(R, \alpha)$ stabilizer of σ_k is $G'_{n-k-1} = \text{EL}_{n+e+r-k-1}(R, \alpha)$, as desired.

Condition (g) says that the quotient \mathbb{X}_n/G'_n is $\lfloor \frac{n-2}{2} \rfloor$ -connected. Conclusion (iii) of Lemma 11.1 says that

$$\mathbb{X}_n/G'_n = \mathbb{B}(R^{n+e,r}) / \text{EL}_{n+e+r}(R, \alpha) \cong \mathbb{B}((R/\alpha)^{n+e,r}).$$

Lemma 8.10 says that R/α satisfies (SR_r) , so we can apply Conclusion (iv) of Lemma 11.1 to see that $\mathbb{B}((R/\alpha)^{n+e,r})$ is the large ordering of a weakly Cohen–Macaulay complex of dimension $\lfloor \frac{n+e+1}{2} \rfloor$. By Theorem 3.8, this implies that $\mathbb{B}((R/\alpha)^{n+e,r})$ is

$$\left\lfloor \frac{n+e+1}{2} \right\rfloor - 1 = \left\lfloor \frac{n+e-1}{2} \right\rfloor \geq \left\lfloor \frac{n-2}{2} \right\rfloor$$

connected. Here we use the fact that $e = \max(2d - 1, m - r, 0)$ is nonnegative.

Finally, the key condition (h) says that for $k \geq 0$ and $n \geq 2k + 2$, the action of G_n on $\text{H}_k(G'_n; M_n)$ induced by the conjugation action of G_n on G'_n fixes pointwise the image of the stabilization map

$$\text{H}_k(G'_{n-1}; M_{n-1}) \rightarrow \text{H}_k(G'_n; M_n).$$

This is the content of the following claim, which is the heart of our proof. Note that this claim is even stronger since it holds for all $G_n = \text{EL}_{n+e+r}(R)$, not just those where $n \geq 2k + 2$.

Claim. *For all $n \geq 1$, the group $\text{EL}_n(R)$ acts trivially on the image of the stabilization map*

$$(14.1) \quad \text{H}_k(\text{EL}_{n-1}(R, \alpha); M(R^{n-1})) \rightarrow \text{H}_k(\text{EL}_n(R, \alpha); M(R^n)).$$

The proof of this claim generalizes a beautiful argument of Charney from [8]. For distinct $1 \leq i, j \leq n$ and $r \in R$, let $e_{ij}^r \in \text{EL}_n(R)$ denote the elementary matrix obtained from the identity by putting r at position (i, j) . Define A to be the subgroup of $\text{EL}_n(R)$ generated by $\{e_{nj}^r \mid 1 \leq j \leq n-1, r \in R\}$ and let B be the subgroup generated by $\{e_{in}^r \mid 1 \leq i \leq n-1, r \in R\}$. Using the Steinberg relation

$$e_{ij}^r = [e_{ik}^r, e_{kj}^1] \quad \text{for distinct } 1 \leq i, j, k \leq n \text{ and } r \in R,$$

we see that $\text{EL}_n(R)$ is generated by $A \cup B$. It is thus enough to prove that A and B act trivially on the image of (14.1). The arguments for A and B are similar, so we will give the details for A and leave B to the reader.

The group A is the abelian subgroup of $\text{GL}_n(R)$ consisting of matrices that differ from the identity only in their n^{th} row. In particular, A is isomorphic to the abelian group R^{n-1} . Moreover, letting Γ be the subgroup of $\text{EL}_n(R)$ generated by A and $\text{EL}_{n-1}(R)$, the group Γ is a sort of ‘‘affine group’’ and in particular

$$(14.2) \quad \Gamma = A \rtimes \text{EL}_{n-1}(R).$$

We would like to imitate this for $\text{EL}_n(R, \alpha)$.

