Hopf's theorem via geometry

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Abstract

We show that elementary ideas about bordism allow a simple and natural proof of Hopf's theorem in group homology.

Let G be a group. Recall that the homology groups of G are defined to be those of an Eilenberg–MacLane space for G. The following theorem of Hopf is perhaps the first nontrivial theorem about group homology. Write $G = F/R$, where F is a free group.

Theorem 0.1 (Hopf, [H])**.** $\mathrm{H}_2(G) \cong \frac{R \cap [F,F]}{[F,R]}$.

There are now many proofs of this theorem, perhaps the most efficient of which derives it from the five-term exact sequence in group homology associated to the short exact sequence

$$1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1,$$

which itself is probably most naturally derived from the Hochschild–Serre spectral sequence. See [B] for more details. The purpose of this note is to explain a proof of Theorem 0.1 using elementary ideas about bordism which is longer than these more abstract proofs, but that I think sheds light on its geometric meaning.

Constructing a homomorphism, I. We begin by constructing a homomorphism

$$\phi\colon R \cap [F,F] \longrightarrow \mathrm{H}_2(G).$$

Let BG be a fixed Eilenberg–MacLane space for G. For $w \in F$, let $\overline{w} \in G$ be the associated element of G. Consider $r \in R \cap [F,F]$. Since $r \in [F,F]$, we can write

$$r = [a_1, b_1] \cdots [a_g, b_g] \qquad (a_1, b_1, \ldots, a_g, b_g \in F). \tag{0.1}$$

The element r is a relation for G, so

$$[\overline{a}_1, \overline{b}_1] \cdots [\overline{a}_g, \overline{b}_g] = 1.$$

In other words, we have a surface relation inside G. Let Σ_g denote a closed oriented genus g surface and let $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ denote the usual generators for $\pi_1(\Sigma_g)$, so

$$[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = 1.$$

We thus have a homomorphism $\pi_1(\Sigma_g) \to G$ taking α_i and β_i to $\overline{a}_i$ and $\overline{b}_i$. Since Σ_g is itself an Eilenberg–MacLane space, this homomorphism is induced by a based map $f\colon \Sigma_g \to BG$ that is unique up to based homotopy. The surface Σ_g has a fundamental class $[\Sigma_g] \in \mathrm{H}_2(\Sigma_g) \cong \mathbb{Z}$, and we define $\phi(r) = f([\Sigma_g]) \in \mathrm{H}_2(G)$. Of course, this appears to depend on the choice of expression (0.1) for r. However, we have the following claim:

Claim 1. *$\phi(r)$ does not depend on the choice of expression (0.1) and the map $\phi\colon F \cap [R,R] \to \mathrm{H}_2(G)$ is a homomorphism.*

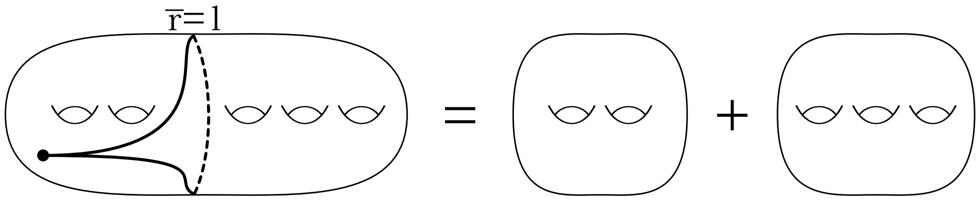

Figure 1: *There is g genus to the right of the curve and g' genus to the left. The indicated curve maps to the element $\overline{r} = 1$ of $\pi_1(BG) = G$, so it extends to a map of a disc. We can thus extend it over a disc and separate the two parts into maps from Σ_g and $\Sigma_{g'}$ to BG, showing that the map from the left hand side surface $\Sigma_{g+g'}$ takes the fundamental class to the sum of the images of the fundamental classes of the Σ_g and $\Sigma_{g'}$ on the right hand side.*

