Burnside's $p^a q^b$-theorem

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Abstract

We prove Burnside's theorem saying that a group of order $p^a q^b$ for primes p and q is solvable.

In this note, we discuss the proof of the following theorem of Burnside [1].

Theorem A. *Let G be a group with $|G| = p^a q^b$ for primes p and q. Then G is solvable.*

The key to the proof is showing that such a group must contain a nontrivial normal subgroup. The subgroups we will construct will be of the following form. All representations in this note are finite-dimensional and defined over $\mathbb{C}$.

Definition 0.1. Let V be a representation of a group G. The *V-central subgroup* of G, denoted $Z(V)$, is the subgroup consisting of all $g \in G$ that act on V by homotheties, i.e. such that there exists some $\lambda \in \mathbb{C}$ such that $g \cdot \vec{v} = \lambda \vec{v}$ for all $\vec{v} \in V$. □

Remark. It is clear that $Z(V)$ is a normal subgroup of G; indeed, for $g \in Z(V)$ and λ as above, for all $h \in G$ we have

$$h^{-1}gh \cdot \vec{v} = h^{-1} \cdot (\lambda(h \cdot \vec{v})) = \lambda(h^{-1}h) \cdot \vec{v} = \lambda \vec{v} \qquad \text{for all } \vec{v} \in V.$$

so $h^{-1}gh \in Z(G)$.

How can we detect nontrivial $Z(V)$? With respect to an appropriate basis, the matrices representing the action of elements of G on V are algebraic integers (see, e.g., [2]). It follows that for $g \in Z(V)$, the action of g on V is a homothety with scaling factor an algebraic integer. Letting χ be the character of V, this scaling factor is precisely $\chi(g)/n$, where $n = \dim(V)$. This must therefore be an algebraic integer. The following lemma is a converse to this:

Lemma 0.2. *Let G be a finite group and let V be an n-dimensional representation of G with character χ. For some $g \in G$, assume that $\chi(g)/n$ is a nonzero algebraic integer. Then $g \in Z(V)$.*

Proof. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be the eigenvalues of the action of g on V. Our goal is to prove that the λ_i are all equal. Assume otherwise. Each of the λ_i is a root of unity, so this implies that

$$|\chi(g)| = |\lambda_1 + \cdots + \lambda_n| < n.$$

This can be improved as follows. Let $\mathbf{k}/\mathbb{Q}$ be a Galois extension containing each of the λ_i. For each $\phi \in \mathrm{Gal}(\mathbf{k}/\mathbb{Q})$, the elements $\phi(\lambda_1), \ldots, \phi(\lambda_n) \in \mathbb{C}$ are also roots of unity that are not all equal, so we have

$$|\phi(\chi(g))| = |\phi(\lambda_1) + \cdots + \phi(\lambda_n)| < n.$$

We thus see that

$$\prod_{\phi \in \mathrm{Gal}(\mathbf{k}/\mathbb{Q})} |\phi\left(\frac{\chi(g)}{n}\right)| < 1.$$

Since $\chi(g)/n$ is a nonzero algebraic integer contained in $\mathbf{k}$, the left hand side lies in $\mathbb{Z}$, and thus must be 0, a contradiction. □

The following technical lemma will help us verify the hypotheses of Lemma 0.2. For $g \in G$, let $\mathrm{cl}(g)$ denote the conjugacy class of g.

Lemma 0.3. *Let G be a finite group and let V be an n-dimensional irreducible representation of G with character χ. For all $g \in G$, the number $|\,\mathrm{cl}(g)|\chi(g)/n$ is an algebraic integer.*

Proof. Fix some $g \in G$, and define

$$\omega = \sum_{h \in \mathrm{cl}(g)} h \in \mathbb{C}[G].$$

The element ω lies in the center of the ring $\mathbb{C}[G]$, so it acts on V by $\mathbb{C}[G]$-module endomorphisms. Since V is irreducible, Schur's Lemma says that $\mathrm{End}_{\mathbb{C}[G]}(V) = \mathbb{C}$, so there exists some $\lambda \in \mathbb{C}$ such that $\omega \cdot \vec{v} = \lambda \vec{v}$ for all $\vec{v} \in V$. Taking traces, we see that

$$n\lambda = \chi(\omega) = \sum_{h \in \mathrm{cl}(g)} \chi(h) = |\,\mathrm{cl}(g)|\chi(g),$$

so

$$\lambda = |\,\mathrm{cl}(g)|\chi(g)/n.$$

With respect to an appropriate basis, the entries of the matrices representing the action of G on V are algebraic integers (see, e.g., [2]). It follows that λ is an algebraic integer, and the lemma follows. □