Define A_α to be the subgroup of $\text{EL}_n(R, \alpha)$ generated by $\{e_{nj}^r \mid 1 \leq j \leq n-1, r \in \alpha\}$. As an abelian group, we have $A_\alpha \cong \alpha^{n-1}$. Define Γ_α to be the subgroup of $\text{EL}_n(R, \alpha)$ generated by A_α and $\text{EL}_{n-1}(R, \alpha)$, so just like (14.2) we have

$$(14.3) \quad \Gamma_\alpha = A_\alpha \rtimes \text{EL}_{n-1}(R, \alpha).$$

The stabilization map (14.1) factors through the map

$$\text{H}_k(\Gamma_\alpha; M(R^n)) \rightarrow \text{H}_k(\text{EL}_n(R, \alpha); M(R^n))$$

induced by the inclusion $\Gamma_\alpha \hookrightarrow \text{EL}_n(R, \alpha)$.

The conjugation action of A on $\text{EL}_n(R, \alpha)$ takes Γ_α to itself. It is thus enough to prove that the conjugation action of A on $\text{H}_k(\Gamma_\alpha; M(R^n))$ is trivial. Define Γ'_α to be the subgroup of $\text{EL}_n(R)$ generated by A and $\text{EL}_{n-1}(R, \alpha)$. Since inner automorphisms act trivially on homology (even with twisted coefficients; see [7, Proposition III.8.1]), the conjugation action of A on $\text{H}_k(\Gamma'_\alpha; M(R^n))$ is trivial. It is thus enough to prove that the inclusion $\Gamma_\alpha \hookrightarrow \Gamma'_\alpha$ induces an isomorphism $\text{H}_k(\Gamma_\alpha; M(R^n)) \cong \text{H}_k(\Gamma'_\alpha; M(R^n))$.

For this, observe that we have a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & A_\alpha & \longrightarrow & \Gamma_\alpha & \longrightarrow & \text{EL}_{n-1}(R, \alpha) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow = \\ 1 & \longrightarrow & A & \longrightarrow & \Gamma'_\alpha & \longrightarrow & \text{EL}_{n-1}(R, \alpha) \longrightarrow 1 \end{array}$$

with exact rows. The first row is the split exact sequence corresponding to (14.3), and the second row is the split exact sequence corresponding to the similar semidirect product decomposition of Γ'_α . This induces a morphism between the Hochschild–Serre spectral sequences computing the homology of Γ_α and Γ'_α with coefficients in $M(R^n)$. On the E^2 -page, this map of spectral sequences takes the form

$$(14.4) \quad \text{H}_p(\text{EL}_{n-1}(R, \alpha); \text{H}_q(A_\alpha; M(R^n))) \rightarrow \text{H}_p(\text{EL}_{n-1}(R, \alpha); \text{H}_q(A; M(R^n))).$$

The abelian group A_α is a finite-index subgroup of the abelian group A , and since we assumed that elementary matrices act unipotently on $M(R^n)$ we can appeal to Lemma 13.1 to see that the action of A on $M(R^n)$ is a unipotent action. Lemma 13.4 therefore implies that the inclusion $A_\alpha \hookrightarrow A$ induces isomorphisms

$$\text{H}_q(A_\alpha; M(R^n)) \cong \text{H}_q(A; M(R^n)) \quad \text{for all } q.$$

We deduce that (14.4) is an isomorphism for all p and q . The spectral sequence comparison theorem therefore implies that the inclusion $\Gamma_\alpha \hookrightarrow \Gamma'_\alpha$ induces an isomorphism

$$H_k(\Gamma_\alpha; M(R^n)) \cong H_k(\Gamma'_\alpha; M(R^n)),$$

as desired. \square

14.2. Non-elementary subgroups. We now generalize Theorem 14.1 by extending it to subgroups other than the elementary ones. This causes us to have a worse range of stability.

