CALCULATING THE SECOND RATIONAL COHOMOLOGY GROUP OF THE TORELLI GROUP

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Abstract. Minahan and the author recently proved results that allow the calculation of the second rational cohomology group of the Torelli group. This builds on two key ingredients: Hain's calculation of the image of the cup product pairing on the first cohomology group, and Kupers–Randal-Williams's calculation of the maximal algebraic subrepresentation of the second cohomology group. This paper gives an exposition of both of these results, including prerequisite material about the Johnson homomorphism.

Contents

1. Introduction 1

Part 1. Image of cup product pairing 5

2. Representation theory of SL_n 5
3. Free groups and free Lie algebras 8
4. Johnson filtration and homomorphisms 10
5. Johnson homomorphism calculations 14
6. Image of the Johnson homomorphism 19
7. Representation theory of Sp_{2g} 21
8. Projection maps and representation theory 24
9. First Johnson homomorphism 27
10. Second Johnson homomorphism 28
11. Ruling out trivial factors 31
12. Cup products and the Johnson homomorphism 32
13. Calculation of image of cup product pairing on surfaces with boundary 33
14. Calculation of image of cup product pairing on punctured surfaces 40
15. Calculation of image of cup product pairing on closed surfaces 41

Part 2. Cup products span 44

16. Representation stability and the degrees of the factors of $\mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q})$ 44
17. Twisted cohomology of the mapping class group and uniform degree bounds 47
18. Cup products span on surfaces with boundary 50
19. Cup products span on punctured surfaces 53
20. Cup products span on closed surfaces 55

References 56

1. Introduction

Let $\Sigma_{g,p}^b$ be an oriented genus g surface with p punctures and b boundary components. Assume that $p+b \leq 1$. We omit p or b if they vanish. The mapping class group $\mathrm{Mod}_{g,p}^b$ is the group of isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_{g,p}^b$ that fix each puncture and boundary component pointwise. The group $\mathrm{Mod}_{g,p}^b$ acts on $\mathrm{H}_1(\Sigma_{g,p}^b)$ and fixes

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the algebraic intersection form. Since $p+b \leq 1$, the algebraic intersection form is symplectic and this action gives a surjection $\text{Mod}_{g,p}^b \to \text{Sp}_{2g}(\mathbb{Z})$ whose kernel $\mathcal{I}_{g,p}^b$ is the Torelli group:

$$1 \longrightarrow \mathcal{I}_{g,p}^b \longrightarrow \text{Mod}_{g,p}^b \longrightarrow \text{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1.$$

In the early 1980's, Johnson [30] calculated $\text{H}^1(\mathcal{I}_{g,p}^b)$ for $g \gg 0$. Building on Minahan's PhD thesis [41], Minahan and the author [42] proved results that allowed them to calculate $\text{H}^2(\mathcal{I}_{g,p}^b;\mathbb{Q})$ for $g \gg 0$. We were able to do the following (see Corollary D below):

(i) Completely determine $\text{H}^2(\mathcal{I}_{g,p}^b;\mathbb{Q})$ as a vector space over $\mathbb{Q}$. In fact, we even describe it as a representation of $\text{Sp}_{2g}(\mathbb{Z})$, which acts in a way we will describe below.

(ii) Prove that all $\text{H}^2(\mathcal{I}_{g,p}^b;\mathbb{Q})$ is in the image of the cup product pairing on $\text{H}^1(\mathcal{I}_{g,p}^b;\mathbb{Q})$.

This calculation depends on previous work of Hain [19] and Kupers–Randal-Williams ([33]; see also its sequel [53]) that we will describe below. The goal of this paper is to explain how to do (i) and (ii) above given the results of [42], and in particular to give an exposition of this work of Hain and Kupers–Randal-Williams

1.1. Finite-dimensionality and algebraicity.

Recall our standing assumption that $p+b \leq 1$. The group $\text{Mod}_{g,p}^b$ acts on $\mathcal{I}_{g,b}^p$ via conjugation. Since inner automorphisms act trivially on group cohomology, we get an induced action of $\text{Mod}_{g,p}^b / \mathcal{I}_{g,p}^b = \text{Sp}_{2g}(\mathbb{Z})$ on each cohomology group $\text{H}^k(\mathcal{I}_{g,p}^b;\mathbb{Q})$. In other words, $\text{H}^k(\mathcal{I}_{g,p}^b;\mathbb{Q})$ is a representation of the arithmetic group $\text{Sp}_{2g}(\mathbb{Z})$.

Set $H = \text{H}_1(\Sigma_{g,p}^b;\mathbb{Q})$. Johnson [30] proved that for $g \geq 3$ we have

$$\text{H}^1(\mathcal{I}_{g,p}^b;\mathbb{Q}) \cong \begin{cases} \wedge^3 H & \text{if } p=1 \text{ or } b=1, \\ (\wedge^3 H)/H & \text{if } p=b=0. \end{cases}$$

Here H is embedded in $\wedge^3 H$ via the map $h \mapsto h \wedge \omega$, where $\omega \in \wedge^2 H$ represents the algebraic intersection pairing.[1] The above isomorphism is an isomorphism of representations of $\text{Sp}_{2g}(\mathbb{Z})$. From it, we deduce that the following two things hold for $g \geq 3$:

- $\text{H}^1(\mathcal{I}_{g,p}^b;\mathbb{Q})$ is finite-dimensional; and
- $\text{H}^1(\mathcal{I}_{g,p}^b;\mathbb{Q})$ is an algebraic representation of $\text{Sp}_{2g}(\mathbb{Z})$, i.e., the action of $\text{Sp}_{2g}(\mathbb{Z})$ on it extends to a representation of the algebraic group $\text{Sp}_{2g}(\mathbb{Q})$.

The main result of [42] extends this to the second rational cohomology group:

Theorem A (Minahan–Putman, [42, Theorem B])**.** *Fix some $p, b \geq 0$ with $p+b \leq 1$. Then $\text{H}^2(\mathcal{I}_{g,p}^b;\mathbb{Q})$ is finite-dimensional for $g \geq 5$ and an algebraic representation of $\text{Sp}_{2g}(\mathbb{Z})$ for $g \geq 6$.*

As we said above, we will assume the truth of Theorem A and describe how to calculate $\text{H}^2(\mathcal{I}_{g,p}^b;\mathbb{Q})$ as in (i) and (ii) above.

Remark 1.1. Dualizing, Theorem A implies that $\text{H}_2(\mathcal{I}_{g,p}^b;\mathbb{Q})$ is finite-dimensional for $g \geq 5$. It is still not known if $\text{H}_2(\mathcal{I}_{g,p}^b)$ is finitely generated or if $\mathcal{I}_{g,p}^b$ is finitely presented for $g \gg 0$. ☐

1.2. Cup product pairing.

The cup product pairing is an alternating pairing

$$\mathfrak{c}\colon \wedge^2 \text{H}^1(\mathcal{I}_{g,p}^b;\mathbb{Q}) \to \text{H}^2(\mathcal{I}_{g,p}^b;\mathbb{Q}).$$

Hain [19] calculated the image of this cup product pairing. To state his result, we must introduce some representation theoretic language. The irreducible algebraic representations

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[1] The algebraic intersection pairing ω is an alternating bilinear form on H, i.e., an element of the dual space $(\wedge^2 H)^* = \text{Hom}(\wedge^2 H, \mathbb{Q})$. The form ω identifies H with H^*, and thus $(\wedge^2 H)^*$ with $\wedge^2 H$. If $\{a_1, b_1, \ldots, a_g, b_g\}$ is a symplectic basis for H, then the resulting $\omega \in \wedge^2 H$ is $\omega = a_1 \wedge b_1 + \cdots + a_g \wedge b_g$.

of $\mathrm{Sp}_{2g}(\mathbb{Z})$ are indexed by partitions σ with at most g parts (see [16, §17]; we will describe this in more detail later). Let $\mathbf{V}_{\sigma}(g)$ be the representation corresponding to σ, so $\mathbf{V}_1(g) = H$ and $\mathbf{V}_{1^3}(g) = (\wedge^3 H)/H$. The domain of the cup product pairing is

$$\begin{aligned}\wedge^2 \mathrm{H}^1(\mathcal{I}_g;\mathbb{Q}) &\cong \wedge^2((\wedge^3 H)/H),\\ \wedge^2 \mathrm{H}^1(\mathcal{I}_g^1;\mathbb{Q}) &\cong \wedge^2(\wedge^3 H),\\ \wedge^2 \mathrm{H}^1(\mathcal{I}_{g,1};\mathbb{Q}) &\cong \wedge^2(\wedge^3 H).\end{aligned}$$

These decompose as[2]

$$\wedge^2((\wedge^3 H)/H) \cong \wedge^2 \mathbf{V}_{1^3}(g) \cong \mathbf{V}_0(g) \oplus \mathbf{V}_{1^2}(g) \oplus \mathbf{V}_{2^2}(g) \oplus \mathbf{V}_{1^4}(g) \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g)$$

and

$$\wedge^2(\wedge^3 H) \cong \mathbf{V}_0(g)^{\oplus 2} \oplus \mathbf{V}_{1^2}(g)^{\oplus 3} \oplus \mathbf{V}_{2^2}(g) \oplus \mathbf{V}_{2,1^2}(g) \oplus \mathbf{V}_{1^4}(g)^{\oplus 2} \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g).$$

Since the cup product pairing is equivariant, its image is isomorphic to a subrepresentation of this. Hain calculated it as follows:[3]

Theorem B (Hain, [19])**.** *Let $g \geq 6$. Fix some $p, b \geq 0$ with $p + b \leq 1$. Then the image of the cup product pairing $\mathfrak{c}\colon \wedge^2 \mathrm{H}^1(\mathcal{I}_{g,p}^b;\mathbb{Q}) \to \mathrm{H}^2(\mathcal{I}_{g,p}^b;\mathbb{Q})$ is isomorphic to the following representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$:*

- *For $\mathcal{I}_g$, the representation $\mathbf{V}_{1^2}(g) \oplus \mathbf{V}_{1^4}(g) \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g)$.*
- *For $\mathcal{I}_g^1$, the representation $\mathbf{V}_{1^2}(g)^{\oplus 2} \oplus \mathbf{V}_{2,1^2}(g) \oplus \mathbf{V}_{1^4}(g)^{\oplus 2} \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g)$.*
- *For $\mathcal{I}_{g,1}$, the representation $\mathbf{V}_0(g) \oplus \mathbf{V}_{1^2}(g)^{\oplus 2} \oplus \mathbf{V}_{2,1^2}(g) \oplus \mathbf{V}_{1^4}(g)^{\oplus 2} \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g)$.*

Remark 1.2. The discrete group $\mathrm{Sp}_{2g}(\mathbb{Z})$ is Zariski dense in the algebraic group $\mathrm{Sp}_{2g}(\mathbb{Q})$. If W is an algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$, it follows that representation-theoretic properties of W as a representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$ are the same as the corresponding properties of W as a representation of $\mathrm{Sp}_{2g}(\mathbb{Q})$. For instance, if W is an irreducible algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Q})$ then it is also an irreducible representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$, and similarly for the decomposition of W into a direct sum of irreducible representations. □

1.3. **Maximal algebraic subrepresentation.** For $g \geq 12$, Kupers–Randal-Williams ([33]; see also its sequel [53]) proved that the fragment of $\mathrm{H}^2(\mathcal{I}_{g,p}^b;\mathbb{Q})$ identified by Theorem B is the maximal algebraic subrepresentation of $\mathrm{H}^2(\mathcal{I}_{g,p}^b;\mathbb{Q})$. They proved this without knowing that $\mathrm{H}^2(\mathcal{I}_{g,p}^b;\mathbb{Q})$ is finite-dimensional, much less an algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$. Roughly speaking, they prove that a larger algebraic subrepresentation would contradict known calculations of $\mathrm{H}^2(\mathrm{Mod}_{g,p}^b;\mathbf{V})$ with $\mathbf{V}$ an algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$.

Remark 1.3. The work of Kupers–Randal-Williams does not depend on Hain's calculation of the image of the cup product pairing.[4] What they actually prove is that the maximal algebraic subrepresentation of $\mathrm{H}^2(\mathcal{I}_{g,p}^b;\mathbb{Q})$ is isomorphic to the representation identified by Theorem B. It then follows that it must be isomorphic to the image of the cup product pairing. If there was cohomology that was not in the image of the cup product pairing, their techniques would detect it. □

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[2] This calculation can easily be done using the program "LiE"; see [35].

[3] Hain also worked out the image for $g \in \{3,4,5\}$. We restrict to $g \geq 6$ since the work of Minahan–Putman [42] only works in this range and it allows a more uniform proof.

[4] Our proof of their theorem will use Hain's calculation, which we will use to give a lower bound on the size of $\mathrm{H}^2(\mathcal{I}_{g,p}^b;\mathbb{Q})$.

Using the fact that we now know that $\mathrm{H}^2(\mathcal{I}_{g,p}^b;\mathbb{Q})$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$ for $g \geq 6$ (Theorem A), we will give a simplified proof of Kupers–Randal-Williams's theorem. In fact, we will use not only Theorem A, but also some of the details from its proof. We will also use some work of Patzt [47] on representation stability to control certain stabilization maps. We will prove the following, whose statement combines Kupers–Randal-Williams's theorem with Theorem A of Minahan and the author:

Theorem C. *Let $g \geq 12$. Fix some $p, b \geq 0$ with $p + b \leq 1$. Then the image of the cup product pairing $\mathfrak{c}\colon \wedge^2 \mathrm{H}^1(\mathcal{I}_{g,p}^b;\mathbb{Q}) \to \mathrm{H}^2(\mathcal{I}_{g,p}^b;\mathbb{Q})$ spans $\mathrm{H}^2(\mathcal{I}_{g,p}^b;\mathbb{Q})$.*

Remark 1.4. The main way our proof is simpler than that of [33] is that we are able to avoid much of their complicated combinatorics, as well the need to carefully develop properties of the twisted Morita–Mumford classes. The paper [33] proves many other results beyond the ones we discuss, and for these their machinery seems essential. ☐

Remark 1.5. The published version of [33] claims to work for $g \geq 6$, but contains an error; see [34]. Their new range $g \geq 12$ is exactly what is needed to apply recent work of Miller–Patzt–Petersen–Randal-Williams [40] to deduce uniform twisted homological stability for $\mathrm{H}^2(\mathrm{Mod}_g^1;\mathbf{V})$. Here "uniform" means that the stable range does not depend on the algebraic representation $\mathbf{V}$ of $\mathrm{Sp}_{2g}(\mathbb{Z})$. We will also use the results of [40]. ☐

1.4. **Combining the results.** Combining Theorems B and C, we deduce the following corollary:

Corollary D (Minahan–Putman, [42])**.** *Let $g \geq 12$. Fix some $p, b \geq 0$ with $p + b \leq 1$. Then the $\mathrm{Sp}_{2g}(\mathbb{Z})$-representation $\mathrm{H}^2(\mathcal{I}_{g,p}^b;\mathbb{Q})$ is as follows:*

- *For $\mathcal{I}_g$, the representation $\mathbf{V}_{1^2}(g) \oplus \mathbf{V}_{1^4}(g) \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g)$.*
- *For $\mathcal{I}_g^1$, the representation $\mathbf{V}_{1^2}(g)^{\oplus 2} \oplus \mathbf{V}_{2,1^2}(g) \oplus \mathbf{V}_{1^4}(g)^{\oplus 2} \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g)$.*
- *For $\mathcal{I}_{g,1}$, the representation $\mathbf{V}_0(g) \oplus \mathbf{V}_{1^2}(g)^{\oplus 2} \oplus \mathbf{V}_{2,1^2}(g) \oplus \mathbf{V}_{1^4}(g)^{\oplus 2} \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g)$.*

In all three cases, $\mathrm{H}^2(\mathcal{I}_{g,p}^b;\mathbb{Q})$ is spanned by the image of the cup product pairing on $\mathrm{H}^1(\mathcal{I}_{g,p}^b;\mathbb{Q})$.

1.5. **Remark on representation theory.** The proofs of Theorems B and C rely on fairly intricate representation-theoretic calculations. Courses on representation theory often focus on theoretical aspects of the subject and do not prepare students to make explicit calculations. A secondary goal of this paper is to give many examples of concrete representation-theoretic calculations.

1.6. **Johnson homomorphisms.** The Johnson homomorphisms are a key tool in the proof of Theorem B. They were introduced by Johnson [26, 27] and further developed by Morita [44] and many others. We will give a fairly complete account (with proofs) of the part of this theory needed for Theorem B. In addition to making this paper more self-contained, this will also give us the opportunity to give some easier representation-theoretic calculations before moving on to the more difficult calculations in Theorem B.

1.7. **Outline of paper.** This paper has two parts: Part 1 proves Theorem B, and Part 2 proves Theorem C. Each part begins with an outline.

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Part 1. Image of cup product pairing

In this first part of the paper, we calculate the image of the cup product pairing. Most of this part is devoting to calculating this for $\mathcal{I}_g^1$, and then at the end we show how to deal with $\mathcal{I}_{g,1}$ and $\mathcal{I}_g$. We start by discussing the representation theory of the algebraic group SL_n in §2. Using this, in §3 we discuss the free Lie algebra and its connection to the lower central series of a free group.

With these preliminaries out of the way, we can now discuss our main tool: the generalized Johnson homomorphism. We introduce it in §4. We then make a number of calculations with it in §5 and prove a theorem of Morita about its image in §6. We then have two final sections of preliminary results in §7 – §8, which discuss the representation theory of the algebraic group Sp_{2g}. Using this, we calculate the images of the the first and second Johnson homomorphisms in §9 and §10.

We then turn to the image of the cup product pairing. In §11 we prove this image has no trivial subrepresentations. Next, in §12, we prove that this image can be calculated in terms of the first Johnson homomorphism. We then perform this calculation in §13. The second Johnson homomorphism plays an important role here, where it is used to compute an upper bound for the image of the cup product pairing. Everything we have done up until now is for $\mathcal{I}_g^1$, and we close this part in §14 and §15 by showing how to deal with $\mathcal{I}_{g,1}$ and $\mathcal{I}_g$.

2. REPRESENTATION THEORY OF SL_n

We begin with some preliminaries about the representation theory of the algebraic group SL_n. Fix a field $\mathbf{k}$ of characteristic 0. We will regard SL_n as being defined over $\mathbf{k}$, and all representations of SL_n will be finite-dimensional and defined over $\mathbf{k}$. Everything we discuss can be found in [5], and in the slightly different language of Lie algebra representations can also be found in [16].

2.1. **Maximal split torus.** Let $\mathbf{T}$ be the subgroup of SL_n consisting of diagonal matrices, so $\mathbf{T}(\mathbf{k}) \cong (\mathbf{k}^{\times})^{n-1}$. This is what is called a split algebraic torus, and is a maximal split torus in SL_n. For $t_1, \ldots, t_n \in \mathbf{k}$ with $t_1 \cdots t_n = 1$, let $\mathrm{diag}(t_1, \ldots, t_n) \in \mathbf{T}(\mathbf{k})$ be the corresponding diagonal matrix. A *character* of $\mathbf{T}$ is a homomorphism $\chi\colon \mathbf{T} \to \mathrm{GL}_1$. For some $d_1, \ldots, d_n \in \mathbb{Z}$, such a character can be written as

$$\chi(\mathrm{diag}(t_1, \ldots, t_n)) = t_1^{d_1} \cdots t_n^{d_n} \quad \text{for all } \mathrm{diag}(t_1, \ldots, t_n) \in \mathbf{T}(\mathbf{k}).$$

Since $t_1 \cdots t_n = 1$, we can assume that $d_n = 1$, in which case this representation is unique. Let $\chi(\mathbf{T})$ be the set of all characters of $\mathbf{T}$. This is an abelian group, and by the above we have $\chi(\mathbf{T}) \cong \mathbb{Z}^{n-1}$. For $1 \leq i \leq n$, let $E_i \in \chi(\mathbf{T})$ be character defined by

$$E_i(\mathrm{diag}(t_1, \ldots, t_n)) = t_i.$$

The characters $\{E_1, \ldots, E_n\}$ generate $\chi(\mathbf{T})$ and satisfy the single relation $E_1 + \cdots + E_n = 0$.

2.2. **Weight decomposition.** Let W be a representation of SL_n. For $\chi \in \chi(\mathbf{T})$, let

$$W_\chi = \{w \in W \mid D \cdot w = \chi(D)w \text{ for all } D \in \mathbf{T}(\mathbf{k})\}.$$

If $W_\chi \neq 0$, then χ is a *weight* of W and W_χ is its corresponding *weight space*. A nonzero vector $w \in W_\chi$ is a *weight vector* with weight χ. We have a direct sum decomposition

$$W = \bigoplus_{\chi \in \chi(\mathbf{T})} W_\chi$$

called the *weight decomposition* of W.

Example 2.1. Let $\{e_1, \ldots, e_n\}$ be the standard basis for the standard representation $\mathbf{k}^n$ of SL_n. The weight decomposition of $\mathbf{k}^n$ is then

$$\mathbf{k}^n = \bigoplus_{i=1}^{n} (\mathbf{k}^n)_{E_i} \quad \text{with } (\mathbf{k}^n)_{E_i} = \langle e_i \rangle.$$

Similarly, for $n \geq 2$ the weight decomposition of $\wedge^2 \mathbf{k}^n$ is

$$\wedge^2 \mathbf{k}^n = \bigoplus_{1 \leq i < j \leq n} (\wedge^2 \mathbf{k}^n)_{E_i + E_j} \quad \text{with } (\wedge^2 \mathbf{k}^n)_{E_i + E_j} = \langle e_i \wedge e_j \rangle$$

and the weight decomposition of $\mathrm{Sym}^2(\mathbf{k}^n)$ is

$$\mathrm{Sym}^2(\mathbf{k}^n) = \left(\bigoplus_{i=1}^{n} \mathrm{Sym}^2(\mathbf{k}^n)_{2E_i} \right) \oplus \left(\bigoplus_{1 \leq i < j \leq n} \mathrm{Sym}^2(\mathbf{k}^n)_{E_i + E_j} \right)$$

with

$$\mathrm{Sym}^2(\mathbf{k}^n)_{2E_i} = \langle e_i \cdot e_i \rangle \quad \text{and} \quad \mathrm{Sym}^2(\mathbf{k}^n)_{E_i + E_j} = \langle e_i \cdot e_j \rangle. \qquad \square$$

2.3. **Highest weight vectors.** Let $\mathbf{U}$ be the subgroup of SL_n consisting of strictly upper triangular matrices. If W is a representation of SL_n, then a *highest weight vector* in W with weight $\chi \in \chi(\mathbf{T})$ is a nonzero vector $w \in W$ such that

- w is a weight vector with weight χ, so $w \neq 0$ and $w \in W_\chi$; and
- w is fixed by all elements of $\mathbf{U}(\mathbf{k})$.

A *highest weight* of W is the weight of a highest weight vector in W.

The integer points $\mathbf{U}(\mathbb{Z})$ are Zariski dense in $\mathbf{U}(\mathbf{k})$, so to check that a weight vector $w \in W$ is a highest weight vector it is enough to check that it is invariant under $\mathbf{U}(\mathbb{Z})$. In fact, it is enough to check this on generators for this group, which are as follows. Let $\{e_1, \ldots, e_n\}$ be the standard basis for $\mathbf{k}^n$.

- For $1 \leq i < j \leq n$, the matrix $E_{ij} \in \mathbf{U}(\mathbf{k})$ that fixes all elements of $\{e_1, \ldots, e_n\}$ except for e_j, on which it does the following:

$$E_{ij}(e_j) = e_j + e_i.$$

We remark that these are examples of elementary matrices.

Example 2.2. Let $\{e_1, \ldots, e_n\}$ be the standard basis for the standard representation $\mathbf{k}^n$ of SL_n. The vector $e_1 \in \mathbf{k}^n$ is a highest weight vector with weight E_1. For $1 \leq k \leq n$, the vector $e_1 \wedge \cdots \wedge e_k \in \wedge^k \mathbf{k}^n$ is a highest weight vector with weight $E_1 + \cdots + E_k$. Also, for $k \geq 1$ the vector $e_1 \cdot \ldots \cdot e_1 \in \mathrm{Sym}^k(\mathbf{k}^n)$ is a highest weight vector with weight kE_1. One can check that up to scaling these are the only highest weight vectors in these representations. $\square$

2.4. **Theorem of the highest weight.** The representation theory of SL_n is controlled by highest weight vectors. Indeed, the theorem of the highest weight says that:

(a) All representations of SL_n decompose as direct sums of irreducible representations.
(b) Up to scaling, each irreducible representation W of SL_n contains a unique highest weight vector.
(c) If W is an arbitrary representation of SL_n and $w \in W$ is a highest weight vector, then the smallest subrepresentation containing w is irreducible.
(d) If W and W' are irreducible representations of SL_n with highest weight vectors $w \in W$ and $w' \in W'$ and if the weights of w and w' are the same, then W and W' are isomorphic representations.

Example 2.3. It follows from the above that the following are all distinct irreducible representations of SL_n:

- $\wedge^k \mathbf{k}^n$ for $1 \leq k \leq n$; and
- $\mathrm{Sym}^k(\mathbf{k}^n)$ for $k \geq 1$. ☐

2.5. **Dominant weights.** To complete the picture of the representation theory of SL_n, we must identify the weights that can appear as highest weights of irreducible representations of SL_n. A *dominant weight* is a character

$$\chi = k_1 E_1 + \cdots + k_n E_n \quad \text{with } k_1, \ldots, k_n \in \mathbb{Z}$$

such that the following hold:

- $k_1 \geq \cdots \geq k_n$; and
- $k_n = 0$, so in particular $k_i \geq 0$ for all $1 \leq i \leq n$. We remark that since the E_i satisfy the single relation $E_1 + \cdots + E_n = 0$, any character χ can be written uniquely as $\chi = k_1 E_1 + \cdots + k_n E_n$ with $k_n = 0$.

The dominant weights are exactly the weights of irreducible representations of SL_n. We will write $\mathbf{W}_\chi(n)$ for the irreducible representation of SL_n with highest weight a given dominant weight χ.

Recall that a *partition* of an integer d of length $m \geq 0$ is a tuple $\sigma = (k_1, \ldots, k_m)$ with $k_1 \geq \cdots \geq k_m \geq 1$ and $k_1 + \cdots k_m = d$. The dominant weights for SL_n are in bijection with partitions of integers with length at most $n-1$. Given such a partition $\sigma = (k_1, \ldots, k_m)$, the corresponding dominant weight is

$$\chi = k_1 E_1 + \cdots + k_m E_m.$$

We will also write $\mathbf{W}_\sigma(n)$ for $\mathbf{W}_\chi(n)$.

Convention 2.4. We will denote multiplicities in partitions using superscripts. For instance, if $\sigma = (5, 4, 4, 4, 1, 1)$ then we will write $\mathbf{W}_{5,4^3,1^2}(n)$ for $\mathbf{W}_\sigma(n)$. ☐

Here are some examples of this notation:

Example 2.5. For SL_n, we have the following:

- $\mathbf{W}_k(n) = \mathrm{Sym}^k(\mathbf{k}^n)$ for $k \geq 1$; and
- $\mathbf{W}_{1^k}(n) = \wedge^k \mathbf{k}^n$ for $1 \leq k < n$.

It is not the case that $\mathbf{W}_{1^n}(n) = \wedge^n \mathbf{k}^n$. Indeed, according to our conventions 1^n does not correspond to a dominant weight. Instead $\wedge^n \mathbf{k}^n$ is isomorphic to the trivial representation, so $\wedge^n \mathbf{k}^n = \mathbf{W}_0(n)$. ☐

2.6. **Stable decompositions.** Let $\sigma = (k_1, \ldots, k_m)$ be a partition of d with at most $n-1$ parts. We will call d the *degree* of the partition. We will also say that d is the degree of the irreducible representation $\mathbf{W}_\sigma(n)$. Schur–Weyl duality implies that the irreducible representation $\mathbf{W}_\sigma(n)$ appears in in $(\mathbf{k}^n)^{\otimes d}$, and moreover that all irreducible representations that appear in $(\mathbf{k}^n)^{\otimes d}$ have degree at most d.

A classical observation is that for $n \geq d+1$, the decomposition of $(\mathbf{k}^n)^{\otimes d}$ into irreducible factors is independent of the parameter n in the following sense:

- If
$$(\mathbf{k}^n)^{\otimes d} = \mathbf{W}_{\sigma_1}(n) \oplus \cdots \mathbf{W}_{\sigma_k}(n)$$
for partitions $\sigma_1, \ldots, \sigma_k$, then we also have
$$(\mathbf{k}^{n+1})^{\otimes d} = \mathbf{W}_{\sigma_1}(n+1) \oplus \cdots \mathbf{W}_{\sigma_k}(n+1).$$

Here are some examples of this:[5]

Example 2.6. Consider $(\mathbf{k}^n)^{\otimes 2}$. This decomposes into irreducible representations in the following ways:

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[5] All these decompositions are calculated using the program "LiE"; see [35].

- For $n = 2$, as $(\mathbf{k}^2)^{\otimes 2} = \mathbf{W}_0(2) \oplus \mathbf{W}_2(2)$.
- For $n \geq 3$, as $(\mathbf{k}^n)^{\otimes 3} = \mathbf{W}_{1^2}(n) \oplus \mathbf{W}_2(n)$.

For $n \geq 3$, this is the decomposition into antisymmetric and symmetric tensors

$$(\mathbf{k}^n)^{\otimes 3} = \mathbf{W}_{1^2}(n) \oplus \mathbf{W}_2(n) = \left(\wedge^2 \mathbf{k}^n\right) \oplus \mathrm{Sym}^2(\mathbf{k}^n).$$

For $n = 2$, this decomposition is degenerate since $\wedge^2 \mathbf{k}^2$ is the trivial representation $\mathbf{W}_0$. $\square$

Example 2.7. Consider $(\mathbf{k}^n)^{\otimes 3}$. This decomposes into irreducible representations in the following ways:

- For $n = 2$, as $(\mathbf{k}^2)^{\otimes 3} = \mathbf{W}_1(2)^{\oplus 2} \oplus \mathbf{W}_3(2)$.
- For $n = 3$, as $(\mathbf{k}^3)^{\otimes 3} = \mathbf{W}_0(3) \oplus \mathbf{W}_{2,1}(3)^{\oplus 2} \oplus \mathbf{W}_3(3)$.
- For $n \geq 4$, as $(\mathbf{k}^n)^{\otimes 3} = \mathbf{W}_{1^3}(n) \oplus \mathbf{W}_{2,1}(n)^{\oplus 2} \oplus \mathbf{W}_3(n)$.

Here for $n \geq 4$ two of the factors are $\mathbf{W}_{1^3}(n) = \wedge^3 \mathbf{k}^n$ and $\mathbf{W}_3(n) = \mathrm{Sym}^3(\mathbf{k}^n)$, while the other factor $\mathbf{W}_{2,1}(n)$ which appears with multiplicity 2 is harder to interpret. Let $\{e_1, \ldots, e_n\}$ be the standard basis for $\mathbf{k}^n$. Embedding $\wedge^2 \mathbf{k}^n$ into $(\mathbf{k}^n)^{\otimes 2}$ in the usual way, the following are highest weight vectors with weight $2E_1 + E_2$:

$$\begin{aligned} w &= e_1 \otimes (e_1 \wedge e_2) = e_1 \otimes e_1 \otimes e_2 - e_1 \otimes e_2 \otimes e_1, \\ w' &= (e_1 \wedge e_2) \otimes e_1 = e_1 \otimes e_2 \otimes e_1 - e_2 \otimes e_1 \otimes e_1. \end{aligned}$$

The subrepresentations of $(\mathbf{k}^n)^{\otimes 3}$ spanned by w and w' are the two copies of $\mathbf{W}_{2,1}(n)$. These are not unique, and any nontrivial linear combination of w and w' is also a highest weight vector of weight $2E_1 + E_2$. $\square$

More generally, if U is a representation of SL_n constructed from the standard representation $\mathbf{k}^n$ using tensor powers, exterior powers, and symmetric powers, then U naturally embeds into $(\mathbf{k}^n)^{\oplus d}$ for a $d \geq 1$ called its degree, and the decomposition of U into irreducible factors is independent of n as long as $n \geq d + 1$. For example:

Example 2.8. Let $U = \mathrm{Sym}^2(\wedge^3 \mathbf{k}^n) \otimes \mathbf{k}^n$. Then U embeds into $(\mathbf{k}^n)^{\otimes 7}$ and has degree 7, and the decomposition of U into irreducible representations is independent of n once $n \geq 8$. $\square$

Convention 2.9. In light of all of this, we can decompose representations of SL_n for $n \gg 0$ by using a computer to make this decomposition for n larger than the degree of the representation. This will be done silently throughout the remainder of the paper. $\square$

Remark 2.10. The idea of stable decompositions of representations has since been subsumed into Church–Farb's notion of representation stability [10]. See [13] for a survey. $\square$

3. FREE GROUPS AND FREE LIE ALGEBRAS

Our next topic is the Lie algebra associated to the lower central series of a group. Everything we discuss here without reference can be found in [56].

3.1. Lower central series.

Let G be a group. The *lower central series* of G is the following inductively defined sequence of subgroups $\gamma_d(G)$:

$$\gamma_1(G) = G \quad \text{and} \quad \gamma_{d+1}(G) = [G, \gamma_d(G)] \text{ for } d \geq 1.$$

By definition, G is nilpotent of degree at most d if $\gamma_{d+1}(G) = 1$. Each quotient group $\gamma_d(G)/\gamma_{d+1}(G)$ is abelian. In fact, even more is true: the conjugation action of G on its normal subgroup $\gamma_d(G)$ descends to a trivial action of G on $\gamma_d(G)/\gamma_{d+1}(G)$. This subsumes being abelian since it implies in particular that the conjugation action of $\gamma_d(G)$ on itself descends to the trivial action on $\gamma_d(G)/\gamma_{d+1}(G)$.

