Classifying spaces and Brown representability

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Abstract

We sketch the proof of the Brown representability theorem and give a few applications of it, the most important being the construction of the classifying space for principal G-bundles.

Let G be a topological group and let CW be the homotopy category of based connected CW-complexes. A *classifying space* for G is a space BG such that for all $X \in \text{CW}$, there exists a bijection between the set of based principal G-bundles on X and the set $[X, BG]$ of all homotopy classes of basepoint-preserving maps from X to BG. This determines the homotopy groups of BG in the following way.

Theorem 0.1. *For all $n \geq 1$, we have $\pi_n(BG) \cong \pi_{n-1}(G)$.*

Remark. For instance, if G is a discrete group, we deduce that BG is a $K(G, 1)$.

Proof of Theorem 0.1. A based map $S^n \to BG$ is the same as a based principal G-bundle on S^n. Via the clutching construction, this is the same as a based map $S^{n-1} \to G$. □

Theorem 0.1 might suggest to the reader that the following is true.

Theorem 0.2. *The group G is homotopy equivalent to ΩBG.*

Proof. Let us give two equivalent descriptions of the map $f\colon G \to \Omega BG$.

- A map $f\colon G \to \Omega BG$ is the same as a map $\Sigma G \to BG$, which is the same as a based principal G-bundle on ΣG. There is an obvious choice of such a bundle, again arising from the clutching construction.
- More directly, we define $f\colon G \to \Omega BG$ as follows. Consider $g \in G$. Define $f(g) \in \Omega BG$ to be the based loop $f(g)\colon S^1 \to BG$ that classifies the G-bundle on S^1 obtained by gluing the ends of the trivial G-bundle on $[0, 1]$ together using g.

To prove that f is a homotopy equivalence, we show that it induces an isomorphism on all homotopy groups. In other words, for all $n \geq 1$ the map $[S^n, G] \to [S^n, \Omega BG]$ induced by f is an isomorphism. Now, using the clutching construction an element of $[S^n, G]$ is the same as a principle G-bundle on S^{n+1}, i.e. an element of $[S^{n+1}, BG] = [\Sigma S^n, BG] = [S^n, \Omega BG]$. It is easy to see that this bijection is induced by f. □

Of course, it is not obvious that a classifying space BG for G exists! There are several explicit constructions of BG, the first being due to Milnor [M]. Our next task is to give an abstract nonsense reason why BG must exist. The key is the following theorem, which was first proved by Brown [Bro].

Theorem 0.3 (Brown representability)**.** *Let $F\colon CW \to Set$ be a contravariant functor satisfying the following two properties.*

1. *Given any collection $\{X_\alpha\}$ of elements of CW, we have $F(\vee_\alpha X_\alpha) = \prod_\alpha F(X_\alpha)$.*

2. *Let X be an object of CW. Consider a cover $X = Y \cup Z$ by subcomplexes such that $Y, Z, Y \cap Z \in CW$. Then for all $y \in F(Y)$ and $z \in F(Z)$ that restrict to the same element of $F(Y \cap Z)$, there exists some $x \in F(X)$ that restricts to $y \in F(Y)$ and $z \in F(Z)$.*

Then there exists some $C \in CW$ and some $c \in F(c)$ such that for all $X \in CW$, the map $[X, C] \to F(X)$ taking $f: X \to C$ to $f^*(c)$ is a bijection.

Remark. It is absolutely necessary for us to consider based connected CW-complexes. The theorem is false without these assumptions; see [Bra].

Before we prove Theorem 0.3, let us give several examples of how it can be used.

Example. If G is a topological group, then we can apply Theorem 0.3 to the functor taking X to the set of based principal G-bundles; the result is the classifying space BG for G.

Example. For all $n \geq 1$, we can apply Theorem 0.3 to the cohomology functor $H^n(\cdot, A)$; the result is a $K(A, n)$ (as can be seen by plugging spheres into the statement).

Example. If T is a based connected topological space, then we can apply Theorem 0.3 to the functor $[\cdot, T]$. The result is a CW-approximation for T, i.e. a based connected CW-complex C such that $[X, C] = [X, T]$ for all $X \in \mathrm{CW}$. We remark that the image in $[C, T]$ of the identity in $[C, C]$ is the usual map that arises in a CW-approximation theorem.