Theorem 14.2. *Let R be a ring satisfying (SR_r) , let $\mathcal{K} \subset K_1(R)$ be a subgroup, let \mathbb{k} be a field of characteristic 0, and let M be a $\text{VIC}(R)$ -module over \mathbb{k} that is polynomial of degree $d \geq -1$ starting at $m \geq 0$. Assume furthermore that $M(R^n)$ is a finite-dimensional vector space over \mathbb{k} for all $n \geq 0$ and that each elementary matrix in $\text{GL}_n(R)$ acts unipotently on $M(R^n)$. Then for all two sided ideals α of R such that $|R/\alpha| < \infty$ and all subgroups $\mathcal{K}' \subset K_1(R, \alpha)$ that map to finite-index subgroups of \mathcal{K} under the map $K_1(R, \alpha) \rightarrow K_1(R)$, the map*

$$H_k(\text{GL}_n^{\mathcal{K}'}(R, \alpha); M(R^n)) \rightarrow H_k(\text{GL}_n^{\mathcal{K}}(R); M(R^n))$$

is an isomorphism for $n \geq 2k + \max(2d + r - 1, m, r) + r - 1$.

Proof. Consider some $n \geq 2k + \max(2d + r - 1, m, r) + r - 1$. Using Theorem 9.3, we have a commutative diagram³⁴

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \text{EL}_n(R, \alpha) & \longrightarrow & \text{GL}_n^{\mathcal{K}'}(R, \alpha) & \longrightarrow & \mathcal{K}' & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \text{EL}_n(R) & \longrightarrow & \text{GL}_n^{\mathcal{K}}(R) & \longrightarrow & \mathcal{K} & \longrightarrow & 1 \end{array}$$

with exact rows. This induces a map between the Hochschild–Serre spectral sequences computing the homology of $\text{GL}_n^{\mathcal{K}'}(R, \alpha)$ and $\text{GL}_n^{\mathcal{K}}(R)$ with coefficients in $M(R^n)$. On the E^2 -page, this map of spectral sequences is of the form

$$(14.5) \quad H_p(\mathcal{K}'; H_q(\text{EL}_n(R, \alpha); M(R^n))) \rightarrow H_p(\mathcal{K}; H_q(\text{EL}_n(R); M(R^n))).$$

We analyze this via a sequence of two claims.

Claim. *For all $q \leq k$, the map $H_q(\text{EL}_n(R, \alpha); M(R^n)) \rightarrow H_q(\text{EL}_n(R); M(R^n))$ is an isomorphism.*

Proof of claim. Since $r \geq 2$ (see Example 8.5), we have $2k + \max(2d + r - 1, m, r) + r - 1 \geq 2k + \max(2d + r - 1, m, r) + 1$. The claim is now immediate from Theorem 14.1. \square

Claim. *For all $q \leq k$, the action of the group \mathcal{K} (resp. \mathcal{K}') on $H_q(\text{EL}_n(R); M(R^n))$ (resp. $H_q(\text{EL}_n(R, \alpha); M(R^n))$) is trivial.*

Proof of claim. The previous claim implies that it is enough to prove that all of $K_1(R)$ acts trivially on $H_q(\text{EL}_n(R); M(R^n))$. Since

$$n - (r - 1) \geq (2k + \max(2d + r - 1, m, r) + r - 1) - (r - 1) = 2k + \max(2d + r - 1, m, r),$$

Theorem 12.1 implies that the map³⁵

$$(14.6) \quad H_q(\text{EL}_{n-(r-1)}(R); M(R^{n-(r-1)})) \rightarrow H_q(\text{EL}_n(R); M(R^n))$$

³⁴The first two columns are injections, but the third is not. If we wanted all the columns to be injective we could have used Lemma 9.7, but at the cost of having to deal with the slightly more complicated group $\overline{\text{EL}}_n^{\mathcal{K}'}(R, \alpha)$. Since our spectral sequence argument does not need the columns to be injective, this is unnecessary.

