Representation theory without character theory

Andrew Putman

Abstract

We give short, direct proofs that if G is a finite group, then the group ring $\mathbb{C}[G]$ decomposes as a direct sum of $\dim(V)$ copies of every irreducible representation V of G and that the number of irreducible representations of G is the same as the number of conjugacy classes of G.

Let G be a finite group. In this note, a *representation* of G means a finite-dimensional complex vector space V upon which G acts linearly. The following results are two of the early highlights of a basic treatment of representation theory:

Theorem 0.1. *The group ring $\mathbb{C}[G]$ decomposes into a direct sum of $\dim(V)$ copies of every irreducible representation V of G.*

Theorem 0.2. *The number of irreducible representations of G equals the number of conjugacy classes in G.*

Perhaps influenced by Serre's book [S], most treatments of the subject derive these results from character theory. While character theory is an essential part of the subject, I find this unsatisfying: basic theorems about the representation theory of G should be able to be proved using actual representations! The purpose of this note is to explain short and direct proof of the above two theorems. Though I have not seen these proofs elsewhere, I am sure they must be known to the experts.

The only two results I will need are Maschke's theorem and Schur's lemma. To assure the reader that I am not slipping anything past them, I will start with proofs of these results.

Theorem 0.3 (Maschke's Theorem)**.** *Every representation of G decomposes into a direct sum of irreducible representations.*

Proof. Let V be a representation of G and let W be a subrepresentation of V. It is enough to find another subrepresentation W' of V such that $V = W \oplus W'$. We can find a G-invariant Hermitian inner product on V by averaging an arbitrary one. Once we have done this, for W' we can simply take the orthogonal complement of W. □

Lemma 0.4 (Schur's Lemma)**.** *Let V and W be irreducible representations of G. Then*

$$\operatorname{Hom}_G(V, W) = \begin{cases} 0 & \text{if } V \text{ is not isomorphic to } W, \\ \mathbb{C} & \text{if } V \cong W. \end{cases}$$

Proof. The kernel of an element of $\operatorname{Hom}_G(V, W)$ is a subrepresentation of V, and thus by irreducibility must either be 0 or V. Similarly, the cokernel of an element of $\operatorname{Hom}_G(V, W)$ must either be 0 or W. We deduce that every nonzero element of $\operatorname{Hom}_G(V, W)$ must be an isomorphism. This implies that $\operatorname{Hom}_G(V, W) = 0$ if V is not isomorphic to W. To deal with the other case, we must prove that $\operatorname{Hom}_G(V, V) \cong \mathbb{C}$. Consider $f \in \operatorname{Hom}_G(V, V)$. The linear map f must have an eigenvalue $\lambda \in \mathbb{C}$. Since $f - \lambda \in \operatorname{Hom}_G(V, V)$ has a nontrivial kernel (namely, an eigenvector), it must be 0, so $f = \lambda$. □

Schur's Lemma has the following consequence.

Lemma 0.5. *Let V and W be representations of G. Assume that W is irreducible and that $V = V_1 \oplus \cdots \oplus V_k$ with each V_i irreducible. Then the dimension of* $\mathrm{Hom}_G(V, W)$ *equals the number of V_i factors that are isomorphic to W.*

Proof. An element of $\mathrm{Hom}_G(V, W)$ is determined by its restriction to each V_i, and Schur's Lemma implies that

$$\mathrm{Hom}_G(V_i, W) = \begin{cases} \mathbb{C} & \text{if } V_i \cong W, \\ 0 & \text{otherwise.} \end{cases}$$ □

We now prove Theorem 0.1.

Proof of Theorem 0.1. Let V be an irreducible representation of G. Letting $e \in G$ be the identity, a G-equivariant map $\phi\colon \mathbb{C}[G] \to V$ is completely determined by $\phi(e) \in V$, and any vector in V occurs as $\phi(e)$ for some G-equivariant map $\phi\colon \mathbb{C}[G] \to V$. It follows that $\mathrm{Hom}_G(\mathbb{C}[G], V)$ is $\dim(V)$-dimensional. The theorem now follows from Lemma 0.5. □

We now turn to counting irreducible representations of G. This requires the following lemma. Let $\mathrm{Mat}_k(\mathbb{C})$ be algebra of $k \times k$ complex matrices.

