# Yet another book on topology II: covering spaces and the fundamental group

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# Contents

# Preface

This is the second volume of what I anticipate will be a four-volume textbook on topology:

- **Vol. 1**: Point-set topology
- **Vol. 2**: Covering spaces and the fundamental group
- **Vol. 3**: Homology
- **Vol. 4**: Cohomology and Poincaré duality

This preface describes the overall goals for this series of books. For a discussion of the mathematical choices I made in this particular volume and how they differ from other topology books, see the introduction for the instructor. That introduction also explains how this book can be used for different kinds of courses. There is also an introduction for the student that sets the stage for the various topics in this book.

### Structure of books

Each volume of this series of books attempts to do three things:

(a) give a linear, well-structured treatment of the subject that is suitable for a first course; and
(b) explain a large number of advanced topics and examples; and
(c) describe (with complete proofs!) how the tools developed in that volume can be applied in other areas of mathematics.

These goals are not compatible with each other.

The solution I have adopted is to divide each volume into two parts. The first part ("Basic topics") represents what I believe should be covered in a first course on the subject. These chapters are intended to be read linearly. Even here there are options, and some chapters and sections are marked with an $*$ to indicate that a first-time reader might want to skip them.

The second part of each volume is a collection of essays on various topics: examples, applications, extensions, topological lore that does not fit neatly into a course, etc. These essays are less self-contained than the basic topics, though they include many proofs and detailed examples. They are not intended to be read linearly, though some refer to other essays for details.[1] They could be used for various kinds of advanced short courses or to enrich a course based on the first part of the volume.

### Details and precision

Topology has an overwhelming amount of machinery. Many theorems have multiple versions with subtly different hypotheses and conclusions. As a writer, it is easy to get lost in technical details. The result is a book that is more of an encyclopedia for experts than a textbook.

However, there is also an opposite danger. Students find it frustrating if they have trouble finding precise definitions and statements of theorems. It can also be frustrating to have to flip between multiple sections to figure out what the author is trying to say.

I therefore tried to strike a balance between technical precision and narrative. For instance, I have not hesitated to repeat definitions for things that last appeared many pages ago. I also try to state theorems in their most natural level of generality, always with an eye to what is useful in applications.

I also try to include proofs that are detailed enough that a typical PhD student can read them without suffering unduly. Of course, a balance has to be struck here: too many details can make it

[1] I promise that there are no circular chains of references!

hard for a reader to figure out the main idea. However, I think that overall I include more details than many authors. This is partially responsible for the lengths of these books.

### Pictures

I include many pictures, and freely use colors to make those pictures easier to understand. Typically these pictures are included in-line in the middle of a paragraph rather than in a separate stand-alone figure box. One technical issue here is that students with various forms of color blindness might have trouble interpreting the pictures. To help alleviate this, I used the palettes from the article [**3**], which I recommend to anyone who wants to use color but is worried about accessibility.

### Audience

Volume 1 (point-set topology) is geared to advanced undergraduates, though it also includes many special topics that are hard to find in the literature and might be of interest to more advanced readers. The subsequent three volumes are aimed at beginning graduate students, roughly at the level of our first-year students at Notre Dame. The level of sophistication does rise as the books progress, largely driven by the increasing difficulty of the material.

### Smooth manifolds

A lot of the early literature on topology focused on spaces that were triangulated. This is reflected in the textbook literature, which often uses combinatorial/piecewise-linear tools. However, outside of algebraic topology smooth manifolds are far more important. There are now many excellent textbooks on smooth manifolds, and I think it makes sense to replace simplicial/piecewise-linear techniques with smooth ones whenever possible.

The level of familiarity with smooth manifolds that I assume gradually increases through the books. Volume 1 (point-set topology) does not assume that students have seen manifolds of any kind before. Volume 2 (covering spaces and the fundamental group) mentions smooth manifolds, but only in remarks and essays. Subsequent volumes assume more, roughly at the level of the classic books by Milnor [**2**] or Guillemin–Pollack [**1**]. This is compatible with the sequence of courses taken by PhD students at most institutions I am familiar with.

### Abstraction vs. concreteness

The following two quotations paraphrase things told to me by topologists whose work I admire:

(a) "I don't care about abstract nonsense or category theory at all. It's a distraction from what really matters. Fundamentally, I don't feel like I understand something until I've figured out a picture that illustrates what is going on."
(b) "I'm not interested in spaces for their own sake. What I care about are definitions and analogies. Category theory is, for me, the central framework for understanding mathematics as a unified whole."

I find myself in the unusual position of being sympathetic to both of these points of view. I love concrete geometry and abstract machinery, and think the best mathematics involves a mixture of both.

I do think that some students come to topology with an unhealthy attitude towards category theory. For some parts of topology and algebra, it has a marvelous unifying power. I cannot imagine ignoring category theory while teaching those subjects. However, it is not magic and needs to be combined with other tools to give interesting results. Students must learn to be eclectic and not expect any single tool to unify all of mathematics.

In keeping with the above, I try to use categorical language (functors, natural transformations, universal properties, etc) when appropriate, but not to over-emphasize it. Topology has many beautiful things to say about concrete geometric questions, and I hope even readers with a taste for abstraction will come to appreciate how all these tools can be used to prove results that seemingly have little to do with high-powered machinery.

## Stylistic goals

I tried hard to obey the following stylistic rules:

(a) Each chapter is short (typically at most 8 pages, not including exercises), does exactly one thing, and is broken up into short sections with descriptive titles.
(b) I avoid overly long unbroken blocks of text. More concretely, I try to avoid having more than 6 lines of text appear without a paragraph break or math display.
(c) I give examples of each new concept, and try to illustrate major results with multiple examples.
(d) I include a large number of exercises in almost every chapter. These are an essential part of the text. Exercises marked with an $*$ are more difficult than the others, and the particularly difficult exercises are marked with $**$.

## Acknowledgements

This is where I thank my many teachers, mentors, collaborators, friends, etc. It will be written later.

## Bibliography

[1] V. W. Guillemin and A. Pollack, *Differential topology*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1974. (Cited on page 2.)
[2] J. W. Milnor, *Topology from the differentiable viewpoint*, revised reprint of the 1965 original, Princeton Landmarks in Mathematics, Princeton Univ. Press, Princeton, NJ, 1997. (Cited on page 2.)
[3] B. Wong, Points of view: Color blindness, Nat. Methods 8 (2011), 441. (Cited on page 2.)

# Introduction for the instructor

This books covers covering spaces and the fundamental group. To help explain the mathematical choices I made when writing it, this introduction first describes the history of the subject. I then describe the overall structure of this book and how it differs from other books on these topics. I then describe the individual chapters and how they can be used in different kinds of courses.

# Introduction for the student

# Part 1

# Basic topics

# Covering spaces

CHAPTER 1

# Definition and basic examples of covering spaces

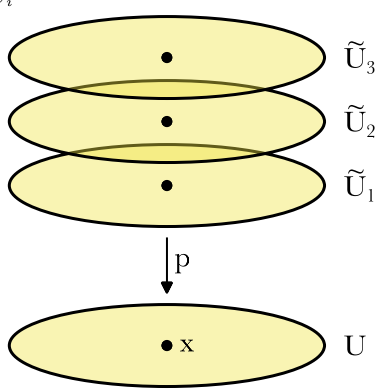
Our first main topic is the theory of covering spaces. This chapter contains some basic definitions and a large number of examples. A first-time reader might be tempted to skip the examples and focus on the theory. This would be a mistake. The richness of the examples is what gives this subject its flavor, and it is impossible to understand the theoretical aspects of covering spaces without having absorbed a large store of these examples.

## 1.1. Definition and examples

Recall that a *local homeomorphism* is a map $p\colon Z \to X$ such that all $z \in Z$ have open neighborhoods $V$ with $U = p(V)$ open and $p|_V\colon V \to U$ a homeomorphism. Roughly speaking, in a covering space this condition is strengthened by adding a uniformity condition to these open neighborhoods. The definition is as follows:

DEFINITION 1.1.1. A *covering space* or simply a *cover* of a space $X$ is a space $\widetilde{X}$ equipped with a map $p\colon \widetilde{X} \to X$ such that for all $x \in X$, there is an open neighborhood $U$ of $x$ satisfying:

- the preimage $p^{-1}(U)$ is the disjoint union of open subsets $\{\widetilde{U}_i\}_{i\in\mathcal{I}}$ of $\widetilde{X}$ such that for all $i \in \mathcal{I}$, the restriction $p|_{\widetilde{U}_i}\colon \widetilde{U}_i \to U$ is a homeomorphism.




We call $U$ a *trivialized neighborhood*[1] of $x$ (or just a *trivialized open set* if we do not want to emphasize $x$) and each $\widetilde{U}_i$ a *sheet* of $\widetilde{X}$ over $U$. We will also often call the map $p\colon \widetilde{X} \to X$ a covering space, and refer to $X$ as the *base* of the cover. □

REMARK 1.1.2. We allow $p^{-1}(U) = \emptyset$. In particular, for any space $X$ the map $p\colon \emptyset \to X$ is a covering space. This convention is controversial, and some authors require the maps in covering spaces to be surjective. □

Here are some easy examples:

EXAMPLE 1.1.3 (Trivial cover). For a space $X$, the identity map $\mathbb{1}_X\colon X \to X$ is a covering space. More generally, for any discrete set $\mathcal{I}$ the projection map $p\colon X \times \mathcal{I} \to X$ is a covering space. We will call these the *trivial covers* of $X$. □

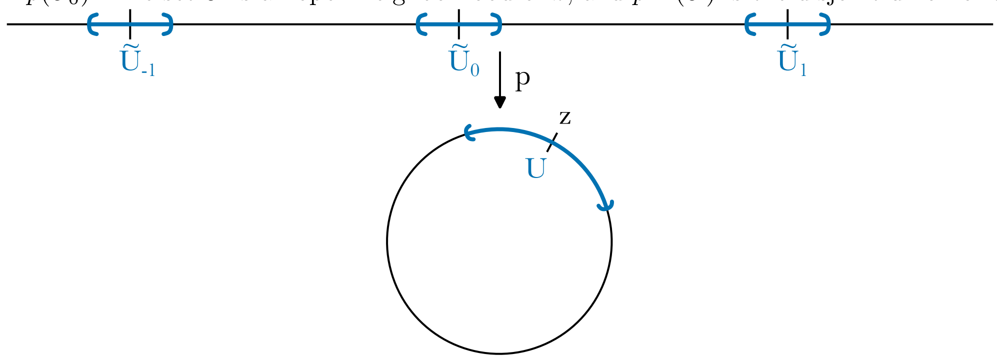
EXAMPLE 1.1.4 (Universal cover of circle). Regard $\mathbb{S}^1$ as the unit circle in the complex plane $\mathbb{C}$. Let $p\colon \mathbb{R} \to \mathbb{S}^1$ be the map $p(\theta) = e^{2\pi i\theta}$. This is a covering space. Indeed, consider $z \in \mathbb{S}^1$. Write

[1] It is also often called an *evenly covered neighborhood*, though we will not use that terminology.