³⁵Here we are applying Theorem 12.1 to $\text{GL}_{n-(r-1)}^0(R)$, which by Example 9.4 we can identify with $\text{EL}_{n-(r-1)}(R)$. This uses the fact that $n - (r - 1) \geq r$.

is surjective. Theorem 9.3 together with Lemma 8.7 implies that the subgroup $\mathrm{GL}_{r-1}(R)$ of $\mathrm{GL}_n(R)$ surjects onto $\mathrm{K}_1(R)$ under the map $\mathrm{GL}_n(R) \rightarrow \mathrm{K}_1(R)$ whose kernel is $\mathrm{EL}_n(R)$. The group $\mathrm{GL}_{r-1}(R)$ is embedded in $\mathrm{GL}_n(R)$ using the upper-left-hand matrix embedding, but we can conjugate it by any element of $\mathrm{GL}_n(R)$ and it will still surject onto $\mathrm{K}_1(R)$. We can thus use the lower-right-hand embedding, which makes $\mathrm{GL}_{r-1}(R)$ commute with $\mathrm{EL}_{n-(r-1)}(R)$. This implies that $\mathrm{K}_1(R)$ acts trivially on the image of (14.6), and hence on $\mathrm{H}_q(\mathrm{EL}_n(R); M(R^n))$. \square

By the second Claim, for $q \leq k$ we can rewrite our map of spectral sequences (14.5) as

$$(14.7) \quad \mathrm{H}_p(\mathcal{K}'; \mathbb{k}) \otimes \mathrm{H}_q(\mathrm{EL}_n(R, \alpha); M(R^n)) \rightarrow \mathrm{H}_p(\mathcal{K}; \mathbb{k}) \otimes \mathrm{H}_q(\mathrm{EL}_n(R); M(R^n)).$$

We assumed that the map $\mathcal{K}' \rightarrow \mathcal{K}$ has finite cokernel, and Lemma 9.8 implies that it has finite kernel. Since \mathbb{k} is a field of characteristic 0, this implies that the map $\mathrm{H}_p(\mathcal{K}'; \mathbb{k}) \rightarrow \mathrm{H}_p(\mathcal{K}; \mathbb{k})$ is an isomorphism for all p . Combining this with the first Claim, we see that (14.7) is an isomorphism for all p and q with $q \leq k$. By the spectral sequence comparison theorem, we deduce that the map

$$\mathrm{H}_k(\mathrm{GL}_n^{\mathcal{K}'}(R, \alpha); M(R^n)) \rightarrow \mathrm{H}_k(\mathrm{GL}_n^{\mathcal{K}}(R); M(R^n))$$

is an isomorphism, as desired. \square

14.3. The general case: removing unipotence. We now prove the following theorem:

Theorem 14.3. *Let R be a ring satisfying (SR_r) , let $\mathcal{K} \subset \mathrm{K}_1(R)$ be a subgroup, let \mathbb{k} be a field of characteristic 0, and let M be a $\mathrm{VIC}(R)$ -module over \mathbb{k} that is polynomial of degree $d \geq -1$ starting at $m \geq 0$. Assume furthermore that $M(R^n)$ is a finite-dimensional vector space over \mathbb{k} for all $n \geq 0$. Then for all two sided ideals α of R such that $|R/\alpha| < \infty$ and all subgroups $\mathcal{K}' \subset \mathrm{K}_1(R, \alpha)$ that map to finite-index subgroups of \mathcal{K} under the map $\mathrm{K}_1(R, \alpha) \rightarrow \mathrm{K}_1(R)$, the map*

$$(14.8) \quad \mathrm{H}_k(\mathrm{GL}_n^{\mathcal{K}'}(R, \alpha); M(R^n)) \rightarrow \mathrm{H}_k(\mathrm{GL}_n^{\mathcal{K}}(R); M(R^n))$$

is an isomorphism for $n \geq \max(m, 2k + 2d + 2r - 1)$.