Lemma 0.6. *Let V be a representation of G. Write*

$$V = V_1^{\oplus k_1} \oplus \cdots \oplus V_n^{\oplus k_n},$$

where the V_i are mutually nonisomorphic irreducible representations of G. Then we have an isomorphism

$$\mathrm{End}_G(V) = \mathrm{Mat}_{k_1}(\mathbb{C}) \oplus \cdots \oplus \mathrm{Mat}_{k_n}(\mathbb{C})$$

of algebras.

Proof. Schur's lemma implies that

$$\mathrm{End}_G(V) = \mathrm{End}_G(V_1^{\oplus k_1}) \oplus \cdots \oplus \mathrm{End}_G(V_n^{\oplus k_n}),$$

so it is enough to prove that $\mathrm{End}_G(V_i^{\oplus k_i}) \cong \mathrm{Mat}_{k_i}(\mathbb{C})$ for all $1 \leq i \leq n$. For this, observe that

$$\mathrm{End}_G(V_i^{\oplus k_i}) \cong \mathrm{Mat}_{k_i}(\mathrm{End}_G(V_i)),$$

where the the isomorphism takes $\phi \in \mathrm{End}_G(V_i^{\oplus k_i})$ to the matrix whose (p, q)-entry is the composition

$$V_i \hookrightarrow V_i^{\oplus k_i} \xrightarrow{\phi} V_i^{\oplus k_i} \longrightarrow V_i.$$

Here the first arrow is the inclusion of the q^{th} factor and the third arrow is the projection onto the p^{th} factor. Schur's lemma says that $\mathrm{End}_G(V_i) \cong \mathbb{C}$, and the lemma follows. □

Proof of Theorem 0.2. Let m be the number of conjugacy classes of G and let n be the number of irreducible G-representations. We will prove that both m and n equal the dimension of the center $Z(\mathbb{C}[G])$:

- The center $Z(\mathbb{C}[G])$ consists of linear combinations of elements of G whose coefficients are constant on conjugacy classes of G. In particular, $Z(\mathbb{C}[G])$ is m-dimensional.

- By Theorem 0.1, we can write

$$\mathbb{C}[G] = V_1^{\oplus k_1} \oplus \cdots \oplus V_n^{\oplus k_n}, \tag{0.1}$$

where $V_1, \ldots, V_n$ are all the irreducible G-representations and $k_i = \dim(V_i)$ for $1 \leq i \leq n$. Using Lemma 0.6, it follows that

$$\mathrm{End}_G(\mathbb{C}[G]) \cong \mathrm{Mat}_{k_1}(\mathbb{C}) \times \cdots \times \mathrm{Mat}_{k_n}(\mathbb{C}). \tag{0.2}$$

Now define an algebra homomorphism $\psi\colon \mathbb{C}[G] \to \mathrm{End}_G(\mathbb{C}[G])$ via the formula

$$\Psi(g)(z) = zg^{-1} \qquad (g \in G, z \in \mathbb{C}[G]).$$

We must multiply z on the right so that $\Psi(g)$ is equivariant with respect to the left G-action on $\mathbb{C}[G]$. It is clear that Ψ is injective. Moreover, (0.1) implies that $\mathbb{C}[G]$ is $k_1^2 + \cdots + k_n^2$ dimensional and (0.2) implies that $\mathrm{End}_G(\mathbb{C}[G])$ is $k_1^2 + \cdots + k_n^2$ dimensional. We conclude that Ψ is an isomorphism. Now, the center of the algebra of $k \times k$ complex matrices is precisely the 1-dimensional set of scalar matrices. It follows that

$$\begin{aligned} \dim Z\left(\mathbb{C}[G]\right) &= \dim\left(Z\left(\mathrm{End}_G\left(\mathbb{C}[G]\right)\right)\right) \\ &= \dim\left(Z\left(\mathrm{Mat}_{k_1}(\mathbb{C})\right)\right) + \cdots + \dim\left(Z\left(\mathrm{Mat}_{k_n}(\mathbb{C})\right)\right) \\ &= 1 + \cdots + 1 \\ &= n, \end{aligned}$$

as desired. $\square$

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References

[S] J.-P. Serre, *Linear representations of finite groups*, translated from the second French edition by Leonard L. Scott, Springer-Verlag, New York, 1977.

Andrew Putman
Department of Mathematics
University of Notre Dame
255 Hurley Hall
Notre Dame, IN 46556
andyp@nd.edu