3.2. **Lie ring associated to the lower central series.** For a group G and a field $\mathbf{k}$, define

$$\mathcal{L}_d(G;\mathbf{k}) = \gamma_d(G)/\gamma_{d+1}(G) \otimes \mathbf{k} \quad \text{for } d \geq 1.$$

The commutator bracket $[-,-]$ on G descends to a bilinear map $\mathcal{L}_d(G;\mathbf{k}) \times \mathcal{L}_e(G;\mathbf{k}) \to \mathcal{L}_{d+e}(G;\mathbf{k})$ that we will also denote by $[-,-]$. Set

$$\mathcal{L}(G;\mathbf{k}) = \bigoplus_{d=1}^{\infty} \mathcal{L}_d(G;\mathbf{k}).$$

The bracket discussed above turns $\mathcal{L}(G;\mathbf{k})$ into a graded Lie algebra over $\mathbf{k}$ that is generated by its degree-1 elements $\mathcal{L}_1(G;\mathbf{k}) \cong G^{\text{ab}} \otimes \mathbf{k}$.

3.3. **Free groups.** Let F_n be the free group on n letters $\{x_1,\ldots,x_n\}$. We thus have

$$\mathcal{L}_1(F_n;\mathbf{k}) = F_n^{\text{ab}} \otimes \mathbf{k} \cong \mathbf{k}^n.$$

For $1 \leq i \leq n$, let $e_i \in \mathcal{L}_1(F_n;\mathbf{k})$ be the image of x_i. It follows from work of Magnus and Witt that the graded Lie algebra $\mathcal{L}(F_n;\mathbf{k})$ is isomorphic to the free Lie algebra $\text{FLie}(\mathbf{k}^n)$ over $\mathbf{k}$ generated by $\{e_1,\ldots,e_n\}$.

3.4. **Free Lie algebra in tensor algebra.** Another way of viewing the free Lie algebra is as follows. Let

$$\mathcal{T}(\mathbf{k}^n) = \bigoplus_{d\geq 1} (\mathbf{k}^n)^{\otimes d}$$

be the tensor algebra generated by $\mathbf{k}^n$. This can be turned into a graded Lie algebra using the bracket

$$[t,t'] = tt' - t't \quad \text{for } t,t' \in \mathcal{T}(\mathbf{k}^n).$$

We then have that $\text{FLie}(\mathbf{k}^n)$ is isomorphic to the sub-Lie algebra of $\mathcal{T}(\mathbf{k}^n)$ generated by its elements of degree 1, i.e., generated by the standard basis $\{e_1,\ldots,e_n\}$ for $\mathbf{k}^n$. In fact, $\mathcal{T}(\mathbf{k}^n)$ is what is called the universal enveloping algebra of the free Lie algebra.

3.5. **Lie representation.** Let $\mathbf{k}$ be a field of characteristic 0. The action of $\text{SL}_n(\mathbf{k})$ on $\mathbf{k}^n$ induces an action of $\text{SL}_n(\mathbf{k})$ on $\text{FLie}(\mathbf{k}^n)$. Each graded term $\text{FLie}_d(\mathbf{k}^n)$ is a finite-dimensional algebraic representation of $\text{SL}_n(\mathbf{k})$. In fact, using the embedding of the free Lie algebra into the tensor algebra we see that $\text{FLie}_d(\mathbf{k}^n)$ is a subrepresentation of $(\mathbf{k}^n)^{\otimes d}$. It is often called the d^{th} Lie representation of $\text{SL}_n(\mathbf{k})$. See [32] for a complete description of the decomposition of the Lie representation into irreducible factors.

3.6. **Low degree Lie representations.** We will only need to understand the first few terms of the Lie representation, which can easily be worked out by hand. Let $E_1,\ldots,E_n$ be the characters of the maximal split torus of SL_n, so

$$E_i(\text{diag}(t_1,\ldots,t_n)) = t_i \quad \text{for all } t_1,\ldots,t_n \in \mathbf{k}^\times \text{ such that } t_1\cdots t_n = 1.$$

To simplify our notation in the calculations below, we will omit the $\otimes$ symbols when writing elements of tensor powers of $\mathbf{k}^n$. For instance, $e_1e_2e_1$ stands for $e_1 \otimes e_2 \otimes e_1$. Just like before, all the representation theory calculations in the examples below were performed with LiE; see [35].

Example 3.1. For $n \geq 2$, we have $\text{FLie}_1(\mathbf{k}^n) = \mathbf{k}^n \cong \mathbf{W}_1(n)$. □

Example 3.2. For $n \geq 3$, we have $\text{FLie}_2(\mathbf{k}^n) = \wedge^2 \mathbf{k}^n \cong \mathbf{W}_{1^2}(n)$. This reflects the fact that the Lie bracket is alternating. Regarding $\text{FLie}_2(\mathbf{k}^n)$ as a subspace of $(\mathbf{k}^n)^{\otimes 2}$, it has the following highest weight vector with weight $E_1 + E_2$:

$$[e_1,e_2] = e_1e_2 - e_2e_1. \qquad \square$$

Example 3.3. For $n \geq 4$, we have

$$\mathrm{FLie}_3(\mathbf{k}^n) = \frac{\mathbf{k}^n \otimes \wedge^2 \mathbf{k}^n}{\wedge^3 \mathbf{k}^n} \cong \mathbf{W}_{2,1}(n).$$

This isomorphism arises from the surjective Lie bracket map from

$$\mathbf{k}^n \otimes \mathrm{FLie}_2(\mathbf{k}^n) = \mathbf{k}^n \otimes \wedge^2 \mathbf{k}^n \cong \mathbf{W}_{2,1}(n) \oplus \mathbf{W}_{1^3}(n)$$

to $\mathrm{FLie}_3(\mathbf{k}^n)$. The fact that $\wedge^3 \mathbf{k}^n \cong \mathbf{W}_{1^3}(n)$ is in the kernel of this Lie bracket map follows from the Jacobi identity. Regarding $\mathrm{FLie}_3(\mathbf{k}^n)$ as a subspace of $(\mathbf{k}^n)^{\otimes 3}$, it has the following highest weight vector with weight $2E_1 + E_2$:

$$[e_1, [e_1, e_2]] = [e_1, e_1 e_2 - e_2 e_1] = e_1 e_1 e_2 - 2 e_1 e_2 e_1 + e_2 e_1 e_1. \qquad \square$$

Example 3.4. For $n \geq 5$, we have

$$\mathrm{FLie}_4(\mathbf{k}^n) \cong \mathbf{W}_{3,1}(n) \oplus \mathbf{W}_{2,1^2}(n).$$

To see this, note that just like in degree 3 the Lie bracket map gives a surjection from

$$\mathbf{k}^n \otimes \mathrm{FLie}_3(\mathbf{k}^n) \cong \mathbf{k}^n \otimes \mathbf{W}_{2,1}(n) \cong \mathbf{W}_{3,1}(n) \oplus \mathbf{W}_{2,1^2}(n) \oplus \mathbf{W}_{2,2}(n)$$

to $\mathrm{FLie}_4(\mathbf{k}^n)$. To see that this Lie bracket map takes $\mathbf{k}^n \otimes \mathrm{FLie}_3(\mathbf{k}^n)$ to a representation containing $\mathbf{W}_{3,1}(n)$ and $\mathbf{W}_{2,1^2}(n)$, note that its image contains the following highest weight vectors:

$$\begin{aligned}
[e_1, [e_1, [e_1, e_2]]] &= [e_1, e_1 e_1 e_2 - 2 e_1 e_2 e_1 + e_2 e_1 e_1] \\
&= e_1 e_1 e_1 e_2 - 3 e_1 e_1 e_2 e_1 + 3 e_1 e_2 e_1 e_1 - e_2 e_1 e_1 e_1, \\
[[e_1, e_2], [e_1, e_3]] &= [e_1 e_2 - e_2 e_1, e_1 e_3 - e_3 e_1] \\
&= (e_1 e_2 e_1 e_3 - e_1 e_3 e_1 e_2) - (e_1 e_2 e_3 e_1 - e_3 e_1 e_1 e_2) \\
&\quad - (e_2 e_1 e_1 e_3 - e_1 e_3 e_2 e_1) + (e_2 e_1 e_3 e_1 - e_3 e_1 e_2 e_1).
\end{aligned}$$

On the other hand, the kernel of the Lie bracket map contains a subrepresentation isomorphic to $\mathbf{W}_{2,2}(n)$. Indeed, the domain of the Lie bracket map is

$$\mathbf{k}^n \otimes \mathrm{FLie}_3(\mathbf{k}^n) \cong \frac{(\mathbf{k}^n)^{\otimes 2} \otimes (\wedge^2 \mathbf{k}^n)}{\mathbf{k}^n \otimes \wedge^3 \mathbf{k}^n}.$$

The representation $(\mathbf{k}^n)^{\otimes 2} \otimes (\wedge^2 \mathbf{k}^n)$ contains the subrepresentation $\mathrm{Sym}^2(\wedge^2 \mathbf{k}^n)$, whose image in $\mathbf{k}^n \otimes \mathrm{FLie}_3(\mathbf{k}^n)$ lies in the kernel of the Lie bracket map since the Lie bracket map is alternating. Since $\mathrm{Sym}^2(\wedge^2 \mathbf{k}^n)$ is not contained in $\mathbf{k}^n \otimes \wedge^3 \mathbf{k}^n$, this give a nonzero subrepresentation in the kernel of the Lie bracket map. The only possibility for it is $\mathbf{W}_{2,2}(n)$. In fact,

$$\mathrm{Sym}^2(\wedge^2 \mathbf{k}^n) \cong \mathbf{W}_{2,2}(n) \oplus \mathbf{W}_{1^4}(n) \cong \mathbf{W}_{2,2}(n) \oplus \wedge^4 \mathbf{k}^n,$$

so it must be the case that the image of $\mathrm{Sym}^2(\wedge^2 \mathbf{k}^n)$ in $\mathbf{k}^n \otimes \mathrm{FLie}_3(\mathbf{k}^n)$ is

$$\frac{\mathrm{Sym}^2(\wedge^2 \mathbf{k}^n)}{\wedge^4 \mathbf{k}^n} \cong \mathbf{W}_{2,2}(n). \qquad \square$$

4. JOHNSON FILTRATION AND HOMOMORPHISMS

The first Johnson homomorphism was introduced by Johnson [26] following earlier work by Sullivan [57]. Johnson [27] later extended this to the higher Johnson homomorphisms. This theory was then developed further by Morita [44] and many others. See [20, 55] for surveys. This section discusses the basic properties of the Johnson homomorphisms.

4.1. **Johnson filtration.** Recall that Σ_g^1 is a genus-g surface with one boundary component. Fix a basepoint $* \in \partial \Sigma_g^1$ and let $\pi = \pi_1(\Sigma_g^1, *)$. Since the mapping class group Mod_g^1 fixes $\partial \Sigma_g^1$ pointwise, the group Mod_g^1 acts on π. This action preserves the lower central series $\gamma_k(\pi)$, so for each $d \geq 1$ we get an action of Mod_g^1 on the d-step nilpotent group

$$N_d(\pi) = \pi / \gamma_{d+1}(\pi).$$

The d^{th} term of the *Johnson filtration*, denoted $\mathcal{I}_g^1[d]$, is the kernel of the action of Mod_g^1 on $N_d(\pi)$. Since $N_1(\pi) = \pi^{\text{ab}} = \mathrm{H}_1(\Sigma_g^1; \mathbb{Z})$, the first term of the Johnson filtration is the Torelli group $\mathcal{I}_g^1$. The Johnson filtration thus forms a descending sequence of groups

$$\mathcal{I}_g^1 = \mathcal{I}_g^1[1] \rhd \mathcal{I}_g^1[2] \rhd \mathcal{I}_g^1[3] \rhd \cdots .$$

The Dehn–Nielsen–Baer theorem [14, Chapter 8] says that Mod_g^1 acts faithfully on π. Since $\cap_{d=1}^{\infty} \gamma_{d+1}(\pi) = 1$ (see [15, Lemma 4.2] or [38, Theorem 1.2]), it follows that $\cap_{d=1}^{\infty} \mathcal{I}_g^1[d] = 1$.

4.2. **Basic properties.** We will soon prove that each $\mathcal{I}_g^1[d] / \mathcal{I}_g^1[d+1]$ is abelian; indeed, the d^{th} Johnson homomorphism is a homomorphism from $\mathcal{I}_g^1[d]$ to an abelian group whose kernel is $\mathcal{I}_g^1[d+1]$. Before doing this, we explain some basic properties of the Johnson filtration.

As we said above, $\mathcal{I}_g^1[1]$ is the ordinary Torelli group. Building on work of Birman [1], Powell ([48]; see [23, 49] for modern proofs) proved that $\mathcal{I}_g^1 = \mathcal{I}_g^1[1]$ is generated by two kinds of elements. For a simple closed curve γ on Σ_g^1, let T_γ denote the left Dehn twist about γ. Powell's generating set then includes:

- Separating twists, that is, Dehn twists T_z such that z is a simple closed separating curve on Σ_g^1.
- Bounding pair maps, that is products $T_x T_y^{-1}$ such that x and y are disjoint nonseparating simple closed curves on Σ_g^1 such that $x \cup y$ separates Σ_g^1.

Johnson ([29]; see [50] for a modern proof) proved that $\mathcal{I}_g^1[2]$ is the subgroup of Torelli group generated by Dehn twists about separating curves. For $d \geq 3$, no explicit generating set for $\mathcal{I}_g^1[d]$ is known, though [8] proves that for each $d \geq 1$ there is some h_d such that $\mathcal{I}_g^1[d]$ is generated by mapping classes supported on subsurfaces of genus at most h_d for all $g \geq h_d$.

As far as finiteness properties go, Johnson [28] proved that $\mathcal{I}_g^1 = \mathcal{I}_g^1[1]$ is finitely generated for $g \geq 3$. This is in contrast to $\mathcal{I}_2^1$, which McCullough–Miller [39] proved is not finitely generated. More recently, Ershov–He [12] and Church–Ershov–Putman [9] proved that $\mathcal{I}_g^1[2]$ is finitely generated for $g \geq 4$. Ershov–Franz [11] gave an enormous explicit finite generating set for $\mathcal{I}_g^1[2]$, though finding one of a reasonable size is still an open question. For $d \geq 3$, Church–Ershov–Putman [9] proved that $\mathcal{I}_g^1[d]$ is finitely generated for $g \geq 2d-1$.

4.3. **Johnson homomorphism preliminaries, I.** Set $H = \mathrm{H}_1(\Sigma_g^1; \mathbb{Q})$. Recall that $\mathrm{FLie}_d(H)$ is the d^{th} graded piece of the free Lie algebra generated by H. Our next goal is to construct a homomorphism $\tau_d \colon \mathcal{I}_g^1[d] \to H \otimes \mathrm{FLie}_{d+1}(H)$ called the Johnson homomorphism such that $\ker(\tau_d) = \mathcal{I}_g^1[d+1]$. The construction we give originated in [27] and was elaborated on by Morita [44]. Recall from §3 that

$$\gamma_{d+1}(\pi)/\gamma_{d+2}(\pi) \otimes \mathbb{Q} \cong \mathrm{FLie}_{d+1}(H).$$

For $w \in \gamma_{d+1}(\pi)$, let $[\![w]\!]_{d+1}$ be its image in $\mathrm{FLie}_{d+1}(H)$.

Consider $f \in \mathrm{Mod}_g^1$. For $x \in \pi$, set $f_\ell(x) = f(x)x^{-1}$. We thus have

$$f(x) = f_\ell(x) x \quad \text{for all } x \in \pi.$$

The ℓ stands for "left". If $f \in \mathcal{I}_g^1[d]$, then by definition we have $f_\ell(x) \in \gamma_{d+1}(\pi)$ for all $x \in \pi$. It thus makes sense to talk about $[\![f_\ell(x)]\!]_{d+1} \in \text{FLie}_{d+1}(H)$. These elements satisfy the following key lemma:

Lemma 4.1. *Let the notation be as above, and consider $f \in \mathcal{I}_g^1[d]$. Then:*

- *(i) For $x, y \in \pi$, we have $[\![f_\ell(xy)]\!]_{d+1} = [\![f_\ell(x)]\!]_{d+1} + [\![f_\ell(y)]\!]_{d+1}$.*
- *(ii) For $z \in [\pi, \pi]$, we have $[\![f_\ell(z)]\!]_{d+1} = 0$.*

Proof. Conclusion (ii) follows from conclusion (i) since conclusion (i) implies that the map

$$z \mapsto [\![f_\ell(z)]\!]_{d+1} \quad \text{for all } z \in \pi$$

is a homomorphism from π to the abelian group $\text{FLie}_{d+1}(H)$. We must therefore only prove (i). Consider $x, y \in \pi$. We have $f(xy) = f_\ell(xy)xy$ and

$$f(xy) = f(x)f(y) = f_\ell(x)xf_\ell(y)y = f_\ell(x)xf_\ell(y)x^{-1}xy,$$

so $f_\ell(xy) = f_\ell(x)xf_\ell(y)x^{-1}$. It follows that

$$[\![f_\ell(xy)]\!]_{d+1} = [\![f_\ell(x)xf_\ell(y)x^{-1}]\!]_{d+1} = [\![f_\ell(x)]\!]_{d+1} + [\![xf_\ell(y)x^{-1}]\!]_{d+1} = [\![f_\ell(x)]\!]_{d+1} + [\![f_\ell(y)]\!]_{d+1}.$$

Here the final equality follows from the fact that the conjugation action of π on $\gamma_{d+1}(\pi)$ descends to the trivial action on $\gamma_{d+1}(\pi)/\gamma_{d+2}(\pi)$. □

4.4. **Johnson homomorphism preliminaries, II.** For $f \in \mathcal{I}_g^1[d]$, define $\widehat{\tau}_d(f)\colon \pi \to \text{FLie}_{d+1}(H)$ via the formula

$$\widehat{\tau}_d(f)(x) = [\![f_\ell(x)]\!]_{d+1} \quad \text{for all } x \in \pi.$$

Lemma 4.1 implies that $\widehat{\tau}_d(f)$ is a homomorphism. The collection of homomorphisms $\pi \to \text{FLie}_{d+1}(H)$ forms a $\mathbb{Q}$-vector space, so it makes sense to add two of them. We then have:

Lemma 4.2. *Let the notation be as above. For $f, h \in \mathcal{I}_g^1[d]$, we have $\widehat{\tau}_d(fh) = \widehat{\tau}_d(f) + \widehat{\tau}_d(h)$.*

Proof. Consider $x \in \pi$. By definition,

$$fh(x) = f(h_\ell(x)x) == f(h_\ell(x))f(x) = f_\ell(h_\ell(x))h_\ell(x)f_\ell(x)x.$$

It follows that

$$\begin{aligned}
\widehat{\tau}_d(fh)(x) &= [\![f_\ell(h_\ell(x))h_\ell(x)f_\ell(x)]\!]_{d+1} \\
&= [\![f_\ell(h_\ell(x))]\!]_{d+1} + [\![h_\ell(x)]\!]_{d+1} + [\![f_\ell(x)]\!]_{d+1} \\
&= 0 + \widehat{\tau}_d(f)(x) + \widehat{\tau}_d(h)(x).
\end{aligned}$$

Here the final equality uses the fact that

$$h_\ell(x) \in \gamma_{d+1}(\pi) \subset [\pi, \pi],$$

so by conclusion (ii) of Lemma 4.1 we have $[\![f_\ell(h_\ell(x))]\!]_{d+1} = 0$. □

4.5. **The Johnson homomorphism.** Lemma 4.2 implies that we can define a homomorphism $\widehat{\tau}_d\colon \mathcal{I}_g^1[d] \to \text{Hom}(\pi, \text{FLie}_{d+1}(H))$ via the formula

$$\widehat{\tau}_d(f) = [\![f_\ell(x)]\!]_{d+1} \quad \text{for all } x \in \pi.$$

Since $\text{FLie}_{d+1}(H)$ is a $\mathbb{Q}$-vector space and $H = \pi^{\text{ab}} \otimes \mathbb{Q}$, we have a canonical identification

$$\text{Hom}(\pi, \text{FLie}_{d+1}(H)) = \text{Hom}(H, \text{FLie}_{d+1}(H)).$$

Let ω be the algebraic intersection form on $H = \text{H}_1(\Sigma_g^1; \mathbb{Q})$. This is a symplectic form, and thus establishes an isomorphism between H and its dual $H^* = \text{Hom}(H, \mathbb{Q})$. Using ω, we can identify

$$\text{Hom}(H, \text{FLie}_{d+1}(H)) = H^* \otimes \text{FLie}_{d+1}(H) = H \otimes \text{FLie}_{d+1}(H).$$

This identification identifies $x \otimes \kappa \in H \otimes \mathrm{FLie}_{d+1}(H)$ with the following map $H \to \mathrm{FLie}_{d+1}(H)$:

$$h \mapsto \omega(x,h)\kappa \quad \text{for } h \in H.$$

Combining all of our identifications, we define $\tau_d \colon \mathcal{I}_g^1[d] \to \mathrm{FLie}_{d+1}(H) \otimes H$ to be the composition

$$\mathcal{I}_g^1[d] \xrightarrow{\widehat{\tau}_d} \mathrm{Hom}(\pi, \mathrm{FLie}_{d+1}(H)) = \mathrm{Hom}(H, \mathrm{FLie}_{d+1}(H)) = H \otimes \mathrm{FLie}_{d+1}(H).$$

This is the d^{th} Johnson homomorphism.

4.6. **Basic properties of Johnson homomorphism.** We close this section by discussing two basic properties of the Johnson homomorphism. The first is as follows:

Lemma 4.3. *For all $d \geq 1$, the kernel of $\tau_d \colon \mathcal{I}_g^1[d] \to H \otimes \mathrm{FLie}_{d+1}(H)$ is $\mathcal{I}_g^1[d+1]$. Consequently, $\mathcal{I}_g^1[d]/\mathcal{I}_g^1[d+1]$ is free abelian.*

Proof. By construction, the image of τ_d lies in the integer points of $H \otimes \mathrm{FLie}_{d+1}(H)$, so the second conclusion follows from the first. To prove the first conclusion, consider $f \in \mathcal{I}_g^1[d]$. We have $\tau_d(f) = 0$ if and only if $[\![f_\ell(x)]\!]_{d+1} = 0$ for all $x \in \pi$. By definition, $[\![f_\ell(x)]\!]_{d+1}$ is the image in

$$\mathrm{FLie}_{d+1}(H) = \gamma_{d+1}(\pi)/\gamma_{d+2}(\pi) \otimes \mathbb{Q}$$

of $f_\ell(x) \in \gamma_{d+1}(\pi)$. Since π is a free group, $\gamma_{d+1}(\pi)/\gamma_{d+2}(\pi)$ is free abelian (see [56]). It follows that $[\![f_\ell(x)]\!]_{d+1} = 0$ for all $x \in \pi$ if and only if $f_\ell(x) \in \gamma_{d+2}(\pi)$ for all $x \in \pi$. Since $f_\ell(x) = f(x)x^{-1}$, this holds if and only if f acts trivially on $\pi/\gamma_{d+2}(\pi)$, i.e., if and only if $f \in \mathcal{I}_g^1[d+1]$. □

For the second, the action of Mod_g^1 on $H = \mathrm{H}_1(\Sigma_g^1;\mathbb{Q})$ induces an action of Mod_g^1 on $H \otimes \mathrm{FLie}_{d+1}(H)$. For $\kappa \in H \otimes \mathrm{FLie}_{d+1}(H)$ and $\phi \in \mathrm{Mod}_g^1$, write $\phi_*(\kappa)$ for the image of κ under ϕ. We then have:

Lemma 4.4. *Let $d \geq 1$. For all $f \in \mathcal{I}_g^1[d]$ and $\phi \in \mathrm{Mod}_g^1$, we have $\tau_d(\phi f \phi^{-1}) = \phi_*(\tau_d(f))$.*

Proof. Regard $\tau_d(\phi f \phi^{-1})$ as a homomorphism $\pi \to \mathrm{FLie}_{d+1}(H)$. By definition, for $x \in \pi$ we have

$$\tau_d(\phi f \phi^{-1})(x) = [\![(\phi f \phi^{-1})_\ell(x)]\!]_{d+1}.$$

We have

$$(\phi f \phi^{-1})_\ell(x) = \phi(f(\phi^{-1}(x)))x^{-1} = \phi\left(f(\phi^{-1}(x))\phi^{-1}(x)\right) = \phi\left(f_\ell(\phi^{-1}(x))\right),$$

so

$$\tau_d(\phi f \phi^{-1})(x) = [\![\phi\left(f_\ell(\phi^{-1}(x))\right)]\!]_{d+1} = \phi[\![f_\ell(\phi^{-1}(x))]\!]_{d+1} = \phi\left(\tau_d(f)(\phi^{-1}(x))\right).$$

This is exactly how ϕ is supposed to act on homomorphisms $\pi \to \mathrm{FLie}_{d+1}(H)$. □

This has the following corollary. The action of Mod_g^1 on $H = \mathrm{H}_1(\Sigma_g^1)$ preserves the algebraic intersection form ω, which is a symplectic form. As we discussed in the introduction, the resulting homomorphism $\mathrm{Mod}_g^1 \to \mathrm{Sp}_{2g}(\mathbb{Z})$ is surjective, and by definition $\mathcal{I}_g^1$ is its kernel:

$$1 \longrightarrow \mathcal{I}_g^1 \longrightarrow \mathrm{Mod}_g^1 \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1.$$

The action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on H extends to the algebraic group $\mathrm{Sp}_{2g}(\mathbb{Q})$. It follows that the action of Mod_g^1 on $H \otimes \mathrm{FLie}_{d+1}(H)$ also comes from an action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ that extends to the algebraic group $\mathrm{Sp}_{2g}(\mathbb{Q})$. We then have:

Corollary 4.5. *For all $g, d \geq 1$, the $\mathbb{Q}$-span of the image of $\tau_d \colon \mathcal{I}_g^1 \to H \otimes \mathrm{FLie}_{d+1}(H)$ is an $\mathrm{Sp}_{2g}(\mathbb{Q})$-subrepresentation of $H \otimes \mathrm{FLie}_{d+1}(H)$.*

Proof. Lemma 4.4 implies that the $\mathbb{Q}$-span of the image of τ_d is an $\mathrm{Sp}_{2g}(\mathbb{Z})$-subrepresentation. Since $\mathrm{Sp}_{2g}(\mathbb{Z})$ is Zariski dense in $\mathrm{Sp}_{2g}(\mathbb{Q})$, all $\mathrm{Sp}_{2g}(\mathbb{Z})$-subrepresentations are actually $\mathrm{Sp}_{2g}(\mathbb{Q})$-subrepresentations. The corollary follows. □

This corollary implies that to understand the image of τ_d, we need to understand the representation theory of the algebraic group $\mathrm{Sp}_{2g}(\mathbb{Q})$. We will discuss this in §7 below after first performing a number of calculations.

5. Johnson homomorphism calculations

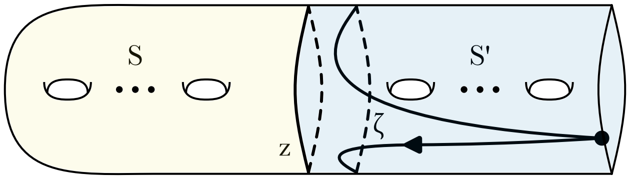
We next calculate the images under the first and second Johnson homomorphisms of some basic elements. Set $H = \mathrm{H}_1(\Sigma_g^1; \mathbb{Q})$ and $\pi = \pi_1(\Sigma_g^1, *)$ with $* \in \partial \Sigma_g^1$.

5.1. **Separating twists, first Johnson homomorphism.**

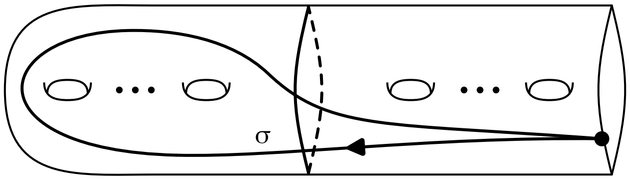
We start with the following. Recall that the first Johnson homomorphism takes the form $\tau_1 \colon \mathcal{I}_g^1[1] \to H \otimes \mathrm{Lie}_2(H)$. Since $\mathrm{Lie}_2(H) \cong \wedge^2 H$ (see Example 3.2), the target of this is $H \otimes \wedge^2 H$.

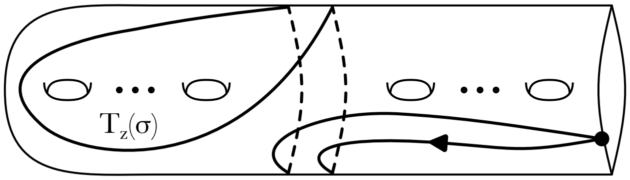
Lemma 5.1. *For some $g \geq 1$, let T_z be a separating twist in $\mathcal{I}_g^1 = \mathcal{I}_g^1[1]$. Then $\tau_1(T_z) = 0$.*

Proof. Let S and S' be the subsurfaces of Σ_g^1 on either side of z. Order them such that $\partial \Sigma_g^1 \subset S'$, so $S \cong \Sigma_h^1$ for some $1 \leq h \leq g$. Let $\zeta \in \pi$ be the following curve:



Regard $\tau_1(T_z)$ as a map $H \to \mathrm{FLie}_2(H)$. To prove that $\tau_1(T_z) = 0$, it is enough to prove that $\tau_1(T_z)$ vanishes on the images of $\mathrm{H}_1(S)$ and $\mathrm{H}_1(S')$ in H. This is trivial for the image of $\mathrm{H}_1(S')$, so we must only prove it for the image of $\mathrm{H}_1(S)$. Identify $\mathrm{H}_1(S)$ with its image in H. As the following shows, an element $u \in \mathrm{H}_1(S)$ can be written as $u = [\sigma]$, where $\sigma \in \pi$ satisfies $T_z(\sigma) = \zeta \sigma \zeta^{-1}$:



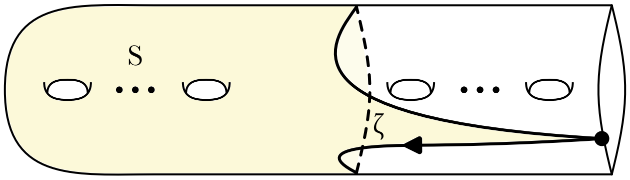


We have $(T_z)_\ell(\sigma) = \zeta \sigma \zeta^{-1} \sigma^{-1} = [\zeta, \sigma]$, so $\tau_1(T_z)(u) = [\![[\zeta, \sigma]]\!]_2$. Since $\zeta \in [\pi, \pi]$ we have $[\zeta, \sigma] \in \gamma_3(\pi)$, so $[\![[\zeta, \sigma]]\!]_2 = 0$, as desired. □

Remark 5.2. Lemma 5.1 implies that separating twists lie in $\mathcal{I}_g^1[2]$. See §5.6 for how to calculate their image under $\tau_2 \colon \mathcal{I}_g^1[2] \to H \otimes \mathrm{FLie}_3(H)$. □

5.2. **Separating simple closed curves.**

Before we perform our next Johnson homomorphism calculation, we need a preliminary result. As in the following figure, let $\zeta \in \pi$ be a simple closed curve that bounds on its right a subsurface S with $S \cong \Sigma_h^1$ for some $h \geq 1$:



We have $\zeta \in [\pi, \pi]$, so it makes sense to discuss $[\![\zeta]\!]_2 \in \mathrm{FLie}_2(H) \cong \wedge^2 H$. Our goal is to calculate $[\![\zeta]\!]_2$.

Identify $\mathrm{H}_1(S; \mathbb{Q})$ with its image in H. The subspace $\mathrm{H}_1(S; \mathbb{Q})$ is a *symplectic subspace* of H, i.e., a subspace on which the algebraic intersection pairing restricts to a nondegenerate form. Let V be a symplectic subspace of H and let ω_V be the restriction of the algebraic intersection form to V. Since ω_V is nondegenerate, it identifies V with its dual V^*. Using this identification, we can identify ω_V with an element

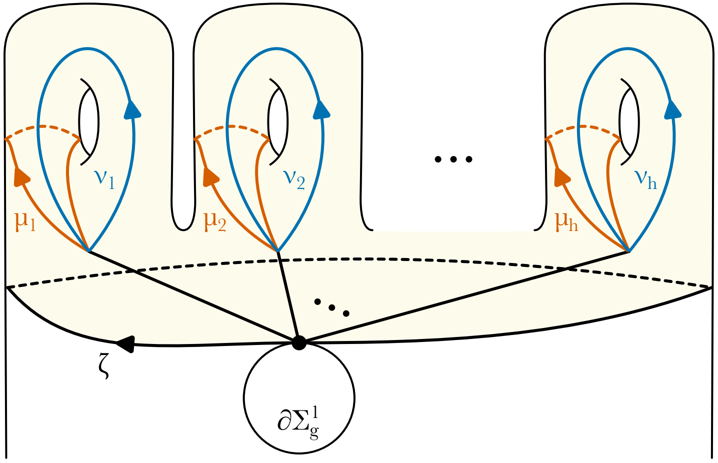
$$\omega_V \in (\wedge^2 V)^* = \wedge^2 V \subset \wedge^2 H.$$

If $\{u_1, v_1, \ldots, u_h, v_h\}$ is a symplectic basis for V, then $\omega_V = u_1 \wedge v_1 + \cdots + u_h \wedge v_h$. We then have the following:

Lemma 5.3. *Let $\zeta \in \pi$ be a simple closed curve that bounds on its right a subsurface S with $S \cong \Sigma_h^1$ for some $h \geq 1$. Set $V = \mathrm{H}_1(S; \mathbb{Q})$. We then have $[\![\zeta]\!]_2 = \omega_V$.*

Proof. As the figure below shows, we can write $\zeta = [\mu_1, \nu_1] \cdots [\mu_h, \nu_h]$ for simple closed curves $\mu_1, \nu_1, \ldots, \mu_h, \nu_h \in \pi$ such that the following holds:

- Letting $u_i = [\mu_i] \in H$ and $v_i = [\nu_i] \in H$ be the homology classes of $\mu_i, \nu_i \in \pi$, the vectors $\{u_1, v_1, \ldots, u_h, v_h\}$ form a symplectic basis for $V = \mathrm{H}_1(S; \mathbb{Q})$.