$z = e^{2\pi i \theta_0}$. Pick $\epsilon > 0$ with $\epsilon < 1/2$. For $n \in \mathbb{Z}$, set

$$\widetilde{U}_n = (\theta_0 + n - \epsilon, \theta_0 + n + \epsilon) \subset \mathbb{R}.$$

Let $U = p(\widetilde{U}_0)$. The set $U$ is an open neighborhood of $z$, and $p^{-1}(U)$ is the disjoint union of the $\widetilde{U}_n$:




Each $\widetilde{U}_n$ projects homeomorphically to $U$, so $U$ is a trivialized neighborhood of $z$ and the $\widetilde{U}_n$ are the sheets over $U$. The covering space $p\colon \mathbb{R} \to \mathbb{S}^1$ is called the *universal cover* of $\mathbb{S}^1$. See §1.5 below for why it has this name. □

EXAMPLE 1.1.5. Let $X$ be a locally connected space and let $U \subset X$ be a union of connected components of $X$. Since $X$ is locally connected, $U$ is open. The inclusion $\iota\colon U \to X$ is a covering space. Indeed, consider $x \in X$ and let $C$ be the connected component of $X$ containing $x$. Since $X$ is locally connected, $C$ is open. If $C \subset U$ then $\iota^{-1}(C) = C$ and $\iota|_C\colon C \to C$ is the identity; otherwise, $\iota^{-1}(C) = \emptyset$. It follows that $C$ is a trivialized neighborhood of $x$. We remark that if $U \neq X$ then $\iota\colon U \to X$ is not surjective. □

REMARK 1.1.6. Let $p\colon \widetilde{X} \to X$ be a covering space. From the definitions, it is straightforward to see that $p$ is an open map and that $p(\widetilde{X})$ is a union of connected components of $X$ (see Exercises 1.4 and 1.5). □

REMARK 1.1.7. Covering spaces are local homeomorphisms, but the converse does not hold. For instance, the map $p\colon (0,2) \to \mathbb{S}^1$ defined by $p(\theta) = e^{2\pi i \theta}$ is a local homeomorphism but does not have a trivialized neighborhood around $x = 1$ (see Exercise 1.3). However, if $\widetilde{X}$ and $X$ are compact Hausdorff spaces then all local homeomorphisms $p\colon \widetilde{X} \to X$ are covering spaces. We will say more about this in §1.6*. □

## 1.2. Degree of a cover

Let $p\colon \widetilde{X} \to X$ be a covering space. The preimages $p^{-1}(x) \subset \widetilde{X}$ of points $x \in X$ are called the *fibers* of $p\colon \widetilde{X} \to X$. For $x \in X$, the fiber $p^{-1}(x)$ is called the *fiber over* $x$. The first main property of covering spaces is that if $X$ is connected, then the cardinalities of its fibers are all equal. More generally:

LEMMA 1.2.1. *Let $p\colon \widetilde{X} \to X$ be a covering space. Then the cardinal-number-valued function*

$$x \mapsto |p^{-1}(x)| \quad \textit{for } x \in X$$

*is locally constant. In particular, if $X$ is connected then all fibers $p^{-1}(x)$ have the same cardinality.*

PROOF. Consider $x \in X$. Let $U$ be a trivialized neighborhood of $x$ and let $\{\widetilde{U}_i\}_{i \in \mathcal{I}}$ be the sheets of $\widetilde{X}$ over $U$. For $y \in U$, the preimage $p^{-1}(y)$ consists of one point in each $\widetilde{U}_i$, so $|p^{-1}(y)| = |\mathcal{I}|$. The lemma follows. □

This suggests the following definition:

DEFINITION 1.2.2. Let $p\colon \widetilde{X} \to X$ be a covering space. If all the fibers of $p$ have the same cardinality, then the common cardinality of those fibers is the *degree* of the cover. For an integer $n \geq 0$, we will call a degree-$n$ cover an *$n$-sheeted* or an *$n$-fold* cover. If the degree of $p$ is an infinite cardinal, then we will often simply say that $p$ has degree infinity or is an infinite-degree cover. □

For instance, the degree of the universal cover $p\colon \mathbb{R} \to \mathbb{S}^1$ is infinity. Lemma 1.2.1 implies that if $X$ is connected, then every covering space $p\colon \widetilde{X} \to X$ has a degree. This is false for non-connected $X$ (see Example 1.1.5 and Exercise 1.7).

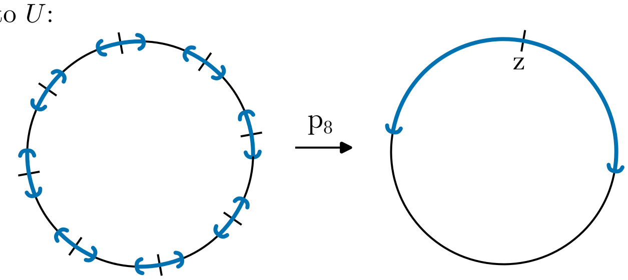
## 1.3. More examples of covering spaces

Here are some more examples of covering spaces:

EXAMPLE 1.3.1 (Degree $n$ cover of circle). Regard $\mathbb{S}^1$ as the unit circle in $\mathbb{C}$. Fix some $n \geq 1$, and define $p_n\colon \mathbb{S}^1 \to \mathbb{S}^1$ via the formula $p_n(z) = z^n$. This is a degree $n$ covering space. Indeed, consider $z \in \mathbb{S}^1$. The preimage $p_n^{-1}(z)$ consists of $n$ distinct points: writing $z = e^{2\pi i \theta_0}$, we have

$$p_n^{-1}(z) = \left\{ e^{2\pi i(\theta_0+m)/n} \mid m \text{ is an integer with } 0 \leq m < n \right\}.$$

Fix some $\epsilon > 0$ with $\epsilon < 1/2$, and let $U = \left\{ e^{2\pi i \theta} \mid \theta \in (\theta_0 - \epsilon, \theta_0 + \epsilon) \right\}$. The set $U$ is an open neighborhood of $z$, and $p_n^{-1}(U)$ is the disjoint union of $n$ subsets of $\mathbb{S}^1$ each of which projects homeomorphically onto $U$:




Thus $U$ is a trivialized neighborhood of $z$ and the components of $p_n^{-1}(U)$ are the sheets over $U$. □

EXAMPLE 1.3.2 (Cosets of discrete subgroups). Let $\mathbf{G}$ be a topological group, i.e., a group that is a topological space such that the product map $\mathbf{G} \times \mathbf{G} \to \mathbf{G}$ and inversion map $\mathbf{G} \to \mathbf{G}$ are continuous. Let $\mathbf{H}$ be a discrete subgroup of $\mathbf{G}$. Here are two examples to keep in mind:

- $\mathbf{G}$ is the additive group $\mathbb{R}^n$, and $\mathbf{H} = \mathbb{Z}^n$; and
- $\mathbf{G} = \mathrm{SL}_n(\mathbb{R})$ and $\mathbf{H} = \mathrm{SL}_n(\mathbb{Z})$.

Endow the set $\mathbf{G}/\mathbf{H} = \{g\mathbf{H} \mid g \in \mathbf{G}\}$ of left cosets with the quotient topology. The quotient map $p\colon \mathbf{G} \to \mathbf{G}/\mathbf{H}$ is an open map.[2] We claim that $p\colon \mathbf{G} \to \mathbf{G}/\mathbf{H}$ is a cover of degree $|H|$. Indeed, consider a point $g_0\mathbf{H}$ of $\mathbf{G}/\mathbf{H}$. Since $\mathbf{H}$ is a discrete subgroup of $\mathbf{G}$, we can find an open neighborhood $V$ of $1 \in \mathbf{G}$ whose translates $\{Vh \mid h \in \mathbf{H}\}$ are all disjoint.[3] Set $U = p(g_0V)$, which is open since $p$ is an open map. We have

$$p^{-1}(U) = \bigsqcup_{h \in \mathbf{H}} g_0 V h.$$

These are disjoint open sets. For $h \in \mathbf{H}$, the map $p|_{g_0Vh}\colon g_0Vh \to U$ is an open bijection, and hence a homeomorphism. It follows that $U$ is a trivialized neighborhood and the sets $g_0Vh$ with $h \in \mathbf{H}$ are the sheets above $U$. □

EXAMPLE 1.3.3. Two of our previous examples are special cases of Example 1.3.2:

- The universal cover $p\colon \mathbb{R} \to \mathbb{S}^1$. Indeed, the additive topological group $\mathbb{R}$ contains the discrete subgroup $\mathbb{Z}$. We have $\mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$, and this homeomorphism fits into a commutative diagram of the following form:

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[2]For an open set $W \subset \mathbf{G}$, the set $p(W) \subset \mathbf{G}/\mathbf{H}$ is open since $p^{-1}(p(W)) = \cup_{h \in \mathbf{H}} Wh$ is a union of open sets.

