Tychonoff's Theorem

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Abstract

We give a simple and direct proof of Tychonoff's Theorem.

1 Introduction

Recall that Tychonoff's theorem is as follows:

Theorem A. *The product of arbitrarly many compact spaces is compact.*

In this theorem, a product of spaces is topologized using product topology, which is defined as follows:

- For a finite product $\prod_{i=1}^{n} X_i$, the topology is generated by the basis

$$\{\prod_{i=1}^{n} U_i \mid U_i \subset X_i \text{ open for } 1 \leq i \leq n\}. \tag{1.1}$$

- For an infinite product $\prod_{i \in I} X_i$, the topology is generated by the basis

$$\{U \times \prod_{i \in I \setminus J} X_i \mid J \subset I \text{ finite and } U \subset \prod_{j \in J} X_j \text{ open}\}. \tag{1.2}$$

  Here $\prod_{j \in J} X_j$ is given the topology from (1.1). We will call the elements of (1.2) the *finitely supported open sets.*

There are many proofs of Theorem A, but all the ones I am aware of suffer from one of the following defects:

- they use a nonstandard characterization of compactness instead of the usual open cover characterization, or
- they use esoteric set-theoretic constructions like nets and ultrafilters.

The one that comes closest to avoiding the above defects uses the Alexander Subbase Theorem, but even that strikes me as somewhat indirect. In this note, I discuss a simple proof that I came up with several years ago. Though I have not seen it in the literature, I can't imagine that I am the first to discover it.

It directly generalizes a standard proof of Tychonoff's theorem for finite products. To set the stage, here is a sketch of that special case (with the full details postponed until they are needed in the proof of the general case). Let $\mathcal{U}$ be an open cover of a product $\prod_{i=1}^n X_i$ of compact spaces. Assume that $\mathcal{U}$ does not contain a finite subcover. It is then easy to find some $p_1 \in X_1$ such that no finite subset of $\mathcal{U}$ covers $p_1 \times \prod_{i=2}^n X_i$, then to find some $p_2 \in X_2$ such that no finite subset of $\mathcal{U}$ covers $p_1 \times p_2 \times \prod_{i=3}^n X_i$, etc. At the end of this process, we have produced a single point

$$p = (p_1, \ldots, p_n) \in X_1 \times \cdots \times X_n$$

such that no finite subset of $\mathcal{U}$ covers p, which is clearly nonsense.

With a bit of care, only one new idea is needed to go from this to the general case! Indeed, the argument used to find the p_i is given in detail in Case 1 below, while the new idea is given in Case 2.

Proof of Theorem A. Let $\{X_i\}_{i \in I}$ be a collection of compact sets and let $X = \prod_{i \in I} X_i$. Consider an open cover $\mathcal{U}$ of X. Our goal is to prove that $\mathcal{U}$ has a finite subcover.

Recall that we defined the finitely supported open sets in (1.2). We claim that we can assume without loss of generality that each element of $\mathcal{U}$ is finitely supported. Indeed, each element of $\mathcal{U}$ is a union of finitely supported open sets, and if we can find a finite subcover of

$$\{V \mid \text{for some } U \in \mathcal{U}, \text{ the set } V \text{ is a finitely supported subset of } U\}, \tag{1.3}$$

then we can clearly find a finite subcover of $\mathcal{U}$. We can thus replace $\mathcal{U}$ with (1.3), and the claim follows. We remark that we will not use this property of $\mathcal{U}$ until Case 2 below.

Assume for the sake of contradiction that $\mathcal{U}$ does not contain a finite subcover. Choose a well-ordering on I, and identify I with the resulting ordinal. For the reader not comfortable with ordinal numbers, we suggest focusing on the case where $I = \{1, 2, \ldots\}$. The ordinals less than or equal to I are then $\emptyset$, the finite sets $\{1, \ldots, n\} \subset I$ (the *successor ordinals*) and the whole set I (the *limit ordinal*).

For all $i \in I$, we will find a point p_i satisfying the following condition: for all ordinals $J \leq I$, no finite subset of $\mathcal{U}$ covers

$$\prod_{j \in J} p_j \times \prod_{i \in I \setminus J} X_i \subset X.$$

A special case of this is that no finite subset of $\mathcal{U}$ covers the single point $(p_i)_{i \in I} \in X$, which is absurd.

We will do this by transfinite induction on ordinals $J \leq I$. The base case is $J = \emptyset$, where the desired fact follows from our assumption that no finite subset of $\mathcal{U}$ covers $\prod_{i \in I} X_i$. Assume now that $\emptyset < J \leq I$ and that for all $J' < J$, we have constructed $p_{j'} \in X_{j'}$ for all $j' \in J'$ such that no finite subset of $\mathcal{U}$ covers

$$\prod_{j' \in J'} p_{j'} \times \prod_{i \in I \setminus J'} X_i \subset X.$$

We must construct p_j for any $j \in J$ such that p_j has not yet been constructed, and then verify that no finite subset of $\mathcal{U}$ covers

$$\prod_{j \in J} p_j \times \prod_{i \in I \setminus J} X_i \subset X.$$

We divide this into two cases.

Case 1. *J is a successor ordinal, so $J = J' \sqcup \{n\}$ for some $J' \subset I$ and $n \in I$.*

The only p_j that needs to be constructed is p_n. Assume for the sake of contradiction that no $p_n \in X_n$ satisfying the desired condition exists. In other words, for all $q \in X_n$, there is a finite subset of $\mathcal{U}$ covering

$$\prod_{j' \in J'} p_{j'} \times q \times \prod_{i \in I \setminus J} X_i \subset X.$$

Let $V_q \subset X$ be the union of the sets in this finite subcover. Letting $\pi \colon X \to X_n$ be the projection, the set $\{\pi(V_q) \mid q \in X_n\}$ is an open cover of the compact space X_n. We can thus find $q_1, \ldots, q_m \in X_n$ such that

$$X_n = \pi(V_{q_1} \cup \cdots \cup V_{q_m}).$$

Setting $V = V_{q_1} \cup \cdots \cup V_{q_m}$, the set V is a finite union of elements of $\mathcal{U}$ containing

$$\prod_{j' \in J'} p_{j'} \times X_n \times \prod_{i \in I \setminus J} X_i \subset X,$$

contradicting our assumption that no finite subset of $\mathcal{U}$ covers this set.

Case 2. *J is a limit ordinal, so for every $j \in J$, there exists some ordinal $J' < J$ with $j \in J'$.*

In this case, there is no need to construct any new p_j. Assume for the sake of contradiction that there is a finite subset of $\mathcal{U}$ covering

$$\prod_{j \in J} p_j \times \prod_{i \in I \setminus J} X_i \subset X.$$

Let V be the union of the elements of this finite subcover. Since every element of $\mathcal{U}$ is finitely supported (see (1.2) for the definition) and V is a finite union of elements of $\mathcal{U}$, it follows that V is finitely supported. This implies that there exists a finite subset $K \subset J$ and an open set $W \subset \prod_{k \in K} X_k$ with $(p_k)_{k \in K} \in W$ such that

$$V = W \times \prod_{i \in I \setminus K} X_i.$$

Let J' be the smallest ordinal containing all the elements of K. Since K is finite, it follows that $J' < J$. What is more, V is the union of finitely many elements of $\mathcal{U}$ such that

$$\prod_{j' \in J'} p_{j'} \times \prod_{i \in I \setminus J'} X_i \subset V,$$

contradicting our assumption that no finite subset of $\mathcal{U}$ covers this set for such ordinals. $\square$