Remark 14.4. This implies Theorem C. Indeed, let R and α and \mathbb{k} and M be as in Theorem 14.3. By Lemma 9.8, the cokernel of the map $\mathrm{K}_1(R, \alpha) \rightarrow \mathrm{K}_1(R)$ is finite. We can thus apply Theorem 14.3 with $\mathcal{K} = \mathrm{K}_1(R)$ and $\mathcal{K}' = \mathrm{K}_1(R, \alpha)$. With these choices, we have by definition that

$$\mathrm{GL}_n^{\mathcal{K}'}(R, \alpha) = \mathrm{GL}_n(R, \alpha) \quad \text{and} \quad \mathrm{GL}_n^{\mathcal{K}}(R) = \mathrm{GL}_n(R).$$

Theorem 14.3 thus says that the map

$$\mathrm{H}_k(\mathrm{GL}_n(R, \alpha); M(R^n)) \rightarrow \mathrm{H}_k(\mathrm{GL}_n(R); M(R^n))$$

is an isomorphism for $n \geq \max(m, 2k + 2d + 2r - 1)$, as claimed by Theorem C. \square

Proof. Lemma 13.2 says that there exists some $u \geq 0$ such that for $n \geq u$, all elementary matrices in $\mathrm{GL}_n(R)$ act unipotently on $M(R^n)$. The *parameters* of M are the triple (d, m, u) . They satisfy $d \geq -1$ and $m, u \geq 0$. However, to avoid having to treat $m = 0$ and $u = 0$ separately we will allow m and u to be arbitrary integers, interpreted in the obvious way (so for instance if $u = -3$ then all elementary matrices in $\mathrm{GL}_n(R)$ act unipotently on $M(R^n)$ for all $n \geq 0$).

We will prove the theorem by induction on the parameters (d, m, u) . There are two base cases:

- The first is where $d \geq 0$ and $m, u \leq 0$. In this case, Theorem 14.2 says that (14.8) is an isomorphism for

$$n \geq 2k + \max(2d + r - 1, 0, r) + (r - 1).$$

Since $d \geq 0$, we have $2d + r \geq \max(2d + r - 1, 0, r)$, so this is even better than the bound

$$n \geq 2k + (2d + r) + (r - 1) = \max(0, 2k + 2d + 2r - 1)$$

we are trying to prove.

- The second is where $d = -1$ and $m, u \in \mathbb{Z}$ are arbitrary. In this case, by the definition of a $\text{VIC}(R)$ -module being polynomial of degree -1 starting at m we have for $n \geq m$ that $M(R^n) = 0$ and hence

$$\mathbb{H}_k(\text{GL}_n^{\mathcal{K}'}(R, \alpha); M(R^n)) = \mathbb{H}_k(\text{GL}_n^{\mathcal{K}}(R); M(R^n)) = 0.$$

In other words, for $n \geq m$ the domain and codomain of (14.8) are both 0, so it is trivially an isomorphism.

Now consider parameters (d, m, u) that do not lie in our base cases. They thus satisfy the following:

- $d \geq 0$, and either $m \geq 1$ or $u \geq 1$ (possibly both hold: $m \geq 1$ and $u \geq 1$).

Moreover, we can assume as an inductive hypothesis that the theorem is true for all M with parameters (d', m', u') satisfying the following:

- Either $d' \leq d - 1$, or $d' = d$ and both $m' \leq m - 1$ and $u' \leq u - 1$.

Note that it is *not* enough to shrink just one of m and u since otherwise we might never reach one of our base cases.

As in Definition 1.16, let ΣM be the shifted $\text{VIC}(R)$ -module and DM be the derived $\text{VIC}(R)$ -module. For $n \geq m$, we have a short exact sequence

$$(14.9) \quad 0 \longrightarrow M(R^n) \longrightarrow \Sigma M(R^n) \longrightarrow DM(R^n) \longrightarrow 0$$

of $\mathbb{k}[\text{GL}_n(R)]$ -modules. The $\text{VIC}(R)$ -module ΣM has parameters $(d, m - 1, u - 1)$, and the $\text{VIC}(R)$ -module DM has parameters $(d - 1, m - 1, u - 1)$.³⁶

To simplify our notation, for all $\mathbb{k}[\text{GL}_n(R)]$ -modules N , we will denote

- $\mathbb{H}_k(\text{GL}_n^{\mathcal{K}}(R); N)$ by $\mathbb{H}_k(N)$.
- $\mathbb{H}_k(\text{GL}_n^{\mathcal{K}'}(R, \alpha); N)$ by $\mathbb{H}_k(\alpha, N)$.