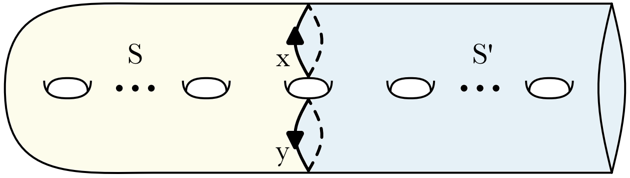


We then have

$$\begin{aligned} [\![\zeta]\!]_2 &= [\![[\mu_1, \nu_1] \cdots [\mu_h, \nu_h]]\!]_2 = [\![[\mu_1, \nu_1]]\!]_2 + \cdots + [\![[\mu_h, \nu_h]]\!]_2 \\ &= u_1 \wedge v_1 + \cdots + u_h \wedge v_h = \omega_V. \end{aligned}$$

□

5.3. **Bounding pair maps, I.** Recall that a bounding pair map is a product $T_x T_y^{-1} \in \mathcal{I}_g^1$, where x and y are disjoint nonseparating curves on Σ_g^1 such that $x \cup y$ separates Σ_g^1. Let S and S' be the subsurfaces on either side of $x \cup y$, ordered such that $\partial \Sigma_g^1 \subset S'$. We therefore have $S \cong \Sigma_h^2$ and $S' \cong \Sigma_{g-h-1}^3$ for some $1 \leq h \leq g - 1$:



As in this figure, orient x and y such that S is to the left of x and the right of y. We call this the *canonical orientation* of x and y. Let $U \subset H$ and $U' \subset H$ be the images of $\mathrm{H}_1(S;\mathbb{Q})$ and $\mathrm{H}_1(S';\mathbb{Q})$. We call the ordered pair (U,U') the *bounded subspaces* of $x \cup y$. Let ω be the algebraic intersection pairing on H. For a subspace $W \subset H$, let

$$W^{\perp} = \{h \in H \mid \omega(h,w) = 0 \text{ for all } w \in W\}.$$

We have $U + U' = \langle [x] \rangle^{\perp}$ and $U \cap U' = \langle [x] \rangle$. Finally, say that an *integral dual* to $[x]$ is an element $h \in H$ with the following two properties:

- $\omega(h,[x]) = 1$; and
- $h \in \mathrm{H}_1(\Sigma_g^1;\mathbb{Z}) \subset H$, i.e., h is an integral point of H.

This implies that $\langle [x],h \rangle^{\perp}$ is a symplectic subspace of H. Letting $V = U \cap \langle [x],h \rangle^{\perp}$ and $V' = U' \cap \langle [x],h \rangle^{\perp}$, it also implies that V and V' are symplectic subspaces of H such that $\langle [x],h \rangle^{\perp} = V \oplus V'$. This direct sum decomposition is orthogonal with respect to ω. We call (V,V') the *bounded splitting* of the pair $(x \cup y, h)$. With these preliminaries, we have the following:

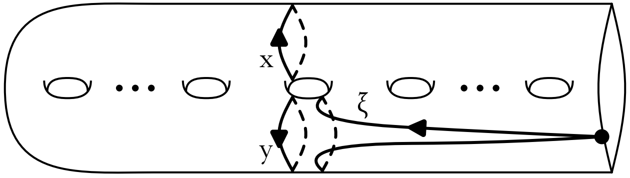
Lemma 5.4. *Let $T_x T_y^{-1} \in \mathcal{I}_g^1$ be a bounding pair map. Give x and y their canonical orientations, and let (U,U') be the bounded subspaces of $x \cup y$. Regard $\tau_1(T_x T_y^{-1})$ as a map $H \to \wedge^2 H$. The following hold:*

(i) For $u \in U$, we have $\tau_1(T_x T_y^{-1})(u) = -[x] \wedge u$.
(ii) For $u' \in U'$, we have $\tau_1(T_x T_y^{-1})(u') = 0$.
(iii) For an integral dual $h \in H$ to $[x]$, let (V,V') be the bounded splitting of the pair $(x \cup y, h)$. We then have $\tau_1(T_x T_y^{-1})(h) = -\omega_V$.

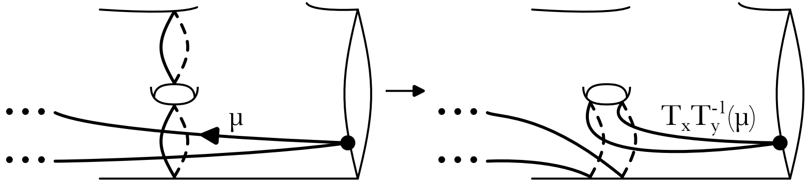
Proof. We prove the three parts separately:

Claim 1. *For $u \in U$, we have $\tau_1(T_x T_y^{-1})(u) = -[x] \wedge u$.*

Let $\xi \in \pi$ be the following curve, so $[\xi] = -[y] = -[x]$; see here:



It is enough to prove the claim for u an integral point of U. This implies that $u = [\mu]$ for a curve $\mu \in \pi$ with $T_x T_y^{-1}(\mu) = \xi \mu \xi^{-1}$; see here:



This implies that $(T_x T_y^{-1})_{\ell}(\mu) = \xi \mu \xi^{-1} \mu^{-1} = [\xi,\mu]$, so

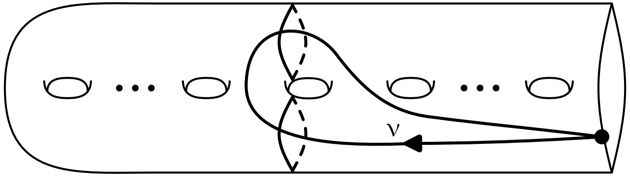
$$\tau_1(T_x T_y^{-1})(u) = [\![[\xi,\mu]]\!]_2 = [\xi] \wedge [\mu] = -[x] \wedge u.$$

Claim 2. *For $u' \in U'$, we have $\tau_1(T_x T_y^{-1})(u') = 0$.*

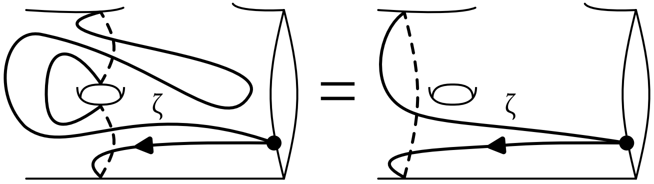
It is enough to prove the claim for u' an integral point of U'. This implies that $u' = [\mu']$ for a curve $\mu' \in \pi$ that is fixed by $T_x T_y^{-1}$. For such a u', the claim is obvious.

Claim 3. *For an integral dual $h \in H$ to $[x]$, let (V,V') be the bounded splitting of the pair $(x \cup y, h)$. We then have $\tau_1(T_x T_y^{-1})(h) = -\omega_V$.*

To make our pictures easier to follow, we will invert $T_x T_y^{-1}$ and prove that $\tau_1(T_y T_x^{-1}) = \omega_V$. Since h is an integral dual to $[x]$, we can find a simple closed curve $\nu \in \pi$ with $h = [\nu]$ that intersects both x and y once; see here:



Set $\zeta = (T_y T_x^{-1})_\ell(\nu) = \left(T_y T_x^{-1}(\nu)\right)\nu^{-1}$; see here:



As this figure shows, ζ is a simple closed separating curve that bounds on its right a surface T with $\mathrm{H}_1(T;\mathbb{Q}) = V$. It therefore follows from Lemma 5.3 that

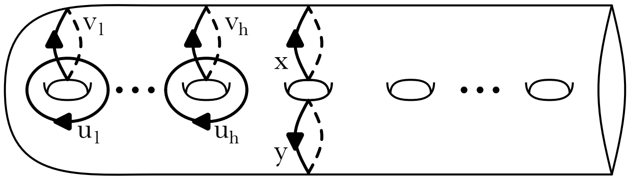
$$\tau_1(T_y T_x^{-1})(h) = [\![(T_y T_x^{-1})_\ell(\nu)]\!]_2 = [\![\zeta]\!]_2 = \omega_V. \qquad \square$$

5.4. **Bounding pair maps, II.** We next translate Lemma 5.4 into a more useful form. The first Johnson homomorphism is of the form $\tau_1 \colon \mathcal{I}_g^1 \to H \otimes \wedge^2 H$. Its target $H \otimes \wedge^2$ contains a copy of $\wedge^3 H$, namely the image of the map

$$h_1 \wedge h_2 \wedge h_3 \in \wedge^3 H \mapsto h_1 \otimes (h_2 \otimes h_3) - h_2 \otimes (h_1 \wedge h_3) + h_3 \otimes (h_1 \wedge h_2) \in H \otimes \wedge^3 H.$$

We then have the following.

Lemma 5.5. *Let $T_x T_y^{-1} \in \mathcal{I}_g^1$ be a bounding pair map. Give x and y their canonical orientations, and let $\{u_1, v_1, \ldots, u_h, v_h\}$ be the homology classes of loops as in the following figure:*



Then $\tau_1(T_x T_y^{-1}) = [x] \wedge (u_1 \wedge v_1 + \cdots + u_h \wedge v_h) \in \wedge^3 H \subset H \otimes \wedge^2 H$.

Proof. Complete $\{u_1, v_1, \ldots, u_h, v_h\}$ to a symplectic basis $\{u_1, v_1, \ldots, u_g, v_g\}$ for H with $v_{h+1} = [x]$ and u_{h+1} an integral dual to $[x]$. Lemma 5.4 says that regarded as a map $H \to \wedge^2 H$, we have that $\tau_1(T_x T_y^{-1})$ is given by the following formulas:

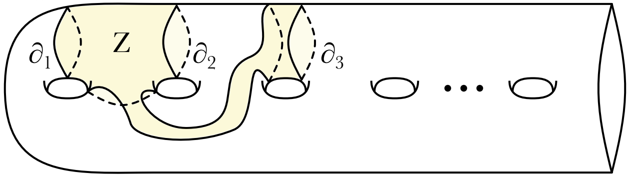
- $\tau_1(T_x T_y^{-1})(u_i) = -[x] \wedge u_i$ and $\tau_1(T_x T_y^{-1})(v_i) = -[x] \wedge v_i$ for $1 \leq i \leq h$.
- $\tau_1(T_x T_y^{-1})(u_{h+1}) = -(u_1 \wedge v_1 + \cdots + u_h \wedge v_h)$.
- $\tau_1(T_x T_y^{-1})$ vanishes on all other elements of $\{u_1, v_1, \ldots, u_g, v_g\}$.

Translating this into an element of $H \otimes \wedge^2 H$, we get

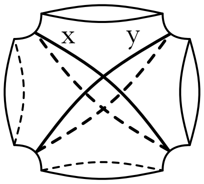
$$\left(\sum_{i=1}^{h} v_i \otimes ([x] \wedge u_i) - u_i \otimes ([x] \wedge v_i)\right) + [x] \otimes (u_1 \wedge v_1 + \cdots + u_h \wedge v_h)$$

$$= \sum_{i=1}^{h} \left([x] \otimes (u_i \wedge v_i) - u_i \otimes ([x] \wedge v_i) + v_i \otimes ([x] \wedge u_i)\right) = \sum_{i=1}^{h} [x] \wedge u_i \wedge v_i. \qquad \square$$

5.5. **Simply intersecting pair maps.** Let $\partial_1, \partial_2, \partial_3$ be three disjoint oriented simple closed curves on Σ_g^1. Our next goal is to construct $f \in \mathcal{I}_g^1$ with $\tau_1(f) = \pm \partial_1 \wedge \partial_2 \wedge \partial_3$ for some choice of sign. Assume that Z is a subsurface of Σ_g^1 such that $Z \cong \Sigma_0^4$ and such that the ∂_i are each boundary components of Z:



As in the following figure, let x and y be simple closed curves on Z that intersect twice with opposite signs, so their algebraic intersection number is 0:



Since the algebraic intersection number of x and y is 0, their actions on H commute and $[T_x, T_y] \in \mathcal{I}_g^1$. We will call $[T_x, T_y] \in \mathcal{I}_g^1$ a *simply intersecting pair map* supported on Z. These have the desired image under τ_1:

Lemma 5.6. *Let Z be a subsurface of Σ_g^1 such that $Z \cong \Sigma_0^4$ and let $[T_x, T_y] \in \mathcal{I}_g^1$ be a simply intersecting pair map supported on Z. Let $\partial_1, \partial_2, \partial_3$ be three boundary components of Z. Orienting the ∂_i arbitrarily, we then have*

$$\tau_1([T_x, T_y]) = \pm[\partial_1] \wedge [\partial_2] \wedge [\partial_3] \in \wedge^3 H \subset H \otimes \wedge^2 H.$$

Proof. Since we have already made a number of calculations, we leave this one to the reader. It can be done by carefully studying the action of $[T_x, T_y]$ on π, or alternatively by factoring $[T_x, T_y]$ as a product of bounding pair maps. See [7] for the details of this calculation. □

Remark 5.7. Let Z and $\partial_1, \partial_2, \partial_3$ be as in Lemma 5.6 and let ∂_4 be the fourth boundary component of Z. It might seem confusing that ∂_4 plays no role in Lemma 5.6. The reason for this is as follows. Orienting ∂_4 arbitrarily, since $\partial_1 \cup \cdots \cup \partial_4$ bounds Z there exists $\epsilon_1, \ldots, \epsilon_4 \in \{\pm 1\}$ such that

$$\epsilon_1[\partial_1] + \epsilon_2[\partial_2] + \epsilon_3[\partial_3] + \epsilon_4[\partial_4] = 0.$$

This implies that replacing one of $\partial_1, \partial_2, \partial_3$ with ∂_4 would not not change $\pm[\partial_1] \wedge [\partial_2] \wedge [\partial_3]$. □

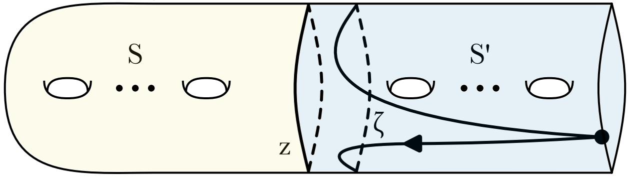
5.6. **Separating twists, second Johnson homomorphism.** Lemma 5.1 implies that separating twists T_z lie in $\mathcal{I}_g^1[2]$. In fact, Johnson [29] proved that separating twists generate $\mathcal{I}_g^1[2]$, which is the domain of the second Johnson homomorphism $\tau_2 \colon \mathcal{I}_g^1[2] \to H \otimes \mathrm{FLie}_3(H)$. Our final goal in this section is to calculate

$$\tau_2(T_z) \in H \otimes \mathrm{FLie}_3(H) \cong H \otimes \frac{H \otimes \wedge^2 H}{\wedge^3 H} \cong \frac{H^{\otimes 2} \otimes \wedge^2 H}{H \otimes \wedge^3 H}.$$

Here the second equality was explained in Example 3.3. We have the following:

Lemma 5.8. *Let T_z be a separating twist in $\mathcal{I}_g^1[2]$. Let S and S' be the subsurfaces of Σ_g^1 on either side of z. Order them such that $\partial \Sigma_g^1 \subset S'$, so $S \cong \Sigma_h^1$ for some $1 \leq h \leq g$. Let V be the image of $\mathrm{H}_1(S; \mathbb{Q})$ in H. Then $\tau_2(T_z)$ is the image of $-\omega_V \otimes \omega_V \in (\wedge^2 H) \otimes (\wedge^2 H)$ in $(H^{\otimes 2} \otimes \wedge^2 H)/(H \otimes \wedge^3 H)$.*

Proof. Let $\zeta \in \pi$ be the following curve:



By the proof of Lemma 5.3, we can write $\zeta = [\mu_1, \nu_1] \cdots [\mu_h, \nu_h]$ for simple closed curves $\mu_1, \nu_1, \ldots, \mu_h, \nu_h \in \pi$ such that the following holds:

- Letting $u_i = [\mu_i] \in H$ and $v_i = [\nu_i] \in H$ be the homology classes of $\mu_i, \nu_i \in \pi$, the vectors $\{u_1, v_1, \ldots, u_h, v_h\}$ form a symplectic basis for $V = \mathrm{H}_1(S; \mathbb{Q})$.

As in Lemma 5.3, the image of ζ in $\mathrm{FLie}_2(H) \cong \wedge^2 H$ is $\omega_V = u_1 \wedge v_1 + \cdots + u_h \wedge v_h$.

Just like in the proof of Lemma 5.8, we have $T_z(\mu_i) = \zeta \mu_i \zeta^{-1}$ and $T_z(\nu_i) = \zeta \nu_i \zeta^{-1}$ for $1 \leq i \leq h$. This implies that $(T_z)_\ell(\mu_i) = \zeta \mu_i \zeta^{-1} \mu_i^{-1} = [\mu_i, \zeta]^{-1}$ and $(T_z)_\ell(\nu_i) = \zeta \nu_i \zeta^{-1} \nu_i^{-1} = [\nu_i, \zeta]^{-1}$. Considered as a map

$$H \to \mathrm{FLie}_3(H) = \frac{H \otimes \wedge^2 H}{\wedge^3 H},$$

we deduce that $\tau_2(T_z)$ vanishes on the image of $\mathrm{H}_1(S')$ and has the following behavior on $\mathrm{H}_1(S)$: for $1 \leq i \leq h$, we have

$$\tau_2(T_z)(u_i) = [\![[\mu_i, \zeta]^{-1}]\!]_3 = -[u_i, \omega_V],$$
$$\tau_2(T_z)(v_i) = [\![[\nu_i, \zeta]^{-1}]\!]_3 = -[v_i, \omega_V].$$

Translating this into an element of $H \otimes \mathrm{Lie}_3(H) = (H^{\otimes 2} \otimes \wedge^2 H)/(H \otimes \wedge^3 H)$, this is exactly the image of

$$-\sum_{i=1}^{h} \left((u_i \otimes v_i) \otimes \omega_V - (v_i \otimes u_i) \otimes \omega_V \right) = -\sum_{i=1}^{h} (u_i \wedge v_i) \otimes \omega_V = -\omega_V \otimes \omega_V$$

in $(H^{\otimes 2} \otimes \wedge^2 H)/(H \otimes \wedge^3 H)$. □

6. IMAGE OF THE JOHNSON HOMOMORPHISM

For some $g \geq 1$, let $H = \mathrm{H}_1(\Sigma_g^1; \mathbb{Q})$. From the calculations in the previous section, it is immediate that the Johnson homomorphism $\tau_d \colon \mathcal{I}_g^1[d] \to H \otimes \mathrm{FLie}_{d+1}(H)$ is not surjective. Our goal in this section is to prove the following theorem of Morita which explains this:

Theorem 6.1 (Morita, [44, Corollary 3.2])**.** *For all $g, d \geq 1$ the image of $\tau_d \colon \mathcal{I}_g^1[d] \to H \otimes \mathrm{FLie}_{d+1}(H)$ is contained in the kernel of the Lie algebra bracket map $H \otimes \mathrm{FLie}_{d+1}(H) \to \mathrm{FLie}_{d+2}(H)$.*

Before proving this, we discuss the extent to which it characterizes the image of τ_d:

- For $d = 1$, by Example 3.2 we have $\mathrm{FLie}_2(H) \cong \wedge^2 H$. We observed in Example 3.3 that the kernel of the Lie bracket map from $H \otimes \mathrm{FLie}_2(H) = H \otimes \wedge^2 H$ to $\mathrm{Lie}_3(H)$ is $\wedge^3 H$, so by Theorem 6.1 the $\mathbb{Q}$-span of the image of τ_1 is contained in $\wedge^3 H$. Johnson [26] proved that the $\mathbb{Q}$-span of the image of τ_1 equals $\wedge^3 H$. We describe this calculation in §9 below.
- For $d = 2$, by Example 3.3 we have

  $$\mathrm{FLie}_3(H) = \frac{H \otimes \wedge^2 H}{\wedge^3 H}.$$

  We observed in Example 3.4 that the kernel of the Lie bracket map from

  $$H \otimes \mathrm{FLie}_3(H) = \frac{H^{\otimes 2} \otimes \wedge^2 H}{H \otimes \wedge^3 H}$$

to $\text{Lie}_4(H)$ is isomorphic to $\text{Sym}^2(\wedge^2 H)/\wedge^4 H$, so by Theorem 6.1 the $\mathbb{Q}$-span of the image of τ_2 is contained in $\text{Sym}^2(\wedge^2 H)/\wedge^4 H$. Morita [43] proved that the $\mathbb{Q}$-span of the image of τ_2 equals $\text{Sym}^2(\wedge^2 H)/\wedge^4 H$. We describe this calculation in §10 below. Morita [44] proved that for $d \geq 3$ there are restrictions on the image of τ_d beyond Theorem 6.1. There is now an enormous literature on the image of the higher Johnson homomorphisms. See [20, 45, 46, 55] for discussions of this literature.

Proof of Theorem 6.1. Fix a basepoint $* \in \partial\Sigma_g^1$ and set $\pi = \pi_1(\Sigma_g^1, *)$. Let $\partial \in \pi$ be the loop around the boundary component, oriented such that the surface is to its right. Since Mod_g^1 fixes $\partial\Sigma_g^1$ pointwise, the action of Mod_g^1 on π fixes the curve ∂. As we will see, this will ultimately be responsible for this theorem.

As in the proof of Lemma 5.3, we can write $\partial = [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]$ for simple closed curves $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \in \pi$. Letting $a_i = [\alpha_i] \in H$ and $b_i = [\beta_i] \in H$ be the homology classes of these curves, the elements $\{a_1, b_1, \ldots, a_g, b_g\}$ form a symplectic basis for H.

Consider $f \in \mathcal{I}_g^1[d]$. For $1 \leq i \leq g$, write $f_\ell(\alpha_i) = \overline{\alpha}_i \in \gamma_{d+1}(\pi)$ and $f_\ell(\beta_i) = \overline{\beta}_i \in \gamma_{d+1}(\pi)$. Regarding $\tau_d(f)$ as an element of $\text{Hom}(H, \text{FLie}_{d+1}(H))$, for $1 \leq i \leq g$ we have

$$\tau_d(f)(a_i) = [\![\overline{\alpha}_i]\!]_{d+1},$$
$$\tau_d(f)(b_i) = [\![\overline{\beta}_i]\!]_{d+1}.$$

It follows that as an element of $H \otimes \text{FLie}_{d+1}(H)$, we have

$$\tau_d(f) = \sum_{i=1}^{g} \left(a_i \otimes [\![\overline{\beta}_i]\!]_{d+1} - b_i \otimes [\![\overline{\alpha}_i]\!]_{d+1}\right).$$

From this, we see that the following element of $\gamma_{d+2}(\pi)$ maps to the image of $\tau_d(f)$ under the bracket map $H \otimes \text{FLie}_{d+1}(H) \to \text{FLie}_{d+2}(H)$:

$$\zeta = \prod_{i=1}^{g} [\alpha_i, \overline{\beta}_i][\overline{\alpha}_i, \beta_i]$$

To prove the theorem, we must prove that $[\![\zeta]\!]_{d+2} = 0$. Writing $\equiv$ for equality modulo $\gamma_{d+3}(\pi)$, we must equivalently prove that $\zeta \equiv 1$.

To see this, observe that

$$f([\alpha_i, \beta_i]) = [\overline{\alpha}_i\alpha_i, \overline{\beta}_i\beta_i]. \tag{6.1}$$

For a group G and $x, y \in G$, our commutator convention is $[x, y] = xyx^{-1}y^{-1}$. For $x, y, z \in G$, we thus have

$$[xy, z] = x[y, z]x^{-1}[x, z] \quad \text{and} \quad [x, yz] = [x, y]y[x, z]y^{-1}.$$

Applying this to (6.1), we see that

$$\begin{aligned} f([\alpha_i, \beta_i]) &= \overline{\alpha}_i[\alpha_i, \overline{\beta}_i\beta_i]\overline{\alpha}_i^{-1}[\overline{\alpha}_i, \overline{\beta}_i\beta_i] \\ &= \left(\overline{\alpha}_i[\alpha_i, \overline{\beta}_i]\overline{\alpha}_i^{-1}\right)\left(\overline{\alpha}_i\overline{\beta}_i[\alpha_i, \beta_i]\overline{\beta}_i^{-1}\overline{\alpha}_i^{-1}\right)[\overline{\alpha}_i, \overline{\beta}_i]\left(\overline{\beta}_i[\overline{\alpha}_i, \beta_i]\overline{\beta}_i^{-1}\right). \end{aligned}$$

Since $\overline{\alpha}_i, \overline{\beta}_i \in \gamma_{d+1}(\pi)$, the following hold:

- $\overline{\alpha}_i\overline{\beta}_i[\alpha_i, \beta_i]\overline{\beta}_i^{-1}\overline{\alpha}_i^{-1} \equiv [\alpha_i, \beta_i]$; and
- $[\overline{\alpha}_i, \overline{\beta}_i] \equiv 1$; and
- $[\alpha_i, \overline{\beta}_i]$ and $[\overline{\alpha}_i, \beta_i]$ commute with everything modulo $\gamma_{d+3}(\pi)$.

Applying these to the above identity, we deduce that

$$f([\alpha_i, \beta_i]) \equiv [\alpha_i, \beta_i][\alpha_i, \overline{\beta}_i][\overline{\alpha}_i, \beta_i].$$

Keeping in mind that $f(\partial) = \partial$ and $\partial = [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]$, we can multiply all of these terms together and commute the terms $[\alpha_i, \overline{\beta}_i]$ and $[\overline{\alpha}_i, \beta_i]$ around and see that

$$\partial = f(\partial) \equiv \prod_{i=1}^{g} [\alpha_i, \beta_i][\alpha_i, \overline{\beta}_i][\overline{\alpha}_i, \beta_i] \equiv \left(\prod_{i=1}^{g} [\alpha_i, \beta_i] \right) \left(\prod_{i=1}^{g} [\alpha_i, \overline{\beta}_i][\overline{\alpha}_i, \beta_i] \right) = \partial \cdot \zeta.$$

Rearranging this, we see that $\zeta \equiv 1$, as desired. □

7. Representation theory of Sp_{2g}

Before describing the images of τ_1 and τ_2, we pause to describe the representation theory of the algebraic group Sp_{2g} over a field $\mathbf{k}$ of characteristic 0. As in our discussion of SL_n in §2, everything we discuss can be found in [5], and in the slightly different language of Lie algebra representations can also be found in [16].

7.1. **Symplectic basis.** Let $\{a_1, b_1, \ldots, a_g, b_g\}$ be the standard symplectic basis for $\mathbf{k}^{2g}$. As is standard, we will describe elements of $\mathrm{Sp}_{2g}(\mathbf{k})$ using matrices whose first g columns give the images of $a_1, \ldots, a_g$ and whose last g columns gives the images of $b_1, \ldots, b_g$. Letting

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

the group $\mathrm{Sp}_{2g}(\mathbf{k})$ thus consists of $2g \times 2g$ matrices M such that $M^t J M = J$.

7.2. **Self-duality and the symplectic form.** Let $H = \mathbf{k}^{2g}$ be the standard representation of $\mathrm{Sp}_{2g}(\mathbf{k})$ and let ω be the symplectic form on H that is preserved by $\mathrm{Sp}_{2g}(\mathbf{k})$. The form ω is an alternating bilinear form $\omega \colon H \times H \to \mathbf{k}$ that is nondegenerate in the sense that it identifies H with its dual $H^* = \mathrm{Hom}(H, \mathbf{k})$. Since $\mathrm{Sp}_{2g}(\mathbf{k})$ preserves ω, it follows that H and H^* are isomorphic representations of $\mathrm{Sp}_{2g}(\mathbf{k})$. It follows that

$$\omega \in (\wedge^2 H)^* \cong \wedge^2 H^* \cong \wedge^2 H.$$

We will henceforth identify ω with its image in $\wedge^2 H$. The element $\omega \in \wedge^2 H$ is invariant under $\mathrm{Sp}_{2g}(\mathbf{k})$, so it spans a trivial subrepresentation of $\wedge^2 H$. With respect to the symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$, we have

$$\omega = a_1 \wedge b_1 + \cdots + a_g \wedge b_g.$$

7.3. **Maximal split torus.** Let $\mathbf{T}$ be the subgroup of Sp_{2g} consisting of diagonal matrices. This notation conflicts with our notation for the corresponding subgroup of SL_n, but context should make it clear which group we are talking about. For $t_1, \ldots, t_g \in \mathbf{k}^\times$, set

$$\mathrm{diag}_{\mathrm{Sp}}(t_1, \ldots, t_g) = \mathrm{diag}(t_1, \ldots, t_g, t_1^{-1}, \ldots, t_g^{-1}) \in \mathbf{T}(\mathbf{k}). \tag{7.1}$$

The group $\mathbf{T}(\mathbf{k})$ is exactly the group of matrices of the form $\mathrm{diag}_{\mathrm{Sp}}(t_1, \ldots, t_g)$, so

$$\mathbf{T}(\mathbf{k}) \cong (\mathbf{k}^\times)^g.$$

For $1 \leq i \leq g$, let $E_i \in \chi(\mathbf{T})$ be the character

$$E_i(\mathrm{diag}_{\mathrm{Sp}}(t_1, \ldots, t_g)) = t_i.$$

The group of characters $\chi(\mathbf{T})$ is then isomorphic to $\mathbb{Z}^g$ and is generated by $\{E_1, \ldots, E_g\}$.

7.4. **Weight decomposition.** Using $\mathbf{T}$, we can talk about weight vectors for representations of Sp_{2g}. The inverses in (7.1) make this slightly more complicated than for SL_n. Here are some examples of weight vectors:

Example 7.1. In H, the vector a_i is a weight vector with weight E_i and the vector b_i is a weight vector with weight $-E_i$. $\square$

Example 7.2. In $\wedge^2 H$, for $1 \leq i < j \leq g$ the vector $a_i \wedge b_j$ is a weight vector with weight $E_i - E_j$. When $i = j$, things degenerate and $a_i \wedge b_i$ is a weight vector with weight 0, i.e., it is fixed by $\mathbf{T}(\mathbf{k})$. The sum of all of these is $\omega = a_1 \wedge b_1 + \cdots + a_g \wedge b_g$, which is fixed not only by $\mathbf{T}(\mathbf{k})$ but by all of $\mathrm{Sp}_{2g}(\mathbf{k})$. $\square$

7.5. **Standard unipotent subgroups.** For representations of SL_n, we defined highest weight vectors to be weight vectors that are fixed by the unipotent subgroup of upper triangular matrices. The corresponding notion of Sp_{2g} is more complicated since the subgroup corresponding to "upper triangular matrices" is slightly more complicated.

The *standard embedding* of $\mathrm{GL}_g(\mathbf{k})$ into $\mathrm{Sp}_{2g}(\mathbf{k})$ is as follows:

$$M \in \mathrm{GL}_g(\mathbf{k}) \mapsto \begin{pmatrix} M & 0 \\ 0 & (M^t)^{-1} \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbf{k}).$$

The image of the standard embedding is exactly the subgroup of $\mathrm{Sp}_{2g}(\mathbf{k})$ consisting of symplectic matrices preserving the subspaces $\langle a_1, \ldots, a_g \rangle$ and $\langle b_1, \ldots, b_g \rangle$. Define $\mathbf{U}_1 < \mathrm{Sp}_{2g}$ to be the image of the group of strictly upper triangular matrices in GL_g under the standard embedding.

Next, for a $g \times g$ matrix M over $\mathbf{k}$ with $M^t = M$, we have a corresponding element of $\mathrm{Sp}_{2g}(\mathbf{k})$:

$$\begin{pmatrix} I_g & M \\ 0 & I_g \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{Z}).$$

Define $\mathbf{U}_2$ to be the subgroup of Sp_{2g} consisting of such matrices.

Remark 7.3. The subgroup $\mathbf{U}$ of Sp_{2g} generated by $\mathbf{U}_1$ and $\mathbf{U}_2$ is what plays the role of strictly upper triangular matrices, but it is convenient to separate out the roles played by $\mathbf{U}_1$ and $\mathbf{U}_2$. $\square$

7.6. **Highest weight vectors.** If V is a representation of Sp_{2g}, then a *highest weight vector* in V with weight $\chi \in \chi(\mathbf{T})$ is a nonzero vector $v \in V$ such that:

- v is a weight vector with weight χ, so $v \neq 0$ and $v \in V_\chi$; and
- v is fixed by all elements of $\mathbf{U}_1(\mathbf{k})$ and $\mathbf{U}_2(\mathbf{k})$.

A *highest weight* of V is the weight of a highest weight vector in V.

