Homotopy groups of spheres and low-dimensional topology

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Abstract

We give a modern account of Pontryagin's approach to calculating $\pi_{n+1}(S^n)$ and $\pi_{n+2}(S^n)$ using techniques from low-dimensional topology.

Contents

1 Introduction

When introducing the homotopy groups of spheres, one standard approach deduces many initial calculations using techniques from smooth manifolds and geometric topology.

- The fact that the universal cover of S^1 is $\mathbb{R}$ implies immediately that

$$\pi_{k+1}(S^1) = \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ 0 & \text{if } k > 0. \end{cases}$$

- If $1 \le m < n$, then an arbitrary continuous map $f : S^m \to S^n$ can be homotoped to be smooth and transverse to a point $p \in S^n$. The fact that $m < n$ implies that $f^{-1}(p) = \varnothing$. We deduce that the image of f lies in the contractible set $S^n \setminus \{p\}$, so f is nullhomotopic. The upshot is that $\pi_m(S^n) = 0$ for $1 \le m < n$.

- Homotopy classes of maps $f : S^n \to S^n$ are classified by their *degree*. Choosing a smooth representative of f, this is the signed count of $f^{-1}(p)$ for a regular value $p \in S^n$. This leads to the calculation $\pi_n(S^n) \cong \mathbb{Z}$ (see §2 below).

Beyond this, the flavor of the subject changes dramatically. In his thesis, Serre introduced powerful algebraic tools that have come to dominate homotopy theory. They quickly lead to remarkable results and calculations, but the above primitive geometry recedes into the distance.

However, low-dimensional topology has more to say about homotopy theory than the above results. The beautiful work of Pontryagin in the 1950's contained geometric derivations of

$$\pi_{n+1}(S^n) = \begin{cases} \mathbb{Z} & \text{if } n = 2, \\ \mathbb{Z}/2 & \text{if } n \geq 3 \end{cases}$$

and

$$\pi_{n+2}(S^n) = \mathbb{Z}/2 \qquad (n \geq 2).$$

This was continued by Rochlin, who managed to prove that $\pi_{n+3}(S^n) \cong \mathbb{Z}/24$ for $n \gg 0$. The details of these proofs are not well-known today largely because it seems very difficult to push them any further than this.

In these notes, we give a detailed account of the aforementioned calculations of Pontryagin. The main source for them is his book [14]. This book is not easy to read today for two reasons. First, it is intended also as a textbook on smooth manifold theory, so it spends a huge amount of time developing well-known foundational results from a somewhat archaic point of view. Second, it avoids using any algebraic topology at all, preferring to develop things from scratch. As an extreme example of this, in [14, §15B] a complicated equivalence relation on immersed submanifolds of a surface is given and the equivalence classes are called the *connectivity group* of a surface, which Pontryagin denotes Δ^1. In fact, Δ^1 is actually the first homology group of the surface with $\mathbb{Z}/2$-coefficients, but it is hard to prove this from Pontryagin's definitions (which do not generalize to spaces other than surfaces). We do not discuss Rochlin's later work; see [4] for a modern discussion of it.

The outline of these notes is as follows. As a warm-up, in §2 we discuss the degree of a map to an n-sphere. In §3 we introduce the Pontryagin–Thom construction. These first two sections cover material that is discussed in many textbooks, so we only sketch some of the proofs. In the later sections, our proofs are more complete. In §4, we use the Pontryagin–Thom construction to prove the Freudenthal suspension theorem. In §5, we calculate $\pi_{n+1}(S^n)$. In §6, we introduce linking numbers. In §7, we discuss the Hopf invariant of a map $S^{2n-1} \to S^n$. In §8, we introduce framed surgery (but to simplify things we only discuss how to attach 1-handles). In §9, we show how to use the Hopf invariant to sharpen the Freudenthal suspension theorem. Finally, in §10 we calculate $\pi_{n+2}(S^n)$. Throughout, we follow the broad outlines of Pontryagin's book, but our treatment of the details is often quite different.

2 The degree of a map and $\pi_n(S^n)$

As a warm-up, this section will discuss the degree of a self-map of S^n, which was introduced by Brouwer [1]. We only sketch the proofs; see [12, §5] for a complete account.

Consider $x \in \pi_n(S^n)$. Fix some $p \in S^n$, and realize x by a smooth map $f : S^n \to S^n$ that is transverse to p (such maps clearly exist). Write $f^{-1}(p) = \{q_1, \ldots, q_k\}$. For $1 \leq i \leq k$, the map on tangent spaces $f_* : (T_{S^n})_{q_i} \to (T_{S^n})_p$ is an isomorphism; define $\epsilon_i = 1$ if it preserves the usual orientation and $\epsilon_i = -1$ if it reverses it. We will call ϵ_i the *sign* of q_i. The *degree* of x is

$$\text{degree}(x) := \sum_{i=1}^{k} \epsilon_i \in \mathbb{Z}.$$

Lemma 2.1. *For $x \in \pi_n(S^n)$, the number degree(x) does not depend on the choice of f or p.*

Proof. We first show that it does not depend on f. Let $f' : S^n \to S^n$ be another smooth realization of x that is transverse to p. Let $F : S^n \times [0,1] \to S^n$ be a homotopy from f to f'. Homotoping F slightly (without changing $F(\cdot, 0)$ or $F(\cdot, 1)$), we can assume that F is transverse to p. The inverse image $F^{-1}(p)$ is then an oriented 1-submanifold of $S^n \times [0,1]$. Each component of $F^{-1}(p)$ is either a circle or an arc connecting two points on $S^n \times \{0,1\}$. Letting α be such an arc, there are three possibilities.

- The endpoints of α are on $S^n \times 0$. Then α connects two points of $f^{-1}(p)$ whose signs are opposite.
- The endpoints of α are on $S^n \times 1$. Then α connects two points of $(f')^{-1}(p)$ whose signs are opposite.
- One endpoint of α is on $S^n \times 0$ and the other is on $S^n \times 1$. Then α connects a point of $f^{-1}(p)$ to a point of $(f')^{-1}(p)$, and the signs of these points are the same.

The above pairwise matching up of points of $f^{-1}(p) \cup (f')^{-1}(p)$ implies that the sum of the signs of $f^{-1}(p)$ equals the sum of the signs of $(f')^{-1}(p)$, as desired.

We now show that degree(x) does not depend on the choice of p. Consider some other point $q \in S^n$, and let $\tau \in \mathrm{SO}_{n+1}(\mathbb{R})$ take p to q. The group $\mathrm{SO}_{n+1}(\mathbb{R})$ is connected, so the map $\tau \circ f : S^n \to S^n$ is homotopic to f (and in particular realizes x). Also, $\tau \circ f$ is transverse to q and satisfies $(\tau \circ f)^{-1}(q) = f^{-1}(p)$. Finally, the signs of the points of $(\tau \circ f)^{-1}(q)$ and the points of $f^{-1}(p)$ are the same. The desired independence follows. $\square$

The map

$$\text{degree} : \pi_n(S^n) \to \mathbb{Z}$$

is easily seen to be a homomorphism. We now prove that it is an isomorphism. This was originally proved by Hopf [7] following earlier work of Brouwer.

Theorem 2.2. *The map degree : $\pi_n(S^n) \to \mathbb{Z}$ is an isomorphism for $n \geq 1$.*

Proof. This is trivial for $n = 1$, so we will assume that $n \geq 2$. It is clear that degree(id) $= 1$, so degree is surjective. To see that it is injective, consider a smooth map $f : S^n \to S^n$ that is transverse to a point $p \in S^n$ and has degree 0. Let $f^{-1}(p) = \{q_1, \ldots, q_k\}$. Since the degree of f is 0, there exist $1 \leq i < j \leq k$ such that the signs of q_i and q_j are opposite. It is easy to see that we can move q_i and q_j until they collide and "cancel" (see [12] for details; this is where we use the fact that $n \geq 2$). Doing this repeatedly, we can assume that $f^{-1}(p) = \varnothing$, so the image of f lies in the contractible set $S^n \setminus \{p\}$. This implies that f is nullhomotopic, as desired. $\square$

Remark. Everything that we discussed above also holds word-for-word for homotopy classes of maps from a compact orientable M^n to S^n.

3 The Pontryagin–Thom construction

In this section, we will describe our main tool for understanding the homotopy groups of spheres. Fix some $n \geq 1$ and $k \geq 0$, and let M^k be a k-dimensional submanifold of $\mathbb{R}^{n+k}$. The restriction to M^k of the tangent bundle of $\mathbb{R}^{n+k}$ is a trivial vector bundle $M^k \times \mathbb{R}^{n+k}$. The tangent bundle T_{M^k} is a subbundle of $M^k \times \mathbb{R}^{n+k}$. Also, the usual Euclidean inner product on the tangent bundle of $\mathbb{R}^{n+k}$ induces a bilinear form on each fiber of $M^k \times \mathbb{R}^{n+k}$. The *normal bundle* of M^k, denoted N_{R^{n+k}/M^k}, is the fiberwise orthogonal complement in $M^k \times \mathbb{R}^{n+k}$ of T_{M^k}. The bundle N_{R^{n+k}/M^k} is an n-dimensional vector bundle. A *framing* on M^k is a vector space isomorphism

$$\mathfrak{f} : M^k \times \mathbb{R}^n \longrightarrow N_{R^{n+k}/M^k}.$$

Of course, this only exists if N_{R^{n+k}/M^k} is a trivial bundle.

A *framed k-manifold* in $\mathbb{R}^{n+k}$ is a closed k-dimensional submanifold M^k of $\mathbb{R}^{n+k}$ together with a fixed framing on M^k. A *framed cobordism* between framed k-manifolds M^k and N^k is a $(k+1)$-dimensional submanifold C^{k+1} of $\mathbb{R}^{n+k} \times [0,1] \subset \mathbb{R}^{n+k+1}$ such that

$$\partial C^{k+1} = C^{k+1} \cap (\mathbb{R}^{n+k} \times \{0,1\}) = (M^k \times 0) \cup (N^k \times 1)$$

together with a framing on C^{k+1} that restricts to the given framings on $M^k \times 0$ and $N^k \times 1$. If such a framed cobordism exists, we will say that M^k and N^k are *framed cobordant*. This defines an equivalence relation on the set of framed k-manifolds in $\mathbb{R}^{n+k}$; let $\Omega_k^{\text{fr}}(\mathbb{R}^{n+k})$ denote the set of equivalence classes. The set $\Omega_k^{\text{fr}}(\mathbb{R}^{n+k})$ is an abelian group under the operation of disjoint union.

The main result concerning $\Omega_k^{\text{fr}}(\mathbb{R}^{n+k})$ is the following theorem of Pontryagin [14].