Proof of claim. For the moment, just regard ϕ as a function taking an expression like (0.1) to an element of $\mathrm{H}_2(G)$ and write

$$\phi(r = [a_1, b_1] \cdots [a_g, b_g]) \in \mathrm{H}_2(G).$$

If

$$r = [a_1, b_1] \cdots [a_g, b_g] \quad \text{and} \quad r' = [a'_1, b'_1] \cdots [a'_{g'}, b'_{g'}]$$

are expressions for elements $r, r' \in R \cap [F, F]$, then as shown in Figure 1 we have

$$\begin{aligned} &\phi(rr' = [a_1, b_1] \cdots [a_g, b_g][a'_1, b'_1] \cdots [a'_{g'}, b'_{g'}]) \\ &\qquad = \phi(r = [a_1, b_1] \cdots [a_g, b_g]) + \phi(r' = [a'_1, b'_1] \cdots [a'_{g'}, b'_{g'}]). \end{aligned}$$

The fact that ϕ is a homomorphism will thus follow once we know that ϕ is well-defined. Now consider two different expressions

$$r = [a_1, b_1] \cdots [a_g, b_g] \quad \text{and} \quad r = [a'_1, b'_1] \cdots [a'_{g'}, b'_{g'}]$$

for the same element $r \in R \cap [F, F]$. We thus have an identity

$$1 = [a_1, b_1] \cdots [a_g, b_g][b'_{g'}, a'_{g'}] \cdots [b'_1, a'_1] \tag{0.2}$$

in the free group F. The map $\Sigma_{g+g'} \to BG$ associated to (0.2) factors through BF, and since $\mathrm{H}_2(BF) = 0$ we deduce that

$$\phi(1 = [a_1, b_1] \cdots [a_g, b_g][b'_{g'}, a'_{g'}] \cdots [b'_1, a'_1]) = 0 \in \mathrm{H}_2(BG).$$

Since this expression also equals

$$\begin{aligned} &\phi(r = [a_1, b_1] \cdots [a_g, b_g]) + \phi(r^{-1} = [b'_{g'}, a'_{g'}] \cdots [b'_1, a'_1]) \\ &\qquad = \phi(r = [a_1, b_1] \cdots [a_g, b_g]) - \phi(r = [a'_1, b'_1] \cdots [b'_{g'}, a'_{g'}]), \end{aligned}$$

we conclude that

$$\phi(r = [a_1, b_1] \cdots [a_g, b_g]) = \phi(r = [a'_1, b'_1] \cdots [b'_{g'}, a'_{g'}]),$$

as desired. $\square$

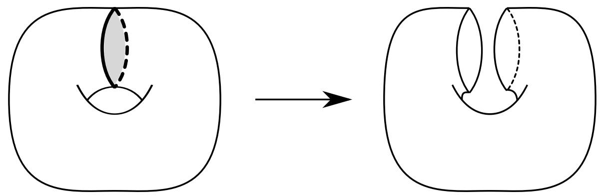

Figure 2: *The curve α_1 is drawn on the right. The $f\colon \Sigma_1 \to BG$ extends over a disc as shown, so we can fill it in and then separate it to form a 2-sphere mapping into BG.*

Constructing the homomorphism, II. Define

$$\mathcal{H}(F,R) = \frac{R \cap [F,F]}{[F,R]}.$$

Our next goal is to prove the following:

Claim 2. *$\phi\colon R \cap [F,F] \to \mathrm{H}_2(G)$ factors through a homomorphism $\psi\colon \mathcal{H}(F,R) \to \mathrm{H}_2(G)$.*

Proof of claim. Consider $r \in R$ and $w \in F$, so $[r,w]$ is a generator for $[F,R]$. We must show that $\phi(r) = 0$. The map $f\colon \Sigma_1 \to BG$ associated to r takes $\alpha_1, \beta_1 \in \pi_1(\Sigma_1)$ to $\overline{r} = 1 \in G$ and $\overline{w} \in G$. As is shown in Figure 2, we can extend f over a disc bounding α_1 and get a map from a 2-sphere to BG, which is nullhomotopic since BG is aspherical. This implies that f extends over a solid torus, and thus that $\phi(r) = f([\Sigma_1]) = 0$. □

Notation 0.2. For $r \in R \cap [F,F]$, we will write $[r]$ for the associated element of $\mathcal{H}(F,R)$. The set $\mathcal{H}(F,R)$ is an abelian group since the relations $[F,R]$ include $[R,R]$, which forces all elements of $R \cap [F,F]$ to commute with one another.