We will use these results to prove the following key proposition:

Proposition 0.4. *Let G be a finite group such that there exists some $g \in G$ with $|\,\mathrm{cl}(g)| = p^k$ for some prime p and some $k \geq 1$. Then there exists some nontrivial representation V of G such that $Z(V) \neq 1$.*

Proof. Let $V_1, \ldots, V_r$ be the irreducible representations of G, ordered such that V_1 is the trivial representation. For $1 \leq i \leq r$, let χ_i be the character of V_i and $n_i = \dim(V_i)$. Since $|\,\mathrm{cl}(g)| \neq 1$, we have $g \neq 1$, so the orthogonality of the columns of the character table says that

$$0 = \sum_{i=1}^{r} \chi_i(g)\overline{\chi_i(1)} = \sum_{i=1}^{r} n_i \chi_i(g) = 1 + \sum_{i=2}^{r} n_i \chi_i(g).$$

Thus

$$\frac{-1}{p} = \sum_{i=2}^{r} \frac{n_i \chi_i(g)}{p}.$$

From this, we see that there must exist some $2 \leq j \leq r$ such that $n_j \chi_j(g)/p$ is not an algebraic integer. Since $\chi_j(g)$ is a sum of roots of unity, it is an algebraic integer. We deduce that p does not divide n_j and that $\chi_j(g) \neq 0$. Sine $|\,\mathrm{cl}(g)| = p^k$ and n_j are coprime, we can find $a, b \in \mathbb{Z}$ such that

$$1 = a|\,\mathrm{cl}(g)| + b n_j.$$

Multiplying this by $\chi_j(g)/n_j$, we get that

$$\frac{\chi_j(g)}{n_j} = a\frac{|\,\mathrm{cl}(g)|\chi_j(g)}{n_j} + b\chi_j(g).$$

Lemma 0.3 implies that this first term is an algebraic integer, and since $\chi_j(g)$ is an algebraic integer the second term is as well. We conclude that $\chi_j(g)/n_j$ is an algebraic integer, so by Lemma 0.2 we have that $g \in Z(V_j)$, as desired. □

Proof of Theorem A. Consider a group G with $|G| = p^a q^b$ for primes p and q. Our goal is to prove that G is solvable. By induction, we can assume that this holds for all such groups of smaller order. We can also assume that G is nonabelian since abelian groups are trivially solvable. Finally, we can assume that p and q are distinct and that $a, b \geq 1$ since otherwise G has prime power order and hence is nilpotent. It is enough to prove that under these circumstances, there exists a normal subgroup $N \lhd G$ that is nontrivial in the sense that $1 \subsetneq N \subsetneq G$. Indeed, our inductive hypothesis will then imply that both N and G/N are solvable, so G is as well.

Let $H < G$ be a Sylow q-subgroup. Since $|H| = q^b$, the group H is a q-group and thus is nilpotent. In particular, its center $Z(H)$ satisfies $Z(H) \neq 1$. Let $g \in Z(H)$ be nontrivial. The centralizer $C_G(g)$ thus contains H, so

$$|\operatorname{cl}(g)| = |G|/|C_G(g)| = p^k \qquad \text{for some } k \leq a.$$

If $k = 0$, then $g \in Z(G)$, so $Z(G)$ is our desired nontrivial normal subgroup. If $k \geq 1$, then instead Proposition 0.4 says that there exists a nontrivial representation V of G such that $Z(V) \neq 1$. If V is not a faithful representation of G, then its kernel is the desired nontrivial normal subgroup. If V is faithful, then $Z(V) \neq G$ since G is nonabelian, so $Z(V)$ is the desired nontrivial normal subgroup. □

References

[1] W. Burnside, On Groups of Order $p^\alpha q^\beta$, Proc. London Math. Soc. (2) 1 (1904), 388–392.

[2] A. Putman, Algebraicity of matrix entries of representations, informal note.

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