Theorem 3.1. *For all $n \geq 1$ and $k \geq 0$, we have $\Omega_k^{fr}(\mathbb{R}^{n+k}) \cong \pi_{n+k}(S^n)$.*

Proof. Since the proof is cogently described in many textbooks (for instance, Milnor's book [12] and Hirsch's book [6]), we will only sketch it. The proof will be very similar to that of Lemma 2.1 discussed above, which shows that the degree is well-defined. It will have several steps.

Step 1. *We construct a homomorphism $\Phi : \pi_{n+k}(S^n) \to \Omega_k^{fr}(\mathbb{R}^{n+k})$.*

Write $S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\}$ and $S^n = \mathbb{R}^n \cup \{\infty\}$. Fix a point $p \in \mathbb{R}^n$ and an orientation-preserving isomorphism $(T_{\mathbb{R}^n})_p \cong \mathbb{R}^n$ on the level of tangent spaces. Consider some $x \in \pi_{n+k}(S^n)$. Realize x by a smooth map $f : S^{n+k} \to S^n$ such that $f(\infty) = \infty$ and such that f is transverse to p (such maps clearly exist). Then $f^{-1}(p)$ is a smooth k-manifold in $\mathbb{R}^{n+k} \subset S^{n+k}$. Moreover, our fixed isomorphism $(T_{\mathbb{R}^n})_p \cong \mathbb{R}^n$ lifts to a framing on $f^{-1}(p)$. We define $\Psi(x)$ to be the element of $\Omega_k^{\text{fr}}(\mathbb{R}^{n+k})$ corresponding to $f^{-1}(p)$. Once we have shown in the next two steps that this does not depend on the various choices we have made, it will be easy to see that Φ is a homomorphism.

Step 2. *We prove that Φ does not depend on the choice of f.*

Let $f' : S^{n+k} \to S^n$ be another smooth map such that $f'(\infty) = \infty$ and such that f' is transverse to p. We can then choose a homotopy $F : S^{n+k} \times [0,1] \to S^n$ from f to f' which is transverse to p and satisfies $F(\infty, t) = \infty$ for $t \in [0,1]$. The preimage $F^{-1}(p)$ is a framed cobordism from $f^{-1}(p)$ to $(f')^{-1}(p)$.

Step 3. *We prove that Φ does not depend on the choice of p or the orientation-preserving isomorphism $(T_{\mathbb{R}^n})_p \cong \mathbb{R}^n$.*

Consider another $q \in R^n$ together with an orientation-preserving isomorphism $(T_{\mathbb{R}^n})_q \cong \mathbb{R}^n$. We allow the possibility that $q = p$. There exists a compactly supported diffeomorphism $\tau : R^n \to R^n$ with the following properties.

- We have $\tau(p) = q$.
- The composition
$$\mathbb{R}^n \cong (T_{\mathbb{R}^n})_p \xrightarrow{\tau_*} (T_{\mathbb{R}^n})_q \cong \mathbb{R}^n$$
is the identity.
- The map τ is isotopic to the identity through an isotopy whose support is compact.

Such a diffeomorphism is easy to construct. Letting $\widehat{\tau} : S^n \to S^n$ be the extension to the one-point compactification, the map $\widehat{\tau}$ is smooth and isotopic to the identity through an isotopy that fixes ∞. It follows that $\widehat{\tau} \circ f : S^{n+k} \to S^n$ realizes $x \in \pi_{n+k}(S^n)$. The key observation is that $\widehat{\tau} \circ f$ is transverse to q and satisfies $(\widehat{\tau} \circ f)^{-1}(q) = f^{-1}(p)$, where the indicated preimages are given the framings coming from the given isomorphisms $(T_{\mathbb{R}^n})_q \cong \mathbb{R}^n$ and $(T_{\mathbb{R}^n})_p \cong \mathbb{R}^n$. The desired independence follows.

Convention. Since we have shown that p can be arbitrary, from this point onwards we will take $p = 0 \in \mathbb{R}^{n+k} \subset S^{n+k}$.

Step 4. *We prove that Φ is surjective.*

Consider a framed k-manifold M^k in $\mathbb{R}^{n+k}$. Let $U \subset \mathbb{R}^{n+k}$ be a tubular neighborhood of M^k. The framing induces a diffeomorphism $U \cong M^k \times \mathbb{R}^n$; let $g : U \to \mathbb{R}^n$ be the composition of this diffeomorphism with the projection $M^k \times \mathbb{R}^n \to \mathbb{R}^n$. Regarding $S^n = \mathbb{R}^n \cup \{\infty\}$ and $S^{n+k} = \mathbb{R}^{n+k} \cup \{\infty\}$, we can then define a map $f : S^{n+k} \to S^n$ via the formula

$$f(x) = \begin{cases} g(x) & \text{if } x \in U, \\ \infty & \text{otherwise.} \end{cases}$$

It is then clear that Φ takes the element of $\pi_{n+k}(S^n)$ associated to f to M^k. The above construction of f is called the *Pontryagin–Thom construction*.

Step 5. *We prove that Φ is injective.*

Consider $x \in \pi_{n+k}(S^n)$ such that $\Phi(x) = 0$. As in the definition of Φ, let $f : S^{n+k} \to S^n$ be a smooth map realizing x such that $f(\infty) = \infty$ and such that f is transverse to p. Set $M^k = f^{-1}(p)$. It is not hard to show that we can homotope f to the map obtained by applying the Pontryagin–Thom construction to M^k. By assumption, there is a framed cobordism $C^{k+1} \subset \mathbb{R}^{n+k} \times [0,1]$ from M^k to the empty manifold. Applying the Pontryagin–Thom construction to C^{k+1} yields a homotopy from f to the constant map. □

Notation 3.2. If M^k is a framed k-manifold in $\mathbb{R}^{n+k}$, then we will denote by $[\![M^k]\!]$ the associated element of $\pi_{n+k}(S^n)$.

4 The Freudenthal suspension theorem

At this point in these notes, we will begin covering material that is hard to find in textbooks, so our proofs will become more complete. As a first illustration of the power of Theorem 3.1, we will use it to prove the Freudenthal suspension theorem. This theorem will only use the simplest type of framed cobordisms, namely those resulting from ambient isotopies, which we now define. Fix a framed k-manifold M^k in $\mathbb{R}^{n+k}$. If $\phi : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ is a diffeomorphism, then we can define a framed k-manifold $\phi(M^k)$ in $\mathbb{R}^{n+k}$ in the obvious way. An ambient isotopy of M^k is a smooth function $\Phi : \mathbb{R}^{n+k} \times [0,1] \to \mathbb{R}^{n+k}$ with the following two properties.

- For all $t \in [0,1]$, the map $\Phi(\cdot, t)$ is a diffeomorphism, which we will denote Φ_t.
- $\Phi_0 = \mathrm{id}$.

The framed k-manifolds M^k and $\Phi_1(M^k)$ are then framed cobordant via the cobordism

$$C^{k+1} = \{(\Phi_t(x), t) \mid x \in M^k,\ t \in [0,1]\} \subset \mathbb{R}^{n+k} \times [0,1],$$

which can be endowed with a framing in the obvious way. In a similar way, if N^k is obtained from M^k by homotoping the framing, then N^k is framed cobordant to M^k.

We now define the suspension homomorphism $E : \pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$. Consider a framed k-manifold M^k in $\mathbb{R}^{n+k}$ with framing $\mathfrak{f} : M^k \times \mathbb{R}^n \to N_{\mathbb{R}^{n+k}/M^k}$. Define $E(M^k)$ to be the following framed k-manifold in $\mathbb{R}^{n+k+1}$.

- The manifold underlying $E(M^k)$ is the image of M^k under the standard inclusion $\mathbb{R}^{n+k} \hookrightarrow \mathbb{R}^{n+k+1}$.
- The framing $E(\mathfrak{f}) : E(M^k) \times \mathbb{R}^{n+1} \to N_{\mathbb{R}^{n+k+1}/E(M^k)}$ is as follows. Let

  $$\{\vec{a}_1, \ldots, \vec{a}_{n+1}\} \subset \mathbb{R}^{n+1} \quad \text{and} \quad \{\vec{b}_1, \ldots, \vec{b}_{n+k+1}\} \subset \mathbb{R}^{n+k+1}$$

  be the standard bases. Then $E(\mathfrak{f})$ is defined via the formulas

  $$E(\mathfrak{f})(x, \vec{a}_i) = \mathfrak{f}(x, \vec{a}_i) \qquad (x \in E(M^k), 1 \le i \le n)$$

  and

  $$E(\mathfrak{f})(x, \vec{a}_{n+1}) = \vec{b}_{n+k+1} \qquad (x \in E(M^k)),$$

  where we are identifying $E(M^k)$ with M^k, the vector space $\mathbb{R}^n$ with its image in $\mathbb{R}^{n+1}$, and the vector space $\mathbb{R}^{n+k}$ with its image in $\mathbb{R}^{n+k+1}$.

This respects the relation of framed cobordism, and thus induces a map $E : \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k}) \to \Omega_k^{\mathrm{fr}}(\mathbb{R}^{n+k+1})$. By Theorem 3.1, this is the same as a map $E : \pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$.

Remark. Recall that we have a based suspension functor Σ that satisfies $\Sigma S^p \cong S^{p+1}$. It is an easy exercise to see that the above map $E : \pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ is the same as the map that takes a map $S^{n+k} \to S^n$ to the suspended map

$$S^{n+k+1} \cong \Sigma S^{n+k} \longrightarrow \Sigma S^n \cong S^{n+1}.$$

The letter E stands for the German word for suspension, namely Einhängung.

The following theorem is due to Freudenthal [3].

Theorem 4.1 (Freudenthal suspension theorem)**.** *The suspension homomorphism*

$$E : \pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$$

is surjective for $n \geq k+1$ and an isomorphism for $n \geq k+2$.

Proof. We will prove that E is surjective for $n \geq k+1$; to prove that it is an isomorphism for $n \geq k+2$, simply run the proof below for a framed cobordism between two framed k-manifold in $\mathbb{R}^{n+k+1}$ (this requires $n \geq k+2$ since the framed cobordism is $(k+1)$-dimensional). An element of $\pi_{n+k+1}(S^{n+1})$ can be written in the form $[\![M^k]\!]$ for some framed k-manifold in $\mathbb{R}^{n+k+1}$. We will prove that M^k can be isotoped such that it is of the form $E(N^k)$ for some framed k-manifold in $\mathbb{R}^{n+k}$. This will be accomplished in two steps.