We now sketch the proof of Theorem 0.3.

Proof sketch of Theorem 0.3. We begin by observing that it is enough to construct $C \in \mathrm{CW}$ and $c \in F(C)$ that satisfy the conclusion of the theorem for all spheres S^n with $n \geq 1$. Indeed, if $X \in \mathrm{CW}$ is arbitrary, then for $x \in F(X)$ we can construct $f: X \to C$ satisfying $f^*(c) = x$ by the usual "cell by cell" procedure, and similarly if $f, f': X \to C$ satisfy $f^*(c) = (f')^*(c) = x$, then we can construct a homotopy from f to f' cell by cell.

We will construct C as follows. Start with $C_0 = \{*\}$ and c_0 the unique element of $F(C_0)$. Assume that C_{n-1} and $c_{n-1} \in F(C_{n-1})$ has been constructed such that for all $1 \leq k \leq n-1$, we have $[S^k, C_{n-1}] = F(S^k)$ via pullback of c_{n-1}. We will construct a CW-complex C_n containing C_{n-1} as a subcomplex together with $c_n \in F(C_n)$ that restricts to $c_{n-1} \in F(C_{n-1})$. The complex C_n will be obtained from C_{n-1} by attaching n-cells and $(n+1)$-cells, and from this it is easy to see that we still have $[S^k, C_n] = F(S^k)$ via pullback of c_n for all $1 \leq k \leq n-1$. We just have to find the right cells to attach to make this true for S^n as well. There are two parts to this (generators and relations):

- First, we wedge on an n-sphere for each element of $F(S^n)$ to get a complex C'_n. Using the first condition in the theorem, we can extend c_{n-1} to $c'_n \in F(C'_n)$ such that the map $f_x: S^n \to C'_n$ taking S^n to the sphere representing $x \in F(S^n)$ satisfies $(f_x)^*(c'_n) = x$. This implies that the map $[S^n, C'_n] \to F(S^n)$ is surjective.

- We now want to make it injective. Observe that the cogroup structure on S^n (the same one that makes homotopy groups into groups) makes $F(S^n)$ into a group. The map $[S^n, C'_n] \to F(S^n)$ is then a group homomorphism. To construct C_n, we attach cells to C'_n to kill off the kernel. Extending c'_n over C_n requires the second condition in the theorem.

Repeating this procedure, we get an increasing sequence

$$C_0 \subset C_1 \subset C_2 \subset \cdots$$

of based connected CW-complexes. Define

$$C = \cup_{n=0}^{\infty} C_n.$$

We now come to the final subtle point of the proof, namely constructing an element $c \in F(C)$ that restricts to $c_n \in F(C_n)$ for all n. The issue here is that we have not assumed any kind of "continuity" for our functor F. Indeed, there is a map

$$F(C) \to \lim_{\leftarrow} F(C_n),$$

but this map need not be bijective. However, it is surjective, which is good enough for us. To prove that it is surjective, replace C by the telescoping collection of mapping cylinders $M(C_n \to C_{n+1})$ (with the basepoints all collapsed to points so that everything is based). This does not change the homotopy type of C; however, we can now decompose C as $X \cup Y$, where X is the union of the even mapping cylinders $M(C_{2n} \to C_{2n+1})$ and Y is the union of the odd mapping cylinders $M(C_{2n+1} \to C_{2n+2})$. Since we have collapsed basepoints, X is actually the wedge of the spaces $M(C_{2n} \to C_{2n+1})$, and similarly for Y. The space $M(C_n \to C_{n+1})$ is homotopy equivalent to C_{n+1}, so we can view c_{n+1} as an element of $F(M(C_n \to C_{n+1}))$. Using the first condition in the theorem, we can then construct elements $c_x \in F(X)$ and $c_y \in F(Y)$ restricting to the various c_n. It is clear that c_x and c_y restrict to the same element of $F(X \cap Y)$ (here $X \cap Y$ is another wedge!), so the second condition in the theorem allows us to glue c_x and c_y together to an element $c \in F(C)$. □

References

[Bra] Martin Brandenburg (https://mathoverflow.net/users/2841/martin-brandenburg), unpointed brown representability theorem, URL (version: 2017-04-13): https://mathoverflow.net/q/11456

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