Maps of surfaces I: fixed genus. The rest of this note will be devoted to a proof that $\psi\colon \mathcal{H}(F,R) \to \mathrm{H}_2(G)$ is an isomorphism. Define

$$\mathrm{Surf}_g(G) = \{f \mid f\colon \Sigma_g \to BG \text{ homotopy class}\}.$$

We then have the following.

Claim 3. *For all $g \geq 0$, there exists a set map $\zeta_g\colon \mathrm{Surf}_g(G) \to \mathcal{H}(F,R)$ such that the composition*

$$\mathrm{Surf}_g(G) \xrightarrow{\zeta_g} \mathcal{H}(F,R) \xrightarrow{\psi} \mathrm{H}_2(G)$$

takes $f\colon \Sigma_g \to BG$ to $f([\Sigma_g])$.

Proof of claim. For $g = 0$, we define $\zeta_g(f) = 0$. Assume now that $g \geq 1$. Consider an element $f\colon \Sigma_g \to BG$ of $\mathrm{Surf}_g(G)$. Homotoping f, we can assume that it is a based map. Letting $\overline{a}_1, \overline{b}_1, \ldots, \overline{a}_g, \overline{b}_g \in G$ be the images under f of the usual generators for $\pi_1(\Sigma_g)$, we have

$$[\overline{a}_1, \overline{b}_1] \cdots [\overline{a}_g, \overline{b}_g] = 1.$$

Pick lifts $a_1, b_1, \ldots, a_g, b_g \in F$ of $\overline{a}_1, \overline{b}_1, \ldots, \overline{a}_g, \overline{b}_g \in G$ and set $r = [a_1, b_1] \cdots [a_g, b_g]$. We then have $r \in R \cap [F,F]$ and $\psi([r]) = h$. Define $\zeta_g(f) = [r]$.

Of course, this definition depends on several choices, but once we have shown it is independent of those choices it will clearly define a map as in the claim. Those choices are as follows:

1. The choice of lifts $a_1, b_1, \ldots, a_g, b_g \in F$ of $\overline{a}_1, \overline{b}_1, \ldots, \overline{a}_g, \overline{b}_g \in G$. Any other such lift will be of the form $a_1 s_1, b_1 t_1, \ldots, a_g s_g, b_g t_g$ for some $s_1, t_1, \ldots, s_g, t_g \in R$. Set $r' = [a_1 s_1, b_1 t_1] \cdots [a_g s_g, b_g t_g]$. Write $\equiv$ to denote equality modulo $[F, R]$. For each i, we have

$$[a_i s_i, b_i t_i] = a_i s_i b_i t_i s_i^{-1} a_i^{-1} t_i^{-1} b_i^{-1} \equiv a_i b_i a_i^{-1} b_i^{-1} s_i t_i s_i^{-1} t_i^{-1} \equiv [a_i, b_i].$$

This implies that r and r' are equal modulo $[F, R]$, so $[r] = [r']$, as desired.
2. The choice of a based map homotopic to f. A different choice will conjugate the elements $\overline{a}_1, \overline{b}_1, \ldots, \overline{a}_g, \overline{b}_g \in G$ by an element of G. The lifts of these elements to F can then be chosen to be conjugate by an element of F. Modulo $[F, R]$, the resulting r will be the same. □

Maps of surfaces II: general. Now define

$$\mathrm{Surf}(G) = \{f \mid f\colon S \to BG \text{ homotopy class with } S \text{ a compact oriented surface}\}.$$

The surfaces S here are not required to be connected. The disjoint union of surfaces makes $\mathrm{Surf}(G)$ into a commutative monoid. Our next goal is to prove the following:

Claim 4. *There exists a surjective map of commutative monoids $\zeta\colon \mathrm{Surf}(G) \to \mathcal{H}(F, R)$ such that the composition*

$$\mathrm{Surf}(G) \xrightarrow{\zeta} \mathcal{H}(F, R) \xrightarrow{\psi} \mathrm{H}_2(G)$$

takes $f\colon S \to BG$ to $f([S])$.