Step 1. *We isotope M^k such that it lies in $\mathbb{R}^{n+k} \subset \mathbb{R}^{n+k+1}$.*

The usual proof of the Whitney embedding theorem (see, e.g., [6, Theorem 1.3.5]) shows that for a generic nonzero vector $\vec{\zeta}$ in $\mathbb{R}^{n+k+1}$, the projection of M^k onto the orthogonal complement of $\vec{\zeta}$ is a smooth embedding. In particular, we can find such a $\vec{\zeta} = (\zeta_1, \ldots, \zeta_{n+k+1})$ with $\zeta_{n+k+1} > 0$. Let

$$X = \{x \in \mathbb{R}^{n+k+1} \mid \text{there exists } s \in \mathbb{R} \text{ such that } x + s\vec{\zeta} \in M^k\},$$

and define $\phi : X \to \mathbb{R}$ by letting $\phi(x)$ for $x \in X$ be the $(n+k+1)^{\text{st}}$ coordinate of the unique point of M^k of the form $x + s\vec{\zeta}$ for some $s \in \mathbb{R}$. The set X is closed and ϕ is smooth, so we can extend ϕ to a smooth function $\widetilde{\phi} : \mathbb{R}^{n+k+1} \to \mathbb{R}$. For $0 \leq t \leq 1$, define $\Psi_t : \mathbb{R}^{n+k+1} \to \mathbb{R}^{n+k+1}$ via the formula

$$\Psi_t(x) = x - \frac{t\phi(x)}{\zeta_{n+k+1}}\vec{\zeta} \qquad (x \in \mathbb{R}^{n+k+1}).$$

Observe that each Ψ_t is a diffeomorphism of $\mathbb{R}^{n+k+1}$, that $\Psi_0 = \text{id}$, and that Ψ_1 takes M^k to a submanifold of $\mathbb{R}^{n+k}$, as desired.

Step 2. *We isotope the framing on M^k such that $M^k = E(N^k)$ for some framed k-manifold in $\mathbb{R}^{n+k}$.*

Let $\mathfrak{f} : M^k \times \mathbb{R}^{n+1} \to N_{\mathbb{R}^{n+k+1}/M^k}$ be the framing. We will regard each fiber of $N_{\mathbb{R}^{n+k+1}/M^k}$ as a subspace of $\mathbb{R}^{n+k+1}$. Letting

$$\{\vec{a}_1, \ldots, \vec{a}_{n+1}\} \subset \mathbb{R}^{n+1} \quad \text{and} \quad \{\vec{b}_1, \ldots, \vec{b}_{n+k+1}\} \subset \mathbb{R}^{n+k+1}$$

be the standard bases, our goal is to construct a family $\mathfrak{f}_t : M^k \times \mathbb{R}^{n+1} \to N_{\mathbb{R}^{n+k+1}/M^k}$ of framings of M^k with the following properties.

- $\mathfrak{f}_0 = \mathfrak{f}$.
- $\mathfrak{f}_1(x, \vec{a}_{n+1}) = \vec{b}_{n+k+1}$ for all $x \in M^k$.
- $\mathfrak{f}_1(x, \vec{a}_i) \subset \mathbb{R}^{n+k} \subset \mathbb{R}^{n+k+1}$ for all $x \in M^k$ and $1 \le i \le n$.

By Step 1, each fiber of $N_{\mathbb{R}^{n+k+1}/M^k}$ contains $\vec{b}_{n+k+1}$. Define $\phi : M^k \to (\mathbb{R}^{n+1} \smallsetminus \{0\})$ via the formula

$$\mathfrak{f}(x, \phi(x)) = \vec{b}_{n+k+1} \qquad (x \in M^k).$$

Since $k \le n-1$ and $(\mathbb{R}^{n+1} \smallsetminus \{0\})$ is homotopy equivalent to S^n, the map ϕ is nullhomotopic. We can thus find a family $\phi_t : M^k \to (\mathbb{R}^{n+1} \smallsetminus \{0\})$ of maps such that $\phi_1 = \phi$ and $\phi_0(x) = \vec{a}_{n+1}$ for all $x \in M^k$.

Next, for $x \in M^k$ let μ_x be the inner product on $\mathbb{R}^{n+1}$ obtained by pulling back the standard inner product on $N_{\mathbb{R}^{n+k+1}/M^k} \subset \mathbb{R}^{n+k+1}$ via the map $\mathfrak{f}(x, \cdot)$. The space of inner products on $\mathbb{R}^{n+1}$ is contractible, so for $x \in M^k$ and $0 \le t \le 1$ we can find an inner product $\mu_{x,t}$ on $\mathbb{R}^{n+1}$ depending continuously on x and t such that $\mu_{x,1} = \mu_x$ and such that $\mu_{x,0}$ is the usual inner product on $\mathbb{R}^{n+1}$.

Define B to be the vector bundle on $M^k \times [0,1]$ whose fiber over $(x,t) \in M^k \times [0,1]$ is the subspace of $\mathbb{R}^{n+1}$ consisting of the $\mu_{x,t}$-orthogonal complement of $\phi_t(x)$. For $x \in M^k$ and $t = 0$, we have $\phi_t(x) = \vec{a}_{n+1}$ and $\mu_{x,t}$ is the standard inner product on $\mathbb{R}^{n+1}$, so the restriction of B to $M^k \times 0$ is the trivial vector bundle $(M^k \times 0) \times \mathbb{R}^n$. It follows that B itself is a trivial vector bundle. We can therefore find linearly independent sections $\eta_1, \ldots, \eta_n$ of it such that $\eta_i(x, 0) = \vec{a}_i$ for all $x \in M^k$. The desired family of framing of M^k can then be defined via the formula

$$\mathfrak{f}_t(x, \sum_{i=1}^{n+1} c_i \vec{a}_i) = \mathfrak{f}(x, c_{n+1}\phi_t(x) + \sum_{i=1}^{n} c_i \eta_i(x,t)) \qquad (x \in M^k, c_1, \ldots, c_{n+1} \in \mathbb{R}). \qquad \square$$

5 Framed circles and $\pi_{n+1}(S^n)$

In this section, we use geometric methods to show that

$$\pi_{n+1}(S^n) = \begin{cases} \mathbb{Z} & \text{if } n = 2, \\ \mathbb{Z}/2 & \text{if } n \ge 3. \end{cases}$$

This is proven in Theorem 5.5 below. We begin with the following theorem of Hurewicz–Steenrod [10].

Theorem 5.1. *For $k \ge 3$, we have $\pi_k(S^2) \cong \pi_k(S^3)$.*

Proof. Recall that the Hopf fibration is the fiber bundle

$$S^1 \hookrightarrow S^3 \to S^2$$

arising from the restriction of the map $(\mathbb{C}^2)^* \to \mathbb{CP}^1 \cong S^2$ to $S^3 \subset (\mathbb{C}^2)^*$. The associated long exact sequence of homotopy groups contains the segment

$$\pi_k(S^1) \to \pi_k(S^3) \to \pi_k(S^2) \to \pi_{k-1}(S^1).$$

Since $\pi_k(S^1) = \pi_{k-1}(S^1) = 0$ for $k \ge 3$, we deduce that $\pi_k(S^2) \cong \pi_k(S^3)$. $\square$

Since $\pi_3(S^3) \cong \mathbb{Z}$, this implies the following theorem of Hopf [8].

Corollary 5.2. *We have* $\pi_3(S^2) = \mathbb{Z}$.

To deal with $\pi_{n+1}(S^n)$ for $n \geq 3$, we will construct an invariant of framed circles in $\mathbb{R}^{n+1}$. This will require an understanding of the fundamental group of $\mathrm{SL}_n(\mathbb{R})$.

Lemma 5.3. *We have*

$$\pi_1(\mathrm{SL}_n(\mathbb{R})) = \begin{cases} \mathbb{Z} & \text{if } n = 2, \\ \mathbb{Z}/2 & \text{if } n \geq 3. \end{cases}$$

Proof. The Lie group $\mathrm{SL}_n(\mathbb{R})$ deformation retracts onto its maximal compact subgroup $\mathrm{SO}_n(\mathbb{R})$ (for instance, via the Gram–Schmidt process), so it is enough to prove that

$$\pi_1(\mathrm{SO}_n(\mathbb{R})) = \begin{cases} \mathbb{Z} & \text{if } n = 2, \\ \mathbb{Z}/2 & \text{if } n \geq 3. \end{cases}$$

The cases $n = 2$ and $n = 3$ follow from the fact that $\mathrm{SO}_2(\mathbb{R}) \cong S^1$ and $\mathrm{SO}_3(\mathbb{R}) \cong \mathbb{RP}^3$. For the cases $n \geq 4$, the long exact sequence associated to the fiber bundle

$$\mathrm{SO}_{n-1}(\mathbb{R}) \hookrightarrow \mathrm{SO}_n(\mathbb{R}) \to S^{n-1}$$

contains the segment

$$\pi_2(S^{n-1}) \to \pi_1(\mathrm{SO}_{n-1}(\mathbb{R})) \to \pi_1(\mathrm{SO}_n(\mathbb{R})) \to \pi_1(S^{n-1}).$$

Since $n \geq 4$, we have $\pi_2(S^{n-1}) = \pi_1(S^{n-1}) = 0$, so $\pi_1(\mathrm{SO}_n(R)) \cong \pi_1(\mathrm{SO}_{n-1}(\mathbb{R}))$ and the lemma follows. □

Now consider a framed 1-manifold M^1 in $\mathbb{R}^{n+1}$ with $n \geq 2$. Let $\mathfrak{f} : M^1 \times \mathbb{R}^n \to N_{\mathbb{R}^{n+1}/M^1}$ be the framing. Define a function

$$\rho_{M^1} : M^1 \to \mathrm{SL}_{n+1}(\mathbb{R})$$

as follows. Recall that each fiber of $N_{\mathbb{R}^{n+1}/M^1}$ is a subspace of $\mathbb{R}^{n+1}$. Let

$$\{\vec{a}_1, \ldots, \vec{a}_n\} \subset \mathbb{R}^n$$

be the standard basis. For $x \in M^1$, there exists a unique vector $\tau(x) \in \mathbb{R}^{n+1}$ in the tangent space to M^1 at x such that the $(n+1) \times (n+1)$ matrix

$$\rho_{M^1}(x) := \begin{pmatrix} \tau(x) & \mathfrak{f}(x, \vec{a}_1) & \mathfrak{f}(x, \vec{a}_2) & \cdots & \mathfrak{f}(x, \vec{a}_n) \end{pmatrix}$$

has determinant 1.