[3]Here are some more details. Since $\mathbf{H}$ is discrete, we can find an open neighborhood $W$ of $1 \in \mathbf{G}$ such that $W \cap \mathbf{H} = \{1\}$. Let $f\colon \mathbf{G} \times \mathbf{G} \to \mathbf{G}$ be the map $f(x,y) = y^{-1}x$. Since $f$ is continuous, the set $f^{-1}(W)$ is an open neighborhood of $(1,1)$ and thus we can find open neighborhoods $V_1$ and $V_2$ of 1 such that $V_1 \times V_2 \subset f^{-1}(W)$. Letting $V = V_1 \cap V_2$, we then have $f(V \times V) \subset W$. We now claim that the sets $\{Vh \mid h \in \mathbf{H}\}$ are all disjoint. Indeed, if $h_1, h_2 \in \mathbf{H}$ are such that $(Vh_1) \cap (Vh_2) \neq \emptyset$, then we can find $v_1, v_2 \in V$ with $v_1h_1 = v_2h_2$, and hence

$$h_2h_1^{-1} = v_2^{-1}v_1 \in f(V \times V) \cap \mathbf{H} \subset W \cap \mathbf{H} = \{1\}.$$

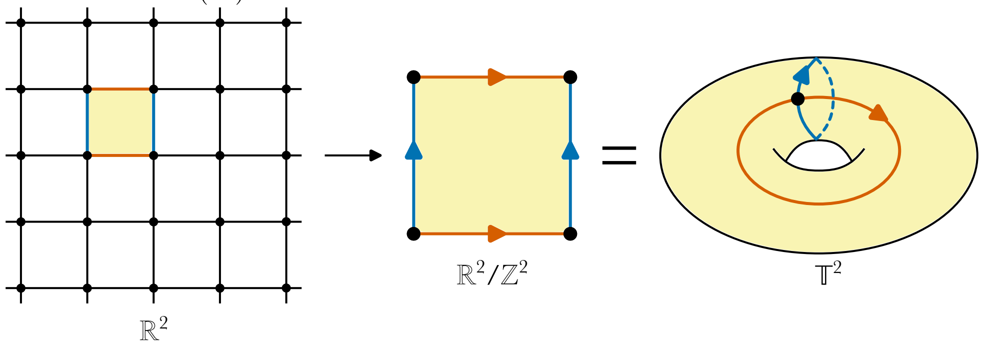
In other words, $h_1 = h_2$.

$$\begin{tikzcd} & \mathbb{R} \arrow[dl] \arrow[dr, "p"] & \\ \mathbb{R}/\mathbb{Z} \arrow[rr, "\cong"] & & \mathbb{S}^1 \end{tikzcd}$$

Using this, we can identify the covers $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ and $p\colon \mathbb{R} \to \mathbb{S}^1$.

- The covers $p_n\colon \mathbb{S}^1 \to \mathbb{S}^1$ defined by $p_n(z) = z^n$. Indeed, $\mathbb{S}^1 \subset \mathbb{C}$ is a topological group under multiplication, and it contains the discrete group $\mu_n$ of $n^{\text{th}}$ roots of unity. The quotient $\mathbb{S}^1/\mu_n$ is homeomorphic to $\mathbb{S}^1$, and just like above we can identify the covers $\mathbb{S}^1 \to \mathbb{S}^1/\mu_n$ and $p_n\colon \mathbb{S}^1 \to \mathbb{S}^1$.

As another example, as we noted in Example 1.3.2 the additive group $\mathbb{R}^n$ contains the discrete subgroup $\mathbb{Z}^n$. As the following figure illustrates, the quotient $\mathbb{R}^n/\mathbb{Z}^n$ is homeomorphic to an $n$-dimensional torus $\mathbb{T}^n = (\mathbb{S}^1)^{\times n}$:

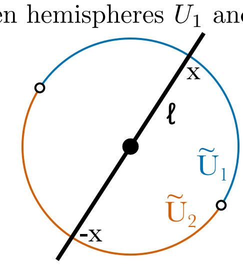



This figure shows the case $n = 2$, and in it the indicated region is a fundamental domain which in the quotient becomes a square with sides identified as indicated. Identifying $\mathbb{R}^n/\mathbb{Z}^n$ with $\mathbb{T}^n$, we get an infinite-degree cover $p\colon \mathbb{R}^n \to \mathbb{T}^n$. □

Example 1.3.4 (Real projective space). Let $\mathbb{RP}^n$ be $n$-dimensional real projective space, that is, the set of lines through the origin in $\mathbb{R}^{n+1}$. Topologize $\mathbb{RP}^n$ as follows:

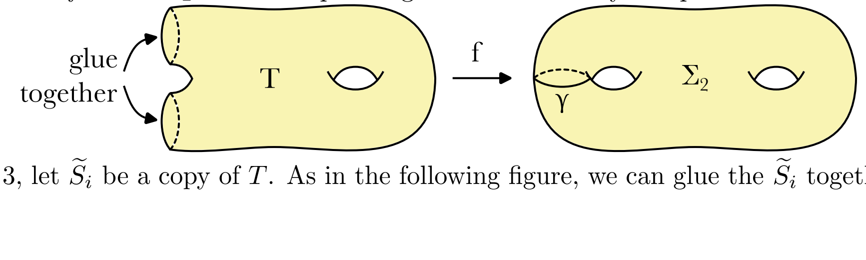
- Let $\pi\colon \mathbb{R}^{n+1} \setminus 0 \to \mathbb{RP}^n$ be the map taking a nonzero point $x \in \mathbb{R}^{n+1}$ to the line determined by 0 and $x$. Give $\mathbb{RP}^n$ the quotient topology determined by $\pi$, so a set $U \subset \mathbb{RP}^n$ is open if and only if $\pi^{-1}(U)$ is open.

We have $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. Let $p\colon \mathbb{S}^n \to \mathbb{RP}^n$ be the restriction of $\pi$ to $\mathbb{S}^n$. This is a degree 2 covering space. Indeed, consider $\ell \in \mathbb{RP}^n$. The line $\ell$ intersects $\mathbb{S}^n$ in two antipodal points $x, -x \in \mathbb{S}^n$. Let $U \subset \mathbb{RP}^n$ be the set of lines $\ell'$ that are *not* orthogonal to $\ell$. This is an open set, and the preimage $p^{-1}(U)$ is the disjoint union of two open hemispheres $\widetilde{U}_1$ and $\widetilde{U}_2$ centered at $x$ and $-x$, respectively:

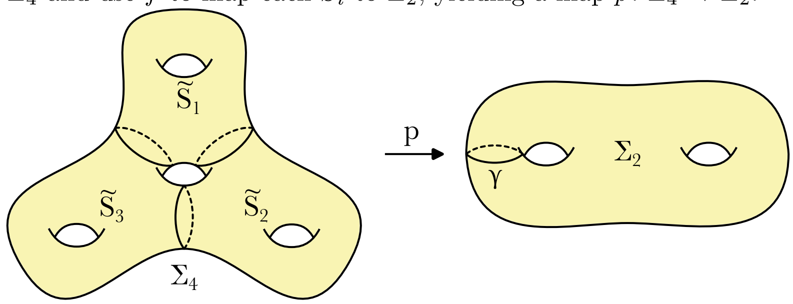



Each $\widetilde{U}_i$ projects homeomorphically to $U$, so $U$ is a trivialized neighborhood and the $\widetilde{U}_i$ are the sheets over $U$. □

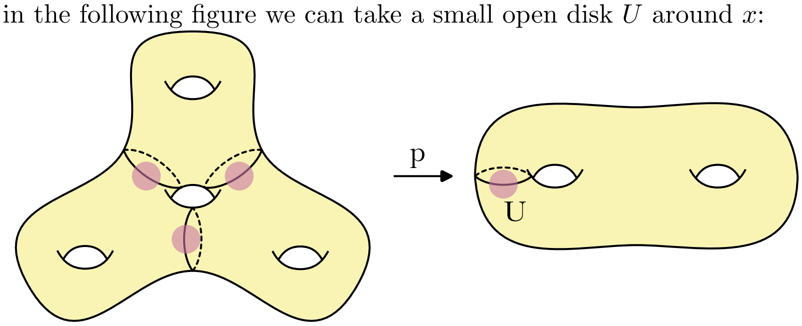
Example 1.3.5. Let $\Sigma_2$ be a genus 2 surface and let $T$ be a genus 1 surface with two boundary components. Let $f\colon T \to \Sigma_2$ be the map that glues the boundary components to form a loop $\gamma$:




For $1 \le i \le 3$, let $\widetilde{S}_i$ be a copy of $T$. As in the following figure, we can glue the $\widetilde{S}_i$ together to form a genus 4 surface $\Sigma_4$ and use $f$ to map each $\widetilde{S}_i$ to $\Sigma_2$, yielding a map $p\colon \Sigma_4 \to \Sigma_2$:




Each of the three black loops in $\Sigma_4$ maps homeomorphically onto the black loop $\gamma$ in $\Sigma_2$. The map $p\colon \Sigma_4 \to \Sigma_2$ is a degree 3 covering space. Indeed, consider a point $x \in \Sigma_2$. If $x \notin \gamma$, then for our trivialized neighborhood we can take $U = \Sigma_2 \setminus \gamma$. The sheets above $U$ are the $\operatorname{Int}(\widetilde{S}_i)$. If instead $x \in \gamma$, then as in the following figure we can take a small open disk $U$ around $x$:




The three disks shown in $\Sigma_4$ are each mapped homeomorphically to $U$, so $U$ is a trivialized neighborhood. $\square$

## 1.4. Isomorphisms between covering spaces

We would like to classify the covers of a space $X$. To do this, we must first define what it means for two covers to be the same, i.e., we must say what it means to have an isomorphism between two covers of $X$. The definition is as follows:

Definition 1.4.1. Let $X$ be a space and let $p_1\colon \widetilde{X}_1 \to X$ and $p_2\colon \widetilde{X}_2 \to X$ be two covers of $X$. A *covering space isomorphism* from $\widetilde{X}_1$ to $\widetilde{X}_2$ is a homeomorphism $f\colon \widetilde{X}_1 \to \widetilde{X}_2$ such that the diagram

$$\begin{tikzcd} \widetilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \widetilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

commutes, i.e., such that $p_2 \circ f = p_1$. If a covering space isomorphism from $\widetilde{X}_1$ to $\widetilde{X}_2$ exists, we say that $\widetilde{X}_1$ and $\widetilde{X}_2$ are *isomorphic* covers of $X$. This is clearly an equivalence relation. $\square$

Remark 1.4.2. This can be rephrased using categorical language as follows. Recall that $\mathtt{Top}$ is the category of topological spaces and continuous maps. For a space $X$, let $\mathtt{Top}_{/X}$ be the category whose objects are spaces $Y$ equipped with maps $\phi\colon Y \to X$ and whose morphisms from $\phi_1\colon Y_1 \to X$ to $\phi_2\colon Y_2 \to X$ are maps $f\colon Y_1 \to Y_2$ such that the diagram

$$\begin{tikzcd} Y_1 \arrow[rr, "f"] \arrow[dr, "\phi_1"'] & & Y_2 \arrow[dl, "\phi_2"] \\ & X & \end{tikzcd}$$

commutes. A covering space $p\colon \widetilde{X} \to X$ is an object of $\mathtt{Top}_{/X}$, and a covering space isomorphism is an isomorphism in $\mathtt{Top}_{/X}$ between two covering spaces. $\square$

Here are two basic examples.