The integer points $\mathbf{U}_1(\mathbb{Z})$ and $\mathbf{U}_2(\mathbb{Z})$ are Zariski dense in $\mathbf{U}_1(\mathbf{k})$ and $\mathbf{U}_2(\mathbf{k})$, so to check that a weight vector $v \in V$ is a highest weight vector it is enough to check that it is invariant under $\mathbf{U}_1(\mathbb{Z})$ and $\mathbf{U}_2(\mathbb{Z})$. In fact, it is enough to check this on generators for these groups, which are as follows:

- For $1 \leq i < j \leq g$, the matrix $X_{ij} \in \mathbf{U}_1(\mathbf{k})$ that fixes all elements of $\{a_1, b_1, \ldots, a_g, b_g\}$ except for a_j and b_i, on which it does the following:

$$X_{ij}(a_j) = a_j + a_i,$$
$$X_{ij}(b_i) = b_i - b_j.$$

- For $1 \leq i \leq g$, the matrix $Y_i \in \mathbf{U}_2(\mathbf{k})$ that fixes all elements of $\{a_1, b_1, \ldots, a_g, b_g\}$ except for b_i, on which it does the following:

$$Y_i(b_i) = b_i + a_i.$$

- For $1 \leq i < j \leq g$, the matrix $Z_{ij} \in \mathbf{U}_2(\mathbf{k})$ that fixes all elements of $\{a_1, b_1, \ldots, a_g, b_g\}$ except for b_i and b_j, on which it does the following:

$$\begin{aligned} Z_{ij}(b_i) &= b_i + a_j, \\ Z_{ij}(b_j) &= b_j + a_i. \end{aligned}$$

We remark that these are examples of elementary symplectic matrices.

Example 7.4. For H, the vector a_1 is a highest weight vector with weight E_1. One can check that up to scaling this is the unique highest weight vector. □

Example 7.5. In $\wedge^2 H$, the vector $a_1 \wedge a_2$ is a highest weight vector with weight $E_1 + E_2$. However, it is not the only highest weight vector: the vector

$$\omega = a_1 \wedge b_1 + \cdots + a_g \wedge b_g$$

is also a highest weight vector, this time with weight 0. In fact, as we we noted above ω is fixed by $\mathrm{Sp}_{2g}(\mathbf{k})$, not just by its subgroups $\mathbf{U}_1(\mathbf{k})$ and $\mathbf{U}_2(\mathbf{k})$. It is enlightening to prove this directly by examining the action of the elements X_{ij} and Y_i and Z_{ij} on ω. □

7.7. **Theorem of the highest weight.** Just like for SL_n, the representation theory of Sp_{2g} is controlled by highest weight vectors. Indeed, the theorem of the highest weight says that the following hold:

(a) All representations of Sp_{2g} decompose as direct sums of irreducible representations.
(b) Up to scaling, each irreducible representation V of Sp_{2g} contains a unique highest weight vector.
(c) If V is an arbitrary representation of Sp_{2g} and $v \in V$ is a highest weight vector, then the smallest subrepresentation containing v is irreducible.
(d) If V and V' are irreducible representations of Sp_{2g} with highest weight vectors $v \in V$ and $v' \in V'$ and if the weights of v and v' are the same, then V and V' are isomorphic representations.

7.8. **Dominant weights.** To complete the picture of the representation theory of Sp_{2g}, we must identify the weights that can appear as highest weights of irreducible representations of Sp_{2g}. A *dominant weight* is a character

$$\chi = k_1 E_1 + \cdots + k_g E_g \quad \text{with } k_1, \ldots, k_g \in \mathbb{Z}$$

such that $k_1 \geq \cdots \geq k_g \geq 0$. Note that unlike for SL_n, there are no relations between the E_i, so we cannot ensure that the last coordinate is 0. The dominant weights are exactly the weights of irreducible representations of Sp_{2g}. If the χ above is a dominant weight, we will denote by $\mathbf{V}_\chi(g)$ the irreducible representation of Sp_{2g} with highest weight χ.

Recall that a *partition* of an integer d of length $m \geq 0$ is a tuple $\sigma = (k_1, \ldots, k_m)$ with $k_1 \geq \cdots \geq k_m \geq 1$ and $k_1 + \cdots k_m = d$. The dominant weights for Sp_{2g} are in bijection with partitions of integers with length at most g. Given such a partition $\sigma = (k_1, \ldots, k_m)$, the corresponding dominant weight is

$$\chi = k_1 E_1 + \cdots + k_m E_m.$$

We will also write $\mathbf{V}_\sigma(g)$ for $\mathbf{V}_\chi(g)$.

Convention 7.6. We will denote multiplicities in partitions using superscripts. For instance, if $\sigma = (5, 4, 4, 4, 1, 1)$ then we will write $\mathbf{V}_{5,4^3,1^2}(g)$ for $\mathbf{V}_\sigma(g)$. □

Here are some examples of this notation. Recall that $H \cong \mathbf{k}^{2g}$ is the standard representation of $\mathrm{Sp}_{2g}(\mathbf{k})$.

Example 7.7. We have $\mathbf{V}_k(g) \cong \operatorname{Sym}^k(H)$. The highest weight vector in $\operatorname{Sym}^k(H)$ with weight kE_1 is $a_1 \cdot \ldots \cdot a_1$. ☐

Example 7.8. Unlike for SL_n, in most cases $\wedge^k H$ is not irreducible. Indeed, for $1 \leq k \leq g$ we have

$$\wedge^k H = \begin{cases} \mathbf{V}_{1^k}(g) \oplus \mathbf{V}_{1^{k-2}}(g) \oplus \cdots \oplus \mathbf{V}_0(g) & \text{if } k \text{ is even,} \\ \mathbf{V}_{1^k}(g) \oplus \mathbf{V}_{1^{k-2}}(g) \oplus \cdots \oplus \mathbf{V}_1(g) & \text{if } k \text{ is odd.} \end{cases}$$

The highest weight vectors corresponding to these decompositions are as follows. As in §7.2 let $\omega = a_1 \wedge b_1 + \cdots + a_g \wedge b_g$. Then for all $d \geq 0$ such that $k - 2d \geq 0$ the vector

$$a_1 \wedge a_2 \wedge \cdots \wedge a_{k-2d} \wedge \omega \wedge \cdots \wedge \omega \in \wedge^k H$$

with d factors of ω is a highest weight vector with weight $E_1 + \cdots + E_{k-2d}$. We will explain how to justify these decompositions with a computer below. ☐

7.9. **Stable decompositions.** Just like for SL_n, there is a notion of stability for representations of $\mathrm{Sp}_{2g}(\mathbf{k})$. Let $\sigma = (k_1, \ldots, k_m)$ be a partition of d with at most g parts. We will call d the *degree* of the partition. We will also say that d is the degree of the irreducible representation $\mathbf{V}_\sigma(g)$. Schur–Weyl duality implies that the irreducible representation $\mathbf{V}_\sigma(g)$ appears in in $H^{\otimes d}$, and moreover that all irreducible representations that appear in $H^{\otimes d}$ have degree at most d.

A classical observation is that for $g \geq d$, the decomposition of $H^{\otimes d}$ into irreducible factors is independent of the parameter g in the following sense. Since we will need to distinguish between $H = \mathbf{k}^{2g}$ for different values of g, we will write $H(g)$ for H.

- If
$$H(g)^{\otimes d} = \mathbf{V}_{\sigma_1}(g) \oplus \cdots \mathbf{V}_{\sigma_k}(g)$$
for partitions $\sigma_1, \ldots, \sigma_k$, then we also have
$$H(g+1)^{\otimes d} = \mathbf{V}_{\sigma_1}(g+1) \oplus \cdots \mathbf{V}_{\sigma_k}(g+1).$$

More generally, if U is a representation of Sp_{2g} constructed from the standard representation H using tensor powers, exterior powers, and symmetric powers, then U naturally embeds into $H(g)^{\oplus d}$ for a $d \geq 1$ called its degree, and the decomposition of U into irreducible factors is independent of g as long as $g \geq d$.

Convention 7.9. In light of all of this, we can decompose representations of Sp_{2g} for $g \gg 0$ by using a computer to make this decomposition for g at least the degree of the representation. This will be done silently throughout the remainder of the paper. ☐

8. PROJECTION MAPS AND REPRESENTATION THEORY

We now explore some more features of the representation theory of Sp_{2g} over a field $\mathbf{k}$ of characteristic 0. Let $H = \mathbf{k}^{2g}$ be the standard representation, let $\omega \in \wedge^2 H$ be the symplectic form, and let $\{a_1, b_1, \ldots, a_g, b_g\}$ be the standard symplectic basis for H.

8.1. **Projection maps.** For $k \geq 0$, define a projection map $q_k \colon \wedge^{k+2} H \to \wedge^k H$ via the following formula:

$$q_k(h_1 \wedge \cdots \wedge h_{k+2}) = \sum_{1 \leq i < j \leq k+2} (-1)^{i+j+1} \omega(h_i, h_k) h_1 \wedge \cdots \widehat{h_i} \wedge \cdots \wedge \widehat{h_j} \wedge \cdots \wedge h_{k+2}.$$

This formula makes sense since the right hand side changes sign when two of the h_i are swapped. We claim that q_k is surjective. To prove this, it is enough to prove that its image contains all highest weight vectors in $\wedge^k H$. As we saw in Example 7.8, these are exactly scalar multiples of the vectors of the following form for $0 \leq d \leq k/2$:

$$w_{k,d} = a_1 \wedge \cdots \wedge a_{k-2d} \wedge \omega \wedge \cdots \wedge \omega \in \wedge^k H$$

In $w_{k,d}$, there are d factors of ω. It is then an enlightening calculation to show that $q_k(w_{k+2,d})$ is a nonzero multiple[6] of $w_{k,d}$, so $w_{k,d}$ is in the image of q_k. The kernel of q_k contains the highest weight vector $a_1 \wedge \cdots \wedge a_{k+2}$ generating the irreducible factor $\mathbf{V}_{1^{k+2}}(g)$ of $\wedge^{k+2} H$. Since this is the only irreducible factor of $\wedge^{k+2} H$ not accounted for by the surjective map $q_k \colon \wedge^{k+2} H \to \wedge^k H$, this must be the entire kernel. In other words, we have a short exact sequence of representations

$$0 \longrightarrow \mathbf{V}_{1^{k+2}}(g) \longrightarrow \wedge^{k+2} H \xrightarrow{q_k} \wedge^k H \longrightarrow 0.$$

8.2. **Splitting H.** We will be particularly interested in $q_1 \colon \wedge^3 H \to H$. Let $\iota \colon H \hookrightarrow \wedge^3 H$ be the embedding $\iota(h) = h \wedge \omega$. It is almost the case that q_1 splits ι. Indeed, consider a nonzero $u \in H$. We can find a symplectic basis $\{u_1, v_1, \ldots, u_g, v_g\}$ for H such that $u = u_1$. We then have

$$\begin{aligned} q_1(\iota(u)) &= q_1(u_1 \wedge (u_1 \wedge v_1 + \cdots + u_g \wedge v_g)) \\ &= q_1(u_1 \wedge u_2 \wedge v_2 + \cdots + u_1 \wedge u_g \wedge v_g) = (g-1)u. \end{aligned}$$

In other words, $q_1 \circ \iota \colon H \to H$ is multiplication by $(g-1)$.

8.3. **Splitting the quotient by H.** Regard H as a subspace of $\wedge^3 H$ via the inclusion ι. Let $p \colon \wedge^3 H \to (\wedge^3 H)/H$ be the projection, so we have a short exact sequence of Sp_{2g}-representations

$$0 \longrightarrow H \xrightarrow{\iota} \wedge^3 H \xrightarrow{p} (\wedge^3 H)/H \longrightarrow 0.$$

This splits via the map $\sigma \colon (\wedge^3 H)/H \to \wedge^3 H$ taking $\kappa \in (\wedge^3 H)/H$ to the unique $\sigma(\kappa) \in \wedge^3 H$ with $p(\sigma(\kappa)) = \kappa$ and $q_1(\sigma(\kappa)) = 0$. Since $q_1 \circ \iota$ is multiplication by $(g-1)$, we can make this more explicit as follows:

- Consider $\kappa \in (\wedge^3 H)/H$. Let $\widetilde{\kappa} \in \wedge^3 H$ be any element with $p(\widetilde{\kappa}) = \kappa$. Set $h = q_1(\widetilde{\kappa})$. Then
$$\sigma(\kappa) = \widetilde{\kappa} - \frac{1}{g-1} h \wedge \omega.$$
Since $q_1(h \wedge \omega) = (g-1)h$, this is the unique element of $\ker(q_1)$ projecting to κ.

For $\kappa \in \wedge^3 H$, let $\overline{\kappa} = p(\kappa)$. Below are two examples of $\sigma(\overline{\kappa})$ for different $\kappa \in \wedge^3 H$.

Example 8.1. For $1 \leq i < j < k \leq g$, we have $\sigma(\overline{a_i \wedge a_j \wedge a_k}) = a_i \wedge a_j \wedge a_k$ since $q_1(a_i \wedge a_j \wedge a_k) = 0$. □

Example 8.2. For distinct $1 \leq i, j \leq g$, we have

$$\sigma(\overline{a_i \wedge a_j \wedge b_j}) = a_i \wedge (a_j \wedge b_j - \frac{1}{g-1}\omega).$$

Indeed, since $q_1(a_i \wedge a_j \wedge b_j) = a_i \in H$, the above recipe shows that

$$\sigma(\overline{a_i \wedge a_j \wedge b_j}) = a_i \wedge a_j \wedge b_j - \frac{1}{g-1} a_i \wedge \omega = a_i \wedge (a_j \wedge b_j - \frac{1}{g-1}\omega). \qquad \square$$

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[6] We will write out the details of this for $k = 1$ in §8.2 below. This will be the only case we need in the remainder of the paper.

8.4. **Generating subrepresentations.** As we will soon see, for our proofs the most important representation of Sp_{2g} is $\wedge^2 \wedge^3 H$. We will need to certify that subrepresentations of its quotient $\wedge^2((\wedge^3 H)/H)$ contain specific irreducible factors. Rather than try to prove a general result, we will prove two lemmas that contain exactly what we need.[7]

Lemma 8.3. *Let $g \geq 6$ and let $\{a_1, b_1, \ldots, a_g, b_g\}$ be the standard symplectic basis for H. Let V be the subrepresentation of $\wedge^2((\wedge^3 H)/H)$ generated by $\overline{\theta} = (\overline{a_1 \wedge a_2 \wedge a_3}) \wedge (\overline{a_4 \wedge a_5 \wedge b_5})$. Then V contains a copy of $\mathbf{V}_{1^4}(g)$.*

Proof. Let $\sigma\colon (\wedge^3 H)/H \to \wedge^3 H$ be the section of the projection $p\colon \wedge^3 H \to (\wedge^3 H)/H$ discussed above and let $\phi\colon \wedge^2 \wedge^3 H \to \wedge^6 H$ be the following map:

$$\phi((u_1 \wedge v_1 \wedge w_1) \wedge (u_2 \wedge v_2 \wedge w_2)) = u_1 \wedge v_1 \wedge w_1 \wedge u_2 \wedge v_2 \wedge w_2.$$

Consider the composition

$$\wedge^2(\wedge^3 H)/H \xrightarrow{\wedge^2\sigma} \wedge^2 \wedge^3 H \xrightarrow{\phi} \wedge^6 H \xrightarrow{q_4} \wedge^4 H.$$

Since $a_1 \wedge \cdots \wedge a_4 \in \wedge^4 H$ is a highest weight vector for a copy of $\mathbf{V}_{1^4}(g)$, it is enough to prove that this composition takes $\overline{\theta}$ to a nonzero multiple of $a_1 \wedge \cdots \wedge a_4$. In fact, to make our formulas clearer we will prove this not for $\overline{\theta}$ but for $(1-g)\overline{\theta}$. We calculate the image of $(1-g)\overline{\theta}$ under the above composition as follows. First, we calculate the image under $\wedge^2\sigma$:

$$\begin{aligned}\wedge^2\sigma((1-g)\overline{\theta}) &= (1-g)(a_1 \wedge \wedge a_2 \wedge a_3) \wedge (a_4 \wedge (a_5 \wedge b_5 - \frac{1}{g-1}\omega)) \\ &= (a_1 \wedge \wedge a_2 \wedge a_3) \wedge (a_4 \wedge ((1-g)a_5 \wedge b_5 + \omega)).\end{aligned}$$

Next, we apply ϕ and get

$$\begin{aligned}&a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge ((1-g)a_5 \wedge b_5 + \omega) \\ =&a_1 \wedge a_2 \wedge a_3 \wedge a_4 \wedge ((2-g)a_5 \wedge b_5 + a_6 \wedge b_6 + \cdots + a_g \wedge b_g).\end{aligned}$$

Finally, we apply q_4 and get

$$((2-g) + (g-5))\, a_1 \wedge a_2 \wedge a_3 \wedge a_4 = -3a_1 \wedge a_2 \wedge a_3 \wedge a_4. \qquad \square$$

Lemma 8.4. *Let $g \geq 6$ and let $\{a_1, b_1, \ldots, a_g, b_g\}$ be the standard symplectic basis for H. Let V be the subrepresentation of $\wedge^2((\wedge^3 H)/H)$ generated by $\overline{\theta} = (\overline{a_1 \wedge a_4 \wedge b_4}) \wedge (\overline{a_2 \wedge a_3 \wedge b_3})$. Then V contains a copy of $\mathbf{V}_{1^2}(g)$.*

Proof. Let $\sigma\colon (\wedge^3 H)/H \to \wedge^3 H$ be the section of the projection $p\colon \wedge^3 H \to (\wedge^3 H)/H$ discussed above and let $\phi\colon \wedge^2 \wedge^3 H \to \wedge^6 H$ be the following map:

$$\phi((u_1 \wedge v_1 \wedge w_1) \wedge (u_2 \wedge v_2 \wedge w_2)) = u_1 \wedge v_1 \wedge w_1 \wedge u_2 \wedge v_2 \wedge w_2.$$

Consider the composition

$$\wedge^2(\wedge^3 H)/H \xrightarrow{\wedge^2\sigma} \wedge^2 \wedge^3 H \xrightarrow{\phi} \wedge^6 H \xrightarrow{q_4} \wedge^4 H \xrightarrow{q_2} \wedge^2 H.$$

Since $a_1 \wedge a_2 \in \wedge^2 H$ is a highest weight vector for a copy of $\mathbf{V}_{1^2}(g)$, it is enough to prove that this composition takes $\overline{\theta}$ to a nonzero multiple of $a_1 \wedge a_2$. In fact, to make our formulas clearer we will prove this not for $\overline{\theta}$ but for $(1-g)^2\overline{\theta}$. We calculate the image of $(1-g)^2\overline{\theta}$ under the above composition as follows. First, we calculate the image under $\wedge^2\sigma$:

$$\begin{aligned}\wedge^2\sigma((1-g)^2\overline{\theta}) &= (1-g)^2(a_1 \wedge (a_4 \wedge b_4 - \frac{1}{g-1}\omega)) \wedge (a_2 \wedge (a_3 \wedge b_3 - \frac{1}{g-1}\omega)) \\ &= (a_1 \wedge ((1-g)a_4 \wedge b_4 + \omega)) \wedge (a_2 \wedge ((1-g)a_3 \wedge b_3 + \omega)).\end{aligned}$$

---

[7] These results might seem unmotivated on a first reading, but we promise the reader that they will prove to be crucial.

Next, we apply ϕ and get

$$\begin{aligned} &a_1 \wedge ((1-g)a_4 \wedge b_4 + \omega) \wedge a_2 \wedge ((1-g)a_3 \wedge b_3 + \omega) \\ =&a_1 \wedge a_2 \wedge (a_3 \wedge b_3 + (2-g)a_4 \wedge b_4 + a_5 \wedge b_5 + \cdots + a_g \wedge b_g) \\ &\quad \wedge ((2-g)a_3 \wedge b_3 + a_4 \wedge b_4 + a_5 \wedge b_5 + \cdots + a_g \wedge b_g). \end{aligned}$$

Finally, we apply $q_2 \circ q_4 \colon \wedge^6 H \to \wedge^2 H$. It is convenient to break this up into the sum of four terms:

$$\begin{aligned} &q_2 \circ q_4(a_1 \wedge a_2 \wedge (a_3 \wedge b_3 + \cdots + a_g \wedge b_g) \wedge (a_3 \wedge b_3 + \cdots + a_g \wedge b_g)) \\ &\quad = 2(g-2)(g-3)a_1 \wedge a_2, \end{aligned}$$

and

$$\begin{aligned} &q_2 \circ q_4(a_1 \wedge a_2 \wedge ((1-g)a_4 \wedge b_4) \wedge (a_5 \wedge b_5 + \cdots + a_g \wedge b_g)) \\ &\quad = 2(1-g)(g-4)a_1 \wedge a_2, \end{aligned}$$

and

$$\begin{aligned} &q_2 \circ q_4(a_1 \wedge a_2 \wedge (a_5 \wedge b_5 + \cdots + a_g \wedge b_g) \wedge ((1-g)a_3 \wedge b_3)) \\ &\quad = 2(1-g)(g-4)a_1 \wedge a_2, \end{aligned}$$

and

$$\begin{aligned} &q_2 \circ q_4(a_1 \wedge a_2 \wedge ((1-g)(a_4 \wedge b_4)) \wedge ((1-g)(a_3 \wedge b_3))) \\ &\quad = 2(1-g)(1-g)a_1 \wedge a_2. \end{aligned}$$

Adding these all up, we get

$$\left(2(g-2)(g-3) + 4(1-g)(g-4) + 2(1-g)(1-g)\right) a_1 \wedge a_2 = (6g-2)a_1 \wedge a_2. \qquad \square$$

9. First Johnson homomorphism

Fix some $g \geq 3$ and let $H = \mathrm{H}_1(\Sigma_g^1; \mathbb{Q})$. Recall from Example 3.2 that $\mathrm{FLie}_2(H) \cong \wedge^2 H \cong \mathbf{V}_0(g) \oplus \mathbf{V}_{1^2}(g)$. We observed in Example 3.3 that the kernel of the Lie bracket map from

$$H \otimes \mathrm{FLie}_2(H) = H \otimes \wedge^2 H \cong \mathbf{V}_{1^3}(g) \oplus \mathbf{V}_1(g)^{\oplus 2} \oplus \mathbf{V}_{2,1}(g)$$

to $\mathrm{Lie}_3(H)$ is

$$\wedge^3 H \cong \mathbf{V}_{1^3}(g) \oplus \mathbf{V}_1(g).$$

Theorem 6.1 therefore implies that the $\mathbb{Q}$-span of the image of $\tau_1 \colon \mathcal{I}_g^1 \to H \otimes \mathrm{FLie}_2(H)$ is contained in $\wedge^3 H$. Johnson [26] proved that the $\mathbb{Q}$-span of the image of τ_1 equals $\wedge^3 H$. This section proves this and uses it to describe $\mathrm{H}_1(\mathcal{I}_g^1; \mathbb{Q})$.

In fact, Johnson proved something stronger:

Theorem 9.1 (Johnson, [26])**.** *For some $g \geq 3$, let $H_{\mathbb{Z}} = \mathrm{H}_1(\Sigma_g^1; \mathbb{Z}) \subset H$. Then the image of $\tau_1 \colon \mathcal{I}_g^1 \to H \otimes \wedge^2 H$ is $\wedge^3 H_{\mathbb{Z}}$. In particular, the $\mathbb{Q}$-span of the image of τ_1 equals $\wedge^3 H$.*

Proof. Since it will illustrate our representation-theoretic tools, we will prove that the $\mathbb{Q}$-span of the image of τ_1 equals $\wedge^3 H$ and leave the proof that the image of τ_1 is $\wedge^3 H_{\mathbb{Z}}$ as an exercise to the reader.[8] As in the following figure, let $\{a_1, b_1, \ldots, a_g, b_g\}$ be the standard symplectic basis for H:

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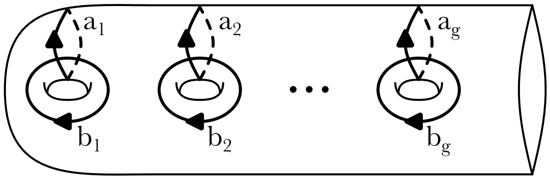
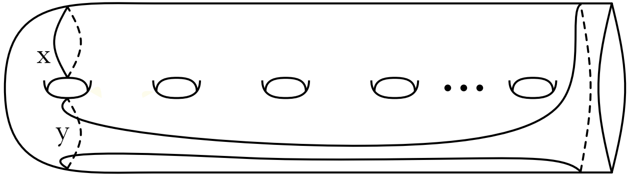
[8] Here is a hint. Letting $S = \{a_1, b_1, \ldots, a_g, b_g\}$ be the standard symplectic basis for $H_{\mathbb{Z}}$, we have that $\wedge^3 H_{\mathbb{Z}}$ is spanned by $\widehat{S} = \{x \wedge y \wedge z \mid x, y, z \in S \text{ distinct}\}$. Using the formulas for τ_1 from Lemmas 5.5 and 5.6, one can show that each $x \wedge y \wedge z \in \widehat{S}$ is up to signs equal to $\tau_1(\phi)$ for ϕ either a bounding pair map or a simply intersecting pair map.



As we noted above, Theorem 6.1 implies that the $\mathbb{Q}$-span J of the image of τ_1 is contained in $\wedge^3 H$. We must prove that $J = \wedge^3 H$. The subspace J is an $\mathrm{Sp}_{2g}(\mathbb{Q})$-subrepresentation of $\wedge^3 H$ (Corollary 4.5). We have $\wedge^3 H = \mathbf{V}_1(g) \oplus \mathbf{V}_{1^3}(g)$. We must prove that J contains both of these irreducible factors. Let $\omega \in \wedge^2 H$ be the intersection pairing.

Claim 1. *The subspace J contains the subrepresentation $\mathbf{V}_1(g) \cong H$ of $\wedge^3 H$.*

This subrepresentation has the highest weight vector $a_1 \wedge \omega$, so we must prove that $a_1 \wedge \omega \in J$. For this, let $T_x T_y^{-1}$ be the following bounding pair map:

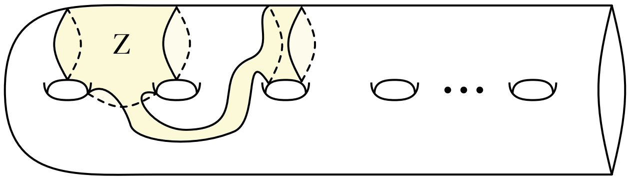


By Lemma 5.5, we have

$$\begin{aligned}\tau_1(T_x T_y^{-1}) &= a_1 \wedge (a_2 \wedge b_2 + \cdots + a_g \wedge b_g) \\ &= a_1 \wedge (a_1 \wedge b_1 + a_2 \wedge b_2 + \cdots + a_g \wedge b_g) = a_1 \wedge \omega.\end{aligned}$$

Claim 2. *The subspace J contains the subrepresentation $\mathbf{V}_{1^3}(g)$ of $\wedge^3 H$.*

This subrepresentation has the highest weight vector $a_1 \wedge a_2 \wedge a_3$, so we must prove that $a_1 \wedge a_2 \wedge a_3 \in J$. For this, let f be a simply intersecting pair map supported on the following subsurface $Z \cong \Sigma_0^4$:



By Lemma 5.6, we have $\tau_1(f) = \pm a_1 \wedge a_2 \wedge a_3$. ☐

Theorem 9.1 implies that τ_1 induces a surjection $(\tau_1)_* \colon \mathrm{H}_1(\mathcal{I}_g^1; \mathbb{Q}) \to \wedge^3 H$. Johnson [29, 30] proved that this is an isomorphism:

Theorem 9.2 (Johnson, [29, 30])**.** *For $g \geq 3$, the first Johnson homomorphism induces an isomorphism $\mathrm{H}_1(\mathcal{I}_g^1; \mathbb{Q}) \cong \wedge^3 H$.*

Remark 9.3. Johnson proves this by first proving that $\ker(\tau_1) = \mathcal{I}_g^1[2]$ is generated by separating twists. This actually holds for all g. For $g \geq 3$, he then proves that squares of separating twists lie in the commutator subgroup of $\mathcal{I}_g^1$, which implies Theorem 9.2. See [50] for another exposition of this calculation. ☐

10. SECOND JOHNSON HOMOMORPHISM

Fix some $g \geq 4$ and let $H = \mathrm{H}_1(\Sigma_g^1; \mathbb{Q})$. Recall from Example 3.3 that

$$\mathrm{FLie}_3(H) \cong \frac{H \otimes \wedge^2 H}{\wedge^3 H} \cong \mathbf{V}_1(g) \oplus \mathbf{V}_{2,1}(g).$$

We observed in Example 3.4 that the kernel of the Lie bracket map from

$$H \otimes \mathrm{FLie}_3(H) = \frac{H^{\otimes 2} \otimes \wedge^2 H}{H \otimes \wedge^3 H} \cong \mathbf{V}_0(g) \oplus \mathbf{V}_{1^2}(g)^{\oplus 2} \oplus \mathbf{V}_{2^2}(g) \oplus \mathbf{V}_{2,1^2}(g) \oplus \mathbf{V}_2(g)^{\oplus 2} \oplus \mathbf{V}_{3,1}(g)$$

to $\mathrm{Lie}_4(H)$ is

$$\frac{\mathrm{Sym}^2(\wedge^2 H)}{\wedge^4 H} \cong \mathbf{V}_0(g) \oplus \mathbf{V}_{1^2}(g) \oplus \mathbf{V}_{2^2}(g).$$

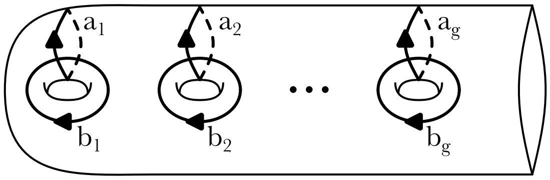
Theorem 6.1 therefore implies that the $\mathbb{Q}$-span of the image of $\tau_2\colon \mathcal{I}_g^1[2] \to H \otimes \mathrm{FLie}_3(H)$ is contained in $\mathrm{Sym}^2(\wedge^2 H)/\wedge^4 H$. Morita [43] proved that the $\mathbb{Q}$-span of the image of τ_2 equals $\mathrm{Sym}^2(\wedge^2 H)/\wedge^4 H$. This section proves this.

Remark 10.1. Unlike for the first Johnson homomorphism, it is not true that the image of τ_2 equals the integer points of $\mathrm{Sym}^2(\wedge^2 H)/\wedge^4 H$. Morita actually proved something more precise than what we will prove, and Yokomizo [59] completed this to give a complete description of the image of τ_2. □

Morita's theorem is as follows:[9]

Theorem 10.2 (Morita, [43, Proposition 1.2])**.** *For $g \geq 4$, the $\mathbb{Q}$-span of the image of $\tau_2\colon \mathcal{I}_g^1[2] \to H \otimes \frac{H^{\otimes 2} \otimes \wedge^2 H}{H \otimes \wedge^3 H}$ equals $\mathrm{Sym}^2(\wedge^2 H)/\wedge^4 H \cong \mathbf{V}_0(g) \oplus \mathbf{V}_{1^2}(g) \oplus \mathbf{V}_{2^2}(g)$.*

Proof. As in the following figure, let $\{a_1, b_1, \ldots, a_g, b_g\}$ be the standard symplectic basis for H:



As we noted above, Theorem 6.1 implies that the $\mathbb{Q}$-span J of the image of τ_2 is contained in $\mathrm{Sym}^2(\wedge^2 H)/\wedge^4 H$. We must prove that $J = \mathrm{Sym}^2(\wedge^2 H)/\wedge^4 H$. The subspace J is an $\mathrm{Sp}_{2g}(\mathbb{Q})$-subrepresentation of $\mathrm{Sym}^2(\wedge^2 H)/\wedge^4 H$ (Corollary 4.5), so we must study this representation. Before doing so, we identify some elements of J.

Claim 1. *Let $\{u_1, v_1, \ldots, u_h, v_h\}$ be a symplectic basis for a symplectic subspace of H. Then the element*

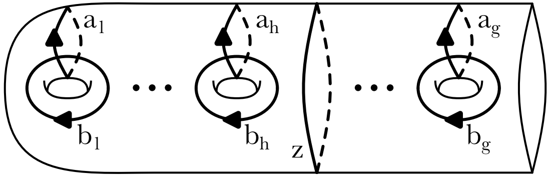
$$(u_1 \wedge v_1 + \cdots + u_h \wedge v_h) \cdot (u_1 \wedge v_1 + \cdots + u_h \wedge v_h) \in \mathrm{Sym}^2(\wedge^2 H)$$

maps to an element of J.

Proof of claim. The group $\mathrm{Sp}_{2g}(\mathbb{Q})$ acts on J. We can find $\phi \in \mathrm{Sp}_{2g}(\mathbb{Q})$ such that $\phi(u_i) = a_i$ and $\phi(v_i) = b_i$ for $1 \leq i \leq h$. Applying ϕ, we reduce ourselves to showing that

$$(a_1 \wedge b_1 + \cdots + a_h \wedge b_h) \cdot (a_1 \wedge b_1 + \cdots + a_h \wedge b_h) \in \mathrm{Sym}^2(\wedge^2 H)$$

maps to an element of J. For this, let z be the following separating curve:



By Lemma 5.8, the element $\tau_2(T_z) \in J$ is the image of the desired element of $\mathrm{Sym}^2(\wedge^2 H)$. □

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[9] The restriction to $g \geq 4$ is to ensure that the decomposition of $\mathrm{Sym}^2(\wedge^2 H)$ is stable.