Orienting M^1 in the direction of $\tau(x)$, we can now define

$$\widehat{\mathcal{P}}(M^1) = (\rho_{M^1})_*([M_1]) \in \mathrm{H}_1(\mathrm{SL}_{n+1}(\mathbb{R}); \mathbb{Z}) \cong \mathbb{Z}/2.$$

This is *almost* an invariant of the framed cobordism class of M^1. The only problem is that if M^1 is the trivially framed unit circle in $\mathbb{R}^2 \subset \mathbb{R}^{n+1}$, then $\widehat{\mathcal{P}}(M^1) = 1$ (easy exercise!); however,

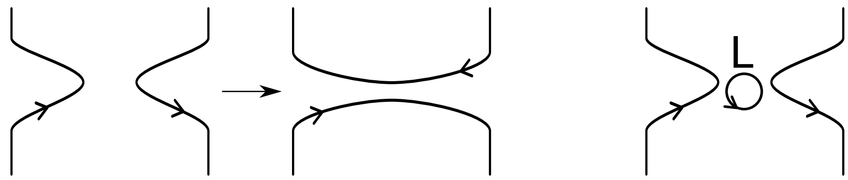

Figure 1: *On the left is the transition from $M^1_{t_0-\epsilon}$ to $M^1_{t_0+\epsilon}$. On the right is the oriented loop L.*

since $[\![M_1]\!] = 0$, we want the answer to be 0. This motivates the following definition: if M^1 is a framed 1-manifold in $\mathbb{R}^{n+1}$ with $n \geq 2$, then

$$\mathcal{P}(M^1) = \widehat{\mathcal{P}}(M^1) + p \in \mathbb{Z}/2,$$

where p is the number of components of M^1. We then have the following lemma, which implies that we can define $\mathcal{P}(x)$ for $x \in \pi_{n+1}(S^n)$ by setting $\mathcal{P}(x) = \mathcal{P}(M^1)$, where M^1 is a framed 1-manifold in $\mathbb{R}^{n+1}$ with $[\![M^1]\!] = x$.

Lemma 5.4. *Fix some $n \geq 2$. Let M^1_0 and M^1_1 be framed 1-manifolds in $\mathbb{R}^{n+1}$ which are framed cobordant. Then $\mathcal{P}(M^1_0) = \mathcal{P}(M^1_1)$.*

Remark. If $n = 1$, then we could define an integer valued invariant $\mathcal{P}(M^1)$ of framed 1-manifolds in $\mathbb{R}^{n+1}$ using the above procedure. However, this would be of little use since this invariant would in fact always vanish (easy exercise!). This reflects the fact that $\pi_k(S^1) = 0$ for $k \geq 2$.

Proof of Lemma 5.4. Let $C^2 \subset \mathbb{R}^{n+1} \times [0,1]$ be a framed cobordism between $M^1_0 \times 0$ and $M^1_1 \times 1$. Perturbing C^2 slightly, we can assume that the projection $\mathbb{R}^{n+1} \times [0,1] \to [0,1]$ restricts to a Morse function $\pi : C^2 \to [0,1]$ whose critical points project to distinct critical values. For all regular values $t \in [0,1]$ of π, define $M^1_t = \pi^{-1}(t)$ and $C^2_{\leq t} = \pi^{-1}([0,t])$. Regarding M^1_t as a 1-dimensional submanifold of $\mathbb{R}^{n+1}$, we can endow M^1_t with a framing on its normal bundle by orthogonally projecting the framing on C^2 to $\mathbb{R}^{n+1}$; with this definition, M^1_0 and M^1_1 agree with the framed manifolds we began with. As t increases from 0 to 1, it is clear that $\mathcal{P}(M^1_t)$ can only change when t passes through a critical value. Letting $t_0 \in [0,1]$ be a critical value and $\epsilon > 0$ be small enough such that the only critical value in $[t_0 - \epsilon, t_0 + \epsilon]$ is t_0, we must prove that $\mathcal{P}(M^1_{t_0-\epsilon}) = \mathcal{P}(M^1_{t_0+\epsilon})$. Let $x \in C^2$ be the critical point with $\pi(x) = t_0$. There are three cases.

- The 2-manifold $C^2_{t_0+\epsilon}$ is obtained from $C^2_{t_0-\epsilon}$ by attaching a 0-handle. In other words, the point x is a local minimum for π. In this case, $M^1_{t_0+\epsilon}$ differs from $M^1_{t_0-\epsilon}$ in that a component V appears. It is clear that $\widehat{\mathcal{P}}(V) = 1$, so $\mathcal{P}(V) = 0$ and $\mathcal{P}(M^1_{t_0-\epsilon}) = \mathcal{P}(M^1_{t_0+\epsilon})$.

- The 2-manifold $C^2_{t_0+\epsilon}$ is obtained from $C^2_{t_0-\epsilon}$ by attaching a 1-handle. In other words, the point x is a saddle point for π. The change from $M^1_{t_0-\epsilon}$ to $M^1_{t_0+\epsilon}$ is as depicted in Figure 1. As is clear from that picture, $M^1_{t_0+\epsilon}$ has either one more or one fewer component than $M^1_{t_0-\epsilon}$ (this depends on whether the two points that are coming together lie on the same component or on different components). Also, the homology classes $\widehat{\mathcal{P}}(M^1_{t_0-\epsilon}) \in \mathrm{H}_1(\mathrm{SL}_{n+1}(\mathbb{R})) \cong \mathbb{Z}/2$ and $\widehat{\mathcal{P}}(M^1_{t_0+\epsilon}) \in \mathrm{H}_1(\mathrm{SL}_{n+1}(\mathbb{R})) \cong \mathbb{Z}/2$ differ by $\widehat{\mathcal{P}}(L)$, where L is the loop shown in Figure 1. The loop L bounds a disc on the framed surface C^2, so $\widehat{\mathcal{P}}(L) = 1$. In summary, $\mathcal{P}(M^1_{t_0+\epsilon}) \in \mathbb{Z}/2$ is obtained from $\mathcal{P}(M^1_{t_0-\epsilon}) \in \mathbb{Z}/2$ by adding 1 (the change in the number of components) and then adding 1 again (the change coming from L). The net effect is that $\mathcal{P}(M^1_{t_0-\epsilon}) = \mathcal{P}(M^1_{t_0+\epsilon})$, as desired.

- The 2-manifold $C^2_{t_0+\epsilon}$ is obtained from $C^2_{t_0-\epsilon}$ by attaching a 2-handle. In other words, the point x is a local maximum for π. This case is similar to that of a 0-handle. □

We now define a special case of the J-homomorphism, which was originally defined by Whitehead [16] following work of Hopf [8]. The homomorphism we need is of the form

$$J_n : \pi_1(\mathrm{SL}_n(\mathbb{R})) \to \pi_{n+1}(S^n)$$

and can be defined as follows. Let $\widehat{L}_2 \subset \mathbb{R}^2$ be the unit circle, oriented in the clockwise direction. We have $N_{\mathbb{R}^2/\widehat{L}_2} \cong \widehat{L}_2 \times \mathbb{R}$. Endowing $N_{\mathbb{R}^2/\widehat{L}_2}$ with the inner product coming from the standard inner product on $\mathbb{R}^2$, we frame $\widehat{L}_2$ with the framing whose value on $x \in \widehat{L}_2$ is the outward facing unit vector. For $n \geq 3$, let $\widehat{L}_n \subset \mathbb{R}^n$ be the result of stabilizing the framed 1-manifold L and let $\mathfrak{f} : \widehat{L}_n \times \mathbb{R}^{n-1} \to N_{\mathbb{R}^n/\widehat{L}_n}$ be the framing. For a map $\theta : \widehat{L}_{n+1} \to \mathrm{SL}_n(\mathbb{R})$, the framing on $\widehat{L}_{n+1}$ obtained by *twisting* $\mathfrak{f}$ by θ is the framing whose value on $(x, \vec{v}) \in \widehat{L}_{n+1} \times \mathbb{R}^n$ is $\mathfrak{f}(x, \theta(x)(\vec{v}))$. Up to homotopy, this only depends on the element of $\pi_1(\mathrm{SL}_n(\mathbb{R}))$ determined by θ, so we will speak of twisting $\mathfrak{f}$ by elements of $\pi_1(\mathrm{SL}_n(\mathbb{R}))$. We will denote by $L_{n+1}(\theta)$ the framed 1-manifold in $\mathbb{R}^{n+1}$ obtained by twisting the framing of $\widehat{L}_{n+1}$ by an element $\theta \in \pi_1(\mathrm{SL}_n(\mathbb{R}))$. Then the J-homomorphism is defined via the formula

$$J_n(\theta) = [\![L_{n+1}(\theta)]\!] \qquad (\theta \in \pi_1(\mathrm{SL}_n(\mathbb{R}))).$$

It is easy to see that this is a homomorphism.

Remark. More generally, the J-homomorphism takes $\pi_k(\mathrm{SL}_n(\mathbb{R}))$ to $\pi_{n+k}(S^n)$.

We now prove the following theorem, which is due to Freudenthal [3].

Theorem 5.5. *For all $n \geq 2$, the J-homomorphism is an isomorphism. In particular,*

$$\pi_{n+1}(S^n) = \begin{cases} \mathbb{Z} & \text{if } n = 2, \\ \mathbb{Z}/2 & \text{if } n \geq 3. \end{cases}$$

Proof. That the J-homomorphism is an isomorphism for $n = 2$ follows easily from the proof of Corollary 5.2. Assume now that $n \geq 3$. Let $E : \pi_3(S^2) \to \pi_{n+1}(S^n)$ be the map obtained by iterating the suspension map. The Freudenthal suspension theorem (Theorem 4.1) implies that E is surjective. Next, let $F : \pi_1(\mathrm{SL}_2(\mathbb{R})) \to \pi_1(\mathrm{SL}_n(\mathbb{R}))$ be the map induced by the usual inclusion $\mathrm{SL}_2(\mathbb{R}) \hookrightarrow \mathrm{SL}_n(\mathbb{R})$. The proof of Lemma 5.3 shows that F is surjective. Combining the evident commutative diagram

$$\begin{tikzcd} \pi_1(\mathrm{SL}_2(\mathbb{R})) \arrow[r, "J_2"] \arrow[d, "F"] & \pi_3(S^2) \arrow[d, "E"] \\ \pi_1(\mathrm{SL}_n(\mathbb{R})) \arrow[r, "J_n"] & \pi_{n+1}(S^n) \end{tikzcd}$$

with the above facts, we see that $J_n : \pi_1(\mathrm{SL}_n(\mathbb{R})) \to \pi_{n+1}(S^n)$ is surjective. Letting $\theta \in \pi_1(\mathrm{SL}_n(\mathbb{R})) \cong \mathbb{Z}/2$ be a generator, it is clear from the construction that $\mathcal{P}(J_n(\theta)) = 1$, so $J_n : \pi_1(\mathrm{SL}_n(\mathbb{R})) \to \pi_{n+1}(S^n)$ is also injective, as desired. □

Remark. If M^1 is a framed 1-manifold in $\mathbb{R}^3$, then $\mathcal{P}(M^1) \in \mathbb{Z}/2$ is the reduction modulo 2 of $[\![M^1]\!] \in \pi_3(S^2) \cong \mathbb{Z}$.

6 Linking numbers

Before we can calculate $\pi_{n+2}(S^n)$, we need four sections of preliminaries. This section reviews the theory of linking numbers.