EXAMPLE 1.4.3. For a real number $\lambda \neq 0$, define $p^\lambda \colon \mathbb{R} \to \mathbb{S}^1$ via the formula $p_\lambda(\theta) = e^{2\pi i \lambda \theta}$. The universal cover of $\mathbb{S}^1$ is thus $p^1 \colon \mathbb{R} \to \mathbb{S}^1$. Each $p^\lambda \colon \mathbb{R} \to \mathbb{S}^1$ is also a covering space, but is isomorphic to the universal cover. Indeed, letting $f \colon \mathbb{R} \to \mathbb{R}$ be the homeomorphism $f(\theta) = \lambda\theta$, the diagram

$$\begin{tikzcd} \mathbb{R} \arrow[rr, "f"] \arrow[dr, "p^\lambda"'] & & \mathbb{R} \arrow[dl, "p^1"] \\ & \mathbb{S}^1 & \end{tikzcd}$$

commutes, so $f$ is a covering space isomorphism from $p^\lambda \colon \mathbb{R} \to \mathbb{S}^1$ to $p^1 \colon \mathbb{R} \to \mathbb{S}^1$. ☐

EXAMPLE 1.4.4. For $n \geq 1$, let $p_n \colon \mathbb{S}^1 \to \mathbb{S}^1$ be the covering space defined by the formula $p_n(z) = z^n$ and let $q_n \colon \mathbb{S}^1 \to \mathbb{S}^1$ be the covering space defined by the formula $q_n(z) = z^{-n}$. The covers $p_n \colon \mathbb{S}^1 \to \mathbb{S}^1$ and $q_n \colon \mathbb{S}^1 \to \mathbb{S}^1$ are isomorphic. Indeed, letting $f \colon \mathbb{S}^1 \to \mathbb{S}^1$ be the homeomorphism $f(z) = z^{-1}$, the diagram

$$\begin{tikzcd} \mathbb{S}^1 \arrow[rr, "f"] \arrow[dr, "p_n"'] & & \mathbb{S}^1 \arrow[dl, "q_n"] \\ & \mathbb{S}^1 & \end{tikzcd}$$

commutes, so $f$ is a covering space isomorphism from $p_n \colon \mathbb{S}^1 \to \mathbb{S}^1$ to $q_n \colon \mathbb{S}^1 \to \mathbb{S}^1$. ☐

## 1.5. Goal

One of our main goals is to classify all the covers of a space up to isomorphism. Remarkably, for a reasonable space $X$ there is a simple *algebraic* classification of covers of $X$. We will describe this classification later after we define the fundamental group of $X$ (see Chapter 11). It resembles the classical Galois correspondence.

To a reasonable path connected space $X$, we will associate a group $G$ called its *fundamental group*. Isomorphism classes of covers $p \colon \widetilde{X} \to X$ with $\widetilde{X}$ path connected will correspond[4] to subgroups $K < G$. The cover corresponding to the trivial subgroup $1 < G$ will be called the *universal cover*, and it will cover all the other covers. Here is an example:

EXAMPLE 1.5.1. The fundamental group of $\mathbb{S}^1$ will turn out to be $\mathbb{Z}$. There are two kinds of subgroups of $\mathbb{Z}$:

- For $n \geq 1$, the subgroup $n\mathbb{Z}$. This will correspond to the cover $p_n \colon \mathbb{S}^1 \to \mathbb{S}^1$ defined by $p_n(z) = z^n$ for $z \in \mathbb{S}^1$.
- The trivial subgroup $0 < \mathbb{Z}$. This will correspond to the universal cover $p \colon \mathbb{R} \to \mathbb{S}^1$ defined by $p(\theta) = e^{2\pi i \theta}$ for $\theta \in \mathbb{R}$.

The universal cover $p \colon \mathbb{R} \to \mathbb{S}^1$ covers the cover $p_n \colon \mathbb{S}^1 \to \mathbb{S}^1$ in the following sense. Define $q_n \colon \mathbb{R} \to \mathbb{S}^1$ via $q_n(\theta) = e^{2\pi i \theta / n}$ for $\theta \in \mathbb{R}$. This is a covering space, and we have a factorization

$$\begin{tikzcd} \mathbb{R} \arrow[r, "q_n"] \arrow[rr, bend right, "p"'] & \mathbb{S}^1 \arrow[r, "p_n"] & \mathbb{S}^1. \end{tikzcd}$$ ☐

## 1.6*. Spaces of polynomials, and covers vs local homeomorphisms

We close this chapter with an important example of a covering space obtained from certain spaces of polynomials. The key input is a criterion for a local homeomorphism to be a covering space. Since this material is a little harder than the rest of this chapter, we have marked it with an $*$ to indicate that a first-time reader might want to skip it.

We start by recalling some definitions from point-set topology. See Volume 1 for more details. Let $Z$ be a space. A *compact neighborhood* of a point $z \in Z$ is a compact set $K \subset Z$ such that $z \in \operatorname{Int}(K)$. The space $Z$ is *locally compact* if for all $z \in Z$ and all open neighborhoods $U$ of $z$, there is a compact neighborhood $K$ of $z$ with $K \subset U$.

---

[4]There is a small issue with basepoints we are ignoring here to simplify our story.

EXAMPLE 1.6*.1. All open subsets of $\mathbb{R}^n$ and all closed subsets of $\mathbb{R}^n$ are locally compact. □

Assume that $Y$ and $Z$ are locally compact Hausdorff spaces. A map $f: Y \to Z$ is *proper* if for all compact subsets $K \subset Z$, the preimage $f^{-1}(K)$ is compact.[5]

EXAMPLE 1.6*.2. The Heine–Borel theorem says that a subset of Euclidean space is compact if and only if it is closed and bounded. If $Y$ and $Z$ are both closed subspaces of Euclidean spaces, it follows that a continuous map $f: Y \to Z$ is proper if and only if preimages of bounded subsets are bounded. Equivalently, if $\{y_k\}_{k\geq 1}$ is a sequence of points in $Y$ with $\lim_{k\to\infty} \|y_k\| = \infty$, then we must have $\lim_{k\to\infty} \|f(y_k)\| = \infty$. □

With these definitions, we have:

LEMMA 1.6*.3. *Let $p: \widetilde{X} \to X$ be a proper local homeomorphism between locally compact Hausdorff spaces. Then $p: \widetilde{X} \to X$ is a covering space.*

PROOF. Consider $x \in X$. Since $p$ is proper, the set $p^{-1}(x)$ is compact. Since $p$ is a local homeomorphism at each point of $p^{-1}(x)$, the set $p^{-1}(x)$ is also discrete. We deduce that $p^{-1}(x)$ is finite. There are two cases:

CASE 1. $p^{-1}(x) \neq \emptyset$.

Enumerate $p^{-1}(x)$ as $p^{-1}(x) = \{\widetilde{x}_1, \ldots, \widetilde{x}_n\}$. For each $1 \leq i \leq n$, there exists a neighborhood $\widetilde{V}_i$ of $\widetilde{x}_i$ such that $p|_{\widetilde{V}_i}$ is a homeomorphism onto its image $V_i \subset X$. Since $\widetilde{X}$ is Hausdorff, we can shrink the $\widetilde{V}_i$ and assume they are all disjoint. Set $U = V_1 \cap \cdots \cap V_n$ and $\widetilde{U}_i = \widetilde{V}_i \cap p^{-1}(U)$. The set $\widetilde{U}_i$ is an open neighborhood of $\widetilde{x}_i$, and $p|_{\widetilde{U}_i}$ is a homeomorphism onto $U$.

By construction, $p^{-1}(U)$ contains $\widetilde{U}_1 \sqcup \cdots \sqcup \widetilde{U}_n$. However, we are not done since $p^{-1}(U)$ might contain points that do not lie in some $\widetilde{U}_i$. We want to shrink $U$ to ensure that this does not happen. Since we need $U$ to be open, we need to delete a closed set $C$ of "bad points" from $U$.

The first step is to shrink $U$ to ensure that $\overline{U}$ is compact. This is possible since $X$ is locally compact: letting $K$ be a compact neighborhood of $x$ with $K \subset U$, we replace $U$ with $\text{Int}(K)$ and each $\widetilde{U}_i$ with $\widetilde{U}_i \cap p^{-1}(\text{Int}(K))$. Since $X$ is Hausdorff the compact set $K$ is closed, so $\overline{U} \subset K$ and thus $\overline{U}$ is compact. Since $p$ is proper $p^{-1}(\overline{U})$ is compact, so since $\widetilde{X}$ is Hausdorff $p^{-1}(\overline{U})$ is closed. Let

$$\widetilde{C} = p^{-1}(\overline{U}) \setminus \bigcup_{i=1}^{n} \widetilde{U}_i.$$

Since $\widetilde{C}$ is a closed subset of the compact set $p^{-1}(\overline{U})$, it follows that $\widetilde{C}$ is compact. This implies that $C = p(\widetilde{C})$ is compact, and hence closed. Since $p^{-1}(x) \subset \cup_{i=1}^{n} \widetilde{U}_i$ we have $x \notin C$. Replacing $U$ with $U \setminus C$ and each $\widetilde{U}_i$ with $\widetilde{U}_i \setminus p^{-1}(C)$, the set $U$ is an open neighborhood of $x$ with $p^{-1}(U) = \widetilde{U}_1 \sqcup \cdots \sqcup \widetilde{U}_n$, as desired.

CASE 2. $p^{-1}(x) = \emptyset$.

Let $K$ be a compact neighborhood of $x$. Since $p$ is proper $p^{-1}(K)$ is compact, so $p(p^{-1}(K)) \subset X$ is compact. Since $X$ is Hausdorff, the compact set $p(p^{-1}(K))$ is closed. Setting $U = \text{Int}(K) \setminus p(p^{-1}(K))$, the set $U$ is open and $p^{-1}(U) = \emptyset$. It follows that $U$ is a trivialized open neighborhood of $x$, as desired. □

Here is an example of how Lemma 1.6*.3 can be used.

EXAMPLE 1.6*.4 (Roots of square-free polynomials). For some $n \geq 1$, let $\text{Poly}_n$ be the space of degree-$n$ monic polynomials over $\mathbb{C}$. Such an $f \in \text{Poly}_n$ can be written as

$$f(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n \quad \text{with } a_1, \ldots, a_n \in \mathbb{C}.$$

[5]This is not quite the right definition if $Y$ and $Z$ are not locally compact Hausdorff spaces. See Chapter 9 of Volume 1 for more details.