Proof of claim. Using the monoid structure on $\mathrm{Surf}(G)$, it is enough to define ζ on elements $f\colon S \to BG$ with S connected. Choose an orientation-preserving diffeomorphism $\Sigma_g \cong S$ and let $\widehat{f}\colon \Sigma_g \to BG$ be the composition of this diffeomorphism with f. We then define $\zeta(f) = \zeta_g(\widehat{f})$. Of course, this depends on the choice of diffeomorphism $\Sigma_g \cong S$, so we must prove that it is independent of this choice; once this has been done, the surjectivity of ζ will be clear. To do this, it is enough to prove that $\zeta_g(\widehat{f}) = \zeta_g(\widehat{f} \circ \rho)$ for an arbitrary orientation-preserving diffeomorphism $\rho\colon \Sigma_g \to \Sigma_g$.

What we have to prove is trivial for $g = 0$, so assume that $g \geq 1$. Since Σ_g is aspherical, we can take Σ_g as our model for $B\pi_1(\Sigma_g)$. The map $\widehat{f}$ then induces a set map $\mathrm{Surf}_g(\pi_1(\Sigma_g)) \to \mathrm{Surf}_g(G)$ taking the identity to $\widehat{f}$. Write $\pi_1(\Sigma_g) = F(\Sigma_g)/R(\Sigma_g)$, where $F(\Sigma_g)$ is the free group on $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ and $R(\Sigma_g)$ is the normal closure of the surface relation $r = [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]$. We then have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Surf}_g(\pi_1(\Sigma_g)) & \xrightarrow{\zeta_g} & \mathcal{H}(F(\Sigma_g), R(\Sigma_g)) & \xrightarrow{\psi} & \mathrm{H}_2(\pi_1(\Sigma_g)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Surf}_g(G) & \xrightarrow{\zeta_g} & \mathcal{H}(F, R) & \xrightarrow{\psi} & \mathrm{H}_2(G)
\end{array}$$

the composition of whose first row takes both elements $\mathrm{id}, \rho \in \mathrm{Surf}_g(\pi_1(\Sigma_g))$ to the generator of $\mathrm{H}_2(\pi_1(\Sigma_g)) \cong \mathbb{Z}$. To prove that $\zeta_g(\widehat{f}) = \zeta_g(\widehat{f} \circ \rho)$, it is thus enough to prove that the map

$$\psi\colon \mathcal{H}(F(\Sigma_g), R(\Sigma_g)) \to \mathrm{H}_2(\pi_1(\Sigma_g)) \cong \mathbb{Z}$$

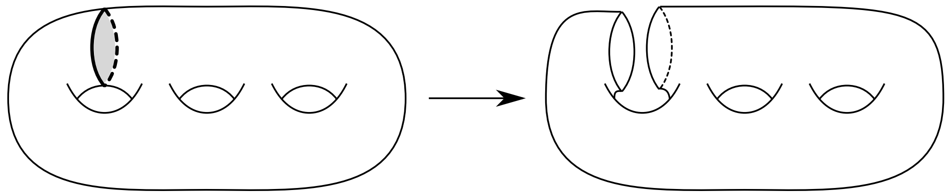

Figure 3: *An elementary bordism*

is an isomorphism (an important special case of Hopf's theorem!).