Let A^k and B^l be disjoint oriented closed submanifolds of $\mathbb{R}^{k+l+1}$ whose dimensions are k and l, respectively. We can then define a smooth function $f : A^k \times B^l \to S^{k+l}$ via the formula

$$f(a,b) = \frac{a-b}{\|a-b\|} \in S^{k+l} \subset \mathbb{R}^{k+l+1}.$$

The *linking number* of A^k and B^l, denoted $\mathrm{Link}(A^k, B^l)$, is the degree of f.

The following lemma shows the effect on their linking number of flipping A^k and B^l.

Lemma 6.1. *Let A^k and B^l be disjoint oriented closed submanifolds of $\mathbb{R}^{k+l+1}$ whose dimensions are k and l, respectively. Then*

$$Link(A^k, B^l) = (-1)^{(k+1)(l+1)} Link(B^l, A^k).$$

Proof. Let $f : A^k \times B^l \to S^{k+l}$ be as in the first definition of the linking pairing, so $\mathrm{Link}(A^k, B^l)$ is the degree of f. Let $\lambda : A^k \times B^l \to B^l \times A^k$ flip the two factors and let $\mu : S^{k+l} \to S^{k+l}$ be the antipodal map. Then λ has degree $(-1)^{kl}$ and μ has degree $(-1)^{k+l+1}$. The number $\mathrm{Link}(B^l, A^k)$ is the degree of $\mu \circ f \circ \lambda$, so

$$\begin{aligned} \mathrm{Link}(B^l, A^k) &= (-1)^{kl}(-1)^{k+l+1}\mathrm{Link}(A^k, B^l) \\ &= (-1)^{k+l+kl+1}\mathrm{Link}(A^k, B^l) \\ &= (-1)^{(k+1)(l+1)}\mathrm{Link}(A^k, B^l). \end{aligned}$$ □

Next, we want to show that linking numbers are unchanged by certain kinds of cobordisms. As a preliminary to this, we need the following result.

Lemma 6.2. *Let X^{n+1} be a compact oriented $(n+1)$-dimensional manifold with boundary and let $g : X^{n+1} \to S^n$ be a smooth map. Then the degree of $g|_{\partial X^{n+1}}$ is 0.*

Proof. Let $p \in S^n$ be a regular value of g. The inverse image $g^{-1}(p)$ is then a properly embedded 1-dimensional submanifold of X^{n+1}. Each component of $g^{-1}(p)$ is either a circle or an arc connecting two points of $(g|_{\partial X^{n+1}})^{-1}(p)$ whose signs are opposite. Because of this, all points of $(g|_{\partial X^{n+1}})^{-1}(p)$ whose sign is positive can be paired with points whose sign is negative, so the degree of $g|_{\partial X^{n+1}}$ is 0. □

The following lemma is the promised result about cobordisms.

Lemma 6.3. *Let A^{k+1} and B^{l+1} be disjoint properly embedded oriented compact submanifolds of $\mathbb{R}^{k+l+1} \times [0,1]$ whose dimensions are k and l, respectively. For $i = 0,1$, let A_i^{k+1} (resp. B_i^{l+1}) be the intersection of A^{k+1} (resp. B^{l+1}) with $\mathbb{R}^{k+l+1} \times i$. Give A_0^{k+1} the orientation coming from A^{k+1} and A_1^{k+1} the opposite orientation, and similarly for B_0^{l+1} and B_1^{l+1}. Then*

$$Link(A_0^{k+1}, B_0^{l+1}) = Link(A_1^{k+1}, B_1^{l+1}).$$

Proof. Deforming A^{k+1} and B^{l+1} slightly (while fixing their boundaries), we can assume that that the projection $\pi : A^{k+1} \sqcup B^{l+1} \to [0,1]$ is a Morse function such that distinct critical points map to distinct critical values. Breaking $[0,1]$ up into small segments, we see that it is enough to prove the special case where the projection $A^{k+1} \sqcup B^{l+1} \to [0,1]$ has a single critical point. Assume without loss of generality that this critical point lies in B^{l+1}. There is therefore an orientation-preserving diffeomorphism $\tau : A_0^{k+1} \times [0,1] \to A^{k+1}$ that takes $A_0^{k+1} \times 0$ to A_0^{k+1}. Letting $\pi' : \mathbb{R}^{k+l+1} \times [0,1] \to \mathbb{R}^{k+l+1}$ be the projection, we can then define a smooth map $F : A_0^{k+1} \times B^{l+1} \to S^{k+l}$ via the formula

$$F(a,b) = \frac{\pi'(\tau(a,\pi(b))) - \pi'(b)}{\|\pi'(\tau(a,\pi(b))) - \pi'(b)\|} \in S^{k+l} \subset \mathbb{R}^{k+l+1}.$$

The key properties of F are as follows.

- $\partial(A_0^{k+1} \times B^{l+1}) = (A_0^{k+1} \times B_0^{l+1}) \sqcup (A_0^{k+1} \times B_1^{l+1})$.
- The restriction of F to $A_0^{k+1} \times B_0^{l+1}$ has degree $\text{Link}(A_0^{k+1}, B_0^{l+1})$.
- The restriction of F to $A_0^{k+1} \times B_1^{l+1}$ has degree $-\text{Link}(A_1^{k+1}, B_1^{l+1})$ (the negative sign occurs here for orientation reasons).

Applying Lemma 6.2 to to F, we conclude that

$$\text{Link}(A_0^{k+1}, B_0^{l+1}) = \text{Link}(A_1^{k+1}, B_1^{l+1}),$$

as desired. □

7 The Hopf invariant

We now discuss the Hopf invariant $\mathcal{H}(x) \in \mathbb{Z}$ of an element $x \in \pi_{2n-1}(S^n)$. This was introduced by Hopf in his paper [9]. Its main purpose for us is that (as we will show in §9) it is precisely the obstruction for improving the bounds in the Freudenthal suspension theorem. This will play an important role in our calculation of $\pi_{n+2}(S^n)$.

Consider an element $x \in \pi_{2n-1}(S^n)$. We will regard S^{2n-1} as $\mathbb{R}^{2n-1} \cup \{\infty\}$ and S^n as $\mathbb{R}^n \cup \{\infty\}$. Fix two distinct points $p, q \in \mathbb{R}^n$. Represent x as a smooth map $\phi : S^{2n-1} \to S^n$ such that $\phi(\infty) = \infty$ and such that both p and q are regular values of ϕ (such maps clearly exist). Observe that

$$\phi^{-1}(p), \phi^{-1}(q) \subset \mathbb{R}^{2n-1} \subset S^{2n-1}$$

are disjoint $(n-1)$-dimensional oriented submanifolds of $\mathbb{R}^{2n-1}$. The *Hopf invariant* of x, denoted $\mathcal{H}(x)$, is $\text{Link}(\phi^{-1}(p), \phi^{-1}(q))$.

Lemma 7.1. *The Hopf invariant does not depend on the choice of ϕ or on the choice of p and q.*

Proof. We begin by proving that $\mathcal{H}(x)$ does not depend on the choice of ϕ. Let $\phi' : S^{2n-1} \to S^n$ be another smooth map representing x such that $\phi'(\infty) = \infty$ and such that both p and q are regular values of ϕ'. There thus exists a smooth map $F : S^{2n-1} \times [0,1] \to S^n$ such that $F(\cdot, 0) = \phi$ and $F(\cdot, 1) = \phi'$. Deforming F slightly (but fixing $F(\cdot, 0)$ and $F(\cdot, 1)$), we can assume that both p and q are regular values of F, and hence that $F^{-1}(p)$ and $F^{-1}(q)$ are submanifolds of $\mathbb{R}^{2n-1} \times [0,1]$. Applying Lemma 6.3 to these submanifolds, we deduce that

$$\operatorname{Link}(\phi^{-1}(p), \phi^{-1}(q)) = \operatorname{Link}((\phi')^{-1}(p), (\phi')^{-1}(q)),$$

as desired.

We now check that $\mathcal{H}(x)$ does not depend on the choice of p and q. Let $p', q' \in \mathbb{R}^n$ be two distinct points. There exists a compactly supported diffeomorphism $\tau : R^n \to R^n$ with the following properties.

- We have $\tau(p) = p'$ and $\tau(q) = q'$.
- The map τ is isotopic to the identity through an isotopy whose support is compact.

Such a diffeomorphism is easy to construct. Letting $\widehat{\tau} : S^n \to S^n$ be the extension to the one-point compactification, the map $\widehat{\tau}$ is smooth and isotopic to the identity through an isotopy that fixes ∞. It follows that $\widehat{\tau} \circ \phi : S^{2n-1} \to S^n$ realizes $x \in \pi_{2n-1}(S^n)$. The key observation is that $\widehat{\tau} \circ \phi$ is transverse to p' and q', and satisfies

$$(\widehat{\tau} \circ \phi)^{-1}(p') = \phi^{-1}(p) \quad \text{and} \quad (\widehat{\tau} \circ \phi)^{-1}(q') = \phi^{-1}(q).$$

The desired independence follows. □

As we said above, we will later prove that the Hopf invariant is the obstruction to improving the range in the Freudenthal suspension theorem. The following is a cheap condition that ensures that the Hopf invariant vanishes.

Lemma 7.2. *If n is even, then $\mathcal{H}(x) = 0$ for all $x \in \pi_{2n-1}(S^n)$.*

Proof. Immediate from Lemma 6.1. □

The final result in this section shows how to calculate $\mathcal{H}(x)$ from a framed manifold representing x.