The topology comes from the coefficients, so $\mathrm{Poly}_n \cong \mathbb{C}^n$. By the fundamental theorem of algebra, such a polynomial has $n$ roots (counted with multiplicity). Define

$$\mathrm{RPoly}_n = \{(f, x) \in \mathrm{Poly}_n \times \mathbb{C} \mid f(x) = 0\}.$$

In other words, $\mathrm{RPoly}_n$ is the space of polynomials equipped with a root. Let $p\colon \mathrm{RPoly}_n \to \mathrm{Poly}_n$ be the map $p(f, x) = f$. For $n \geq 2$ this is not a covering space since the fibers of $p$ have different cardinalities. For example,

$$|p^{-1}(z^n)| = |\{(z^n, 0)\}| = 1 \quad \text{but} \quad |p^{-1}(z^n - 1)| = |\left\{(z^n - 1, \mu) \mid \mu \text{ an } n^{\text{th}} \text{ root of unity}\right\}| = n.$$

As suggested by this, the issue arises because of polynomials with repeated roots. A root $x \in \mathbb{C}$ of a polynomial $f(z)$ is a repeated root if $f'(x) = 0$. Define

$$\mathrm{Poly}_n^{\mathrm{sf}} = \{f \in \mathrm{Poly}_n \mid f \text{ has no repeated roots}\}$$

and

$$\mathrm{RPoly}_n^{\mathrm{sf}} = \left\{(f, x) \in \mathrm{Poly}_n^{\mathrm{sf}} \times \mathbb{C} \mid f(x) = 0\right\}.$$

The "sf" stands for "square-free". The spaces $\mathrm{Poly}_n^{\mathrm{sf}}$ and $\mathrm{RPoly}_n^{\mathrm{sf}}$ are open subsets of $\mathrm{Poly}_n$ and $\mathrm{RPoly}_n$, respectively.[6] The projection $p\colon \mathrm{RPoly}_n^{\mathrm{sf}} \to \mathrm{Poly}_n^{\mathrm{sf}}$ is a degree-$n$ covering space. Indeed, since $p$ is a proper map[7] whose fibers all have cardinality $n$, by Lemma 1.6*.3 it is enough to prove that $p\colon \mathrm{RPoly}_n^{\mathrm{sf}} \to \mathrm{Poly}_n^{\mathrm{sf}}$ is a local homeomorphism. But this is easy: for $(f, x) \in \mathrm{RPoly}_n^{\mathrm{sf}}$, since $f(z)$ has no repeated roots we have $f'(x) \neq 0$, so by the implicit function theorem there is a neighborhood $U \subset \mathrm{Poly}_n^{\mathrm{sf}}$ of $f$ such that around $(f, x)$ the subspace

$$\mathrm{RPoly}_n^{\mathrm{sf}} \subset \mathrm{Poly}_n^{\mathrm{sf}} \times \mathbb{C} \subset \mathbb{C}^n \times \mathbb{C}$$

is the graph of a function $U \to \mathbb{C}$. □

REMARK 1.6*.5. The cover $p\colon \mathrm{RPoly}_n^{\mathrm{sf}} \to \mathrm{Poly}_n^{\mathrm{sf}}$ from Example 1.6*.4 encodes a lot of interesting geometric and algebraic information about roots of polynomials. For instance, the fundamental theorem of algebra is equivalent[8] to the fact that $p\colon \mathrm{RPoly}_n^{\mathrm{sf}} \to \mathrm{Poly}_n^{\mathrm{sf}}$ has degree $n$ for all $n \geq 1$. We remark that there are several topological proofs of the fundamental theorem of algebra. Perhaps the shortest can be found in Chapter 9 of Volume 1, which deduces it from the fact that the map $p\colon \mathrm{RPoly}_n \to \mathrm{Poly}_n$ is a proper map. □

## 1.7. Exercises

EXERCISE 1.1. Carefully prove that the following are covering spaces. Let $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

(a) The map $p\colon \mathbb{C} \to \mathbb{C}^\times$ defined by $p(z) = e^z$.
(b) For $n \in \mathbb{Z} \setminus \{0\}$, the map $p\colon \mathbb{C}^\times \to \mathbb{C}^\times$ defined by $p(z) = z^n$. □

EXERCISE 1.2. Prove that the map $p\colon \mathbb{C} \to \mathbb{C}$ defined by $p(z) = z^2$ is not a covering space. □

EXERCISE 1.3. Do the following:

(a) Define $p\colon (0, 2) \to \mathbb{S}^1$ to be the map $p(\theta) = e^{2\pi i \theta}$ for $\theta \in (0, 2)$. Prove that $p$ is a surjective local homeomorphism but is not a covering space.
(b) Give an example of a local homeomorphism $f\colon X \to Y$ and a subset $A \subset X$ such that $f|_A\colon A \to f(A)$ is not a local homeomorphism. □

EXERCISE 1.4. Let $p\colon \widetilde{X} \to X$ be a covering space. Do the following:

(a) Prove that $p$ is an open map.

[6] This is an elementary exercise (see Exercise 1.17). A sophisticated way to see it is to use the fact that having a repeated root is equivalent to the vanishing of the discriminant, which is a polynomial in the coefficients of the polynomial.

[7] To show that $p\colon \mathrm{RPoly}_n^{\mathrm{sf}} \to \mathrm{Poly}_n^{\mathrm{sf}}$ is proper, it is enough to prove that $p\colon \mathrm{RPoly}_n \to \mathrm{Poly}_n$ is proper. Since $\mathrm{RPoly}_n$ is a closed subset of $\mathrm{Poly}_n \times \mathbb{C} \cong \mathbb{C}^n \times \mathbb{C}$, this is a consequence of the elementary fact that for all $C > 0$, there exists some $D > 0$ such that if $x \in \mathbb{C}$ is a root of $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ and $|a_k| \leq C$ for all $1 \leq k \leq n$, then $|x| \leq D$.

[8] The point here is that a polynomial with repeated roots certainly has roots, so to prove the fundamental theorem of algebra one need only find roots for polynomials without repeated roots.

(b) Assume that $p$ has finite degree. Prove that $p$ is a closed map.
(c) Give an example of a covering space $p\colon \widetilde{X} \to X$ such that $p$ is not a closed map.
(d) Assume that $p$ has degree 1. Prove that it is a homeomorphism. $\square$

EXERCISE 1.5. Let $p\colon \widetilde{X} \to X$ be a covering space. Do the following:

(a) Prove that the image of $p$ is a union of connected components of $X$. Hint: prove that $p(\widetilde{X})$ is both open and closed.
(b) Assume that $\widetilde{X} \neq \emptyset$ and $X$ is connected. Prove that $p$ is surjective.
(c) Assume that $p$ is surjective. Prove that $p$ is a quotient map, i.e., that a set $U \subset X$ is open if and only if $p^{-1}(U)$ is open. $\square$

EXERCISE 1.6. Let $p\colon Y \to X$ be a finite-sheeted cover and let $q\colon Z \to Y$ be an arbitrary cover. Prove that $p \circ q\colon Z \to X$ is a cover. We remark that it is not true that the composition of two covering space maps is a covering space (see Exercise 2.14**). $\square$

EXERCISE 1.7. Let $p_1\colon \widetilde{X}_1 \to X_1$ and $p_2\colon \widetilde{X}_2 \to X_2$ be covering spaces. Define $q\colon \widetilde{X}_1 \sqcup \widetilde{X}_2 \to X_1 \sqcup X_2$ via the formula

$$q(z) = \begin{cases} p_1(z) & \text{if } z \in \widetilde{X}_1, \\ p_2(z) & \text{if } z \in \widetilde{X}_2. \end{cases}$$

Prove that $q\colon \widetilde{X}_1 \sqcup \widetilde{X}_2 \to X_1 \sqcup X_2$ is a covering space. Use this construction to find a surjective covering space over a non-connected base that does not have a degree. $\square$

EXERCISE 1.8. Let $p_1\colon \widetilde{X}_1 \to X_1$ and $p_2\colon \widetilde{X}_2 \to X_2$ be covering spaces. Define $q\colon \widetilde{X}_1 \times \widetilde{X}_2 \to X_1 \times X_2$ via the formula $q(z_1, z_2) = (p_1(z_1), p_2(z_2))$. Prove that $q\colon \widetilde{X}_1 \times \widetilde{X}_2 \to X_1 \times X_2$ is a covering space. $\square$

EXERCISE 1.9. In this exercise, you will prove that Exercise 1.8 is false for infinite products. For $n \geq 1$, let $X_n = \mathbb{S}^1$ and $\widetilde{X}_n = \mathbb{R}$ and let $p_n\colon \widetilde{X}_n \to X_n$ be the universal cover. Give $\prod_{n\geq 1} \widetilde{X}_n$ and $\prod_{n\geq 1} X_n$ the product topologies, and define a map $p\colon \prod_{n\geq 1} \widetilde{X}_n \to \prod_{n\geq 1} X_n$ via the formula

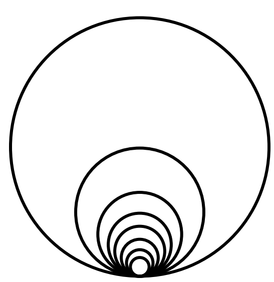
$$p(z_1, z_2, \ldots) = (p_1(z_1), p_2(z_2), \ldots) \quad \text{for all } (z_1, z_2, \ldots) \in \prod_{n\geq 1} \widetilde{X}_n.$$

Prove that $p\colon \prod_{n\geq 1} \widetilde{X}_n \to \prod_{n\geq 1} X_n$ is not a covering space. $\square$

EXERCISE 1.10. Do the following:

(a) Let $p\colon \widetilde{X} \to X$ be a covering space, let $\widetilde{Y}$ be a connected component of $\widetilde{X}$, and let $q = p|_{\widetilde{Y}}$. Assume that $X$ is locally connected.[9] Prove that $q\colon \widetilde{Y} \to X$ is a covering space.
(b) Give an example to show that (a) is false without the assumption that $X$ is locally connected. $\square$

EXERCISE 1.11*. For $n \geq 1$, let $C_n \subset \mathbb{R}^2$ be the circle of radius $1/n$ with center $(0, 1/n)$. Let $X = \cup_{n=1}^{\infty} C_n$, topologized as a subspace of $\mathbb{R}^2$. This is sometimes called the "earring space" or the "shrinking wedge of circles":



Construct a collection of covers $\{p_i\colon \widetilde{X}_i \to X\}_{i\in I}$ with the following property. Let $\widetilde{X}$ be the disjoint

[9]A space $X$ is locally connected if for all $x \in X$ and all open neighborhoods $V$ of $x$, there exists a connected open neighborhood $U$ of $x$ with $U \subset V$. This implies that the connected components of $X$ are both open and closed.

union of the $\widetilde{X}_i$ and let $p\colon \widetilde{X} \to X$ be the map that is $p_i$ on $\widetilde{X}_i$ for all $i \in I$. Then $p\colon \widetilde{X} \to X$ is not a cover. □

EXERCISE 1.12. Prove the following:

(a) Let $p\colon \widetilde{X} \to X$ be a cover and let $Z \subset X$ be a subspace. Define $\widetilde{Z} = p^{-1}(Z)$ and $q = p|_{\widetilde{Z}}$. Prove that $q\colon \widetilde{Z} \to Z$ is a covering space. We will call this the *restriction* of $p$ to $Z$.