Let $\widetilde{J}$ be the $\mathrm{Sp}_{2g}(\mathbb{Q})$-subrepresentation of $\mathrm{Sym}^2(\wedge^2 H)$ spanned by the elements identified in Claim 1. By that claim, it is enough to prove that $\widetilde{J}$ maps surjectively to $\mathrm{Sym}^2(\wedge^2 H)/\wedge^4 H$. We have

$$\begin{aligned}\mathrm{Sym}^2(\wedge^2 H) &= \mathbf{V}_{1^4}(g) \oplus \mathbf{V}_{1^2}(g)^{\oplus 2} \oplus \mathbf{V}_0(g)^{\oplus 2} \oplus \mathbf{V}_{2^2}(g),\\ \wedge^4 H &= \mathbf{V}_{1^4}(g) \oplus \mathbf{V}_{1^2}(g) \oplus \mathbf{V}_0(g).\end{aligned}$$

The next step is to tease apart the copies of $\mathbf{V}_0(g)$ and $\mathbf{V}_{1^4}(g)$ in $\mathrm{Sym}^2(\wedge^2 H)$ coming from $\wedge^4 H$ and the copies that we will find in $\widetilde{J}$. The embedding $\iota\colon \wedge^4 H \to \mathrm{Sym}^2(\wedge^2 H)$ takes the form

$$h_1 \wedge h_2 \wedge h_3 \wedge h_4 \in \wedge^4 H \mapsto (h_1 \wedge h_2)\cdot(h_3 \wedge h_4) - (h_1 \wedge h_3)\cdot(h_2 \wedge h_4) + (h_1 \wedge h_4)\cdot(h_2 \wedge h_3).$$

Letting $\omega = a_1 \wedge b_1 + \cdots + a_g \wedge b_g$ be the symplectic form, the following are highest weight vectors for the three irreducible factors of $\wedge^4 H$:

$$v_0 = a_1 \wedge a_2 \wedge a_3 \wedge a_4 \quad \text{and} \quad v_1 = a_1 \wedge a_2 \wedge \omega \quad \text{and} \quad v_2 = \omega \wedge \omega.$$

The vector $\iota(v_0)$ must be a highest weight vector for the unique copy of $\mathbf{V}_{1^4}(g)$ in $\mathrm{Sym}^2(\wedge^2 H)$. As for the other terms, in $\mathrm{Sym}^2(\wedge^2 H)$ we have highest weight vectors

$$w_1 = (a_1 \wedge a_2)\cdot\omega \quad \text{and} \quad w_2 = \omega\cdot\omega \quad \text{and} \quad w_3 = (a_1 \wedge a_2)\cdot(a_1 \wedge a_2).$$

Since $\iota(v_1)$ is not a multiple of w_1 and $\iota(v_2)$ is not a multiple of w_2, we see that:

- $\iota(v_0)$ is a highest weight vector for the copy of $\mathbf{V}_{1^4}(g)$ in $\mathrm{Sym}^2(\wedge^2 H)$.
- $\iota(v_1)$ and w_1 are highest weight vectors for the two copies of $\mathbf{V}_{1^2}(g)$ in $\mathrm{Sym}^2(\wedge^2 H)$.
- $\iota(v_2)$ and w_2 are highest weight vectors for the two copies of $\mathbf{V}_0(g)$ in $\mathrm{Sym}^2(\wedge^2 H)$.
- w_3 is a highest weight vector for the copy of $\mathbf{V}_{2^2}(g)$ in $\mathrm{Sym}^2(\wedge^2 H)$.

To show that $\widetilde{J}$ maps surjectively to $\mathrm{Sym}^2(\wedge^2 H)/\wedge^4 H$, it is therefore enough to prove the following. Note that we prove the three claims in order of difficulty: first w_2, then w_3, and then finally w_1.

Claim 2. *The highest weight vector* $w_2 = \omega\cdot\omega \in \mathrm{Sym}^2(\wedge^2 H)$ *lies in* $\widetilde{J}$*, so the image of* $\widetilde{J}$ *in* $\mathrm{Sym}^2(\wedge^2 H)/\wedge^4 H$ *contains a copy of* $\mathbf{V}_0(g)$*.*

Since $\omega = a_1 \wedge b_1 + \cdots + a_g \wedge b_g$, we have $\omega\cdot\omega \in \widetilde{J}$ by definition (cf. Claim 1).

Claim 3. *The highest weight vector* $w_3 = (a_1 \wedge a_2)\cdot(a_1 \wedge a_2) \in \mathrm{Sym}^2(\wedge^2 H)$ *lies in* $\widetilde{J}$*, so the image of* $\widetilde{J}$ *in* $\mathrm{Sym}^2(\wedge^2 H)/\wedge^4 H$ *contains a copy of* $\mathbf{V}_{2^2}(g)$*.*

By definition, the following elements all lie in $\widetilde{J}$:

- $(a_1 \wedge b_1)\cdot(a_1 \wedge b_1)$; and
- $(a_1 \wedge (b_1 + a_2))\cdot(a_1 \wedge (b_1 + a_2))$; and
- $(a_1 \wedge (b_1 - a_2))\cdot(a_1 \wedge (b_2 - a_2))$.

Since

$$\begin{aligned}&(a_1 \wedge (b_1 + a_2))\cdot(a_1 \wedge (b_1 + a_2)) + (a_1 \wedge (b_1 - a_2))\cdot(a_1 \wedge (b_2 - a_2))\\ =&2(a_1 \wedge b_1)\cdot(a_1 \wedge b_1) + 2(a_1 \wedge a_2)\cdot(a_1 \wedge a_2),\end{aligned}$$

it follows that $(a_1 \wedge a_2)\cdot(a_1 \wedge a_2) \in \widetilde{J}$.

Claim 4. *The highest weight vector* $w_1 = (a_1 \wedge a_2)\cdot\omega \in \mathrm{Sym}^2(\wedge^2 H)$ *lies in* $\widetilde{J}$*, so the image of* $\widetilde{J}$ *in* $\mathrm{Sym}^2(\wedge^2 H)/\wedge^4 H$ *contains a copy of* $\mathbf{V}_{1^2}(g)$*.*

Observe first that

$$(a_1 \wedge (b_1 + a_2))\cdot\omega = (a_1 \wedge b_1)\cdot\omega + (a_1 \wedge a_2)\cdot\omega.$$

From this, we see that it is enough to prove that both $(a_1 \wedge (b_1 + a_2)) \cdot \omega$ and $(a_1 \wedge b_1) \cdot \omega$ lie in $\widetilde{J}$. Since $\operatorname{Sp}_{2g}(\mathbb{Q})$ fixes ω, these two elements differ by an element of $\operatorname{Sp}_{2g}(\mathbb{Q})$. It is therefore enough to prove that one of them lies in $\widetilde{J}$. We will prove that $(a_1 \wedge b_1) \cdot \omega$ lies in $\widetilde{J}$.

We expand this out:

$$(a_1 \wedge b_1) \cdot \omega = (a_1 \wedge b_1) \cdot (a_1 \wedge b_1) + (a_1 \wedge b_1) \cdot (a_2 \wedge b_2) + \cdots + (a_1 \wedge b_1) \cdot (a_g \wedge b_g).$$

Since $(a_1 \wedge b_1) \cdot (a_1 \wedge b_1)$ lies in $\widetilde{J}$ by definition, we see that it is enough to prove that $(a_1 \wedge b_1) \cdot (a_i \wedge b_i)$ lies in $\widetilde{J}$ for all $2 \leq i \leq g$. Since these elements all differ by an element of $\operatorname{Sp}_{2g}(\mathbb{Q})$, it is enough to prove that one of them lies in $\widetilde{J}$. We will prove that $(a_1 \wedge b_1) \cdot (a_2 \wedge b_2)$ lies in $\widetilde{J}$.

By definition, the following three elements all lie in $\widetilde{J}$:

- $(a_1 \wedge b_1) \cdot (a_1 \wedge b_1)$; and
- $(a_2 \wedge b_2) \cdot (a_2 \wedge b_2)$; and
- $(a_1 \wedge b_1 + a_2 \wedge b_2) \cdot (a_1 \wedge b_1 + a_2 \wedge b_2)$.

Since

$$\begin{aligned} &(a_1 \wedge b_1 + a_2 \wedge b_2) \cdot (a_1 \wedge b_1 + a_2 \wedge b_2) \\ =&(a_1 \wedge b_1) \cdot (a_1 \wedge b_1) + 2(a_1 \wedge b_1) \cdot (a_2 \wedge b_2) + (a_2 \wedge b_2) \cdot (a_2 \wedge b_2), \end{aligned}$$

we conclude that $(a_1 \wedge b_1) \cdot (a_2 \wedge b_2)$ lies in $\widetilde{J}$, as desired. □

11. Ruling out trivial factors

Recall that our goal in this part of the paper is to prove Theorem B. For $\mathcal{I}_g^1$, this theorem calculates the image of the cup product pairing $\mathfrak{c} \colon \wedge^2 \mathrm{H}^1(\mathcal{I}_g^1; \mathbb{Q}) \to \mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q})$. As a prelude to this calculation, this section proves that $\mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q})$ has no trivial subrepresentations.

11.1. **Second cohomology group of Torelli group and mapping class group.** Though it is not absolutely necessary for our proof, to avoid technicalities we start by recalling the following theorem which was discussed in the introduction and which we are assuming throughout this paper:

Theorem A (Minahan–Putman, [42, Theorem B]**).** *The homology group* $\mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q})$ *is finite-dimensional for* $g \geq 5$ *and an algebraic representation of* $\operatorname{Sp}_{2g}(\mathbb{Z})$ *for* $g \geq 6$.

Remark 11.1. As we noted in the introduction, [42] also proves similar results for $\mathcal{I}_g$ and $\mathcal{I}_{g,1}$. In fact, it proves a result for $\mathcal{I}_{g,p}^b$ in general, though it requires care to properly define the Torelli group on a surface with multiple punctures and boundary components. □

For the whole mapping class group, we have the following:

Theorem 11.2 (Harer, [21])**.** *For* $g \geq 3$*, we have* $\mathrm{H}^2(\operatorname{Mod}_g^1; \mathbb{Q}) = \mathbb{Q}$.

Remark 11.3. After many further developments, the ultimate result in this direction was proved by Madsen–Weiss [37], who proved that the cohomology ring $\mathrm{H}^\bullet(\operatorname{Mod}_g^1; \mathbb{Q})$ is isomorphic to a polynomial ring $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$ with $|\kappa_i| = 2i$ in a range of degrees that tends to infinity as $g \mapsto \infty$. □

11.2. **Borel stability theorem.** We will also need the following special case of the classical Borel stability theorem [3, 4] on the cohomology of arithmetic groups. The explicit stable range in the following is due to Tshishiku [58]:

Theorem 11.4 (Borel, [3, 4])**.** *For* $g \geq 2$*, the following hold:*

(i) If $\mathbf{V}_\sigma(g)$ *is a nontrivial irreducible representation of* Sp_{2g}*, then* $\mathrm{H}^k(\operatorname{Sp}_{2g}(\mathbb{Z}); \mathbf{V}_\sigma(g)) = 0$ *for* $k \leq g - 1$.

(ii) In degrees $k \leq g-1$, the cohomology ring $\mathrm{H}^{\bullet}(\mathrm{Sp}_{2g}(\mathbb{Z});\mathbb{Q})$ is isomorphic to a polynomial ring $\mathbb{Q}[c_2, c_6, c_{10}, \ldots]$ with $\deg(c_{4i-2}) = 4i-2$ for $i \geq 1$.

11.3. **No trivial factors.** We can now rule out trivial factors of $\mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q})$:

Lemma 11.5. *For $g \geq 6$, the $\mathrm{Sp}_{2g}(\mathbb{Z})$-representation $\mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q})$ contains no trivial factors.*

Proof. The restriction $g \geq 6$ implies that $\mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q})$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$ (Theorem A). It therefore decomposes as a direct sum of irreducible factors, so the statement of the lemma makes sense. Consider the Hochschild–Serre spectral sequence with coefficients in $\mathbf{V}_0(g) = \mathbb{Q}$ associated to the short exact sequence

$$1 \longrightarrow \mathcal{I}_g^1 \longrightarrow \mathrm{Mod}_g^1 \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1.$$

This spectral sequence takes the form

$$E_2^{pq} = \mathrm{H}^p(\mathrm{Sp}_{2g}(\mathbb{Z}); \mathrm{H}^q(\mathcal{I}_g^1;\mathbb{Q})) \Rightarrow \mathrm{H}^{p+q}(\mathrm{Mod}_g^1;\mathbb{Q}).$$

To understand H^2, we must understand E_2^{pq} for $p+q \leq 2$ as well as E_2^{21} and E_2^{30}. For $q=0$, by Theorem 11.4 we have

$$E_2^{p0} = \mathrm{H}^p(\mathrm{Sp}_{2g}(\mathbb{Z});\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } p = 0, 2, \\ 0 & \text{if } p = 1, 3. \end{cases}$$

Next, let $H = \mathrm{H}_1(\Sigma_g^1;\mathbb{Q})$. Theorem 9.2 says that $\mathrm{H}^1(\mathcal{I}_g^1;\mathbb{Q}) \cong \wedge^3 H$, which has no trivial factors. Theorem 11.4 therefore implies that

$$E_2^{p1} = \mathrm{H}^p(\mathrm{Sp}_{2g}(\mathbb{Z}); \wedge^3 H) = 0 \quad \text{for } p \leq g-1.$$

Finally,

$$E_2^{02} = \mathrm{H}^0(\mathrm{Sp}_{2g}(\mathbb{Z}); \mathrm{Hom}(\mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q}),\mathbb{Q})) \cong \mathrm{Hom}_{\mathrm{Sp}_{2g}}(\mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q}), \mathbf{V}_0(g)).$$

It follows that the dimension of E_2^{02} equals the number of copies of $\mathbf{V}_0(g)$ in $\mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q})$. Summarizing, since $g \geq 6$ the E_2-page of our spectral sequence takes the form

$$\begin{array}{|cccc} E_2^{02} & & & \\ 0 & 0 & 0 & \\ \mathbb{Q} & 0 & \mathbb{Q} & 0 \\ \hline \end{array}$$

It follows that $\mathrm{H}^2(\mathrm{Mod}_g^1;\mathbb{Q}) \cong E_2^{02} \oplus \mathbb{Q}$. Theorem 11.2 says that this equals $\mathbb{Q}$, so we conclude that $E_2^{02} = 0$, i.e., that there are no copies of $\mathbf{V}_0(g)$ in $\mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q})$. □

12. Cup products and the Johnson homomorphism

In this section, we relate the image of the cup product pairing to the Johnson homomorphism. For $H_{\mathbb{Z}} = \mathrm{H}_1(\Sigma_g^1;\mathbb{Z})$, Theorem 9.1 implies that the first Johnson homomorphism can be regarded as a homomorphism $\tau_1 \colon \mathcal{I}_g^1 \to \wedge^3 H_{\mathbb{Z}}$. We then have:

Lemma 12.1. *Let $g \geq 3$. Set $H_{\mathbb{Z}} = \mathrm{H}_1(\Sigma_g^1;\mathbb{Z})$, and let $\tau_1 \colon \mathcal{I}_g^1 \to \wedge^3 H_{\mathbb{Z}}$ be the first Johnson homomorphism. Then as representations of $\mathrm{Sp}_{2g}(\mathbb{Z})$, the images of the following two maps are the same:*

- *The cup product pairing $\mathfrak{c} \colon \wedge^2 \mathrm{H}^1(\mathcal{I}_g^1;\mathbb{Q}) \to \mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q})$.*
- *The map $(\tau_1)_* \colon \mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q}) \to \mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q})$ induced by the first Johnson homomorphism.*

Proof. The first observation in the proof is as follows:

Claim. *The image of the cup product pairing* $\mathfrak{c}\colon \wedge^2 \mathrm{H}^1(\mathcal{I}_g^1;\mathbb{Q}) \to \mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q})$ *is the same as the image of the induced map* $(\tau_1)^*\colon \mathrm{H}^2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q}) \to \mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q})$.

Proof of claim. Let $c\colon \wedge^2 \mathrm{H}^1(\wedge^3 H_{\mathbb{Z}};\mathbb{Q}) \to \mathrm{H}^2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q})$ be the cup product pairing. Since $H_{\mathbb{Z}}$ is a free abelian group, the map c is an isomorphism. By the naturality of the cup product pairing, we have a commutative diagram

$$\begin{tikzcd}
\wedge^2 \mathrm{H}^1(\wedge^3 H_{\mathbb{Z}};\mathbb{Q}) \arrow[r, "\wedge^2(\tau_1)^*"] \arrow[d, "c", "\cong"'] & \wedge^2 \mathrm{H}^1(\mathcal{I}_g^1;\mathbb{Q}) \arrow[d, "\mathfrak{c}"] \\
\mathrm{H}^2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q}) \arrow[r, "(\tau_1)^*"] & \mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q}).
\end{tikzcd}$$

Theorem 9.2 implies that the top row is an isomorphism. The claim follows. ☐

Working now with homology rather than cohomology, we have $\mathrm{H}_1(\wedge^3 H_{\mathbb{Z}};\mathbb{Q}) \cong \wedge^3 H$. This is an isomorphism of representations of $\mathrm{Sp}_{2g}(\mathbb{Z})$. As a representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$, the representation H is isomorphic to its dual. Dualizing, we therefore get that $\mathrm{H}^1(\wedge^3 H_{\mathbb{Z}};\mathbb{Q}) \cong \wedge^3 H$ and thus that $\mathrm{H}^2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q}) \cong \wedge^2 \wedge^3 H$. We deduce that $\mathrm{H}^2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q})$ is also self-dual as a representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$. From this, we see that as representations of $\mathrm{Sp}_{2g}(\mathbb{Z})$ the images of the dual maps

$$(\phi_1)^*\colon \mathrm{H}^2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q}) \to \mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q}),$$
$$(\phi_1)_*\colon \mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q}) \to \mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q})$$

are isomorphic. The lemma follows. ☐

13. CALCULATION OF IMAGE OF CUP PRODUCT PAIRING ON SURFACES WITH BOUNDARY

We now prove Theorem B for surfaces with boundary. The statement of this result is as follows. It is due to Hain [19], but our exposition is also influenced by an unpublished paper of Habegger–Sorger [18].

Theorem B (surface with boundary case). *Let* $g \geq 6$. *The image of the cup product pairing* $\mathfrak{c}\colon \wedge^2 \mathrm{H}^1(\mathcal{I}_g^1;\mathbb{Q}) \to \mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q})$ *is isomorphic to the following representation of* $\mathrm{Sp}_{2g}(\mathbb{Z})$*:*

$$\mathbf{V}_{1^2}(g)^{\oplus 2} \oplus \mathbf{V}_{2,1^2}(g) \oplus \mathbf{V}_{1^4}(g)^{\oplus 2} \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g).$$

Proof. Let $H_{\mathbb{Z}} = \mathrm{H}_1(\Sigma_g^1;\mathbb{Z})$ and $H = \mathrm{H}_1(\Sigma_g^1;\mathbb{Q})$. Theorem 9.1 implies that the first Johnson homomorphism can be regarded as a homomorphism $\tau_1\colon \mathcal{I}_g^1 \to \wedge^3 H_{\mathbb{Z}}$. By Lemma 12.1, it is enough to prove that the image of the induced map

$$(\tau_1)_*\colon \mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q}) \to \mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q}) \tag{13.1}$$

is the indicated representation. Since $\wedge^3 H_{\mathbb{Z}}$ is a free abelian group, we have that the representation $\mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q})$ is isomorphic to[10]

$$\wedge^2 \wedge^3 H \cong \mathbf{V}_0(g)^{\oplus 2} \oplus \mathbf{V}_{1^2}(g)^{\oplus 3} \oplus \mathbf{V}_{2^2}(g) \oplus \mathbf{V}_{2,1^2}(g) \oplus \mathbf{V}_{1^4}(g)^{\oplus 2} \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g). \tag{13.2}$$

We will first prove that the image of the map (13.1) is contained in the desired subrepresentation of this, and after that prove that in fact it is equal to it.

Step 1. *The image of the map* $(\tau_1)_*\colon \mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q}) \to \mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q})$ *is contained in a subrepresentation of* $\mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q}) \cong \wedge^2 \wedge^3 H$ *that is isomorphic to the following:*

$$\mathbf{V}_{1^2}(g)^{\oplus 2} \oplus \mathbf{V}_{2,1^2}(g) \oplus \mathbf{V}_{1^4}(g)^{\oplus 2} \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g). \tag{13.3}$$

---

[10] Here we are using the fact that $g \geq 6$ to ensure that the decomposition of $\wedge^2 \wedge^3 H$ is stable (see §7.9), and thus can be computed with LiE [35].

Let C be the cokernel of the map $(\tau_1)_*\colon \mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q}) \to \mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q})$. We must prove that C contains the subrepresentation $V = \mathbf{V}_0(g)^{\oplus 2} \oplus \mathbf{V}_{1^2}(g) \oplus \mathbf{V}_{2^2}(g)$, which is obtained from (13.2) by deleting the subrepresentation (13.3). Since $\mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q})$ contains no trivial subrepresentations (Lemma 11.5), it is certainly the case that C contains the subrepresentation $\mathbf{V}_0(g)^{\oplus 2}$. It remains to prove that C contains $W = \mathbf{V}_{1^2}(g) \oplus \mathbf{V}_{2^2}(g)$.

The kernel of τ_1 is the second term $\mathcal{I}_g^1[2]$ of the Johnson filtration (Lemma 4.3). We therefore have a short exact sequence of groups

$$0 \longrightarrow \mathcal{I}_g^1[2] \longrightarrow \mathcal{I}_g^1 \xrightarrow{\tau_1} \wedge^3 H_{\mathbb{Z}} \longrightarrow 1.$$

Associated to this is a five-term exact sequence in group homology (see [6, Corollary VII.6.4]). For a group G acting on a vector space M, let M_G be the G-coinvariants, i.e., the quotient $M/\langle m - gm \mid g \in G,\, m \in M\rangle$. The five-term exact sequence then takes the form

$$\mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q}) \xrightarrow{(\tau_1)_*} \mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q}) \longrightarrow \mathrm{H}_1(\mathcal{I}_g^1[2];\mathbb{Q})_{\mathcal{I}_g^1} \longrightarrow \mathrm{H}_1(\mathcal{I}_g^1;\mathbb{Q}) \xrightarrow{(\phi_1)_*} \mathrm{H}_1(\wedge^3 H_{\mathbb{Z}};\mathbb{Q}) \longrightarrow 0.$$

Here the coinvariants $\mathrm{H}_1(\mathcal{I}_g^1[2];\mathbb{Q})_{\mathcal{I}_g^1}$ are with respect to the action of $\mathcal{I}_g^1$ induced by the conjugation action of $\mathcal{I}_g^1$ on its normal subgroup $\mathcal{I}_g^1[2]$. By Theorem 9.2, the map

$$(\phi_1)_*\colon \mathrm{H}_1(\mathcal{I}_g^1;\mathbb{Q}) \to \mathrm{H}_1(\wedge^3 H_{\mathbb{Z}};\mathbb{Q}) = \wedge^3 H$$

is an isomorphism. The above five-term exact sequence thus gives an exact sequence

$$\mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q}) \xrightarrow{(\tau_1)_*} \mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q}) \longrightarrow \mathrm{H}_1(\mathcal{I}_g^1[2];\mathbb{Q})_{\mathcal{I}_g^1} \longrightarrow 0. \tag{13.4}$$

From this, we see that $C \cong \mathrm{H}_1(\mathcal{I}_g^1[2];\mathbb{Q})_{\mathcal{I}_g^1}$. This isomorphism is $\mathrm{Sp}_{2g}(\mathbb{Z})$-equivariant for the action of $\mathrm{Sp}_{2g}(\mathbb{Z}) = \mathrm{Mod}_g^1 / \mathcal{I}_g^1$ on $\mathrm{H}_1(\mathcal{I}_g^1[2];\mathbb{Q})_{\mathcal{I}_g^1}$ induced by the conjugation action of Mod_g^1 on $\mathcal{I}_g^1[2]$.

Our goal, therefore, is to prove that $\mathrm{H}_1(\mathcal{I}_g^1[2];\mathbb{Q})_{\mathcal{I}_g^1}$ contains the subrepresentation $W = \mathbf{V}_{1^2}(g) \oplus \mathbf{V}_{2^2}(g)$. We will detect this with the second Johnson homomorphism $\tau_2\colon \mathcal{I}_g^1[2] \to H \otimes \mathrm{Lie}_3(H)$. This is $\mathcal{I}_g^1$-invariant, so it factors through $\mathrm{H}_1(\mathcal{I}_g^1[2];\mathbb{Q})_{\mathcal{I}_g^1}$. Theorem 10.2 implies that the $\mathbb{Q}$-span of the image of τ_2 is

$$V' = \mathrm{Sym}^2(\wedge^2 H)/\wedge^4 H \cong \mathbf{V}_0(g) \oplus \mathbf{V}_{1^2}(g) \oplus \mathbf{V}_{2^2}(g).$$

Each of these is an irreducible factor of $\mathrm{H}_1(\mathcal{I}_g^1[2];\mathbb{Q})_{\mathcal{I}_g^1}$, so certainly $\mathrm{H}_1(\mathcal{I}_g^1[2];\mathbb{Q})_{\mathcal{I}_g^1}$ contains W.

Before going on, we make a remark. At the beginning of this step, our goal was to to detect $V = \mathbf{V}_0(g)^{\oplus 2} \oplus \mathbf{V}_{1^2}(g) \oplus \mathbf{V}_{2^2}(g)$. We handled the trivial representations, and then used the second Johnson homomorphism to detect $V' = \mathbf{V}_0(g) \oplus \mathbf{V}_{1^2}(g) \oplus \mathbf{V}_{2^2}(g)$.

The only difference between V and V' is that V contains one extra copy of $\mathbf{V}_0(g)$. Morita [43, §4] gave a beautiful geometric interpretation of this extra factor of $\mathbf{V}_0(g)$. He used the Casson invariant of homology 3-spheres to construct a Mod_g^1-invariant map $m\colon \mathcal{I}_g^1[2] \to \mathbb{Q}$ that he calls the "core" of the Casson invariant. It follows from his results that m does not factor through the second Johnson homomorphism, so it provides the missing $\mathbf{V}_0(g)$ factor.

Morita's homomorphism $m\colon \mathcal{I}_g^1[2] \to \mathbb{Q}$ has the following beautiful formula. Recall that Johnson [29] proved that $\mathcal{I}_g^1[2]$ is generated by separating twists. Consider a separating twist T_z. The curve z separates Σ_g^1 into two subsurfaces S and S'. Order them such that $\partial \Sigma_g^1 \subset S'$, so $S \cong \Sigma_h^1$. We then have

$$m(T_z) = 4h(h-1).$$

See [43, Theorem 5.3]. As far as this author is aware, there is no way to prove that this formula gives a homomorphism that does not use either the Casson invariant or other similarly deep facts about the cohomology of the mapping class group.

Step 2. *The image of the map* $(\tau_1)_* \colon \mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q}) \to \mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q})$ *equals the following subrepresentation of* $\mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q}) \cong \wedge^2 \wedge^3 H$*:*

$$\mathbf{V}_{1^2}(g)^{\oplus 2} \oplus \mathbf{V}_{2,1^2}(g) \oplus \mathbf{V}_{1^4}(g)^{\oplus 2} \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g).$$

By Step 1, it is enough to prove that the image of the map

$$(\tau_1)_* \colon \mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q}) \to \mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q}) \cong \wedge^2 \wedge^3 H. \tag{13.5}$$

contains each of the indicated irreducible factors. Before doing this, we expand on the isomorphism $\mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q}) \cong \wedge^2 \wedge^3 H$. Since $H_{\mathbb{Z}}$ is an abelian group, its multiplication map $H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to H_{\mathbb{Z}}$ induces a product structure on $\mathrm{H}_\bullet(\wedge^3 H_{\mathbb{Z}};\mathbb{Q})$ called the Pontryagin product (see [6, §V.5]). This makes $\mathrm{H}_\bullet(\wedge^3 H_{\mathbb{Z}};\mathbb{Q})$ into a graded ring, and

$$\mathrm{H}_\bullet(\wedge^3 H_{\mathbb{Z}};\mathbb{Q}) \cong \wedge^\bullet \wedge^3 H.$$

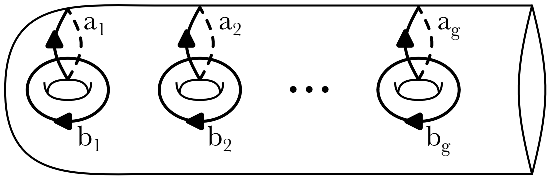
The isomorphism $\mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q}) \cong \wedge^2 \wedge^3 H$ discussed above is a special case of this.

To prove our result, we will need to detect elements of $\wedge^2 \wedge^3 H$ that lie in the image of the map (13.5). We will do with using so-called "abelian cycles". Consider commuting elements $f, h \in \mathcal{I}_g^1$. Since f and h commute, we can define a homomorphism $\Psi_{f,h} \colon \mathbb{Z}^2 \to \mathcal{I}_g^1$ taking the standard basis vectors of $\mathbb{Z}^2$ to f and h. We therefore get an induced map $(\Psi_{f,h})_* \colon \mathrm{H}_2(\mathbb{Z}^2;\mathbb{Q}) \to \mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q})$. We have $\mathrm{H}_2(\mathbb{Z}^2;\mathbb{Q}) \cong \wedge^2\mathbb{Q}^2 \cong \mathbb{Q}$ generated by the fundamental class $[\mathbb{Z}^2] \in \mathrm{H}_2(\mathbb{Z}^2;\mathbb{Q})$, so we have an element $(\Psi_{f,h})_*([\mathbb{Z}^2]) \in \mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q})$.

Define $\mathfrak{C}(f,h) \in \mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q}) = \wedge^2 \wedge^3 H$ to be the image of $(\Psi_{f,h})_*([\mathbb{Z}^2])$ under the map $(\tau_1)_*$. This equals the image of the fundamental class $[\mathbb{Z}^2]$ under the map induced by the map $\tau_1 \circ \Psi_{f,h} \colon \mathbb{Z}^2 \to \wedge^3 H_{\mathbb{Z}}$. The Pontryagin product makes the homology groups of both $\mathbb{Z}^2$ and $\wedge^3 H_{\mathbb{Z}}$ into a ring, and the map on homology induced by $\tau_1 \circ \Psi_{f,h}$ is a ring homomorphism. Since the fundamental class $[\mathbb{Z}^2] \in \mathrm{H}_2(\mathbb{Z}^2;\mathbb{Q})$ is the Pontryagin product of the first homology classes of the two standard generators of $\mathbb{Z}^2$, we deduce the following key formula:

$$\mathfrak{C}(f,h) = \tau_1(f) \wedge \tau_1(h) \in \wedge^2 \wedge^3 H.$$

In other words, given any two commuting elements $f, h \in \mathcal{I}_g^1$ the element $\mathfrak{C}(f,h) = \tau_1(f) \wedge \tau_1(h) \in \wedge^2 \wedge^3 H$ lies in the image of the map (13.5). Using this, we will prove that each of the desired irreducible factors lies in the image of (13.5). We will do the calculations roughly in increasing order of difficulty. As in the following figure, let $\{a_1, b_1, \ldots, a_g, b_g\}$ be the standard symplectic basis for H:



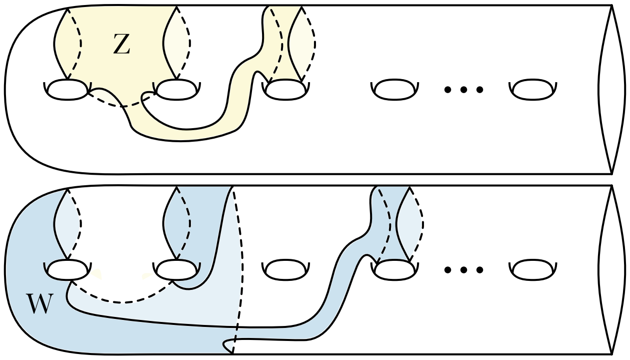
Also, let $\omega \in \wedge^2 H$ be the symplectic form.