Lemma 7.3. *Let M^{n-1} be a framed $(n-1)$-dimensional manifold in $\mathbb{R}^{2n-1}$ with framing $\mathfrak{f} : M^{n-1} \times \mathbb{R}^n \to N_{\mathbb{R}^{2n-1}/M^{n-1}}$. Regarding the fibers of $N_{\mathbb{R}^{2n-1}/M^{n-1}}$ as n-dimensional subspaces of $\mathbb{R}^{2n-1}$, for $\vec{v} \in \mathbb{R}^n$ define*

$$M^{n-1}(\vec{v}) = \{x + \mathfrak{f}(x, \vec{v}) \mid x \in M^{n-1}\}.$$

Then for $\vec{v} \in \mathbb{R}^n$ sufficiently close to 0, the set $M^{n-1}(\vec{v})$ is an $(n-1)$-dimensional submanifold of $\mathbb{R}^{2n-1}$ which is disjoint from M^{n-1} and

$$\mathcal{H}([\![M^{n-1}]\!]) = \mathit{Link}(M^{n-1}, M^{n-1}(\vec{v})).$$

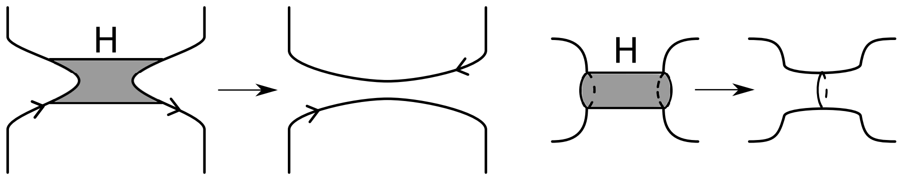

Figure 2: *On the left is a good* 1*-handle H for a framed* 1*-manifold M^1 together with $M^1(H)$. On the right is a good* 1*-handle H for a framed* 2*-manifold M^2 together with $M^2(H)$.*

Proof. It is obvious that for $\vec{v} \in \mathbb{R}^n$ sufficiently close to 0, the set $M^{n-1}(\vec{v})$ is an $(n-1)$-dimensional submanifold of $\mathbb{R}^{2n-1}$ which is disjoint from M^{n-1}. The indicated formula follows from the fact that the sets M^{n-1} and $M^{n-1}(\vec{v})$ are pullbacks of regular values under the map given by the Pontryagin–Thom construction. □

8 Framed surgery

Our promised improvement to the Freudenthal suspension theorem will require more drastic framed cobordisms than the ambient isotopies used to prove the ordinary Freudenthal suspension theorem. The basic tool for constructing these framed cobordisms is *framed surgery*. Thankfully we will only need the simplest case of this.

Let M^k be a framed k-manifold in $\mathbb{R}^{n+k}$. The framing provides an orientation on the normal bundle $N_{\mathbb{R}^{n+k}/M^k}$. The direct sum $T_{M^k} \oplus N_{\mathbb{R}^{n+k}/M^k}$ is the pullback to M^k of $T_{\mathbb{R}^{n+k}}$, which is also oriented, so T_{M^k} acquires an orientation. We will call this orientation on M^k the orientation *induced by the framing*.

For all $p \geq 0$, let D^p denote a p-dimensional closed disc. Observe that $\partial(D^p \times D^q) = (S^{p-1} \times D^q) \cup (D^p \times S^{q-1})$. The subspaces $S^{p-1} \times D^q$ and $D^p \times S^{q-1}$ of $\partial(D^p \times D^q)$ meet along $S^{p-1} \times S^{q-1}$.

A *good* 1*-handle* for M^k is a $(k+1)$-dimensional oriented submanifold H of $\mathbb{R}^{n+k}$ with the following properties (see Figure 2 for examples).

- $H \cong D^1 \times D^k$.
- $H \cap M^k = S^0 \times D^k$.
- The orientation of $S^0 \times D^k \subset \partial(D^1 \times D^k)$ is the same as the orientation this manifold acquires as a codimension 0 submanifold of M^k (which is given the orientation induced by the framing).

Let $Y = S^0 \times D^k \subset \partial H$ and $Z = D^1 \times S^{k-1} \subset \partial H$. As in Figure 2, let $M^k(H)$ be the oriented k-dimensional submanifold of $\mathbb{R}^{n+k}$ obtained by deleting $\mathrm{Int}(Y) \subset H \cap M^k$ from M^k, adding Z, and then "smoothing the corners".

Lemma 8.1. *Let M^k be a framed k-manifold in $\mathbb{R}^{n+k}$ and let $H \subset \mathbb{R}^{n+k}$ be a good* 1*-handle for M^k. Then there exists a framing on $M^k(H)$ such that M^k is framed cobordant to $M^k(H)$.*

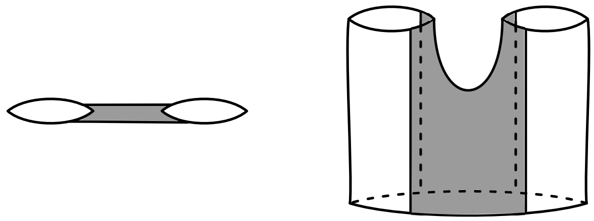

Figure 3: *On the left is a* 1*-manifold with a good* 1*-handle (in bold). On right right is the cobordism constructed in Lemma 8.1; the handle has been "pushed" down.*

Proof. Let $t \in [0,1]$ denote the final coordinate in $\mathbb{R}^{n+k} \times [0,1]$. As is shown in Figure 3, we can obtain a cobordism $C^{k+1} \subset \mathbb{R}^{n+k} \times [0,1]$ between M^k and $M^k(H)$ via the following procedure (which results in a manifold with corners that should be smoothed).

- Begin with
$$\widehat{C}^{k+1} := \left(M^k \smallsetminus \mathrm{Int}\,(Y)\right) \times [0,1].$$
- Glue H to $\widehat{C}^{k+1}$ in the following way, which one can think of as "pushing" $H \times 1$ down into $\mathbb{R}^{n+k} \times [0,1]$.
  - The submanifold Y of $\partial H = Y \cup Z$ is at height $t = 1$.
  - Let $U \subset Z$ be an open collar neighborhood of ∂Z. Then the submanifold $Z \smallsetminus U$ of ∂H is at height $t = 0$.
  - The rest of H lies in $\mathbb{R}^{n+k} \times (0,1)$.
  - The submanifold H of C^{k+1} is orthogonal to the planes $t = 0$ and $t = 1$.

All that remains is to construct a framing on C^{k+1}. We already have a framing on $\widehat{C}^{k+1}$ and on $Y \subset \partial H$ which we must extend over H. To deal with this extension, we will use obstruction theory (see [2, Chapter 7] and [5, p. 415] for two different expositions of this; the point of view of [2, Chapter 7] is more elementary). The first obstruction to extending our framing over H lies in

$$\mathrm{H}^1(H, Y; \pi_0(\mathrm{GL}_n(\mathbb{R}))) \cong \mathrm{H}^1(H, Y; \mathbb{Z}/2) = \mathbb{Z}/2.$$

This is generated by an arc connecting the two components of Y. Our assumption concerning the orientations in the definition of "good 1-handle" implies that this first obstruction vanishes. Since H is contractible and $Y \subset \partial H$ consists of two contractible components, there are no higher obstructions and the desired extension exists. □

9 Improving the Freudenthal suspension theorem

Recall that Theorem 4.1 (the Freudenthal suspension theorem) says that the suspension map $\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ is surjective for $n \geq k+1$ and an isomorphism for $n \geq k+2$. These bounds are sharp in general, i.e. the suspension map $\pi_{2n}(S^n) \to \pi_{2n+1}(S^{n+1})$ need not be surjective. The following theorem (proved by Freudenthal in [3], the same paper that contains the usual Freudenthal suspension theorem) says that the image is exactly the set of elements whose Hopf invariant is 0.

Theorem 9.1. *The image of the suspension homomorphism* $E : \pi_{2n}(S^n) \to \pi_{2n+1}(S^{n+1})$ *is* $\{x \in \pi_{2n+1}(S^{n+1}) \mid \mathcal{H}(x) = 0\}$.

Proof. We first prove that $\mathcal{H}(E(y)) = 0$ for all $y \in \pi_{2n}(S^n)$. Write $y = [\![P^n]\!]$, where P^n is a framed n-dimensional manifold in $\mathbb{R}^{2n}$. Let $\{\vec{b}_1, \ldots, \vec{b}_{2n+1}\}$ be the standard basis for $\mathbb{R}^{2n+1}$. Lemma 7.3 implies that $\mathcal{H}(E(y)) = \operatorname{Link}(E(P^n), E(P^n) + \vec{b}_{2n+1})$. This linking number is clearly 0.

Now consider $x \in \pi_{2n+1}(S^{n+1})$ such that $\mathcal{H}(x) = 0$. We will prove that x is in the image of E. Write $x = [\![M^n]\!]$, where M^n is a framed n-dimensional manifold in $\mathbb{R}^{2n+1}$. Our goal is to prove that M^n is framed cobordant to $E(N^n)$, where N^n is a framed n-dimensional manifold in $\mathbb{R}^{2n}$. Just like the proof of Theorem 4.1 (the Freudenthal suspension theorem), this will be accomplished in two steps.

Step 1. *We find a framed cobordism from* M^n *to a submanifold of* $\mathbb{R}^{2n} \subset \mathbb{R}^{2n+1}$.

We remark that this step does *not* use our assumption about the Hopf invariant. It follows from the proof of the Whitney embedding theorem (see [6, Theorem 1.3.5] and the discussion on [6, p. 27]) that for a generic nonzero vector $\vec{\zeta}$ in $\mathbb{R}^{2n+1}$, the projection of M^n onto the orthogonal complement of $\vec{\zeta}$ is an immersion (not necessarily an embedding!) without triple points. Let $\vec{\zeta} = (\zeta_1, \ldots, \zeta_{2n+1})$ be such a vector with $\zeta_{2n+1} > 0$. We will prove that M^n can be modified by a framed cobordism such that the projection of M^n onto the orthogonal complement of $\vec{\zeta}$ is an embedding. We will then be able to follow Step 1 of the proof of Theorem 4.1 line-for-line and deduce the desired result.

The fact that the projection is an immersion implies that there are finitely many lines parallel to $\vec{\zeta}$ that pass through more than one point of M^n. Let these lines be $\ell_1, \ldots, \ell_m$. The fact the projection has no triple points implies that each ℓ_i passes through exactly two points of M^n. If $m = 0$, then we are done, so assume that $m \geq 1$. Let $\ell_1 \cap M^n = \{p, q\}$. We can then find a neighborhood U of ℓ_1 that does not intersect any other ℓ_i and an orientation-preserving diffeomorphism $\Psi : \mathbb{R}^{2n+1} \to U$ with the following properties.

- The image of $\{t\vec{b}_{2n+1} \mid t \in \mathbb{R}\}$ is ℓ_1.
- The image of any line parallel to $\vec{b}_{2n+1}$ is a line parallel to ℓ_1.
- Set
$$V = \{\vec{v} \mid \vec{v} \in \langle \vec{b}_1, \ldots, \vec{b}_n \rangle\} \quad \text{and} \quad W = \{\vec{v} + \vec{b}_{2n+1} \mid \vec{v} \in \langle \vec{b}_{n+1}, \ldots, \vec{b}_{2n} \rangle\}.$$
Then $\Psi(V \cup W) = M^n \cap U$.
- The orientation on V induced by the framing on M^n is defined by the ordered basis $\{\vec{b}_1, \ldots, \vec{b}_n\}$.
- The orientation on W induced by the framing on M^n is opposite to that defined by the ordered basis $\{\vec{b}_{n+1}, \ldots, \vec{b}_{2n}\}$.