(b) Let $X$ be a locally connected space with connected components $\{X_j\}_{j\in J}$. For each $j \in J$, let $q_j\colon Y_j \to X_j$ be a covering space. Define
$$Y = \bigsqcup_{j\in J} Y_j,$$
and let $q\colon Y \to X$ be the map that for $j \in J$ and $y \in Y_j$ satisfies $q(y) = q_j(y) \in X_j \subset X$. Prove that $q\colon Y \to X$ is a covering space.

(c) Part (b) is false if $X$ is not assumed to be locally connected. Construct a non-locally connected counterexample in which each $q_j\colon Y_j \to X_j$ is a homeomorphism. □

EXERCISE 1.13. For spaces $Y$ and $Z$ equipped with maps $f_1\colon Y \to X$ and $f_2\colon Z \to X$ to another space $X$, the *fiber product* of $f_1$ and $f_2$ is the space
$$Y \times_X Z = \{(y,z) \in Y \times Z \mid f_1(y) = f_2(z)\}.$$
Letting $g_1\colon Y \times_X Z \to Y$ and $g_2\colon Y \times_X Z \to Z$ be the projections onto the first and second factors, we have a commutative diagram
$$\begin{tikzcd} Y \times_X Z \arrow[r, "g_1"] \arrow[d, "g_2"'] & Y \arrow[d, "f_1"] \\ Z \arrow[r, "f_2"] & X. \end{tikzcd}$$
Prove the following:[10]

(a) If $f_1$ is injective, then $g_2$ is injective.
(b) If $f_1$ is surjective, then $g_2$ is surjective.
(c) If $f_1$ is a homeomorphism, then $g_2$ is a homeomorphism.
(d) Assume that $f_2\colon Z \to X$ is the inclusion of a subspace. Set $W = f_1^{-1}(Z)$. Prove that we can identify $g_2\colon Y \times_X Z \to Z$ with the restriction of $f_1$ to $W$ in the following sense:
  - There exists a homeomorphism $\phi\colon W \to Y \times_X Z$ such that the following diagram commutes:
$$\begin{tikzcd} W \arrow[r, "\phi", "\cong"'] \arrow[d, "f_1|_W"'] & Y \times_X Z \arrow[d, "g_2"] \\ Z \arrow[r, "\mathbb{1}_Z"] & Z \end{tikzcd}$$
(e) Assume that $f_1\colon Y \to X$ and $f_2\colon Z \to X$ are inclusions of subspaces. Prove that $Y \times_X Z \cong Y \cap Z$.

We will give a more categorical description of the fiber product via a universal property in the exercises of Chapter 17. □

EXERCISE 1.14. Let $p\colon \widetilde{X} \to X$ be a cover. This exercise will use the notion of fiber product from Exercise 1.13. Prove the following:

(a) Let $f\colon Z \to X$ be a map. Let $\widetilde{Z} = \widetilde{X} \times_X Z$ be the fiber product of $\widetilde{X}$ with $Z$ and let $q\colon \widetilde{Z} \to Z$ be the associated map. Prove that $q$ is a covering space. This is often called the *pullback* of $p\colon \widetilde{X} \to X$ along $f\colon Z \to X$.
(b) Let $p'\colon \widetilde{X}' \to X$ be a cover that is isomorphic to $p\colon \widetilde{X} \to X$. Prove that the pullbacks of $p'\colon \widetilde{X}' \to X$ and $p\colon \widetilde{X} \to X$ along a map $f\colon Z \to X$ are isomorphic.

[10]The fiber product is symmetric in $Y$ and $Z$, so all of these questions have analogous versions with the roles of $Y$ and $Z$ swapped.

(c) Let $Z \subset X$ be a subspace. Let $\iota\colon Z \to X$ be the inclusion and let $q\colon \widetilde{Z} \to Z$ be the pullback of $p\colon \widetilde{X} \to X$ along $\iota\colon Z \to X$. Prove that the covering space $q\colon \widetilde{Z} \to Z$ is isomorphic to the restriction of $p\colon \widetilde{X} \to X$ to $Z$ discussed in Exercise 1.12.

(d) Let $f\colon Z \to X$ and $g\colon W \to Z$ be maps. Let $q\colon \widetilde{Z} \to Z$ be the pullback of $p\colon \widetilde{X} \to X$ along $f\colon Z \to X$ and let $r\colon \widetilde{W} \to W$ be the pullback of $q\colon \widetilde{Z} \to Z$ along $g\colon W \to Z$. Finally, let $r'\colon \widetilde{W}' \to W$ be the pullback of $p\colon \widetilde{X} \to X$ along $f \circ g\colon W \to X$. Prove that $r\colon \widetilde{W} \to W$ and $r'\colon \widetilde{W}' \to W$ are isomorphic covering spaces. $\square$

Exercise 1.15. Let $p\colon \widetilde{X} \to X$ be a cover. This exercise shows that many point-set topological properties of $X$ are inherited by $\widetilde{X}$.

(a) If $X$ is Hausdorff, then prove that $\widetilde{X}$ is Hausdorff.

(b) If $X$ is regular, then prove that $\widetilde{X}$ is regular.

(c)$**$ If $X$ is paracompact, then prove that $\widetilde{X}$ is paracompact. Hint: first prove that there is a locally finite collection of closed sets $\{C_i\}_{i \in I}$ of $\widetilde{X}$ such that each $C_i$ is paracompact, and then prove that this implies that $\widetilde{X}$ is paracompact.

(d)$*$ If $X$ is metrizable, then prove that $\widetilde{X}$ is metrizable. Hint: Use the Smirnov metrization theorem, which says that a space is metrizable if and only if it is paracompact and locally metrizable. $\square$

Exercise 1.16*. Let $p\colon \widetilde{X} \to X$ be a covering space such that $p^{-1}(x)$ is finite and nonempty for all $x \in X$. Prove that $X$ is compact Hausdorff if and only if $\widetilde{X}$ is compact Hausdorff. $\square$

Exercise 1.17. Let $n \geq 1$. Prove that $\text{Poly}_n^{\text{sf}}$ and $\text{RPoly}_n^{\text{sf}}$ are open subsets of $\text{Poly}_n$ and $\text{RPoly}_n$. Hint: first reduce to showing that $\text{Poly}_n^{\text{sf}}$ is open in $\text{Poly}_n$. Equivalently, the subspace $\mathcal{R}$ of $\text{Poly}_n$ consisting of polynomials with a repeated root is closed. Let $(f_m)_{m \geq 1}$ be a sequence of polynomials in $\mathcal{R}$ such that $f = \lim_{m \to \infty} f_m$ in $\text{Poly}_n$. We must prove that $f$ has a repeated root. For $m \geq 1$, let $x_m$ be a repeated root of $f_m$. We thus have $f_m(x_m) = f_m'(x_m) = 0$ for all $m \geq 1$. Prove that the sequence $(x_m)_{m \geq 1}$ of points of $\mathbb{C}$ has a convergent subsequence, and use this to prove that $f$ has a repeated root. $\square$

CHAPTER 2

# Deck transformations and regular covers

We now discuss the symmetries of a cover.

## 2.1. Deck transformations

We defined isomorphisms of covers in §1.4. The isomorphisms from a cover to itself are called deck transformations:

Definition 2.1.1. Let $p\colon \widetilde{X} \to X$ be a covering space. A *deck transformation* of $p\colon \widetilde{X} \to X$ is a covering space isomorphism $f\colon \widetilde{X} \to \widetilde{X}$, i.e., a homeomorphism such that the following diagram commutes:

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "f", "\cong"'] \arrow[dr, "p"'] & & \widetilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

The deck transformations of $p\colon \widetilde{X} \to X$ form a group under composition called the *deck group* of $p\colon \widetilde{X} \to X$, denoted $\mathrm{Deck}(p\colon \widetilde{X} \to X)$ or simply $\mathrm{Deck}(\widetilde{X})$. □

For a cover $p\colon \widetilde{X} \to X$, the deck group $\mathrm{Deck}(\widetilde{X})$ acts on $\widetilde{X}$. Here is an example:

Example 2.1.2. Let $p\colon \mathbb{R} \to \mathbb{S}^1$ be the universal cover, so $p(\theta) = e^{2\pi i\theta}$ for all $\theta \in R$. For each $n \in \mathbb{Z}$, we can define a deck transformation $f_n\colon \mathbb{R} \to \mathbb{R}$ via the formula $f_n(\theta) = \theta + n$. □

## 2.2. Determining the deck group

The key to understanding the deck group is the following lemma, which says that in favorable situations deck transformations are completely determined by what they do to a single point. Recall that if $G$ is a group acting on a set $S$, then the action is *free* if for all $s \in S$ the $G$-stabilizer $G_s = \{g \in G \mid g \cdot s = s\}$ is trivial.

Lemma 2.2.1. *Let $p\colon \widetilde{X} \to X$ be a covering space with $\widetilde{X}$ connected. Let $f, g\colon \widetilde{X} \to \widetilde{X}$ be two deck transformations such that there exists some $z_0 \in \widetilde{X}$ with $f(z_0) = g(z_0)$. Then $f = g$. Consequently, $\mathrm{Deck}(\widetilde{X})$ acts freely on $\widetilde{X}$.*

Proof. Let $E = \left\{z \in \widetilde{X} \mid f(z) = g(z)\right\}$. Our goal is to prove that $E = \widetilde{X}$. By assumption $z_0 \in E$, so since $\widetilde{X}$ is connected it is enough to prove that $E$ is both open and closed.[1] Consider $z \in \widetilde{X}$. We must prove that if $z \in E$ (resp. $z \notin E$) then there is an open neighborhood of $z$ contained in $E$ (resp. disjoint from $E$). Let $U$ be a trivialized neighborhood of $p(z)$.