Claim 1. *The image of the map* $(\tau_1)_* \colon \mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q}) \to \mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q})$ *contains a copy of* $\mathbf{V}_{2^2,1^2}(g)$*.*

The vector

$$(a_1 \wedge a_2 \wedge a_3) \wedge (a_1 \wedge a_2 \wedge a_4) \in \wedge^2 \wedge^3 H = \mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q})$$

is a highest weight vector for a copy of $\mathbf{V}_{2^2,1^2}(g)$, so it is enough to prove that this vector lies in the image of $(\tau_1)_*$. Let $Z \cong \Sigma_0^4$ and $W \cong \Sigma_0^4$ be the following subsurfaces:



Let $f \in \mathcal{I}_g^1$ be a simply intersecting pair map supported on Z and let $h \in \mathcal{I}_g^1$ be a simply intersecting pair map supported on W. By Lemma 5.6, we have $\tau_1(f) = \pm a_1 \wedge a_2 \wedge a_3$ and $\tau_1(h) = \pm a_1 \wedge a_2 \wedge a_4$. Since the interiors of Z and W are disjoint, the mapping classes f and h commute. The image of $(\tau_1)_*$ thus contains

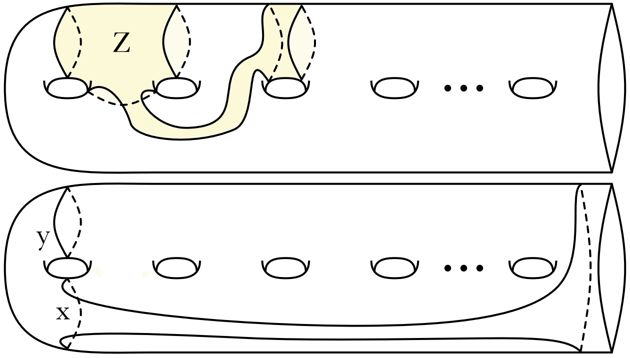
$$\mathfrak{C}(f,h) = \pm(a_1 \wedge a_2 \wedge a_3) \wedge (a_1 \wedge a_2 \wedge a_4).$$

Claim 2. *The image of the map* $(\tau_1)_*\colon \mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q}) \to \mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q})$ *contains a copy of* $\mathbf{V}_{2,1^2}(g)$*.*

Recall that $\omega \in \wedge^2 H$ is the symplectic form. The vector

$$(a_1 \wedge a_2 \wedge a_3) \wedge (a_1 \wedge \omega) \in \wedge^2 \wedge^3 H = \mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q})$$

is a highest weight vector for a copy of $\mathbf{V}_{2,1^2}(g)$, so it is enough to prove that this vector lies in the image of $(\tau_1)_*$. Let $Z \cong \Sigma_0^4$ be the following subsurface and let $T_x T_y^{-1} \in \mathcal{I}_g^1$ be the following bounding pair map:



Let $f \in \mathcal{I}_g^1$ be a simply intersecting pair map supported on Z. By Lemma 5.6 we have $\tau_1(f) = \pm a_1 \wedge a_2 \wedge a_3$, and by Lemma 5.4 we have

$$\tau_1(T_x T_y^{-1}) = a_1 \wedge (a_2 \wedge b_2 + \cdots + a_g \wedge b_g) = a_1 \wedge (a_1 \wedge b_1 + \cdots + a_g \wedge b_g) = a_1 \wedge \omega.$$

Since $x \cup y$ is disjoint from the interior of Z, the mapping classes f and $T_x T_y^{-1}$ commute. The image of $(\tau_1)_*$ thus contains

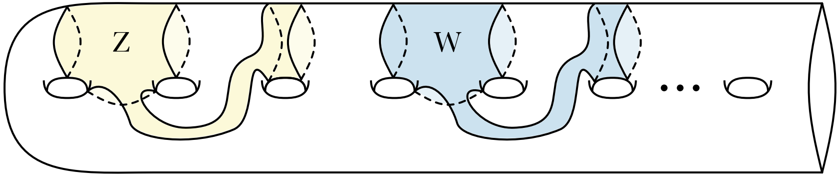
$$\mathfrak{C}(f, T_x T_y^{-1}) = \pm(a_1 \wedge a_2 \wedge a_3) \wedge (a_1 \wedge \omega).$$

Claim 3. *The image of the map* $(\tau_1)_*\colon \mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q}) \to \mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q})$ *contains a copy of* $\mathbf{V}_{1^6}(g)$*.*

It is awkward to find a highest weight vector in $\wedge^2 \wedge^3 H$ for $\mathbf{V}_{1^6}(g)$, so we do something slightly different. Let $\phi\colon \wedge^2 \wedge^3 H \to \wedge^6 H$ be the following map:

$$\phi((u_1 \wedge v_1 \wedge w_1) \wedge (u_2 \wedge v_2 \wedge w_2)) = u_1 \wedge v_1 \wedge w_1 \wedge u_2 \wedge v_2 \wedge w_2.$$

The vector $a_1 \wedge \cdots \wedge a_6 \in \wedge^6 H$ is a highest weight vector for a copy of $\mathbf{V}_{1^6}(g)$ in $\wedge^6 H$. It is therefore enough to construct an element θ in the image of $(\tau_1)_*$ such that $\phi(\theta) = a_1 \wedge \cdots \wedge a_6$. Let $Z \cong \Sigma_0^4$ and $W \cong \Sigma_0^4$ be the following subsurfaces:



Let $f \in \mathcal{I}_g^1$ be a simply intersecting pair map supported on Z and let $h \in \mathcal{I}_g^1$ be a simply intersecting pair map supported on W. By Lemma 5.6, we have $\tau_1(f) = \pm a_1 \wedge a_2 \wedge a_3$ and $\tau_1(h) = \pm a_4 \wedge a_5 \wedge a_6$. Since the interiors of Z and W are disjoint, the mapping classes f and h commute. The image of $(\tau_1)_*$ thus contains

$$\theta = \mathfrak{C}(f, h) = \pm(a_1 \wedge a_2 \wedge a_3) \wedge (a_4 \wedge a_5 \wedge a_6),$$

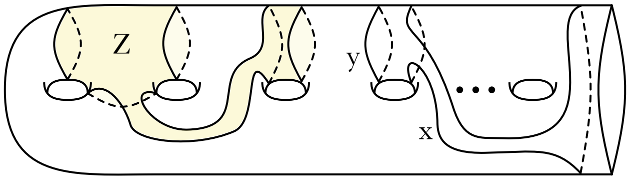
and $\phi(\theta) = \pm a_1 \wedge \cdots \wedge a_6 \in \wedge^6 H$.

Claim 4. *The image of the map* $(\tau_1)_* \colon \mathrm{H}_2(\mathcal{I}_g^1; \mathbb{Q}) \to \mathrm{H}_2(\wedge^3 H_{\mathbb{Z}}; \mathbb{Q})$ *contains a copy of* $\mathbf{V}_{1^4}(g)^{\oplus 2}$.

We will have to be careful to ensure that the two copies of $\mathbf{V}_{1^4}(g)$ we find are genuinely different. Recall that H is embedded in $\wedge^3 H$ via the map $H \hookrightarrow \wedge^3 H$ taking $h \in H$ to $h \wedge \omega$. Let $\mathcal{K}$ be the kernel of the map $p\colon \wedge^2 \wedge^3 H \to \wedge^2((\wedge^3 H)/H)$, so we have a short exact sequence of representations

$$0 \longrightarrow \mathcal{K} \longrightarrow \wedge^2 \wedge^3 H \xrightarrow{p} \wedge^2\left(\left(\wedge^3 H\right)/H\right) \longrightarrow 0.$$

We will first find a copy of $\mathbf{V}_{1^4}(g)$ in the image of $(\tau_1)_*$ that lies in $\mathcal{K}$. Let $\phi\colon \wedge^2 \wedge^3 H \to \wedge^6 H$ be the map from Claim 3. The vector $a_1 \wedge \cdots \wedge a_4 \wedge \omega \in \wedge^6 H$ is a highest weight vector for a copy of $\mathbf{V}_{1^4}(g)$ in $\wedge^6 H$. It is therefore enough to construct an element θ in the image of $(\tau_1)_*$ such that $\theta \in \mathcal{K}$ and $\phi(\theta) = a_1 \wedge \cdots \wedge a_4 \wedge \omega$. Let $Z \cong \Sigma_0^4$ be the following subsurface and let $T_x T_y^{-1} \in \mathcal{I}_g^1$ be the following bounding pair map:



Let $f \in \mathcal{I}_g^1$ be a simply intersecting pair map supported on Z. By Lemma 5.6 we have $\tau_1(f) = \pm a_1 \wedge a_2 \wedge a_3$, and by Lemma 5.4 we have

$$\begin{aligned} \tau_1(T_x T_y^{-1}) &= a_4 \wedge (a_1 \wedge b_1 + \cdots + \widehat{a_4 \wedge b_4} + \cdots + a_g \wedge b_g) \\ &= a_4 \wedge (a_1 \wedge b_1 + \cdots + a_g \wedge b_g) = a_4 \wedge \omega. \end{aligned}$$

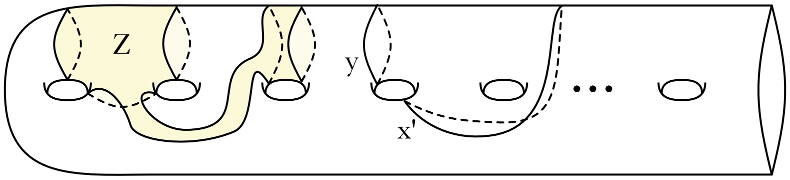
Since $x \cup y$ is disjoint from the interior of Z, the mapping classes f and $T_x T_y^{-1}$ commute. The image of $(\tau_1)_*$ thus contains

$$\theta = \mathfrak{C}(f, T_x T_y^{-1}) = \pm(a_1 \wedge a_2 \wedge a_3) \wedge (a_4 \wedge \omega).$$

We have $\theta \in \mathcal{K}$ since $a_4 \wedge \omega \in H \subset \wedge^3 H$, and $\phi(\theta) = a_1 \wedge \cdots \wedge a_4 \wedge \omega$, as desired.

Since the copy of $\mathbf{V}_{1^4}(g)$ we found lies in the kernel $\mathcal{K}$ of the projection $p\colon \wedge^2 \wedge^3 H \to \wedge^2((\wedge^3 H)/H)$, to find a second copy of $\mathbf{V}_{1^4}(g)$ it is enough to find a copy of $\mathbf{V}_{1^4}(g)$ that maps nontrivially to $\wedge^2((\wedge^3 H)/H)$. For $\kappa \in \wedge^2 \wedge^3 H$, let $\overline{\kappa} = p(\kappa) \in \wedge^2((\wedge^3 H)/H)$. We must find some $\theta' \in \wedge^2 \wedge^3 H$ in the image of $(\tau_1)_*$ such that the subrepresentation of $\wedge^2((\wedge^3 H)/H)$ generated by $\overline{\theta}'$ contains a copy of $\mathbf{V}_{1^4}(g)$.

Let $Z \cong \Sigma_0^4$ be the following subsurface and let $T_{x'}T_y^{-1} \in \mathcal{I}_g^1$ be the following bounding pair map:



Note that Z and y are the same as in the previous figure. Let $f \in \mathcal{I}_g^1$ be a simply intersecting pair map supported on Z. By Lemma 5.6 we have $\tau_1(f) = \pm a_1 \wedge a_2 \wedge a_3$, and by Lemma 5.4 we have $\tau_1(T_{x'}T_y^{-1}) = a_4 \wedge a_5 \wedge b_5$. Since $x' \cup y$ is disjoint from the interior of Z, the mapping classes f and $T_{x'}T_y^{-1}$ commute. The image of $(\tau_1)_*$ thus contains

$$\theta' = \mathfrak{C}(f, T_{x'}T_y^{-1}) = \pm(a_1 \wedge a_2 \wedge a_3) \wedge (a_4 \wedge a_5 \wedge b_5).$$

Lemma 8.3 says that $\overline{\theta}' \in \wedge^2(\wedge^3 H)/H$ generates a subrepresentation containing a copy of $\mathbf{V}_{1^4}(g)$, as desired.

Claim 5. *The image of the map* $(\tau_1)_*\colon \mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q}) \to \mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q})$ *contains a copy of* $\mathbf{V}_{1^2}(g)^{\oplus 2}$.

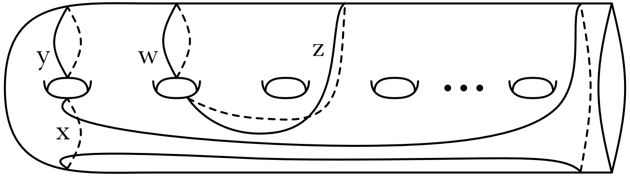
Just like in Claim 4, let $\mathcal{K}$ be the kernel of the map $p\colon \wedge^2\wedge^3 H \to \wedge^2((\wedge^3 H)/H)$, so we have a short exact sequence of representations

$$0 \longrightarrow \mathcal{K} \longrightarrow \wedge^2\wedge^3 H \xrightarrow{p} \wedge^2\left(\left(\wedge^3 H\right)/H\right) \longrightarrow 0.$$

We will first find a copy of $\mathbf{V}_{1^2}(g)$ in the image of $(\tau_1)_*$ that lies in $\mathcal{K}$. Recall that $\omega(-,-)$ is the symplectic form on H. Let $q\colon \wedge^3 H \to H$ be the map defined by the formula

$$q(h_1 \wedge h_2 \wedge h_3) = \omega(h_1,h_2)h_3 - \omega(h_1,h_3)h_2 + \omega(h_2,h_3)h_1 \quad \text{for } h_1,h_2,h_3 \in H$$

and let $\psi\colon \wedge^2\wedge^3 H \to \wedge^2 H$ be the map $\psi = \wedge^2 q$. The vector $a_1 \wedge a_2 \in \wedge^2 H$ is a highest weight vector for a copy of $\mathbf{V}_{1^2}(g)$ in $\wedge^2 H$. It is therefore enough to construct an element θ in the image of $(\tau_1)_*$ such that $\theta \in \mathcal{K}$ and $\phi(\theta)$ is a nonzero multiple of $a_1 \wedge a_2$. Let $T_xT_y^{-1} \in \mathcal{I}_g^1$ and $T_zT_w^{-1} \in \mathcal{I}_g^1$ be the following bounding pair maps:



By Lemma 5.4, we have

$$\begin{aligned}\tau_1(T_xT_y^{-1}) &= a_1 \wedge (a_2 \wedge b_2 + \cdots + a_g \wedge b_g)\\ &= a_1 \wedge (a_1 \wedge b_1 + \cdots + a_g \wedge b_g) = a_1 \wedge \omega,\\ \tau_1(T_zT_w^{-1}) &= a_2 \wedge a_3 \wedge b_3.\end{aligned}$$

Since $x \cup y$ and $z \cup w$ are disjoint, the mapping classes $T_xT_y^{-1}$ and $T_zT_w^{-1}$ commute. The image of $(\tau_1)_*$ thus contains

$$\theta = \mathfrak{C}(T_xT_y^{-1}, T_zT_w^{-1}) = (a_1 \wedge \omega) \wedge (a_2 \wedge a_3 \wedge b_3).$$

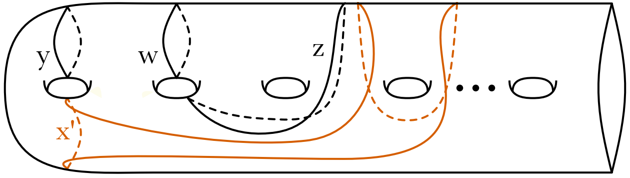
We have $\theta \in \mathcal{K}$ since $a_1 \wedge \omega \in H \subset \wedge^3 H$, and

$$\begin{aligned}\psi(\theta) &= q(a_1 \wedge \omega) \wedge q(a_2 \wedge a_3 \wedge b_3))\\ &= q(a_1 \wedge a_2 \wedge b_2 + a_1 \wedge a_3 \wedge b_3 + \cdots + a_1 \wedge a_g \wedge b_g) \wedge q(a_2 \wedge a_3 \wedge b_3)\\ &= (g-1)a_1 \wedge a_2,\end{aligned}$$

as desired.

Since the copy of $\mathbf{V}_{1^2}(g)$ we found lies in the kernel $\mathcal{K}$ of the projection $p\colon \wedge^2 \wedge^3 H \to \wedge^2((\wedge^3 H)/H)$, to find a second copy of $\mathbf{V}_{1^4}(g)$ it is enough to find a copy of $\mathbf{V}_{1^2}(g)$ that maps nontrivially to $\wedge^2((\wedge^3 H)/H)$. For $\kappa \in \wedge^2 \wedge^3 H$, let $\overline{\kappa} = p(\kappa) \in \wedge^2((\wedge^3 H)/H)$. We must find some $\theta' \in \wedge^2 \wedge^3 H$ in the image of $(\tau_1)_*$ such that the subrepresentation of $\wedge^2((\wedge^3 H)/H)$ generated by $\overline{\theta}'$ contains a copy of $\mathbf{V}_{1^2}(g)$.

Let $T_{x'}T_y^{-1} \in \mathcal{I}_g^1$ and and $T_zT_w^{-1} \in \mathcal{I}_g^1$ be the following bounding pair maps:



Note that y and z and w are the same as in the previous figure. By Lemma 5.4 we have[11]

$$\tau_1(T_{x'}T_y^{-1}) = -a_1 \wedge a_4 \wedge b_4,$$
$$\tau_1(T_zT_w^{-1}) = a_2 \wedge a_3 \wedge b_3.$$

Since $x' \cup y$ and $z \cup w$ are disjoint, the mapping classes $T_{x'}T_y^{-1}$ and $T_zT_w^{-1}$ commute The image of $(\tau_1)_*$ thus contains

$$\theta' = \mathfrak{C}(T_{x'}T_y^{-1}, T_zT_w^{-1}) = -(a_1 \wedge a_4 \wedge b_4) \wedge (a_2 \wedge a_3 \wedge b_3).$$

Lemma 8.4 says that $\overline{\theta}' \in \wedge^2(\wedge^3 H)/H$ generates a subrepresentation containing a copy of $\mathbf{V}_{1^2}(g)$, as desired. □

Before moving on to deal with punctured and closed surfaces, we record a consequence of the above proof. Recall that for a group G acting on a vector space M, we denote by M_G the G-coinvariants, i.e., the quotient $M/\langle m - gm \mid g \in G,\, m \in M\rangle$. The conjugation action of Mod_g^1 on its normal subgroup $\mathcal{I}_g^1[2]$ induces an action of Mod_g^1 on $\mathrm{H}_1(\mathcal{I}_g^1[2];\mathbb{Q})$. Passing to the $\mathcal{I}_g^1$-coinvariants, we get an action of $\mathrm{Mod}_g^1/\mathcal{I}_g^1 \cong \mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathrm{H}_1(\mathcal{I}_g^1[2];\mathbb{Q})_{\mathcal{I}_g^1}$.

Corollary 13.1. *For $g \geq 6$, we have* $\mathrm{H}_1(\mathcal{I}_g^1[2];\mathbb{Q})_{\mathcal{I}_g^1} \cong \mathbf{V}_0(g)^{\oplus 2} \oplus \mathbf{V}_{1^2}(g) \oplus \mathbf{V}_{2^2}(g)$.

Proof. Let $H = \mathrm{H}_1(\Sigma_g^1;\mathbb{Q})$ and $H_{\mathbb{Z}} = \mathrm{H}_1(\Sigma_g^1;\mathbb{Z})$. Recall the following exact sequence (13.4) that we constructed in Step 1 of the proof above:

$$\mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q}) \xrightarrow{(\tau_1)_*} \mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q}) \longrightarrow \mathrm{H}_1(\mathcal{I}_g^1[2];\mathbb{Q})_{\mathcal{I}_g^1} \longrightarrow 0.$$

As we discussed before Step 1, the representation $\mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q})$ is isomorphic to

$$\wedge^2 \wedge^3 H \cong \mathbf{V}_0(g)^{\oplus 2} \oplus \mathbf{V}_{1^2}(g)^{\oplus 3} \oplus \mathbf{V}_{2^2}(g) \oplus \mathbf{V}_{2,1^2}(g) \oplus \mathbf{V}_{1^4}(g)^{\oplus 2} \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g).$$

We proved in Theorem B that the image of $(\tau_1)_*$ is

$$\mathbf{V}_{1^2}(g)^{\oplus 2} \oplus \mathbf{V}_{2,1^2}(g) \oplus \mathbf{V}_{1^4}(g)^{\oplus 2} \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g).$$

Combining the above three math displays gives the corollary. □

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[11] The minus sign is there in the first formula for orientation reasons. It would be cleaner to use $T_yT_{x'}^{-1}$, but we decided that this would be harder to follow.

14. CALCULATION OF IMAGE OF CUP PRODUCT PAIRING ON PUNCTURED SURFACES

We now show how to deal with the Torelli group $\mathcal{I}_{g,1}$ on a genus g surface $\Sigma_{g,1}$ with one puncture:

Theorem B (punctured surface case). *Let $g \geq 6$. The image of the cup product pairing $\mathfrak{c}\colon \wedge^2 \mathrm{H}^1(\mathcal{I}_{g,1};\mathbb{Q}) \to \mathrm{H}^2(\mathcal{I}_{g,1};\mathbb{Q})$ is isomorphic to the following representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$:*

$$\mathbf{V}_{1^2}(g)^{\oplus 2} \oplus \mathbf{V}_{2,1^2}(g) \oplus \mathbf{V}_{1^4}(g)^{\oplus 2} \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g).$$

Proof. Note that the only difference between this and what we proved for surfaces with boundary is the presence of one additional $\mathbf{V}_0(g)$ factor. Let $H = \mathrm{H}_1(\Sigma_g^1;\mathbb{Q})$ and $H_{\mathbb{Z}} = \mathrm{H}_1(\Sigma_g^1;\mathbb{Z})$ and let $b = \partial \Sigma_g^1$. There is a central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Mod}_g^1 \longrightarrow \mathrm{Mod}_{g,1} \longrightarrow 1 \tag{14.1}$$

whose central $\mathbb{Z}$ term is generated by the Dehn twist T_b (see [14, Proposition 3.19]). Since the separating twist T_b lies in $\mathcal{I}_g^1$, the above restricts to a central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{I}_g^1 \longrightarrow \mathcal{I}_{g,1} \longrightarrow 1.$$

In fact, T_b lies in $\mathcal{I}_g^1[2]$. This implies that the first Johnson $\tau_1\colon \mathcal{I}_g^1 \to \wedge^3 H_{\mathbb{Z}}$ factors through a homomorphism $\mathcal{I}_{g,1} \to \wedge^3 H_{\mathbb{Z}}$. We will also call this the first Johnson homomorphism and denote it by τ_1. Since τ_1 detects $\mathrm{H}_1(\mathcal{I}_g^1;\mathbb{Q})$ (Theorem 9.2) and factors through $\mathcal{I}_{g,1}$, it follows that τ_1 also detects $\mathrm{H}_1(\mathcal{I}_{g,1};\mathbb{Q})$, i.e., it induces an isomorphism

$$(\tau_1)_*\colon \mathrm{H}_1(\mathcal{I}_{g,1};\mathbb{Q}) \xrightarrow{\cong} \mathrm{H}_1(\wedge^3 H_{\mathbb{Z}};\mathbb{Q}) = \wedge^3 H.$$

In light of this, the exact same proof we gave in Lemma 12.1 for $\mathcal{I}_g^1$ shows that the image of the cup product pairing $\wedge^2 \mathrm{H}^1(\mathcal{I}_{g,1};\mathbb{Q}) \to \mathrm{H}^2(\mathcal{I}_{g,1};\mathbb{Q})$ is isomorphic to the image of the map $(\tau_1)_*\colon \mathrm{H}_2(\mathcal{I}_{g,1};\mathbb{Q}) \to \mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q})$. Summarizing where we are, our goal is now the following:

(†) Prove that the image of $(\tau_1)_*\colon \mathrm{H}_2(\mathcal{I}_{g,1};\mathbb{Q}) \to \mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q})$ equals the image of $(\tau_1)_*\colon \mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q}) \to \mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q})$ plus one additional $\mathbf{V}_0(g)$ factor.

As notation, let $\mathcal{I}_{g,1}[2]$ be the kernel of the map $\tau_1\colon \mathcal{I}_{g,1} \to \wedge^3 H_{\mathbb{Z}}$. Just like in the proof of Step 1 in §13, the five-term exact sequence in group homology associated to the short exact sequence

$$1 \longrightarrow \mathcal{I}_{g,1}[2] \longrightarrow \mathcal{I}_{g,1} \xrightarrow{\tau_1} \wedge^3 H_{\mathbb{Z}} \longrightarrow 1$$

can be analyzed to give an exact sequence

$$\mathrm{H}_2(\mathcal{I}_{g,1};\mathbb{Q}) \xrightarrow{(\tau_1)_*} \mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q}) \longrightarrow \mathrm{H}_1(\mathcal{I}_{g,1}[2];\mathbb{Q})_{\mathcal{I}_{g,1}} \longrightarrow 0.$$

This fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q}) \arrow[r, "(\tau_1)_*"] \arrow[d] & \mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q}) \arrow[r] \arrow[d, equal] & \mathrm{H}_1(\mathcal{I}_g^1[2];\mathbb{Q})_{\mathcal{I}_g^1} \arrow[r] \arrow[d] & 0 \\
\mathrm{H}_2(\mathcal{I}_{g,1};\mathbb{Q}) \arrow[r, "(\tau_1)_*"] & \mathrm{H}_2(\wedge^3 H_{\mathbb{Z}};\mathbb{Q}) \arrow[r] & \mathrm{H}_1(\mathcal{I}_{g,1}[2];\mathbb{Q})_{\mathcal{I}_{g,1}} \arrow[r] & 0.
\end{tikzcd}$$

In light of this, to prove (†) it is enough to prove the following:

Claim. *The $\mathrm{Sp}_{2g}(\mathbb{Z})$ representation $\mathrm{H}_1(\mathcal{I}_g^1[2];\mathbb{Q})_{\mathcal{I}_g^1}$ equals $\mathrm{H}_1(\mathcal{I}_{g,1}[2];\mathbb{Q})_{\mathcal{I}_{g,1}}$ plus one additional $\mathbf{V}_0(g)$ factor.*

In the central extension (14.1), the kernel $\mathbb{Z}$ is generated by the Dehn twist T_b about the boundary component b. Since this is a separating twist, it lies in $\mathcal{I}_g^1[2]$ and (14.1) restricts to a central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{I}_g^1[2] \longrightarrow \mathcal{I}_{g,1}[2] \longrightarrow 1.$$

Since this is a central extension, we see that the conjugation action of $\mathcal{I}_{g,1}$ on $\mathcal{I}_{g,1}[2]$ lifts to an action of $\mathcal{I}_g^1$ on $\mathcal{I}_{g,1}[2]$. This satisfies

$$\mathrm{H}_1(\mathcal{I}_{g,1}[2];\mathbb{Q})_{\mathcal{I}_g^1} = \mathrm{H}_1(\mathcal{I}_{g,1}[2];\mathbb{Q})_{\mathcal{I}_{g,1}}.$$

We can therefore attempt to compare $\mathrm{H}_1(\mathcal{I}_g^1[2];\mathbb{Q})_{\mathcal{I}_g^1}$ to $\mathrm{H}_1(\mathcal{I}_{g,1}[2];\mathbb{Q})_{\mathcal{I}_g^1}$, which will be easier since we are taking coinvariants with respect to the same group.

Since the above is a central extension, the associated five-term exact sequence in group homology (see [6, Corollary VII.6.4]) contains the segment

$$\mathbb{Q} \longrightarrow \mathrm{H}_1(\mathcal{I}_g^1[2];\mathbb{Q}) \longrightarrow \mathrm{H}_1(\mathcal{I}_{g,1}[2];\mathbb{Q}) \longrightarrow 0.$$

Taking coinvariants is right exact, so this gives an exact sequence

$$\mathbb{Q} \longrightarrow \mathrm{H}_1(\mathcal{I}_g^1[2];\mathbb{Q})_{\mathcal{I}_g^1} \longrightarrow \mathrm{H}_1(\mathcal{I}_{g,1}[2];\mathbb{Q})_{\mathcal{I}_g^1} \longrightarrow 0.$$

We claim that the map from $\mathbb{Q} \cong \mathbf{V}_0(g)$ to $\mathrm{H}_1(\mathcal{I}_g^1[2];\mathbb{Q})_{\mathcal{I}_g^1}$ is injective. Indeed, the second Johnson homomorphism τ_2 induces a map on $\mathrm{H}_1(\mathcal{I}_g^1[2];\mathbb{Q})_{\mathcal{I}_g^1}$, and Lemma 5.8 implies that $\tau_2(T_b) \neq 0$. Since T_b generates the $\mathbb{Z}$ that became the $\mathbb{Q}$ above, the claim follows. We therefore have a short exact sequence

$$0 \longrightarrow \mathbf{V}_0(g) \longrightarrow \mathrm{H}_1(\mathcal{I}_g^1[2];\mathbb{Q})_{\mathcal{I}_g^1} \longrightarrow \mathrm{H}_1(\mathcal{I}_{g,1}[2];\mathbb{Q})_{\mathcal{I}_g^1} \longrightarrow 0,$$

as desired. ☐

Exactly like at the end of §13, the following is an immediate consequence of the above proof:

Corollary 14.1. *For $g \geq 6$, we have* $\mathrm{H}_1(\mathcal{I}_{g,1}[2];\mathbb{Q})_{\mathcal{I}_g^1} \cong \mathbf{V}_0(g) \oplus \mathbf{V}_{1^2}(g) \oplus \mathbf{V}_{2^2}(g)$.

15. Calculation of image of cup product pairing on closed surfaces

We close this part of the paper by showing how to deal with the Torelli group $\mathcal{I}_g$ on a closed genus g surface Σ_g:

Theorem B (closed surface case). *Let $g \geq 6$. The image of the cup product pairing $\mathfrak{c}\colon \wedge^2 \mathrm{H}^1(\mathcal{I}_g;\mathbb{Q}) \to \mathrm{H}^2(\mathcal{I}_g;\mathbb{Q})$ is isomorphic to the following representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$:*

$$\mathbf{V}_{1^2}(g) \oplus \mathbf{V}_{1^4}(g) \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g).$$

Proof. Let $H = \mathrm{H}_1(\Sigma_g^1;\mathbb{Q})$ and $H_{\mathbb{Z}} = \mathrm{H}_1(\Sigma_g^1;\mathbb{Z})$ and let $\pi = \pi_1(\Sigma_g,*)$. The groups $\mathrm{Mod}_{g,1}$ and Mod_g are connected by a Birman exact sequence of the form

$$1 \longrightarrow \pi \longrightarrow \mathrm{Mod}_{g,1} \longrightarrow \mathrm{Mod}_g \longrightarrow 1.$$

See [14, §4.2]. Since $\mathrm{H}_1(\Sigma_{g,1}) = \mathrm{H}_1(\Sigma_g)$, the action of $\mathrm{Mod}_{g,1}$ on $H_{\mathbb{Z}}$ factors through Mod_g. This implies that the kernel π of the Birman exact sequence acts trivially on $H_{\mathbb{Z}}$, so π is contained in the Torelli group. It follows that the Birman exact sequence restricts to an exact sequence

$$1 \longrightarrow \pi \longrightarrow \mathcal{I}_{g,1} \longrightarrow \mathcal{I}_g \longrightarrow 1.$$

Let $\omega \in \wedge^2 H$ be the symplectic form. Since ω restricts to a $\mathbb{Z}$-valued symplectic form on $H_{\mathbb{Z}}$, we can actually regard ω as an element of $\wedge^2 H_{\mathbb{Z}}$. Johnson [26] proved that the first

Johnson homomorphism $\tau_1\colon \mathcal{I}_{g,1} \longrightarrow \wedge^3 H_{\mathbb{Z}}$ constructed in the previous section fits into a commutative diagram

$$\begin{tikzcd}
\pi \arrow[r] \arrow[d, "\tau_1"] & \mathcal{I}_{g,1} \arrow[d, "\tau_1"] \\
H_{\mathbb{Z}} \arrow[r, "\iota"] & \wedge^3 H_{\mathbb{Z}},
\end{tikzcd}$$

where $\iota\colon H_{\mathbb{Z}} \to \wedge^3 H_{\mathbb{Z}}$ takes $h \in H_{\mathbb{Z}}$ to $h \wedge \omega$. Identifying $H_{\mathbb{Z}}$ with its image in $\wedge^3 H_{\mathbb{Z}}$, the above implies that τ_1 induces a homomorphism $\tau_1\colon \mathcal{I}_g \to (\wedge^3 H_{\mathbb{Z}})/H_{\mathbb{Z}}$ fitting into a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \pi \arrow[r] \arrow[d, "\tau_1"] & \mathcal{I}_{g,1} \arrow[r] \arrow[d, "\tau_1"] & \mathcal{I}_g \arrow[r] \arrow[d, "\tau_1"] & 1 \\
0 \arrow[r] & H_{\mathbb{Z}} \arrow[r, "\iota"] & \wedge^3 H_{\mathbb{Z}} \arrow[r] & (\wedge^3 H_{\mathbb{Z}})/H_{\mathbb{Z}} \arrow[r] & 0.
\end{tikzcd}$$

Theorem 9.2 implies that $\tau_1\colon \mathcal{I}_{g,1} \to \wedge^3 H_{\mathbb{Z}}$ induces an isomorphism[12]

$$(\tau_1)_*\colon \mathrm{H}_1(\mathcal{I}_{g,1};\mathbb{Q}) \xrightarrow{\cong} \mathrm{H}_1(\wedge^3 H_{\mathbb{Z}};\mathbb{Q}) = \wedge^3 H,$$

so it follows that $\tau_1\colon \mathcal{I}_g \to (\wedge^3 H_{\mathbb{Z}})/H_{\mathbb{Z}}$ also induces an isomorphism[13]

$$(\tau_1)_*\colon \mathrm{H}_1(\mathcal{I}_g;\mathbb{Q}) \xrightarrow{\cong} \mathrm{H}_1((\wedge^3 H_{\mathbb{Z}})/H_{\mathbb{Z}};\mathbb{Q}) = (\wedge^3 H)/H.$$

In light of this, the exact same proof we gave in Lemma 12.1 for $\mathcal{I}_g^1$ shows that the image of the cup product pairing $\wedge^2 \mathrm{H}^1(\mathcal{I}_g;\mathbb{Q}) \to \mathrm{H}^2(\mathcal{I}_g;\mathbb{Q})$ is isomorphic to the image of the map $(\tau_1)_*\colon \mathrm{H}_2(\mathcal{I}_g;\mathbb{Q}) \to \mathrm{H}_2((\wedge^3 H_{\mathbb{Z}})/H_{\mathbb{Z}};\mathbb{Q})$. Summarizing where we are, our goal is now the following:

(†) Prove that the image of $(\tau_1)_*\colon \mathrm{H}_2(\mathcal{I}_g;\mathbb{Q}) \to \mathrm{H}_2((\wedge^3 H_{\mathbb{Z}})/H_{\mathbb{Z}};\mathbb{Q})$ is isomorphic to $\mathbf{V}_{1^2}(g) \oplus \mathbf{V}_{1^4}(g) \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g)$.