The image H under Ψ of the set

$$\{\vec{v} + t\vec{b}_{2n+1} \mid t \in [0,1],\ \vec{v} = \sum_{i=1}^{n} c_i(\cos(t\frac{\pi}{2})\vec{b}_i + \sin(t\frac{\pi}{2})\vec{b}_{n+i}) \text{ for } (c_1, \ldots, c_n) \in \mathbb{R}^n \text{ w/ } \sum_{i=1}^{n} c_i^2 \leq 1\}$$

is then a good 1-handle for M^n. Define $M^n(H)$ as in §8, so by Lemma 8.1 we have that M^n is framed cobordant to $M^n(H)$. Clearly ℓ_1 is disjoint from $M^n(H)$, the lines $\ell_2, \ldots, \ell_m$ each intersect $M^n(H)$ exactly twice, and no other lines parallel to $\vec{\zeta}$ intersect $M^n(H)$ in more than one place. We now replace M^n with $M^n(H)$. Repeating this procedure, we can complete the proof.

Step 2. *We isotope the framing on M^n such that $M^n = E(N^n)$ for some framed n-manifold in $\mathbb{R}^{2n}$.*

This can be done exactly like Step 2 of the proof of Theorem 4.1. The only difference is that we cannot use our dimension assumptions to prove that the map ϕ that appears there is nullhomotopic. However, using the interpretation of the Hopf invariant in Lemma 7.3, our assumption that $\mathcal{H}(x) = 0$ implies immediately that this map ϕ is nullhomotopic. □

10 Framed surfaces and $\pi_{n+2}(S^n)$

We now turn to $\pi_{n+2}(S^n)$. Our goal is to prove Theorem 10.5 below, which asserts that $\pi_{n+2}(S^n) = \mathbb{Z}/2$ for $n \geq 2$. The key to this will be the construction of a $\mathbb{Z}/2$-valued invariant of framed 2-manifolds in $\mathbb{R}^{n+2}$.

Consider a framed 2-manifold M^2 in $\mathbb{R}^{n+2}$ with framing $\mathfrak{f} : M^2 \times \mathbb{R}^n \to N_{\mathbb{R}^{n+2}/M^2}$. We begin by recalling some conventions introduced earlier. The framing endows $N_{\mathbb{R}^{n+2}/M^2}$ with an orientation. We will regard the fibers of the tangent bundle T_{M^2} and the normal bundle $N_{\mathbb{R}^{n+2}/M^2}$ as subspaces of $\mathbb{R}^{n+2}$, and we will endow them with the inner product obtained by restricting the standard inner product on $\mathbb{R}^{n+2}$. We will give M^2 the orientation for which the direct sum decomposition $T_{M^2} \oplus N_{\mathbb{R}^{n+2}/M^2}$ gives the standard orientation on $\mathbb{R}^{n+2}$.

If L is an oriented 1-submanifold of M^2, then we can frame L as follows. For all $x \in L$, let $n(x)$ be the unique unit tangent vector to M^2 at x with the following properties.

- The vector $n(x)$ is orthogonal to L.
- The vector $n(x)$ points to the right of L (using the orientations on L and M^2).

Also, let $\{\vec{a}_1, \ldots, \vec{a}_{n+1}\}$ be the standard basis for $\mathbb{R}^{n+1}$. Our framing on L is then the function $\mathfrak{f}_L : L \times \mathbb{R}^{n+1} \to N_{\mathbb{R}^{n+2}/L}$ defined via the formula

$$\mathfrak{f}_L(x, \vec{v} + c\vec{a}_{n+1}) = \mathfrak{f}(x, \vec{v}) + cn(x) \qquad (\vec{v} \in \mathbb{R}^n, c \in \mathbb{R}).$$

We will call this the *framing on L induced by M^2*.

We now prove the following lemma.

Lemma 10.1. *Consider a framed 2-manifold M^2 in $\mathbb{R}^{n+2}$. Let L and L' be homologous oriented 1-submanifolds of M^2. Endow L and L' with the framings induced by M^2. Then L and L' are framed cobordant as framed 1-manifolds in $\mathbb{R}^{n+2}$.*

Proof. Since L and L' are homologous, there exists an oriented properly embedded surface Σ^2 in $M^2 \times [0,1]$ such that

$$\partial\Sigma^2 = \Sigma^2 \cap (M^2 \times \{0,1\}) = (L \times 0) \cup (L' \times 1).$$

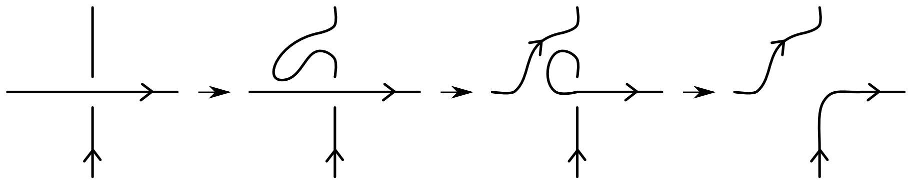

Figure 4: *On the far left is our intersection p_i. The second figure is a transitional one obtained by just homotoping L''. The third figure from the left is L'''. The fourth figure is $L^{(4)}$; eliminating the intersection while keeping the framing from the ambient surface requires eliminating one "twist" from the framing.*

For the reader who has not seen this kind of thing before, Σ can be obtained by taking an immersed surface with the above properties (which clearly exists) and then resolving the self-intersections (a collection of circles) to make it embedded. The 3-manifold $M^2 \times [0,1]$ in $\mathbb{R}^{n+3} \times [0,1]$ acquires a framing from M^2, and using this we can put a framing on Σ^2 just like we did for L. This gives the desired framed cobordism. $\square$

By Lemma 10.1, we can define a function

$$\widehat{q}_{M^2} : \mathrm{H}_1(M^2; \mathbb{Z}) \longrightarrow \mathbb{Z}/2$$

by setting $\widehat{q}_{M^2}(x) = \mathcal{P}(L)$, where L is an oriented 1-submanifold of M^2 representing x endowed with the framing induced by M^2 and $\mathcal{P}(L)$ is the invariant of framed 1-manifolds in $\mathbb{R}^{n+2}$ introduced in §5. One's first impulse might be that $\widehat{q}_{M^2}$ is a homomorphism; indeed, in [13] Pontryagin assumes that this holds and uses it to give a proof of the (false) theorem that $\pi_{n+2}(S^n) = 0$ for $n \geq 3$. The correct statement is as follows. Let $\mathfrak{i}(\cdot,\cdot)$ be the $\mathbb{Z}/2$-valued algebraic intersection pairing on $\mathrm{H}_1(M^2;\mathbb{Z}/2)$. For $x, y \in \mathrm{H}_1(M^2;\mathbb{Z})$, we will let $\mathfrak{i}(x,y)$ denote $\mathfrak{i}(\overline{x},\overline{y})$, where $\overline{x}$ and $\overline{y}$ are the images in $\mathrm{H}_1(M^2;\mathbb{Z}/2)$ of x and y, respectively.

Lemma 10.2. *If M^2 is a framed surface in $\mathbb{R}^{n+2}$ with $n \geq 1$, then*

$$\widehat{q}_{M^2}(x+y) = \widehat{q}_{M^2}(x) + \widehat{q}_{M^2}(y) + \mathfrak{i}(x,y) \qquad (x, y \in \mathrm{H}_1(M^2;\mathbb{Z})).$$

Proof. Let L and L' be oriented 1-submanifolds of M^2 realizing the homology classes x and y, respectively. Homotoping L and L', we can assume that they intersect transversely. Let $L \cap L' = \{p_1, \ldots, p_k\}$, so $k \equiv \mathfrak{i}(x,y)$ modulo 2. Let L'' be the result of taking $L \cup L'$ and pushing one of the two strands meeting at each p_i off the surface into $\mathbb{R}^{n+2}$ as in Figure 4.a (it does not matter how you do this). Give L'' the framing coming from the framings on L and L' induced by M^2. Thus

$$\mathcal{P}(L'') = \mathcal{P}(L) + \mathcal{P}(L') = \widehat{q}_{M^2}(x) + \widehat{q}_{M^2}(y).$$

Let L''' be the result of modifying L'' in a neighborhood of each p_i as shown in Figure 4.b. As is clear from that Figure, L''' is framed cobordant to L'' (indeed, it can be obtained from L'' by a sequence of surgeries on good 1-handles as in Lemma 8.1), so $\mathcal{P}(L''') = \mathcal{P}(L'')$. Finally, let $L^{(4)}$ be the result of "pulling tight" each self-intersection of L''' and giving the result the framing coming from M^2. From Figure 4, it is clear that this requires eliminating one "twist" in the framing for each self-intersection, so

$$\widehat{q}_{M^2}(x+y) = \mathcal{P}(L^{(4)}) = \mathcal{P}(L''') + k = \widehat{q}_{M^2}(x) + \widehat{q}_{M^2}(y) + \mathfrak{i}(x,y),$$

as desired. ∎

Lemma 10.2 implies in particular that $\widehat{q}_{M^2}(2x) = 2\widehat{q}_{M^2}(x) = 0$ for all $x \in \mathrm{H}_1(M^2;\mathbb{Z})$. This implies that $\widehat{q}_{M^2}$ descends to a function

$$q_{M^2} : \mathrm{H}_1(M^2;\mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$$

satisfying

$$q_{M^2}(x+y) = q_{M^2}(x) + q_{M^2}(y) + \mathfrak{i}(x,y) \qquad (x,y \in \mathrm{H}_1(M^2;\mathbb{Z}/2)).$$

Such functions are known as *quadratic refinements* of the nondegenerate symmetric bilinear form $\mathfrak{i}(\cdot,\cdot)$ (this is symmetric since we are working over $\mathbb{Z}/2$). Since we are working over $\mathbb{Z}/2$, such quadratic refinements are classified by their *Arf invariants*, which we briefly describe (see, e.g., [15, Appendix] for details and proofs; a broader discussion of Arf's work on this subject, of which the above is a tiny special case, is in [11]). A *symplectic basis* for $\mathrm{H}_1(M^2;\mathbb{Z}/2)$ is a basis $\{\vec{a}_1, \vec{b}_1, \ldots, \vec{a}_g, \vec{b}_g\}$ such that

$$\mathfrak{i}(\vec{a}_i, \vec{b}_j) = \delta_{ij} \quad \text{and} \quad \mathfrak{i}(\vec{a}_i, \vec{a}_j) = \mathfrak{i}(\vec{b}_i, \vec{b}_j) = 0 \qquad (1 \le i,j \le g),$$

where δ_{ij} is the Kronecker delta. Choosing such a symplectic basis, we then define the Arf invariant of q_{M^2} to be

$$\mathrm{Arf}(q_{M^2}) = \sum_{i=1}^{g} q_{M^2}(a_i) q_{M^2}(b_i) \in \mathbb{Z}/2.$$

This does not depend on the choice of symplectic basis. The relevance of the Arf invariant for us is the following result.