Assume first that $z \in E$. Let $\widetilde{U}$ be the sheet above $U$ containing $f(z) = g(z)$. Set $V = f^{-1}(\widetilde{U}) \cap g^{-1}(\widetilde{U})$, so $V$ is an open neighborhood of $z$ with $f(V), g(V) \subset \widetilde{U}$. For $z' \in V$, both $f(z')$ and $g(z')$ are the unique point of $\widetilde{U}$ projecting to $p(z') \in U$, so in particular $f(z') = g(z')$. This implies that $V \subset E$, as desired.

Assume now that $z \notin E$, so $f(z) \neq g(z)$. Let $\widetilde{U}_1$ and $\widetilde{U}_2$ be the sheets above $U$ with $f(z) \in \widetilde{U}_1$ and $g(z) \in \widetilde{U}_2$. Since $f(z) \neq g(z)$, the sheets $\widetilde{U}_1$ and $\widetilde{U}_2$ are distinct and hence disjoint. Set $W = f^{-1}(\widetilde{U}_1) \cap g^{-1}(\widetilde{U}_2)$, so $W$ is an open neighborhood of $z$ with $f(W) \subset \widetilde{U}_1$ and $g(W) \subset \widetilde{U}_2$. Since $\widetilde{U}_1 \cap \widetilde{U}_2 = \emptyset$, this implies that $f(z') \neq g(z')$ for all $z' \in W$, so $W$ is disjoint from $E$, as desired. □

[1]Note that if $\widetilde{X}$ is Hausdorff (like most spaces in this book) it is automatic that $E$ is closed; indeed, if $Z$ is Hausdorff then for any continuous maps $f, g\colon Y \to Z$ the set of $y \in Y$ with $f(y) = g(y)$ is closed.

The following is a typical example of how to use Lemma 2.2.1 to determine the deck group of a covering space:

EXAMPLE 2.2.2 (Universal cover of circle). Let $p\colon \mathbb{R} \to \mathbb{S}^1$ be the universal cover of $\mathbb{S}^1$. For $n \in \mathbb{Z}$, let $f_n\colon \mathbb{R} \to \mathbb{R}$ be the deck transformation defined by the formula $f_n(\theta) = \theta + n$. We claim that

$$\mathrm{Deck}(p\colon \mathbb{R} \to \mathbb{S}^1) = \{f_n \mid n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

To see this, consider an arbitrary deck transformation $f\colon \mathbb{R} \to \mathbb{R}$. Since $p(f(0)) = p(0)$, we must have $f(0) = n$ for some $n \in \mathbb{Z}$. Since $f(0) = f_n(0)$, Lemma 2.2.1 implies that $f = f_n$. □

## 2.3. Regular covers

Roughly speaking, a regular cover[2] is a cover with a deck group that is as large as possible. Let $p\colon \widetilde{X} \to X$ be a covering space. The group $\mathrm{Deck}(\widetilde{X})$ acts on $\widetilde{X}$. For $x \in X$, the action of $\mathrm{Deck}(\widetilde{X})$ on $\widetilde{X}$ preserves the fiber $p^{-1}(x)$, so $\mathrm{Deck}(\widetilde{X})$ acts on $p^{-1}(x)$. Lemma 2.2.1 implies that if $\widetilde{X}$ is connected then this action is free, so for $z_1, z_2 \in p^{-1}(x)$ there exists at most one $f \in \mathrm{Deck}(\widetilde{X})$ with $f(z_1) = z_2$. A regular cover is a cover where such an $f$ always exists:

DEFINITION 2.3.1. A *regular cover* is a cover $p\colon \widetilde{X} \to X$ such that for all $x \in X$, the group $\mathrm{Deck}(\widetilde{X})$ acts transitively on $p^{-1}(x)$, i.e., for all $\widetilde{x}_1, \widetilde{x}_2 \in p^{-1}(x)$ there exists some $f \in \mathrm{Deck}(\widetilde{X})$ with $f(\widetilde{x}_1) = \widetilde{x}_2$. A cover that is not regular is *irregular*. □

REMARK 2.3.2. If $p\colon \widetilde{X} \to X$ is a regular cover with $\widetilde{X}$ connected, then by Lemma 2.2.1 the deck group acts freely and transitively on the fiber $p^{-1}(x)$ for all $x \in X$. □

EXAMPLE 2.3.3 (Universal cover of circle). The calculation in Example 2.2.2 shows that the universal cover $p\colon \mathbb{R} \to \mathbb{S}^1$ is regular. □

EXAMPLE 2.3.4 (Trivial cover). Let $X$ be a space and $\mathcal{I}$ be a discrete set. Consider the trivial cover $p\colon X \times \mathcal{I} \to X$. Set $G = \mathrm{Deck}(p\colon X \times \mathcal{I} \to X)$. For each bijection $\sigma\colon \mathcal{I} \to \mathcal{I}$, we can define an element $f_\sigma \in G$ via the formula $f_\sigma(x, i) = (x, \sigma(i))$. These elements act transitively on the fibers, so $p\colon X \times \mathcal{I} \to X$ is regular. One can check that all elements of $G$ are of the form $f_\sigma$ if $X$ is connected (see Exercise 2.10). □

REMARK 2.3.5. It is harder to show that a cover is irregular. We will give an example below in Example 2.7.3. □

## 2.4. More examples of regular covers

Most of the covers we have seen so far are regular:

EXAMPLE 2.4.1 (Degree $n$ cover of circle). Let $p_n\colon \mathbb{S}^1 \to \mathbb{S}^1$ be the cover defined by the formula $p_n(z) = z^n$. We claim that $p_n\colon \mathbb{S}^1 \to \mathbb{S}^1$ is a regular cover with deck group isomorphic to the cyclic group $C_n$ of order $n$. Indeed, let $G = \mathrm{Deck}(p_n\colon \mathbb{S}^1 \to \mathbb{S}^1)$. Let $f \in G$ be the map $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ defined by the formula $f(z) = e^{2\pi i/n} z$. The element $f$ has order $n$ and its powers act transitively on the fiber $p_n^{-1}(1)$, which equals the $n^{\text{th}}$ roots of unity. This implies that the cover is regular.

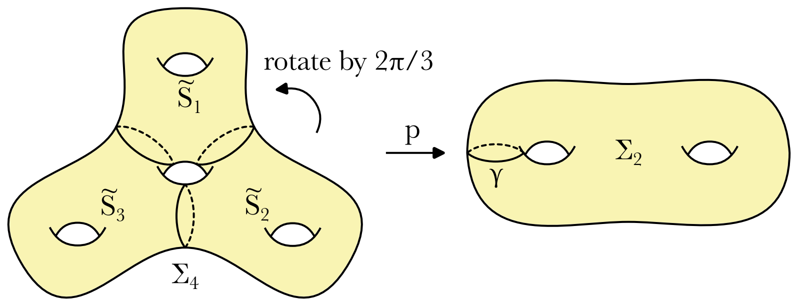
The same argument using Lemma 2.2.1 that we used in Example 2.2.2 shows that $G$ is the cyclic group of order $n$ generated by $f$. We repeat that argument one more time: for $g \in G$, we have $g(1) = e^{2\pi k i/n}$ for some $k \in \mathbb{Z}$ (well-defined modulo $n$). Since $g$ and $f^k$ both take 1 to $e^{2\pi k i/n}$, it follows from Lemma 2.2.1 that $g = f^k$. □

EXAMPLE 2.4.2 (Cosets of discrete subgroups). Let $\mathbf{G}$ be a topological group and let $\mathbf{H} < \mathbf{G}$ be a discrete subgroup. Then the projection $p\colon \mathbf{G} \to \mathbf{G}/\mathbf{H}$ is a regular cover. Indeed, for $h \in \mathbf{H}$ define $f_h\colon \mathbf{G} \to \mathbf{G}$ via the formula $f_h(g) = gh^{-1}$. Then $f_h \in \mathrm{Deck}(p\colon \mathbf{G} \to \mathbf{G}/\mathbf{H})$, and the $f_h$ act transitively on the fibers of $p\colon \mathbf{G} \to \mathbf{G}/\mathbf{H}$. If $\mathbf{G}$ is connected, then by Lemma 2.2.1 this is the entire deck group, so $\mathrm{Deck}(p\colon \mathbf{G} \to \mathbf{G}/\mathbf{H}) \cong \mathbf{H}$. As a special case, the deck group of the cover $p\colon \mathbb{R}^n \to \mathbb{T}^n$ is the group $\mathbb{Z}^n$, which acts on $\mathbb{R}^n$ by translations. □

[2]These are also often called *normal covers*.

EXAMPLE 2.4.3 (Real projective space). Consider $n \geq 1$. The cover $p\colon \mathbb{S}^n \to \mathbb{RP}^n$ is regular. Indeed, the map $f\colon \mathbb{S}^n \to \mathbb{S}^n$ defined by $f(z) = -z$ is an element of the deck group that swaps the two elements in the fiber over any point of $\mathbb{RP}^n$. By Lemma 2.2.1, the deck group of $p\colon \mathbb{S}^n \to \mathbb{RP}^n$ is the cyclic group $C_2$ of order 2 generated by $f$. □

EXAMPLE 2.4.4 (Cover of surface). Consider the covering space $p\colon \Sigma_4 \to \Sigma_2$ from Example 1.3.5. Set $G = \mathrm{Deck}(p\colon \Sigma_4 \to \Sigma_2)$. There is a deck transformation $f \in G$ that rotates $\Sigma_4$ by $2\pi/3$ as follows:



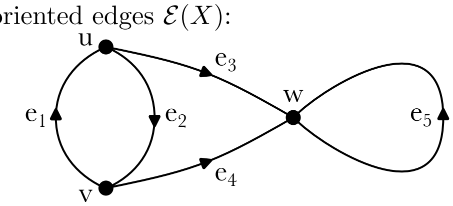

The element $f$ has order 3 and its powers act transitively on all the fibers. This implies that the cover is regular, and also by Lemma 2.2.1 that $G$ is the cyclic group of order 3 generated by $f$. □

EXAMPLE 2.4.5 (Roots of square-free polynomials). The degree-$n$ covering space $p\colon \mathrm{RPoly}_n^{\mathrm{sf}} \to \mathrm{Poly}_n^{\mathrm{sf}}$ discussed in Example 1.6*.4 is regular for $n = 2$ (see Exercise 2.9), but is irregular for $n \geq 3$. We do not have the technology to prove this yet,[3] but it should not be surprising. Indeed, if it were regular then the deck group $G$ would act transitively on the roots of every degree-$n$ polynomial with distinct roots, and if such a canonical group action existed then we would surely teach about it in elementary abstract algebra classes.[4] □

We will meet examples of irregular covers whose irregularity is easy to verify below when we discuss covers of graphs.