As notation, let $\mathcal{I}_g[2]$ be the kernel of the map $\tau_1\colon \mathcal{I}_g \to (\wedge^3 H_{\mathbb{Z}})/H_{\mathbb{Z}}$. Just like in the proof of Step 1 in §13, the five-term exact sequence in group homology associated to the short exact sequence

$$1 \longrightarrow \mathcal{I}_g[2] \longrightarrow \mathcal{I}_g \xrightarrow{\tau_1} (\wedge^3 H_{\mathbb{Z}})/H_{\mathbb{Z}} \longrightarrow 1$$

can be analyzed to give an exact sequence

$$\mathrm{H}_2(\mathcal{I}_g;\mathbb{Q}) \xrightarrow{(\tau_1)_*} \mathrm{H}_2((\wedge^3 H_{\mathbb{Z}})/H_{\mathbb{Z}};\mathbb{Q}) \longrightarrow \mathrm{H}_1(\mathcal{I}_g[2];\mathbb{Q})_{\mathcal{I}_g} \longrightarrow 0. \tag{15.1}$$

We have that the representation $\mathrm{H}_2((\wedge^3 H_{\mathbb{Z}})/H_{\mathbb{Z}};\mathbb{Q})$ is isomorphic to

$$\wedge^2(\wedge^3 H)/H \cong \mathbf{V}_0(g) \oplus \mathbf{V}_{1^2}(g) \oplus \mathbf{V}_{2^2}(g) \oplus \mathbf{V}_{1^4}(g) \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g).$$

In light of this isomorphism and the exact sequence (15.1), to prove (†) it is enough to perform the following three steps:

Step 1. *The image of the map* $(\tau_1)_*\colon \mathrm{H}_2(\mathcal{I}_g;\mathbb{Q}) \to \mathrm{H}_2((\wedge^3 H_{\mathbb{Z}})/H_{\mathbb{Z}};\mathbb{Q})$ *contains the subrepresentation* $\mathbf{V}_{1^2}(g) \oplus \mathbf{V}_{1^4}(g) \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g)$ *of* $\mathrm{H}_2((\wedge^3 H_{\mathbb{Z}})/H_{\mathbb{Z}};\mathbb{Q}) \cong \wedge^2(\wedge^3 H)/H$.

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[12] See the proof in §14 for why this result about $\mathcal{I}_{g,1}$ follows from Theorem 9.2, which concerns $\mathcal{I}_g^1$.

[13] This fact was first proved by Johnson in [29, 30].

We have a commutative diagram

$$\begin{tikzcd}
\mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q}) \arrow[r, "(\tau_1)_*"] \arrow[d] & \mathrm{H}_2((\wedge^3 H_{\mathbb{Z}});\mathbb{Q}) \arrow[r, equal] \arrow[d, two heads] & \wedge^2 \wedge^3 H \arrow[d, two heads] \\
\mathrm{H}_2(\mathcal{I}_g;\mathbb{Q}) \arrow[r, "(\tau_1)_*"] & \mathrm{H}_2((\wedge^3 H_{\mathbb{Z}})/H_{\mathbb{Z}};\mathbb{Q}) \arrow[r, equal] & \wedge^2(\wedge^3 H)/H.
\end{tikzcd}$$

In §13, we proved that the image of the top row is the subrepresentation

$$\begin{aligned}
&\mathbf{V}_{1^2}(g)^{\oplus 2} \oplus \mathbf{V}_{2,1^2}(g) \oplus \mathbf{V}_{1^4}(g)^{\oplus 2} \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g) \\
&\subset \wedge^2 \wedge^3 H = \mathbf{V}_{1^2}(g)^{\oplus 3} \oplus \mathbf{V}_{2^2}(g) \oplus \mathbf{V}_{2,1^2}(g) \oplus \mathbf{V}_{1^4}(g)^{\oplus 2} \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g).
\end{aligned}$$

This maps to the subrepresentation

$$\begin{aligned}
&\mathbf{V}_{1^2}(g) \oplus \mathbf{V}_{1^4}(g) \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g) \\
&\subset \wedge^2 (\wedge^3 H)/H = \mathbf{V}_0(g) \oplus \mathbf{V}_{1^2}(g) \oplus \mathbf{V}_{2^2}(g) \oplus \mathbf{V}_{1^4}(g) \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g).
\end{aligned}$$

The step follows.

Step 2. *The homology group $\mathrm{H}_2(\mathcal{I}_g;\mathbb{Q})$ contains no trivial subrepresentation, so in particular the image of $(\tau_1)_*\colon \mathrm{H}_2(\mathcal{I}_g;\mathbb{Q}) \to \mathrm{H}_2((\wedge^3 H_{\mathbb{Z}})/H_{\mathbb{Z}};\mathbb{Q})$ does not contain the subrepresentation $\mathbf{V}_0(g)$.*

Dualizing, it is enough to prove that the $\mathrm{Sp}_{2g}(\mathbb{Z})$-representation $\mathrm{H}^2(\mathcal{I}_g;\mathbb{Q})$ contains no trivial factors.[14] The proof is very similar to that of Lemma 11.5, which proved that $\mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q})$ contains no trivial factors. We recall the basic idea of that proof. Consider the Hochschild–Serre spectral sequence of the short exact sequence

$$1 \longrightarrow \mathcal{I}_g \longrightarrow \mathrm{Mod}_g \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1.$$

Harer [21] proved that $\mathrm{H}^2(\mathrm{Mod}_g;\mathbb{Q}) = \mathbb{Q}$. Using the Borel stability theorem (Theorem 11.4), we can calculate the portion of the E_2^{pq}-page relevant to computing $\mathrm{H}^2(\mathrm{Mod}_g;\mathbb{Q})$. The desired result falls out of this calculation. We refer the reader to Lemma 11.5 for the details.

Step 3. *$\mathrm{H}_1(\mathcal{I}_g[2];\mathbb{Q})_{\mathcal{I}_g}$ contains the subrepresentation $\mathbf{V}_{2^2}(g)$.*

Just like in §14, it is convenient to lift the conjugation action of $\mathcal{I}_g$ on $\mathcal{I}_g[2]$ to an action of $\mathcal{I}_{g,1}$ on $\mathcal{I}_2[2]$ that factors through $\mathcal{I}_g$. This allows us to rewrite our coinvariants as

$$\mathrm{H}_1(\mathcal{I}_g[2];\mathbb{Q})_{\mathcal{I}_g} = \mathrm{H}_1(\mathcal{I}_g[2];\mathbb{Q})_{\mathcal{I}_{g,1}}.$$

From the beginning of the proof, recall the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \pi \arrow[r] \arrow[d, "\tau_1"] & \mathcal{I}_{g,1} \arrow[r] \arrow[d, "\tau_1"] & \mathcal{I}_g \arrow[r] \arrow[d, "\tau_1"] & 1 \\
0 \arrow[r] & H_{\mathbb{Z}} \arrow[r, "\iota"] & \wedge^3 H_{\mathbb{Z}} \arrow[r] & (\wedge^3 H_{\mathbb{Z}})/H_{\mathbb{Z}} \arrow[r] & 0.
\end{tikzcd}$$

All the vertical maps are surjective. Using the fact that $[\pi,\pi]$ is the kernel of the map $\pi \to H_{\mathbb{Z}}$, we get a short exact sequence

$$1 \longrightarrow [\pi,\pi] \longrightarrow \mathcal{I}_{g,1}[2] \longrightarrow \mathcal{I}_g[2] \longrightarrow 1.$$

See, e.g., [50, Theorem 4.1] for more details. This induces an exact sequence

$$\mathrm{H}_1([\pi,\pi];\mathbb{Q}) \longrightarrow \mathrm{H}_1(\mathcal{I}_{g,1}[2];\mathbb{Q}) \longrightarrow \mathrm{H}_1(\mathcal{I}_g[2];\mathbb{Q}) \longrightarrow 0.$$

---

[14] For this, recall that Theorem A says that $\mathrm{H}^2(\mathcal{I}_g;\mathbb{Q})$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$.

Taking coinvariants is right exact, so we have an exact sequence

$$\mathrm{H}_1([\pi,\pi];\mathbb{Q})_{\mathcal{I}_{g,1}} \longrightarrow \mathrm{H}_1(\mathcal{I}_{g,1}[2];\mathbb{Q})_{\mathcal{I}_{g,1}} \longrightarrow \mathrm{H}_1(\mathcal{I}_g[2];\mathbb{Q})_{\mathcal{I}_{g,1}} \longrightarrow 0.$$

Corollary 14.1 says that

$$\mathrm{H}_1(\mathcal{I}_{g,1}[2];\mathbb{Q})_{\mathcal{I}_g^1} \cong \mathbf{V}_0(g) \oplus \mathbf{V}_{1^2}(g) \oplus \mathbf{V}_{2^2}(g).$$

To prove that $\mathrm{H}_1(\mathcal{I}_g[2];\mathbb{Q})_{\mathcal{I}_{g,1}}$ contains a copy of $\mathbf{V}_{2^2}(g)$, it is therefore enough to prove that $\mathrm{H}_1([\pi,\pi];\mathbb{Q})_{\mathcal{I}_{g,1}}$ does not contain a copy of $\mathbf{V}_{2^2}(g)$. Recall that $\mathcal{I}_{g,1}$ contains as a subgroup the kernel π of the Birman exact sequence. The action of $\mathcal{I}_{g,1}$ on π restricts to the usual action of π on itself by conjugation. From this, we see that[15]

$$\mathrm{H}_1([\pi,\pi];\mathbb{Q})_\pi = [\pi,\pi]/[\pi,[\pi,\pi]] \otimes \mathbb{Q} \cong \wedge^2 H.$$

Since $\mathcal{I}_{g,1}$ acts trivially on H, this implies that

$$\mathrm{H}_1([\pi,\pi];\mathbb{Q})_{\mathcal{I}_{g,1}} \cong \wedge^2 H \cong \mathbf{V}_0(g) \oplus \mathbf{V}_{1^2}(g).$$

In particular, $\mathrm{H}_1([\pi,\pi];\mathbb{Q})_{\mathcal{I}_{g,1}}$ does not contain a copy of $\mathbf{V}_{2^2}(g)$, as desired. □

Part 2. Cup products span

In this second part of the paper, we prove that the second rational cohomology group of the Torelli group is spanned by cup products of elements of H^1. Most of this part is devoting to proving this for $\mathcal{I}_g^1$, and then at the end we show how to deal with $\mathcal{I}_{g,1}$ and $\mathcal{I}_g$. Though we handle the technical details differently, the heart of our proof follows work of Kupers–Randal-Williams ([33]; see also its sequel [53]).

Recall that Theorem A says that $\mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q})$ is a finite-dimensional algebraic representation of Sp_{2g} for $g \geq 6$. In fact, the proof of Theorem A in [42] gives a bit more than this. We start in §16 by combining this extra information with tools from the theory of representation stability to bound the degrees of irreducible factors of of $\mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q})$ for $g \gg G$. The constant G here is not effective.

In §17, we relate the irreducible factors of $\mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q})$ to the twisted cohomology of the mapping class group Mod_g^1. This allows us to apply recent work of Miller–Patzt–Petersen–Randal-Williams [40] to show that we can take G to be 12.

In §18, we use all of this together with computations of Kawazumi [31] of the twisted cohomology of Mod_g^1 to prove that cup products span $\mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q})$ for $g \geq 12$. Finally, in §19 and §20 we show how to derive the corresponding results for $\mathcal{I}_{g,1}$ and $\mathcal{I}_g$.

16. REPRESENTATION STABILITY AND THE DEGREES OF THE FACTORS OF $\mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q})$

Fix some $g \geq 6$. Let $H = \mathrm{H}_1(\Sigma_g^1;\mathbb{Q})$. The first step will be to use tools from the theory of representation stability to bound the degrees of the irreducible factors of $\mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q})$.

16.1. Degree of irreducible representation.

Recall that we defined the degree of an irreducible representation of the algebraic group Sp_{2g} in §7 as follows. An irreducible representation is of the form $\mathbf{V}_\sigma(g)$ with σ a partition with at most g parts. Writing $\sigma = (k_1,\ldots,k_m)$, the degree of $\mathbf{V}_\sigma(g)$ is $d = k_1 + \cdots + k_m$. This is also the minimal d such that $\mathbf{V}_\sigma(g)$ appears in $H^{\otimes d}$.

---

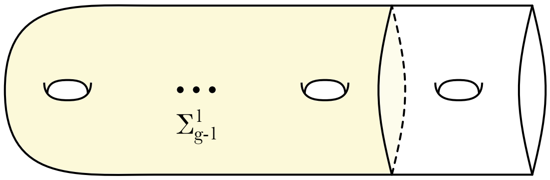
[15] The isomorphism $[\pi,\pi]/[\pi,[\pi,\pi]] \otimes \mathbb{Q} \cong \wedge^2 H$ follows from the fact that $[\pi,\pi]/[\pi,[\pi,\pi]] \otimes \mathbb{Q}$ is the second graded piece of the Lie algebra associated to the lower central series of π, which is the free Lie algebra (see §3). It is therefore isomorphic to $\mathrm{Lie}_2(H) \cong \wedge^2 H$; see Example 3.2.

16.2. **Algebraicity.** Recall that we are assuming the following theorem, which for convenience we state for homology rather than cohomology:

Theorem A (Minahan–Putman, [42, Theorem B]**).** *The homology group* $\mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q})$ *is finite-dimensional for* $g \geq 5$ *and an algebraic representation of* $\mathrm{Sp}_{2g}(\mathbb{Z})$ *for* $g \geq 6$.

Remark 16.1. As we noted in the introduction, [42] also proves similar results for $\mathcal{I}_g$ and $\mathcal{I}_{g,1}$. In fact, it proves a result for $\mathcal{I}_{g,p}^b$ in general, though it requires care to properly define the Torelli group on a surface with multiple punctures and boundary components. ☐

16.3. **Cokernel of stabilization map.** We will need a consequence of the proof of Theorem A. Embed Σ_{g-1}^1 into Σ_g^1 as follows:



By extending mapping classes on Σ_{g-1}^1 to Σ_g^1 by the identity, we get an inclusion $\mathrm{Mod}_{g-1}^1 \hookrightarrow \mathrm{Mod}_g^1$. This restricts to an inclusion $\mathcal{I}_{g-1}^1 \hookrightarrow \mathcal{I}_g^1$. This inclusion induces a map on homology that we will write as

$$f_g\colon \mathrm{H}_2(\mathcal{I}_{g-1}^1;\mathbb{Q}) \to \mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q}).$$

We will call f_g the *stabilization map*. The domain of f_g is a representation of $\mathrm{Sp}_{2(g-1)}(\mathbb{Z})$ and the codomain of f_g is a representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$. Moreover, f_g is $\mathrm{Sp}_{2(g-1)}(\mathbb{Z})$-equivariant. The cokernel $\mathrm{coker}(f_g)$ is therefore a representation of $\mathrm{Sp}_{2(g-1)}(\mathbb{Z})$. We then have:

Theorem 16.2. *Let* $f_g\colon \mathrm{H}_2(\mathcal{I}_{g-1}^1;\mathbb{Q}) \to \mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q})$ *be the stabilization map. The following hold for* $g \geq 5$*:*

- *(i) The cokernel* $\mathrm{coker}(f_g)$ *is a finite-dimensional algebraic representation of* $\mathrm{Sp}_{2(g-1)}(\mathbb{Z})$.
- *(ii) Each irreducible factor of* $\mathrm{coker}(f_g)$ *has degree at most* 5.

Proof. Conclusion (i) is the main technical result that goes into the proof of Theorem A in [42]. The proof actually shows how to write $\mathrm{coker}(f_g)$ in terms of a series of explicit representations.[16] If you carefully go through the argument, you will see that all these explicit representations have degree at most 5. The theorem follows. ☐

Remark 16.3. The paper [42] introduces a new approach for proving these kinds of algebraicity results. If this approach is used in other contexts, it will likely also give some d such that all the irreducible factors of the relevant cokernels have degree at most d. The representation stability argument we give below will then prove that the irreducible factors of the representations in question all eventually have degree at most $d + 1$. ☐

16.4. **Representation stability.** We next prove the following. Using the language of [10], it says that $\mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q})$ satisfies Church–Farb's notion of representation stability. We remark that (iii) is due to Boldsen–Dollerup [2]. We include it here since it forms part of the representation stability package.

Theorem 16.4. *There exists some* $G \geq 1$ *and partitions* $\sigma_1, \ldots, \sigma_n$ *with at most* G *parts such that the following holds for* $g \geq G$*:*

- *(i) We have* $\mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q}) \cong \mathbf{V}_{\sigma_1}(g) \oplus \cdots \oplus \mathbf{V}_{\sigma_n}(g)$.

---

[16] It does not calculate $\mathrm{coker}(f_g)$ completely since for instance some of these representations appear as domains of maps into $\mathrm{coker}(f_g)$, and these maps might be 0. Of course, as the paper you are reading shows once Theorem A is known we can calculate $\mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q})$ for $g \geq 12$, and thus in particular work out $\mathrm{coker}(f_g)$ for $g \gg 0$.

(ii) The stabilization map $f_g\colon \mathrm{H}_2(\mathcal{I}_{g-1}^1;\mathbb{Q}) \to \mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q})$ *is injective.*
(iii) The $\mathrm{Sp}_{2g}(\mathbb{Z})$*-orbit of the image of* f_g *spans* $\mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q})$*.*

Proof. Set $V_g = \mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q})$. This is a representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$, and $f_g\colon V_{g-1} \to V_g$ is $\mathrm{Sp}_{2(g-1)}(\mathbb{Z})$-equivariant. The sequence of representations

$$V_1 \xrightarrow{f_2} V_2 \xrightarrow{f_3} V_3 \xrightarrow{f_4} \cdots$$

has the following key property:

- For all $m \geq n \geq 1$, the subgroup $1 \times \mathrm{Sp}_{2(m-n)}(\mathbb{Z})$ of $\mathrm{Sp}_{2m}(\mathbb{Z})$ acts trivially on the image of the composition

$$V_n \xrightarrow{f_{n+1}} \cdots \xrightarrow{f_m} V_m.$$

As was noted in [47, Proposition 3.6], this is equivalent to the V_g forming an $\mathtt{SI}(\mathbb{Z})$-module in the sense of Putman–Sam [52]. We have the following two key properties:

- For $g \geq 5$, the vector space V_g is finite-dimensional (Theorem A).
- Conclusion (iii) holds for $g \geq 7$. This was proved by Boldsen–Dollerup [2].

In the language of $\mathtt{SI}(\mathbb{Z})$-modules, these two facts imply that the V_g form a finitely generated $\mathtt{SI}(\mathbb{Z})$-module. Since V_g is also an algebraic representation of Sp_{2g} for $g \geq 6$ (Theorem A), it is also a rational $\mathtt{SI}(\mathbb{Z})$-module in the sense of [47]. We remark that [47] requires all the V_g to be algebraic, but for its proofs it is enough for them to be algebraic for g sufficiently large. We can now apply [47, Theorem B], which gives our three conclusions. ☐

16.5. **Branching rule.** The last ingredient we need for the main result of this section is the following theorem describing which irreducible representations appear when we branch an irreducible representation of Sp_{2g} to $\mathrm{Sp}_{2(g-1)}$:

Theorem 16.5 ([17, Theorem 8.1.5])**.** *For some* $g \geq 2$*, let* σ *be a partition with at most* g *parts. Write* $\sigma = (k_1, \ldots, k_m)$*. The following are a complete list of the irreducible factors appearing in the restriction* $\mathrm{Res}^{\mathrm{Sp}_{2g}}_{\mathrm{Sp}_{2(g-1)}} \mathbf{V}_\sigma(g)$*:*

- *Irreducible representations of the form* $\mathbf{V}_{\sigma'}(g-1)$ *such that* σ' *is a partition with at most* $g-1$ *parts satisfying the following condition. Write* $\sigma' = (k_1', \ldots, k_{m'}')$*. We then require that the following holds for all* i*:*

$$k_{i+2} \leq k_i' \leq k_i.$$

Here our convention is that $k_j = 0$ *for* $j \geq m$ *and* $k_j' = 0$ *for* $j \geq m'$*.*

We need the following immediate corollary of this:

Corollary 16.6. *For some* $g \geq 2$*, let* $\mathbf{V}_\sigma(g)$ *be an irreducible representation of* Sp_{2g} *of degree* $d \geq 1$*. Assume that* σ *has at most* $g-1$ *parts. Then* $\mathrm{Res}^{\mathrm{Sp}_{2g}}_{\mathrm{Sp}_{2(g-1)}} \mathbf{V}_\sigma(g)$ *contains at least two irreducible factors:* $\mathbf{V}_\sigma(g-1)$*, and a factor* $\mathbf{V}_{\sigma'}(g-1)$ *of degree* $d-1$*.*

16.6. **Bounding the degrees.** We now come to the main result of this section:

Proposition 16.7. *There exists some* $G \geq 1$ *such that the following hold for* $g \geq G$*:*

- *The* $\mathrm{Sp}_{2g}(\mathbb{Z})$*-representation* $\mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q})$ *is a finite-dimensional algebraic representation of* Sp_{2g} *all of whose irreducible factors have degree at most* 6*.*

Proof. Passing to duals preserves degrees, so it is enough to prove this for homology rather than cohomology. For $g \geq 1$, set $V_g = \mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q})$, and for $g \geq 2$ let $f_g\colon V_{g-1} \to V_g$ be the stabilization map. Combining Theorems 16.2 and 16.4, we can find $G \geq 1$ such that for $g \geq G$ the following all hold:

- There exist irreducible representations $\mathbf{V}_{\sigma_1}(g-1), \ldots, \mathbf{V}_{\sigma_n}(g-1)$ of Sp_{2g-1} such that $V_{g-1} = \mathbf{V}_{\sigma_1}(g-1) \oplus \cdots \oplus \mathbf{V}_{\sigma_n}(g-1)$ and $V_g = \mathbf{V}_{\sigma_1}(g) \oplus \cdots \oplus \mathbf{V}_{\sigma_n}(g)$.
- The stabilization map $f_g \colon V_{g-1} \to V_g$ is injective.
- The cokernel of f_g is an algebraic representation of $\mathrm{Sp}_{2(g-1)}$ all of whose irreducible factors have degree at most 5.

Consider some $g \geq G$, and let the σ_i be as in the first bullet point. We claim that each σ_i has degree at most 6. To see this, set

$$W = \mathrm{Res}^{\mathrm{Sp}_{2g}}_{\mathrm{Sp}_{2(g-1)}} V_g \cong \bigoplus_{i=1}^{n} \mathrm{Res}^{\mathrm{Sp}_{2g}}_{\mathrm{Sp}_{2(g-1)}} \mathbf{V}_{\sigma_i}(g).$$

Corollary 16.6 say that the $\mathrm{Sp}_{2(g-1)}$-representation $\mathbf{V}_{\sigma_i}(g-1)$ appears as one of the irreducible factors of $\mathrm{Res}^{\mathrm{Sp}_{2g}}_{\mathrm{Sp}_2(g-1)} \mathbf{V}_{\sigma_i}(g)$. The representation $\mathrm{coker}(f_g)$ is therefore isomorphic to the one obtained from W by deleting copies of $\mathbf{V}_{\sigma_1}(g-1), \ldots, \mathbf{V}_{\sigma_n}(g-1)$. Each of the remaining irreducible factors has degree at most 5. The proof concludes by noting that Corollary 16.6 also implies that if $\mathbf{V}_{\sigma_i}(g)$ has degree $d \geq 1$, then $\mathrm{Res}^{\mathrm{Sp}_{2g}}_{\mathrm{Sp}_{2(g-1)}} \mathbf{V}_{\sigma_i}(g)$ contains an irreducible factor $\mathbf{V}_{\sigma'}(g-1)$ of degree $d-1$. Necessarily $d-1 \leq 5$, so $d \leq 6$. $\square$

Remark 16.8. In the next section, we will prove that Proposition 16.7 holds for $G = 12$. $\square$

17. Twisted cohomology of the mapping class group and uniform degree bounds

In this section, we relate $\mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q})$ to the cohomology of the mapping class group with twisted coefficients. Using recent breakthrough work of Miller–Patzt–Petersen–Randal-Williams [40] on stability phenomena for these twisted cohomology groups, we will be able to prove that Proposition 16.7 holds for $G = 12$.

17.1. Borel stability theorem.

We will need the following special case of the classical Borel stability theorem [3, 4] on the cohomology of arithmetic groups, which we already stated in §11 but whose statement we recall for convenience The explicit stable range in the following is due to Tshishiku [58]:

Theorem 11.4 (Borel, [3, 4]). *For $g \geq 2$, the following hold:*

(i) If $\mathbf{V}_\sigma(g)$ is a nontrivial irreducible representation of Sp_{2g}, then $\mathrm{H}^k(\mathrm{Sp}_{2g}(\mathbb{Z}); \mathbf{V}_\sigma(g)) = 0$ for $k \leq g-1$.

(ii) In degrees $k \leq g-1$, the cohomology ring $\mathrm{H}^\bullet(\mathrm{Sp}_{2g}(\mathbb{Z}); \mathbb{Q})$ is isomorphic to a polynomial ring $\mathbb{Q}[c_2, c_6, c_{10}, \ldots]$ with $\deg(c_{4i-2}) = 4i-2$ for $i \geq 1$.

17.2. Twisted cohomology and the Torelli group.

We now relate the irreducible factors of $\mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q})$ to the twisted cohomology of Mod_g^1:

Lemma 17.1. *For some $g \geq 12$, let $\mathbf{V}_\sigma(g)$ be a nontrivial irreducible representation of Sp_{2g}. Then the dimension of $\mathrm{H}^2(\mathrm{Mod}_g^1; \mathbf{V}_\sigma(g))$ equals the number of copies of $\mathbf{V}_\sigma(g)$ in $\mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q})$.*

Proof. The restriction $g \geq 12$ implies that $\mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q})$ is a finite-dimensional algebraic representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$ (Theorem A). It therefore decomposes as a direct sum of irreducible factors, so the statement of the lemma makes sense. Consider the Hochschild–Serre spectral sequence with coefficients in $\mathbf{V}_\sigma(g)$ associated to the short exact sequence

$$1 \longrightarrow \mathcal{I}_g^1 \longrightarrow \mathrm{Mod}_g^1 \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1.$$

This spectral sequence takes the form

$$E_2^{pq} = \mathrm{H}^p(\mathrm{Sp}_{2g}(\mathbb{Z}); \mathrm{H}^q(\mathcal{I}_g^1; \mathbf{V}_\sigma(g))) \Rightarrow \mathrm{H}^{p+q}(\mathrm{Mod}_g^1; \mathbf{V}_\sigma(g)).$$

We are interested in H^2, so the relevant terms are those with $p+q \leq 2$. To understand potential differentials, we will also need to understand E_2^{12} and E_2^{03}. We divide the proof into two cases.

Case 1. $\mathbf{V}_\sigma(g) \notin \{\mathbf{V}_1(g), \mathbf{V}_{1^3}(g)\}$.

Since $\mathcal{I}_g^1$ acts trivially on $\mathbf{V}_\sigma(g)$, we have

$$\mathrm{H}^q(\mathcal{I}_g^1; \mathbf{V}_\sigma(g)) = \mathrm{Hom}(\mathrm{H}^q(\mathcal{I}_g^1; \mathbb{Q}), \mathbf{V}_\sigma(g)). \tag{17.1}$$

For $q = 0$ this is just $\mathbf{V}_\sigma(g)$, so by Theorem 11.4 we have

$$E_2^{p0} = \mathrm{H}^p(\mathrm{Sp}_{2g}(\mathbb{Z}); \mathbf{V}_\sigma(g)) = 0 \quad \text{for } p \leq g-1.$$

For $q = 1$, letting $H = \mathrm{H}_1(\Sigma_g^1; \mathbb{Q})$ we can apply Theorem 9.2 to see that (17.1) equals

$$\mathrm{Hom}(\mathrm{H}^1(\mathcal{I}_g^1; \mathbb{Q}), \mathbf{V}_\sigma(g)) = \mathrm{Hom}(\wedge^3 H, \mathbf{V}_\sigma(g)).$$

This decomposes as a direct sum of irreducible representations of Sp_{2g}. Moreover, since $\wedge^3 H = \mathbf{V}_1(g) \oplus \mathbf{V}_{1^3}(g)$ we can use our assumption that $\mathbf{V}_\sigma(g) \notin \{\mathbf{V}_1(g), \mathbf{V}_{1^3}(g)\}$ to see that it has no trivial factors. We can therefore apply Theorem 11.4 to see that

$$E_2^{p1} = \mathrm{H}^p(\mathrm{Sp}_{2g}(\mathbb{Z}); \mathrm{Hom}(\wedge^3 H, \mathbf{V}_\sigma(g))) = 0 \quad \text{for } p \leq g-1.$$

Finally,

$$E_2^{02} = \mathrm{H}^0(\mathrm{Sp}_{2g}(\mathbb{Z}); \mathrm{Hom}(\mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q}), \mathbf{V}_\sigma(g))) \cong \mathrm{Hom}_{\mathrm{Sp}_{2g}}(\mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q}), \mathbf{V}_\sigma(g)).$$

It follows that the dimension of E_2^{02} equals the number of copies of $\mathbf{V}_\sigma(g)$ in $\mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q})$. Summarizing, since $g \geq 12$ the E_2-page of our spectral sequence takes the form

$$\begin{array}{|llll}
E_2^{02} & & & \\
0 & 0 & 0 & \\
0 & 0 & 0 & 0 \\
\hline
\end{array}$$

It follows that

$$\mathrm{H}^2(\mathrm{Mod}_g^1; \mathbf{V}_\sigma(g)) \cong E_\infty^{02} = E_2^{02}.$$

The dimension of this equals the number of copies of $\mathbf{V}_\sigma(g)$ in $\mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q})$, as desired.

Case 2. $\mathbf{V}_\sigma(g) \in \{\mathbf{V}_1(g), \mathbf{V}_{1^3}(g)\}$.

The only difference between this and the previous case is that now $\mathrm{Hom}(\wedge^3 H, \mathbf{V}_\sigma(g))$ has a 1-dimensional trivial representation. Applying Theorem 11.4, we therefore get that

$$E_2^{p1} = \mathrm{H}^p(\mathrm{Sp}_{2g}(\mathbb{Z}); \mathrm{Hom}(\wedge^3 H, \mathbf{V}_\sigma(g))) = \begin{cases} \mathbb{Q} & \text{if } p = 0, 2, \\ 0 & \text{if } p = 1. \end{cases}$$

Our spectral sequence therefore takes the form

$$\begin{array}{|llll}
E_2^{02} & & & \\
\mathbb{Q} & 0 & \mathbb{Q} & \\
0 & 0 & 0 & 0 \\
\hline
\end{array}$$

Since the dimension of E_2^{02} is the number of copies of $\mathbf{V}_\sigma(g)$ in $\mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q})$, to prove the lemma we need to show that the differential

$$d\colon E_2^{02} \to E_2^{21} = \mathbb{Q}$$

is 0. We will sketch a proof of this below, but first we want to point out that a much more general vanishing holds. For this, we refer the reader to Kupers–Randal-Williams's argument in the proof of [33, Theorem 4.1], which uses an elegant criterion to certify that differentials in these kinds of spectral sequences vanish (see [33, Lemma 4.3]).[17]

Here is a sketch of Kupers–Randal-Williams's argument specialized to this case. What we need to show is that the $\mathbb{Q}$ in E_2^{21} survives to give a $\mathbb{Q}$ in $\mathrm{H}^3(\mathrm{Mod}_g^1; V_\sigma(g))$. Since $\mathbf{V}_\sigma(g) \in \{\mathbf{V}_1(g), \mathbf{V}_{1^3}(g)\}$, the irreducible representation $V_\sigma(g)$ is a factor of $H^{\otimes 3}$. This implies that $\mathrm{H}^3(\mathrm{Mod}_g^1; V_\sigma(g))$ is naturally a direct factor of $\mathrm{H}^3(\mathrm{Mod}_g^1; H^{\otimes 3})$. Let

$$M = \bigoplus_{k \geq 0} \mathrm{H}^k(\mathrm{Mod}_g^1; H^{\otimes 3}).$$

The vector space M is a graded module over the graded ring $R = \mathrm{H}^\bullet(\mathrm{Mod}_g^1; \mathbb{Q})$. Madsen–Weiss [37] proved that R is isomorphic to a polynomial ring $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$ with $|\kappa_i| = 2i$ in a range of degrees that tends to infinity as $g \mapsto \infty$.

As we will discuss more detail in §18 below, Kawazumi [31] proved that in a range of degrees M is a free module over R. Our hypothesis $g \geq 12$ implies that every term we care about lies in this stable range, so we can treat M as a free module over R. In the spectral sequence above, the $\mathbb{Q}$ in E_2^{01} necessarily survives to give a $\mathbb{Q}$ in

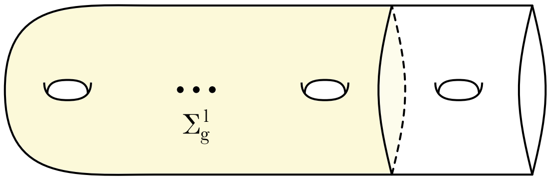
$$\mathrm{H}^1(\mathrm{Mod}_g^1; V_\sigma(g)) \subset \mathrm{H}^1(\mathrm{Mod}_g^1; H^{\otimes 3}) \subset M.$$

Let $b \in M$ generate this copy of $\mathbb{Q}$. For degree reasons, b must be a scalar multiple of one of Kawazumi's free generators. The $\mathbb{Q}$ in

$$\mathrm{H}^3(\mathrm{Mod}_g^1; V_\sigma(g)) \subset \mathrm{H}^3(\mathrm{Mod}_g^1; H^{\otimes 3}) \subset M$$

we are trying to show survives is spanned by[18] $\kappa_1 b$, which is nonzero since b is a free generator. □

17.3. **Uniform twisted stability.** Just like we did in §16, embed Σ_g^1 into Σ_{g+1}^1 as follows:



By extending mapping classes on Σ_g^1 to Σ_{g+1}^1 by the identity, we get an inclusion $\mathrm{Mod}_g^1 \hookrightarrow \mathrm{Mod}_{g+1}^1$. This induces a stabilization map $\mathrm{H}_k(\mathrm{Mod}_g^1) \to \mathrm{H}_k(\mathrm{Mod}_{g+1}^1)$. A classical theorem of Harer [22] says that this is an isomorphism for $g \gg k$. Later Ivanov [24] showed how to incorporate twisted coefficients into this kind of stability result. See [54] for a modern treatment of this and its many generalizations.