Lemma 10.3. *Let M_0^2 and M_1^2 be framed 2-manifolds in $\mathbb{R}^{n+2}$ which are framed cobordant. Then Arf$(q_{M_0^2})$ = Arf$(q_{M_1^2})$.*

Proof. Let $C^3 \subset \mathbb{R}^{n+2} \times [0,1]$ be a framed cobordism between $M_0^2 \times 0$ and $M_1^2 \times 1$. Perturbing C^3 slightly, we can assume that the projection $\mathbb{R}^{n+2} \times [0,1] \to [0,1]$ restricts to a Morse function $\pi : C^3 \to [0,1]$ whose critical points project to distinct critical values. For all regular values $t \in [0,1]$ of π, define $M_t^2 = \pi^{-1}(t)$ and $C^3_{\le t} = \pi^{-1}([0,t])$. Regarding M_t^2 as a 2-dimensional submanifold of $\mathbb{R}^{n+2}$, we can endow M_t^2 with a framing on its normal bundle by orthogonally projecting the framing on C^3 to $\mathbb{R}^{n+2}$; with this definition, M_0^2 and M_1^2 agree with the framed manifolds we began with. As t increases from 0 to 1, it is clear that $\mathrm{Arf}(q_{M_t^2})$ can only change when t passes through a critical value. Letting $t_0 \in [0,1]$ be a critical value and $\epsilon > 0$ be small enough such that the only critical value in $[t_0 - \epsilon, t_0 + \epsilon]$ is t_0, we must prove that $\mathrm{Arf}(q_{M^2_{t_0-\epsilon}}) = \mathrm{Arf}(q_{M^2_{t_0+\epsilon}})$. Let $x \in C^3$ be the critical point with $\pi(x) = t_0$. There are four cases

- The 3-manifold $C^3_{\le t_0+\epsilon}$ is obtained from $C^3_{\le t_0-\epsilon}$ by attaching a 0-handle. In other words, the point x is a local minimum for π. In this case, $M^2_{t_0+\epsilon}$ differs from $M^2_{t_0-\epsilon}$ in that a spherical component V appears. Since $\mathrm{H}_1(V;\mathbb{Z}/2) = 0$, this component contributes nothing to the Arf invariant and $\mathrm{Arf}(q_{M^2_{t_0-\epsilon}}) = \mathrm{Arf}(q_{M^2_{t_0+\epsilon}})$.

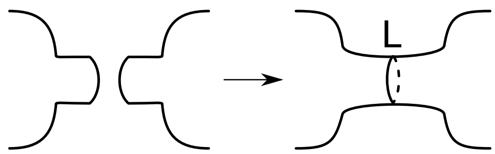

Figure 5: *On the left is the surface $M^2_{t_0-\epsilon}$, on the right is the surface $M^2_{t_0+\epsilon}$. The "core" L of the new tube is depicted.*

- The 3-manifold $C^3_{\le t_0+\epsilon}$ is obtained from $C^3_{\le t_0-\epsilon}$ by attaching a 1-handle. In this case, $M^2_{t_0+\epsilon}$ is obtained from $M^2_{t_0-\epsilon}$ by attaching a tube as in Figure 5. Let L be the oriented 1-submanifold of $M^2_{t_0+\epsilon}$ forming the "core" of this tube, as shown in Figure 5. Endow L with the framing induced by $M^2_{t_0+\epsilon}$. The loop L bounds a disc in the framed 3-manifold $C^3_{\le t_0+\epsilon}$. This disc can be framed (just like the cobordism in Lemma 10.1), so L is framed cobordant to the empty manifold and $\mathcal{P}(L) = 0$. We now can choose a symplectic basis $\{a_1, b_1, \ldots, a_g, b_g\}$ for $\mathrm{H}_1(M^2_{t_0+\epsilon}; \mathbb{Z}/2)$ such that a_g equals the homology class of L and $\{a_1, b_1, \ldots, a_{g-1}, b_{g-1}\}$ comes from $\mathrm{H}_1(M^2_{t_0-\epsilon}; \mathbb{Z}/2)$ in the obvious sense. We then have

$$\mathrm{Arf}(q_{M^2_{t_0+\epsilon}}) = \left(\sum_{i=1}^{g-1} q_{M^2_{t_0-\epsilon}}(a_i)\, q_{M^2_{t_0-\epsilon}}(b_i)\right) + q_{M^2_{t_0+\epsilon}}(a_g) q_{M^2_{t_0+\epsilon}}(b_g) = \mathrm{Arf}(q_{M^2_{t_0-\epsilon}}) + 0,$$

  as desired.

- The 3-manifold $C^3_{\le t_0+\epsilon}$ is obtained from $C^3_{\le t_0-\epsilon}$ by attaching a 2-handle. This is similar to the case of a 1-handle (but in this case $M^2_{t_0-\epsilon}$ can be obtained from $M^2_{t_0+\epsilon}$ by attaching a tube rather than the other way around).

- The 3-manifold $C^3_{\le t_0+\epsilon}$ is obtained from $C^3_{\le t_0-\epsilon}$ by attaching a 3-handle. In other words, the point x is a local maximum for π. This is similar to the case of a 0-handle. □

Using the above invariant, we will prove the following lemma.

Lemma 10.4. *For $n \ge 2$, let $\nu_n : S^{n+1} \to S^n$ represent the generator for $\pi_{n+1}(S^n)$ (see Theorem 5.5). Then $\nu_n \circ \nu_{n+1} : S^{n+2} \to S^n$ represents a nontrivial element of $\pi_{n+2}(S^n)$ for all $n \ge 2$.*

Proof. Let us begin by recalling the construction of ν_n from Theorem 5.5. Let $\widehat{L}_2 \subset \mathbb{R}^2$ be the unit circle, oriented in the clockwise direction. We have $N_{\mathbb{R}^2/\widehat{L}_2} \cong \widehat{L}_2 \times \mathbb{R}$. Endowing $N_{\mathbb{R}^2/\widehat{L}_2}$ with the inner product coming from the standard inner product on $\mathbb{R}^2$, we frame $\widehat{L}_2$ with the framing whose value on $x \in \widehat{L}_2$ is the outward facing unit vector. For $n \ge 3$, let $\widehat{L}_n \subset \mathbb{R}^n$ be the result of stabilizing the framed 1-manifold L. Finally, let $L_n \subset \mathbb{R}^n$ be the result of twisting the framing of $\widehat{L}_n$ by a generator for $\pi_1(\mathrm{SL}_{n-1}(\mathbb{R}))$. Theorem 5.5 says that $\nu_n = [\![L_{n+1}]\!]$.

As in the proof of Theorem 3.1, we can represent ν_n by the map $f_n : S^{n+1} \to S^n$ obtained by applying the Pontryagin–Thom construction to L_{n+1}. Regarding S^n as $\mathbb{R}^n \cup \{\infty\}$, under this construction every point of $\mathbb{R}^n$ is a regular value for f_n and $L_{n+1} \subset \mathbb{R}^{n+1}$ is the inverse image of $0 \in \mathbb{R}^n$ (framed using the framing obtained by pulling back the standard frame at 0). From this, we see that $0 \in \mathbb{R}^n \subset S^n$ is a regular value of $f_n \circ f_{n+1}$. Define $T_n = (f_n \circ f_{n+1})^{-1}(0) \subset \mathbb{R}^{n+2}$ (framed again by pulling back the standard frame at 0), so T_n is a framed 2-manifold in $\mathbb{R}^{n+2}$. It follows from the proof of Theorem 3.1 that $\nu_n \circ \nu_{n+1} = [\![T_n]\!]$, so to prove the lemma it is enough to show that $\mathrm{Arf}(q_{T_n}) = 1$.

It follows from its definition that $T_n = f_{n+1}^{-1}(L_{n+1})$. Moreover, for all $x \in L_{n+1}$ the inverse image $f_{n+1}^{-1}(x)$ is a circle. It follows that T_n is a circle bundle over a circle. Since T_n is orientable, we deduce that T_n is a torus. We now make the following constructions.

- Define $A = f_{n+1}^{-1}(p)$, where p is some point in L_{n+1}. Thus A is a 1-submanifold of T_n. Orient A arbitrarily and endow A with the framing induced by T_n. Then since f_{n+1} represents a generator for $\pi_{n+2}(S^{n+1})$, it follows that $\mathcal{P}(A) = 1$.

- Let $V \subset \mathbb{R}^{n+2}$ be one of the fibers in the tubular neighborhood of L_{n+2} used in the Pontryagin–Thom construction. It follows that $(f_{n+1})|_V$ is a diffeomorphism from V to $\mathbb{R}^{n+1} \subset S^{n+1}$. Define $B = ((f_{n+1})|_V)^{-1}(L_{n+1})$, so B is a 1-submanifold of T_n. Orient B arbitrarily and endow B with the framing induced by T_n. Then since f_n represents a generator for $\pi_{n+1}(S^n)$, it follows that $[\![B]\!] \in \pi_{n+2}(S^{n+1})$ is a generator, and thus that $\mathcal{P}(B) = 1$.

Let $a \in \mathrm{H}_1(T_n;\mathbb{Z}/2)$ and $b \in \mathrm{H}_1(T_n;\mathbb{Z}/2)$ be the homology classes of A and B, respectively. Then $\{a, b\}$ is a symplectic basis for $\mathrm{H}_1(T_n;\mathbb{Z}/2)$, so by definition we have

$$\mathrm{Arf}(q_{T_n}) = q_{T_n}(a) q_{T_n}(b) = \mathcal{P}(A)\mathcal{P}(B) = 1,$$

as desired. □

We now come to the main theorem of this section, which was proved independently by Pontryagin [14] and Whitehead [17]

Theorem 10.5. *For $n \geq 2$, we have $\pi_{n+2}(S^n) = \mathbb{Z}/2$. It is generated by $\nu_n \circ \nu_{n+1}$, where ν_m is as in Lemma 10.4.*

Proof. The proof is by induction on n. The base case $n = 2$ follows from the following facts.

- Theorem 5.1, which implies that $\pi_4(S^2) \cong \pi_4(S^3)$.
- Theorem 5.5, which says that $\pi_4(S^3) \cong \mathbb{Z}/2$.
- Lemma 10.4, which says that $\nu_2 \circ \nu_3$ represents a nontrivial element of $\pi_4(S^2)$.

Now assume that $n \geq 3$. Let $E : \pi_4(S^2) \to \pi_{n+2}(S^n)$ be the composition of the suspension maps. Combining the ordinary Freudenthal suspension theorem (Theorem 4.1) with the improved suspension theorem (Theorem 9.1) and Lemma 7.2, we see that E is surjective. Lemma 10.4 implies that $\pi_{n+2}(S^n)$ is not trivial, so E cannot be the zero map. We conclude that E must be an isomorphism and hence $\pi_{n+2}(S^n) = \mathbb{Z}/2$. Finally, Lemma 10.4 implies that $\nu_n \circ \nu_{n+1}$ represents a generator of this group. □

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