The long exact sequence in $\text{GL}_n^{\mathcal{K}'}(R, \alpha)$ -homology associated to (14.9) maps to the one in $\text{GL}_n^{\mathcal{K}}(R)$ -homology, so for $n \geq m$ and all k we have a commutative diagram

$$\begin{array}{ccccccccc} \mathbb{H}_{k+1}(\alpha, \Sigma M(R^n)) & \rightarrow & \mathbb{H}_{k+1}(\alpha, DM(R^n)) & \rightarrow & \mathbb{H}_k(\alpha, M(R^n)) & \rightarrow & \mathbb{H}_k(\alpha, \Sigma M(R^n)) & \rightarrow & \mathbb{H}_k(\alpha, DM(R^n)) \\ \downarrow g_1 & & \downarrow g_2 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 \\ \mathbb{H}_{k+1}(\Sigma M(R^n)) & \rightarrow & \mathbb{H}_{k+1}(DM(R^n)) & \rightarrow & \mathbb{H}_k(M(R^n)) & \rightarrow & \mathbb{H}_k(\Sigma M(R^n)) & \rightarrow & \mathbb{H}_k(DM(R^n)) \end{array}$$

with exact rows. When $n \geq r$, Lemma 9.9 says that $\text{GL}_n^{\mathcal{K}'}(R, \alpha)$ is a finite-index subgroup of $\text{GL}_n^{\mathcal{K}}(R)$. In that case, since all our coefficients are vector spaces over a field \mathbb{k} of characteristic 0, the transfer map (see [7, Chapter III.9]) implies that all the f_i and g_i are surjections. Our inductive hypothesis says the following about them:

³⁶The reason $u - 1$ appears in both ΣM and DM here is as follows. Consider $n \geq u - 1$. We have $\Sigma M(R^n) = M(R^{n+1})$. Since all elementary matrices in $\text{GL}_{n+1}(R)$ act unipotently on $M(R^{n+1})$, all elementary matrices in its subgroup $\text{GL}_n(R)$ do as well. If a linear operator ϕ acts unipotently on a vector space V and preserves a subspace $W \subset V$, then ϕ also acts unipotently on V/W . It follows that all elementary matrices in $\text{GL}_n(R)$ also act unipotently on the quotient space $DM(R^n)$ of $M(R^{n+1})$.

- Since ΣM has parameters $(d, m - 1, u - 1)$, the map f_2 is an isomorphism for $n \geq \max(m - 1, 2k + 2d + 2r - 1)$ and the map g_1 is an isomorphism for $n \geq \max(m - 1, 2(k + 1) + 2d + 2r - 1) = \max(m - 1, 2k + 2d + 2r + 1)$.
- Since DM has parameters $(d - 1, m - 1, u - 1)$, the map f_3 is an isomorphism for $n \geq \max(m - 1, 2k + 2(d - 1) + 2r - 1) = \max(m - 1, 2k + 2d + 2r - 3)$ and the map g_2 is an isomorphism for $n \geq \max(m - 1, 2(k + 1) + 2(d - 1) + 2r - 1) = \max(m - 1, 2k + 2d + 2r - 1)$.

For $n \geq \max(m, 2k + 2d + 2r - 1)$, the maps g_2 and f_2 and f_3 are isomorphisms and the map g_1 is a surjection (remember, it is always a surjection!), so by the five-lemma the map f_1 is an isomorphism, as desired. \square

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