For twisted coefficients, all of these classical results gives stable ranges that depend strongly on the coefficients. Recently Miller–Patzt–Petersen–Randal-Williams [40] proved a theorem whose range is independent of the coefficients. To set it up, for some $g \geq 1$ let σ be a partition with at most g parts. We then have the irreducible representation $\mathbf{V}_\sigma(g)$ of Sp_{2g}. There is a natural map $\mathbf{V}_\sigma(g) \to \mathbf{V}_\sigma(g+1)$ (cf. the branching rule discussed in Theorem 16.5). This gives a stabilization map $\mathrm{H}_k(\mathrm{Mod}_g^1; \mathbf{V}_\sigma(g)) \to \mathrm{H}_k(\mathrm{Mod}_{g+1}^1; \mathbf{V}_\sigma(g+1))$.

---

[17] The proof of [33, Theorem 4.1] shows that the relevant differentials vanish with coefficients in tensor powers $H^{\otimes d}$, not in $V_\sigma(g)$. Since $V_\sigma(g)$ is a factor of an appropriate $H^{\otimes d}$, this gives the desired result. In fact, since we only care about $\mathbf{V}_\sigma(g) \in \{\mathbf{V}_1(g), \mathbf{V}_{1^3}(g)\}$ it is enough to handle $H^{\otimes 3}$.

[18] Really, the product structure in this spectral sequence comes from an action of $\mathrm{H}^\bullet(\mathrm{Sp}_{2g}(\mathbb{Z}); \mathbb{Q})$, but we can use κ_1 since κ_1 is a scalar multiple of the pullback of the generator $c_2 \in \mathrm{H}^2(\mathrm{Sp}_{2g}(\mathbb{Z}); \mathbb{Q})$ from Theorem 11.4.

The paper [40] proves the following result:[19]

Theorem 17.2 (Miller–Patzt–Petersen–Randal-Williams [40, Theorem 1.1])**.** *Let* $g \geq 1$ *and let* σ *be a partition with* $m \leq g$ *parts. Then the stabilization map* $\mathrm{H}_k(\mathrm{Mod}_g^1; \mathbf{V}_\sigma(g)) \to \mathrm{H}_k(\mathrm{Mod}_{g+1}^1; \mathbf{V}_\sigma(g+1))$ *is an isomorphism for* $g \geq 4k+4$.

17.4. Uniform degree bounds.

We now prove the main result of this section:

Proposition 17.3. *For* $g \geq 12$*, the following holds:*

- *The* $\mathrm{Sp}_{2g}(\mathbb{Z})$*-representation* $\mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q})$ *is a finite-dimensional algebraic representation of* Sp_{2g} *all of whose irreducible factors have degree at most* 6*.*

Proof. Theorem A implies that $\mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q})$ is a finite-dimensional algebraic representation of Sp_{2g} in this range. Moreover, Proposition 16.7 says that there is some $G \geq 1$ such that for $g \geq G$ each irreducible factor of $\mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q})$ has degree at most 6. We must verify that this propagates down to $g \geq 12$.

Consider some $12 \leq g \leq G$. Let $\mathbf{V}_\sigma(g)$ be an irreducible representation of Sp_{2g} of degree greater than 6. Lemma 17.1 implies that the dimension of $\mathrm{H}^2(\mathrm{Mod}_g^1; \mathbf{V}_\sigma(g))$ equals the number of copies of $\mathbf{V}_\sigma(g)$ in $\mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q})$. We therefore must prove that $\mathrm{H}^2(\mathrm{Mod}_g^1; \mathbf{V}_\sigma(g)) = 0$. Theorem 17.2 implies that $\mathrm{H}^2(\mathrm{Mod}_g^1; \mathbf{V}_\sigma(g))$ is isomorphic to $\mathrm{H}^2(\mathrm{Mod}_G^1; \mathbf{V}_\sigma(G))$, which vanishes since all irreducible factors of $\mathrm{H}^2(\mathcal{I}_G^1; \mathbb{Q})$ have degree at most 6. The proposition follows. ☐

18. CUP PRODUCTS SPAN ON SURFACES WITH BOUNDARY

Recall that Theorem C says that for $g \geq 12$ the second rational cohomology of the Torelli group is spanned by cup products of classes in H^1. This section proves this for $\mathcal{I}_g^1$.

18.1. Detecting the decomposition.

Let $H = \mathrm{H}_1(\Sigma_g^1; \mathbb{Q})$. Our proof uses work of Kawazumi [31] describing $\mathrm{H}^\bullet(\mathrm{Mod}_g^1; H^{\otimes d})$ in a stable range. We start by recasting our results from the previous section in these terms:

Lemma 18.1. *Fix* $g \geq 12$ *and* $d \geq 1$*. Set* $H = \mathrm{H}_1(\Sigma_g^1; \mathbb{Q})$*. Let* $t \geq 0$ *be the dimension of the trivial subrepresentation of* $H^{\otimes d}$*. We then have*

$$\dim_{\mathbb{Q}} \mathrm{H}^2(\mathrm{Mod}_g^1; H^{\otimes d}) = t + \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathrm{Sp}_{2g}}(\mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q}), H^{\otimes d}).$$

Proof. Write $H^{\otimes d} = \mathbf{V}_0(g)^{\oplus t} \oplus V$ with V a direct sum of nontrivial irreducible representations of Sp_{2g}. We have

$$\mathrm{H}^2(\mathrm{Mod}_g^1; H^{\otimes d}) = \mathrm{H}^2(\mathrm{Mod}_g^1; \mathbf{V}_0(g))^{\oplus t} \oplus \mathrm{H}^2(\mathrm{Mod}_g^1; V).$$

Theorem 11.2 says that

$$\dim_{\mathbb{Q}} \mathrm{H}^2(\mathrm{Mod}_g^1; \mathbf{V}_0(g))^{\oplus t} = t,$$

and Lemma 17.1 together with Schur's Lemma says that

$$\dim_{\mathbb{Q}} \mathrm{H}^2(\mathrm{Mod}_g^1; V) = \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathrm{Sp}_{2g}}(\mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q}), V).$$

Since $\mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q})$ has no trivial subrepresentations (Lemma 11.5), this is the same as

$$\dim_{\mathbb{Q}} \mathrm{Hom}_{\mathrm{Sp}_{2g}}(\mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q}), \mathbf{V}_0(g)^{\oplus t} \oplus V) = \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathrm{Sp}_{2g}}(\mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q}), H^{\otimes d}).$$

The lemma follows. ☐

---

[19] The statement of [40, Theorem 1.1] gives a range in which the relative twisted homology group $\mathrm{H}_d(\mathrm{Mod}_{g+1}^1, \mathrm{Mod}_g^1; \mathbf{V}_\sigma(g+1), \mathbf{V}_\sigma(g))$ vanishes. To derive stability for H_k we need this relative twisted homology group to vanish for $k \leq d+1$. This explains the range given in our statement.

18.2. **Weighted partitions.** We now turn to Kawazumi's description of $\mathrm{H}^\bullet(\mathrm{Mod}_g^1; H^{\otimes d})$. This requires a preliminary definition. A *weighted partition* of $\{1, \ldots, d\}$ consists of the following data:

- For some $n \geq 1$, a decomposition $\{1, \ldots, d\} = S_1 \sqcup \cdots \sqcup S_n$ where the S_i are disjoint nonempty sets.
- For each $1 \leq i \leq n$, a weight $w_i \geq 0$. If $|S_i| = 1$, then we require $w_i \geq 1$.

The ordering on the S_i is not important, so we identify two weighted partitions if they differ by a permutation of the S_i. Let $\mathcal{P}_d$ be the set of all weighted partitions of $\{1, \ldots, d\}$.

18.3. **Twisted Morita–Mumford classes.** Consider some $P \in \mathcal{P}_d$. Define $k(P) \geq 0$ in the following way. Assume that P consists of the decomposition $\{1, \ldots, d\} = S_1 \sqcup \cdots \sqcup S_n$ and the weights $w_1, \ldots, w_n \geq 0$. We then set[20]

$$k(P) = d + 2\sum_{i=1}^{n}(w_i - 1) \geq 0.$$

This only depends on the weights, not the decomposition. Kawazumi [31, p. 388] defines an associated twisted Morita–Mumford class $m_P \in \mathrm{H}^{k(P)}(\mathrm{Mod}_g^1; H^{\otimes d})$. It is a cup product of classes $m_{S_i,w_i} \in \mathrm{H}^{|S_i|+2(w_i-1)}(\mathrm{Mod}_g^1; H^{\otimes |S_i|})$, where the cup product uses the product map

$$H^{\otimes |S_1|} \otimes \cdots \otimes H^{\otimes |S_n|} \xrightarrow{\cong} H^{\otimes d}$$

coming from the decomposition $S_1 \sqcup \cdots \sqcup S_n = \{1, \ldots, d\}$.

18.4. **Kawazumi's work.** For a fixed $d \geq 0$, the collection of cohomology groups $M_d = \mathrm{H}^\bullet(\mathrm{Mod}_g^1; H^{\otimes d})$ is a graded module over the cohomology ring $R = \mathrm{H}^\bullet(\mathrm{Mod}_g^1; \mathbb{Q})$. Madsen–Weiss [37] proved that R is isomorphic to a polynomial ring $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$ with $|\kappa_i| = 2i$ in a range of degrees that tends to infinity as $g \mapsto \infty$. Kawazumi [31] proved that M_d is a free R-module in a range of degrees.

A version of his theorem is as follows. The stable range in it is not the one in Kawazumi's paper, but it follows from Theorem 17.2 together with the fact that the decomposition of $H^{\otimes d}$ into irreducible factors is stable once $g \geq d$ (see §7.9).

Theorem 18.2 (Kawazumi, [31, Theorem 1.B])**.** *Let $d, g \geq 0$ be such that $g \geq d$. Then in degrees k such that $g \geq 4k + 4$, we have that $\mathrm{H}^\bullet(\mathrm{Mod}_g^1; H^{\otimes d})$ is a free graded module over $\mathrm{H}^\bullet(\mathrm{Mod}_g^1; \mathbb{Q})$ with free basis the twisted Morita–Mumford classes $\{m_P \mid P \in \mathcal{P}_d\}$.*

Remark 18.3. Before Kawazumi's work, Looijenga [36] used Hodge theory to prove a similar theorem for the mapping class group Mod_g of a closed surface. The proofs in [36] and [31] are quite different. □

18.5. **Proof of main theorem.** We close this section by proving Theorem C for surfaces with boundary:

Theorem C (surface with boundary case). *Let $g \geq 12$. The image of the cup product pairing $\mathfrak{c}\colon \wedge^2 \mathrm{H}^1(\mathcal{I}_g^1; \mathbb{Q}) \to \mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q})$ spans $\mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q})$.*

Proof. Proposition 17.3 says that $\mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q})$ is a finite-dimensional algebraic representation of Sp_{2g} all of whose irreducible factors have degree at most 6. Letting $H = \mathrm{H}_1(\Sigma_g^1; \mathbb{Q})$, this implies that all irreducible factors of $\mathrm{H}^2(\mathcal{I}_g^1; \mathbb{Q})$ appear in $H^{\otimes d}$ for some $1 \leq d \leq 6$. We proved in Theorem B that the image of the cup product pairing is isomorphic to

$$\mathcal{V}_{\text{cup}} = \mathbf{V}_{1^2}(g)^{\oplus 2} \oplus \mathbf{V}_{2,1^2}(g) \oplus \mathbf{V}_{1^4}(g)^{\oplus 2} \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g).$$

---

[20] To see that $k(P) \geq 0$, note that it can also be written as $k(P) = 2\sum_{i=1}^{n}(|S_i|/2 + w_i - 1)$. Our condition on the weights ensures that each term of this sum is nonnegative.

This appears as a direct summand of $\mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q})$. To prove that this is all of $\mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q})$, it is enough to check that it accounts for all of $\mathrm{Homs}_{\mathrm{Sp}_{2g}}(\mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q}), H^{\otimes d})$ for $1 \leq d \leq 6$. In other words, we must prove that

$$\dim_{\mathbb{Q}} \mathrm{Homs}_{\mathrm{Sp}_{2g}}(\mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q}), H^{\otimes d}) = \dim_{\mathbb{Q}} \mathrm{Homs}_{\mathrm{Sp}_{2g}}(\mathcal{V}_{\mathrm{cup}}, H^{\otimes d})$$

for $1 \leq d \leq 6$. Let t_d be the dimension of the trivial subrepresentation of $H^{\otimes d}$. By Lemma 18.1, we have

$$\dim_{\mathbb{Q}} \mathrm{H}^2(\mathrm{Mod}_g^1; H^{\otimes d}) = t_d + \dim_{\mathbb{Q}} \mathrm{Homs}_{\mathrm{Sp}_{2g}}(\mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q}), H^{\otimes d}).$$

Combining all of this, we see that we must prove the following:

Claim. *For $1 \leq d \leq 6$, we have* $\dim_{\mathbb{Q}} \mathrm{H}^2(\mathrm{Mod}_g^1; H^{\otimes d}) = t_d + \dim_{\mathbb{Q}} \mathrm{Homs}_{\mathrm{Sp}_{2g}}(\mathcal{V}_{\mathrm{cup}}, H^{\otimes d})$.

Since $g \geq 12$, the decomposition of $H^{\otimes d}$ is stable for $1 \leq d \leq 6$ (see §7.9). We can therefore compute the right hand side of the desired identity using LiE [35]:

d	t_d	$\dim_{\mathbb{Q}} \mathrm{Homs}_{\mathrm{Sp}_{2g}}(\mathcal{V}_{\mathrm{cup}}, H^{\otimes d})$	$t_d + \dim_{\mathbb{Q}} \mathrm{Homs}_{\mathrm{Sp}_{2g}}(\mathcal{V}_{\mathrm{cup}}, H^{\otimes d})$
1	0	0	0
2	1	2	3
3	0	0	0
4	3	17	20
5	0	0	0
6	15	175	190

We now focus on the left hand side $\dim_{\mathbb{Q}} \mathrm{H}^2(\mathrm{Mod}_g^1; H^{\otimes d})$. Set $M_d = \mathrm{H}^{\bullet}(\mathrm{Mod}_g^1; H^{\otimes d})$ and $R = \mathrm{H}^{\bullet}(\mathrm{Mod}_g^1; \mathbb{Q})$. By the Madsen–Weiss theorem [37], in a range of degrees that includes 2 we have that $R \cong \mathbb{Q}[\kappa_1, \kappa_2, \ldots]$ with $\kappa_i \in \mathrm{H}^{2i}(\mathrm{Mod}_g^1;\mathbb{Q})$. Theorem 18.2 says that in the range of degrees we care about, M_d is a free R-module on the classes $\{m_P \mid P \in \mathcal{P}_d\}$. For $P \in \mathcal{P}_d$, recall that $m_P \in \mathrm{H}^{k(m_P)}(\mathrm{Mod}_g^1; H^{\otimes d})$. It follows that the $\mathbb{Q}$-vector space $\mathrm{H}^2(\mathrm{Mod}_g^1; H^{\otimes d})$ has a basis consisting of the following elements:

- m_P for $p \in \mathcal{P}_d$ with $k(m_P) = 2$; and
- $\kappa_1 m_P$ for $p \in \mathcal{P}_d$ with $k(m_P) = 0$.

Using a computer, it is easy to enumerate the weighted partitions $P \in \mathcal{P}_d$ with $k(m_P) \in \{0, 2\}$ for $d \in \{1, \ldots, 6\}$. See [51] for the code.[21] The results are as follows:

d	$\lvert\{P \in \mathcal{P}_d \mid k(m_P) = 0\}\rvert$	$\lvert\{P \in \mathcal{P}_d \mid k(m_P) = 2\}\rvert$	$\dim_{\mathbb{Q}} \mathrm{H}^2(\mathrm{Mod}_g^1; H^{\otimes d})$
1	0	0	0
2	1	2	3
3	0	0	0
4	3	17	20
5	0	0	0
6	15	175	190

Comparing the above tables, the right hand columns are the same.[22] The claim follows. □

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[21] This code was written and tested entirely by hand. No AI tools were used.

[22] In fact, the tables are the same, which is no accident: $\{P \in \mathcal{P}_d \mid k(m_P) = 0\}$ is in bijection with a basis for the trivial subrepresentation of $H^{\otimes d}$.

19. Cup products span on punctured surfaces

We next prove Theorem C for punctured surfaces:

Theorem C (punctured surface case). *Let $g \geq 12$. The image of the cup product pairing $\mathfrak{c}\colon \wedge^2 \mathrm{H}^1(\mathcal{I}_{g,1};\mathbb{Q}) \to \mathrm{H}^2(\mathcal{I}_{g,1};\mathbb{Q})$ spans $\mathrm{H}^2(\mathcal{I}_{g,1};\mathbb{Q})$.*

Proof. Since we already proved in §18 that $\mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q})$ is spanned by the image of the cup product pairing, we know from Theorem B that

$$\mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q}) \cong \mathbf{V}_{1^2}(g)^{\oplus 2} \oplus \mathbf{V}_{2,1^2}(g) \oplus \mathbf{V}_{1^4}(g)^{\oplus 2} \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g).$$

Theorem B also says that the image of the cup product pairing for $\mathcal{I}_{g,1}$ is isomorphic to

$$\mathbf{V}_0(g) \oplus \mathbf{V}_{1^2}(g)^{\oplus 2} \oplus \mathbf{V}_{2,1^2}(g) \oplus \mathbf{V}_{1^4}(g)^{\oplus 2} \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g).$$

The only difference between this and $\mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q})$ is a single copy of the trivial representation $\mathbf{V}_0(g) \cong \mathbb{Q}$. To prove that the image of the cup product pairing for $\mathcal{I}_{g,1}$ is everything, it is thus enough to prove the following claim (in which for convenience we dualize and switch to homology):

Claim. *We have $\dim_{\mathbb{Q}} \mathrm{H}_2(\mathcal{I}_{g,1};\mathbb{Q}) = 1 + \dim_{\mathbb{Q}} \mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q})$.*

Let $b = \partial \Sigma_g^1$. As we noted in the proof of Theorem B for punctured surfaces in §14, there is a central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{I}_g^1 \longrightarrow \mathcal{I}_{g,1} \longrightarrow 1$$

whose central $\mathbb{Z}$ term is generated by the Dehn twist T_b (see [14, Proposition 3.19] for the corresponding exact sequence for the mapping class group). The associated Hochschild–Serre spectral sequence takes the form

$$E^2_{pq} = \mathrm{H}_p(\mathcal{I}_{g,1};\mathrm{H}_q(\mathbb{Z};\mathbb{Q})) \Rightarrow \mathrm{H}_{p+q}(\mathcal{I}_g^1;\mathbb{Q}).$$

Since our extension is central, the action of $\mathcal{I}_{g,1}$ here is trivial and thus

$$E^2_{pq} = \begin{cases} \mathrm{H}_p(\mathcal{I}_{g,1};\mathbb{Q}) & \text{for } q = 0,1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $H = \mathrm{H}_1(\Sigma_{g,1};\mathbb{Q})$. As we noted in the proof of Theorem B for punctured surfaces in §14, we have $\mathrm{H}_1(\mathcal{I}_{g,1};\mathbb{Q}) \cong \wedge^3 H$. This isomorphism is induced by the first Johnson homomorphism. Because of all of this, the portion of our spectral sequence that can contribute to $\mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q})$ looks like this:

$$\begin{array}{|lll} \mathbb{Q} & \wedge^3 H & \\ \mathbb{Q} & \wedge^3 H & \mathrm{H}_2(\mathcal{I}_{g,1};\mathbb{Q}) \\ \hline \end{array}$$

Theorem 9.2 says that $\mathrm{H}_1(\mathcal{I}_g^1;\mathbb{Q}) \cong \wedge^3 H$, so $E^2_{01} = \mathbb{Q}$ must be killed by the differential

$$\mathrm{H}_2(\mathcal{I}_{g,1};\mathbb{Q}) = E^2_{20} \to E^2_{01} = \mathbb{Q}.$$

In the claim below we will prove that $E^2_{11} = \wedge^3 H$ is also killed by a differential. From this, we can conclude that

$$\mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q}) \cong \ker(\mathrm{H}_2(\mathcal{I}_{g,1};\mathbb{Q}) \to \mathbb{Q}),$$

so $\dim_{\mathbb{Q}} \mathrm{H}_2(\mathcal{I}_{g,1};\mathbb{Q}) = 1 + \dim_{\mathbb{Q}} \mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q})$, as desired.

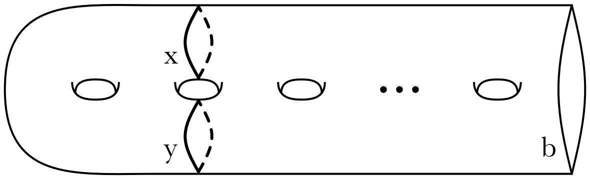
It remains to prove the following:

Claim. *In the above spectral sequence, $E^2_{11} = \wedge^3 H$ is killed by a differential.*

Recall that $\mathrm{H}_1(\mathcal{I}_g^1;\mathbb{Q}) = \wedge^3 H$. For $f \in \mathcal{I}_g^1$, let

$$[f] \in (\mathcal{I}_g^1)^{\mathrm{ab}} \otimes \mathbb{Q} = \mathrm{H}_1(\mathcal{I}_g^1;\mathbb{Q})$$

be the corresponding element. Since $g \geq 3$, it follows from a theorem of Johnson [25] that $\mathcal{I}_g^1$ is generated by bounding pair maps $T_x T_y^{-1}$ such that $x \cup y$ bounds a subsurface homeomorphic to Σ_1^2:



These are called genus 1 bounding pair maps. Letting $T_x T_y^{-1}$ be a genus 1 bounding pair map, it is enough to prove that $[T_x T_y^{-1}] \in \wedge^3 H = E_{11}^2$ is killed by a differential.

Recall that T_b is the Dehn twist about $b = \partial\Sigma_g^1$. This generates the kernel of the central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{I}_g^1 \longrightarrow \mathcal{I}_{g,1} \longrightarrow 1$$

whose Hochschild–Serre spectral sequence we are studying. The subgroup of $\mathcal{I}_g^1$ generated by T_b and $T_x T_y^{-1}$ is isomorphic to $\mathbb{Z}^2$. We have a commutative diagram of central extensions

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, equal] & \mathbb{Z}^2 \arrow[r] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \langle T_b \rangle \arrow[r] \arrow[d, equal] & \langle T_b \rangle \times \langle T_x T_y^{-1} \rangle \arrow[r] \arrow[d] & \langle T_x T_y^{-1} \rangle \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{Z} \arrow[r] & \mathcal{I}_g^1 \arrow[r] & \mathcal{I}_{g,1} \arrow[r] & 1
\end{tikzcd}$$

whose bottom right arrow is induced by the map taking $T_x T_y^{-1}$ to its image in $\mathcal{I}_{g,1}$. Consider the map of Hochschild–Serre spectral sequences from the spectral sequence of the middle extension to the spectral sequence of the bottom extension. The map on E_{11}^2-terms is of the form

$$\mathrm{H}_1(\langle T_x T_y^{-1} \rangle; \mathrm{H}_1(\langle T_b \rangle;\mathbb{Q})) \to \mathrm{H}_1(\mathcal{I}_{g,1}; \mathrm{H}_1(\mathbb{Z};\mathbb{Q})) = \mathrm{H}_1(\mathcal{I}_{g,1};\mathbb{Q}) = \wedge^3 H.$$

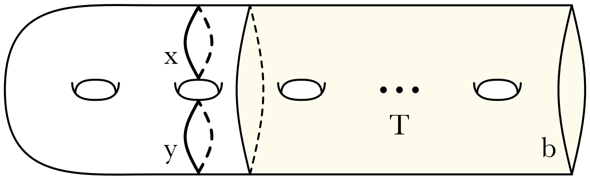
Our goal is to prove that the image of this is killed by a differential. We have

$$\mathrm{H}_1(\langle T_x T_y^{-1} \rangle; \mathrm{H}_1(\langle T_b \rangle;\mathbb{Q})) = \mathrm{H}_1(\langle T_x T_y^{-1} \rangle;\mathbb{Q}) \otimes \mathrm{H}_1(\langle T_b \rangle;\mathbb{Q}) = \mathrm{H}_2(\langle T_b \rangle \times \langle T_x T_y^{-1} \rangle;\mathbb{Q}).$$

From this, we see that our goal is equivalent to showing that the image of the map

$$\mathrm{H}_2(\langle T_b \rangle \times \langle T_x T_y^{-1} \rangle;\mathbb{Q}) \to \mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q}) \tag{19.1}$$

induced by the inclusion $\langle T_b \rangle \times \langle T_x T_y^{-1} \rangle \hookrightarrow \mathcal{I}_g^1$ is zero. Let $T \cong \Sigma_{g-2}^2$ be the following subsurface:



Denote by $\mathcal{I}_g^1(T)$ the subgroup of $\mathcal{I}_g^1$ consisting of mapping classes supported on T. We can factor our inclusion as

$$\langle T_b \rangle \times \langle T_x T_y^{-1} \rangle \hookrightarrow \mathcal{I}_g^1(T) \times \langle T_x T_y^{-1} \rangle \to \mathcal{I}_g^1.$$

Using this, our map (19.1) factors as

$$\begin{aligned}&\mathrm{H}_2(\langle T_b\rangle \times \langle T_x T_y^{-1}\rangle;\mathbb{Q}) = \mathrm{H}_1(\langle T_b\rangle;\mathbb{Q}) \otimes \mathrm{H}_1(\langle T_x T_y^{-1}\rangle;\mathbb{Q})\\ &\quad\to \mathrm{H}_1(\mathcal{I}_g^1(T);\mathbb{Q}) \otimes \mathrm{H}_1(\langle T_x T_y^{-1}\rangle;\mathbb{Q}) \hookrightarrow \mathrm{H}_2(\mathcal{I}_g^1(T) \times \langle T_x T_y^{-1}\rangle;\mathbb{Q}) \to \mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q}).\end{aligned}$$

It is therefore enough to prove that the map $\mathrm{H}_1(\langle T_b\rangle;\mathbb{Q}) \to \mathrm{H}_1(\mathcal{I}_g^1(T);\mathbb{Q})$ is the zero map, i.e., that the separating twist T_b vanishes in

$$\mathrm{H}_1(\mathcal{I}_g^1(T);\mathbb{Q}) = \mathcal{I}_g^1(T)^{\mathrm{ab}} \otimes \mathbb{Q}.$$

In fact, T_b^2 vanishes in the abelianization of $\mathcal{I}_g^1(T)$. This calculation is essentially due to Johnson [29], though in this level of generality it was first written out in [50]. ☐

20. Cup products span on closed surfaces

We close this paper by proving Theorem C for closed surfaces:

Theorem C (closed surface case). *Let $g \geq 12$. The image of the cup product pairing $\mathfrak{c}\colon \wedge^2 \mathrm{H}^1(\mathcal{I}_g;\mathbb{Q}) \to \mathrm{H}^2(\mathcal{I}_g;\mathbb{Q})$ spans $\mathrm{H}^2(\mathcal{I}_g;\mathbb{Q})$.*

Proof. Set $H = \mathrm{H}_1(\Sigma_g;\mathbb{Q})$. Since we already proved in §19 that $\mathrm{H}^2(\mathcal{I}_{g,1};\mathbb{Q})$ is spanned by the image of the cup product pairing, we know from Theorem B that

$$\mathrm{H}^2(\mathcal{I}_{g,1};\mathbb{Q}) \cong \mathbf{V}_0(g) \oplus \mathbf{V}_{1^2}(g)^{\oplus 2} \oplus \mathbf{V}_{2,1^2}(g) \oplus \mathbf{V}_{1^4}(g)^{\oplus 2} \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g).$$

Theorem B also says that the image of the cup product pairing for $\mathcal{I}_g$ is isomorphic to

$$\mathbf{V}_{1^2}(g) \oplus \mathbf{V}_{1^4}(g) \oplus \mathbf{V}_{2^2,1^2}(g) \oplus \mathbf{V}_{1^6}(g)$$

The only difference between this and $\mathrm{H}^2(\mathcal{I}_{g,1};\mathbb{Q})$ is that $\mathrm{H}^2(\mathcal{I}_{g,1};\mathbb{Q})$ contains the following additional representations:

$$\mathbf{V}_0(g) \oplus \mathbf{V}_{1^2}(g) \oplus \mathbf{V}_{2,1^2}(g) \oplus \mathbf{V}_{1^4}(g) \cong \mathbf{V}_0(g) \oplus \left(H \otimes (\wedge^3 H)/H\right).$$

Since $\mathbf{V}_0(g)$ is 1-dimensional, it is thus enough to prove that

$$\dim_{\mathbb{Q}} \mathrm{H}^2(\mathcal{I}_{g,1};\mathbb{Q}) = 1 + \dim_{\mathbb{Q}}\left(H \otimes (\wedge^3 H)/H\right) + \dim_{\mathbb{Q}} \mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q}).$$

In fact, since $\dim_{\mathbb{Q}} \mathrm{H}^2(\mathcal{I}_g^1;\mathbb{Q})$ is at least as large as the image of the cup product pairing we already know that the right hand side is greater than or equal to the left hand side, so we only need to prove the reverse inequality. For this, we will dualize and switch to homology.

Claim. *We have $\dim_{\mathbb{Q}} \mathrm{H}_2(\mathcal{I}_{g,1};\mathbb{Q}) \leq 1 + \dim_{\mathbb{Q}}\left(H \otimes (\wedge^3 H)/H\right) + \dim_{\mathbb{Q}} \mathrm{H}_2(\mathcal{I}_g^1;\mathbb{Q})$.*

Let $\pi = \pi_1(\Sigma_g)$. As we already noted in the proof of Theorem B for closed surfaces in §15, the Birman exact sequence for the mapping class group (see [14, §4.2]) restricts to a short exact sequence

$$1 \longrightarrow \pi \longrightarrow \mathcal{I}_{g,1} \longrightarrow \mathcal{I}_g \longrightarrow 1.$$

The associated Hochschild–Serre spectral sequence takes the form

$$E^2_{pq} = \mathrm{H}_p(\mathcal{I}_g;\mathrm{H}_q(\pi;\mathbb{Q})) \Rightarrow \mathrm{H}_{p+q}(\mathcal{I}_{g,1};\mathbb{Q}).$$

Let $H = \mathrm{H}_1(\pi;\mathbb{Q}) = \mathrm{H}_1(\Sigma_g;\mathbb{Q})$. We also have $\mathrm{H}_2(\pi;\mathbb{Q}) = \mathrm{H}_2(\Sigma_g;\mathbb{Q}) = \mathbb{Q}$. The group $\mathcal{I}_g$ acts trivially on both of these, so

$$\begin{aligned}E^2_{01} &= \mathrm{H}_0(\mathcal{I}_g;\mathrm{H}_1(\pi;\mathbb{Q})) = H,\\ E^2_{02} &= \mathrm{H}_0(\mathcal{I}_g;\mathrm{H}_2(\pi;\mathbb{Q})) = \mathbb{Q}.\end{aligned}$$

We also noted during the proof of Theorem B for closed surfaces in §15 that the first Johnson homomorphism descends to the following isomorphism, which was first proved by Johnson in [29, 30]:

$$\mathrm{H}_1(\mathcal{I}_g;\mathbb{Q}) \cong (\wedge^3 H)/H.$$

We deduce that

$$\begin{aligned} E^2_{10} &= \mathrm{H}_1(\mathcal{I}_g;\mathbb{Q}) \cong (\wedge^3 H)/H, \\ E^2_{11} &= \mathrm{H}_1(\mathcal{I}_g;\mathrm{H}_1(\pi;\mathbb{Q})) = \mathrm{H}_1(\pi;\mathbb{Q}) \otimes \mathrm{H}_1(\mathcal{I}_g;\mathbb{Q}) \cong H \otimes ((\wedge^3 H)/H). \end{aligned}$$

Summarizing all of this, the portion of this spectral sequence that can contribute to $\mathrm{H}_2(\mathcal{I}_{g,1};\mathbb{Q})$ is

$$\begin{array}{|lll} \mathbb{Q} & & \\ H & H \otimes ((\wedge^3 H)/H) & \\ \mathbb{Q} & (\wedge^3 H)/H & \mathrm{H}_2(\mathcal{I}_g;\mathbb{Q}) \\ \hline \end{array}$$

As we noted in §14, Theorem 9.2 implies that

$$\mathrm{H}_1(\mathcal{I}_{g,1};\mathbb{Q}) \cong \wedge^3 H \cong H \oplus ((\wedge^3 H)/H).$$

This implies that the differential

$$H = E^2_{01} \to E^2_{02} = \mathrm{H}_2(\mathcal{I}_g;\mathbb{Q})$$

must vanish. We conclude that all of $E^2_{02} = \mathrm{H}_2(\mathcal{I}_g;\mathbb{Q})$ survives to the E^{∞} page. Potentially[23] there might be nonzero differentials involving E^2_{02} and E^2_{11}, but in any case we conclude that

$$\dim_{\mathbb{Q}} \mathrm{H}_2(\mathcal{I}_{g,1};\mathbb{Q}) \leq 1 + \dim_{\mathbb{Q}} \left(H \otimes (\wedge^3 H)/H\right) + \dim_{\mathbb{Q}} \mathrm{H}_2(\mathcal{I}^1_g;\mathbb{Q}),$$

as desired. □

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[23] Once this proof is complete, since the inequality we are proving is an equality we will know that these differentials vanish too.

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