We now prove this. Since ψ is surjective (we have not yet proved this in general, but we already observed it for surface groups in the previous paragraph!), it is enough to prove that $\mathcal{H}(F(\Sigma_g), R(\Sigma_g))$ is cyclic. Since $R(\Sigma_g) \subset [F(\Sigma_g), F(\Sigma_g)]$, the definition of $\mathcal{H}(F(\Sigma_g), R(\Sigma_g))$ reduces to

$$\mathcal{H}(F(\Sigma_g), R(\Sigma_g)) = \frac{R(\Sigma_g)}{[F(\Sigma_g), R(\Sigma_g)]}.$$

The group $R(\Sigma_g)$ is the normal closure of r, and the relations $[F(\Sigma_g), R(\Sigma_g)]$ force all the $F(\Sigma_g)$-conjugates of r to represent the same element of $\mathcal{H}(F(\Sigma_g), R(\Sigma_g))$. It follows that $\mathcal{H}(F(\Sigma_g), R(\Sigma_g))$ is the cyclic group generated by $[r]$, as desired. □

The kernel bounds. We now characterize elements of the kernel of $\psi\colon \mathcal{H}(F,R) \to \mathrm{H}_2(G)$.

Claim 5. *Let $f\colon S \to BG$ be an element of* $\mathrm{Surf}(G)$*. Assume that $\psi(\zeta(f)) = 0$. Then there exists a compact oriented 3-manifold M^3 with $\partial M^3 = S$ and an extension $F\colon M^3 \to BG$ of f.*

Proof of claim. Since $\psi(\zeta(f)) = f([S]) = 0$, the 2-cycle $f([S])$ is the boundary of a singular 3-chain. Assembling the various singular 3-simplices together, we obtain a 3-dimensional simplicial complex T mapping into BG. If T were an oriented 3-manifold, then it would be the desired M^3. Unfortunately, T is not necessarily a manifold. From its construction, it is clear that T is a 3-manifold in a neighborhood of each point except the vertices. Thickening up the "boundary" S of T, we can assume that the only problematic points are the interior vertices v of T. The neighborhood of such a vertex is homeomorphic to a cone $C(S)$ on a closed oriented connected surface S. If S is not a sphere, then T is not a manifold at v. To fix this, let $H(S)$ be the handlebody whose boundary is S. There is a continuous map $H(S) \to C(S)$ that is a homeomorphism away from the core of $H(S)$ and takes the core of $H(S)$ to the cone point. We can now resolve the singularity v by removing the cone neighborhood and gluing in $H(S)$. Let M^3 be the result of doing this to all the interior vertices of T. The space $\widehat{T}$ is an oriented 3-manifold, and the maps $H(S) \to C(S)$ piece together to give a map $M^3 \to T$. The composition $M^3 \to T \to BG$ is the desired extension of f. □

Endgame. We finally prove that $\psi\colon \mathcal{H}(F,R) \to \mathrm{H}_2(G)$ is an isomorphism.

Claim 6. *The map $\psi\colon \mathcal{H}(F,R) \to \mathrm{H}_2(G)$ is surjective.*

Proof of claim. Consider $h \in \mathrm{H}_2(G)$. Assembling the singular 2-simplices making up a a 2-chain representing h, we obtain a compact oriented surface S and a map $f\colon S \to BG$. We then have $\psi(\zeta(f)) = h$. □

Claim 7. *The map* $\psi\colon \mathcal{H}(F,R) \to \mathrm{H}_2(G)$ *is injective.*

Proof of claim. Consider $[r] \in \ker(\psi)$. Write $[r] = \zeta(f)$ for some $f\colon S \to BG$. By Claim 5, there exists a compact oriented 3-manifold M^3 with $\partial M^3 = S$ and an extension $F\colon M^3 \to BG$ of f. Choosing an appropriate Morse function on M^3, we see that we can convert f into a map $\emptyset \to BG$ via a sequence of the following moves and their inverses:

- Deleting an S^2-component from S.
- Letting γ be a simple closed curves on S such that $f|_\gamma$ extends over a disc, cut S along γ, glue discs to the two resulting boundary components (see Figure 3), and map the resulting surface to BG in evident way.

Neither of these moves changes the homology class represented by f. The first clearly does not change $[r] = \zeta(f)$. As for the second, it clearly does not change $[r] = \zeta(f)$ if γ is a separating curve, and using the relations $[F,R]$ in $\mathcal{H}(F,R)$ like we did in Claim 2 (cf. Figure 2), we see that it does not change it for nonseparating curves either. We conclude that $[r] = \zeta(\emptyset \to BG) = 0$, as desired. □

References

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