## 2.5. Graphs

Graphs[5] provide a rich source of examples of covering spaces. Recall that a graph $X$ is a set of vertices $\mathcal{V}(X)$ connected by oriented edges $\mathcal{E}(X)$:




For each vertex $v \in \mathcal{V}(X)$, the *degree* of $v$, denoted $\deg(v)$, is the number of edges that start or end at $v$. If an edge is a loop based at $v$, it contributes 2 to $\deg(v)$. We call $X$ a *finite* graph if the sets $\mathcal{V}(X)$ and $\mathcal{E}(X)$ are both finite. In this case, it is clear how to regard $X$ as a topological space: the vertices are a discrete set of points, and the edges are copies of $I = [0, 1]$ that are glued to the vertices. More generally, we say that $X$ is *locally finite* if for each $v \in \mathcal{V}(X)$ its degree $\deg(v)$ is finite. For locally finite graphs $X$, it is also clear how to regard $X$ as a topological space.

REMARK 2.5.1. In the general case, we regard $X$ as a 1-dimensional CW complex. See Essay K in Volume 1 for more details. □

[3] Proofs can be found in Exercises 9.9 and 12*.3.

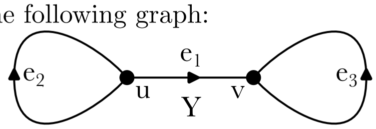
[4] For $n = 2$, this group action just exchanges the two roots.

[5] Our conventions about graphs are that unless otherwise specified they are oriented and we allow multiple edges and loops.

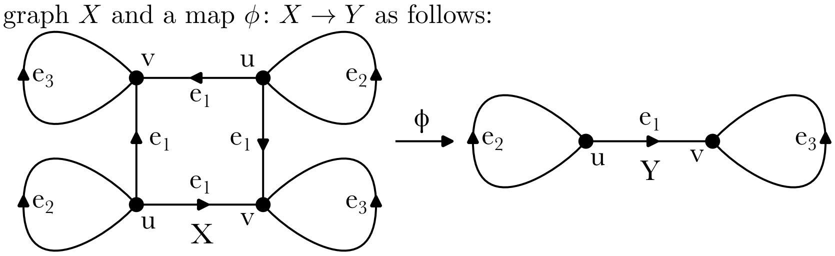
## 2.6. Maps of graphs

Let $X$ and $Y$ be graphs. To define a continuous map $\phi\colon X \to Y$, we must specify where $\phi$ sends each vertex and edge. This is particularly easy to do if we require our map to take vertices to vertices and oriented edges to oriented edges, which will suffice for the examples in this section. For instance:

EXAMPLE 2.6.1. Let $Y$ be the following graph:




We specify a graph $X$ and a map $\phi\colon X \to Y$ as follows:

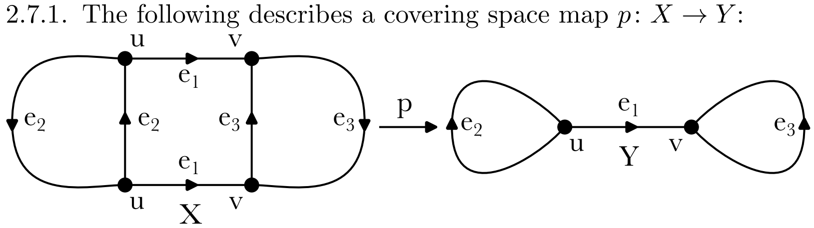



Here we label the vertices and oriented edges of $X$ by the vertices and oriented edges they map to. What this map does is map each edge of the central square of $X$ to the single non-loop edge of $Y$ and map each loop in $X$ to the appropriate loop in $Y$. The interiors of the edges are identified with copies of the open interval $(0, 1)$, and $\phi$ respects these identifications. □

## 2.7. Covers of graphs

The above example is not a covering map. The problem is that it is not a local homeomorphism at the vertices. What is needed for a covering map is informally that for each vertex "the same edges enter and exit as in the target". Here is an example of a covering map with the same $Y$ as above but a different $X$:

EXAMPLE 2.7.1. The following describes a covering space map $p\colon X \to Y$:

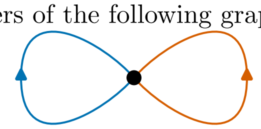



This is a covering space map since:

- for both vertices of $X$ mapping to $u$, one edge exits mapping to $e_1$, one edge exits mapping to $e_2$, and one edge enters mapping to $e_2$; and
- for both vertices of $X$ mapping to $v$, one edge enters mapping to $e_1$, one edge exits mapping to $e_3$, and one edge enters mapping to $e_3$.

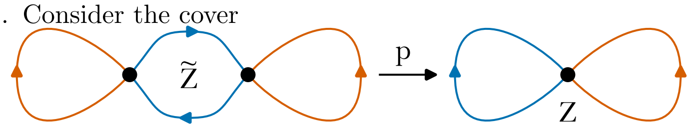
This is a regular cover with deck group isomorphic to $C_2$. The generator of $C_2$ acts on $X$ by the involution that swaps the two vertices labeled $u$, the two vertices labeled $v$, and for $i = 1, 2, 3$ the two oriented edges labeled $e_i$. □

We now give several different covers of the following graph $Z$:



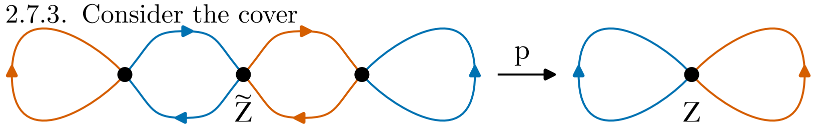
Since $Z$ has only one vertex, there is no need to give it a name; indeed, all vertices of a cover of $Z$ map to that one vertex. We also use colors rather than letters to distinguish the two edges of $Z$, and label the edges in the domain of our covering space maps by coloring them with the appropriate colors.

EXAMPLE 2.7.2. Consider the cover



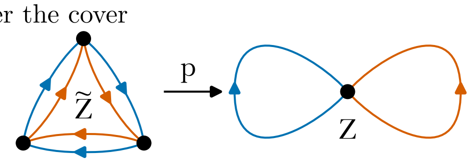

This is a degree 2 regular cover. The deck group is isomorphic to $C_2$, and acts on $\widetilde{Z}$ by the involution that swaps the two vertices, the two orange loops, and the two blue edges. $\square$

EXAMPLE 2.7.3. Consider the cover



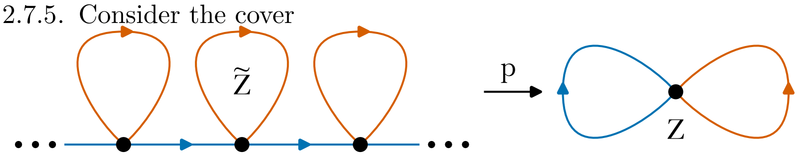

This is an irregular cover with trivial deck group. To see this, note that a deck transformation of $\widetilde{Z}$ must take vertices to vertices and oriented edges to oriented edges, and also must preserve the coloring on the edges. Since there is only one orange loop in $\widetilde{Z}$, any deck transformation must fix that orange loop and therefore be the identity (see Lemma 2.2.1). $\square$

EXAMPLE 2.7.4. Consider the cover



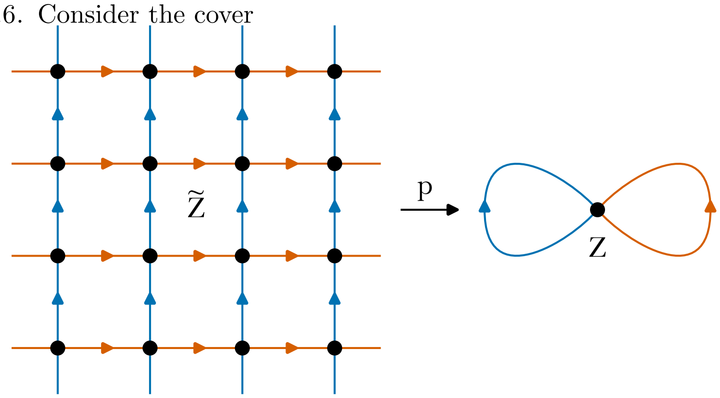

This is a degree 3 regular cover. The deck group is $C_3$, which acts on $\widetilde{Z}$ by rotations. $\square$

EXAMPLE 2.7.5. Consider the cover



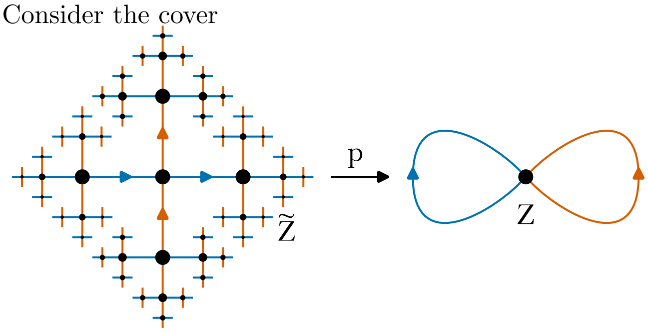

This is an infinite degree regular cover. The deck group is isomorphic to $\mathbb{Z}$, which acts on $\widetilde{Z}$ as translations. $\square$

EXAMPLE 2.7.6. Consider the cover



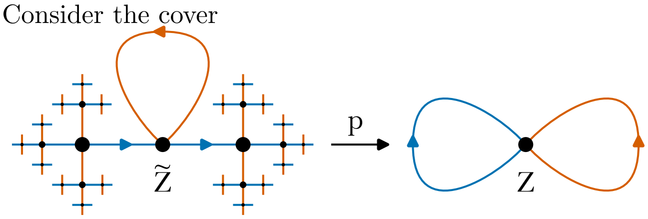

Here $\widetilde{Z}$ is the graph embedded in $\mathbb{R}^2$ whose vertices are at $\mathbb{Z}^2$ and whose edges are horizontal and vertical line segments connecting points of $\mathbb{Z}^2$. This is an infinite degree regular cover. The deck group is isomorphic to $\mathbb{Z}^2$, which acts on $\widetilde{Z} \subset \mathbb{R}^2$ via the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ by integer translations. $\square$

Example 2.7.7. Consider the cover




The indicated pattern in the domain repeats infinitely often, making it an infinite 4-valent tree.[6] The horizontal edges are oriented going right, and the vertical edges are oriented going up. This is an infinite degree regular cover. Indeed, for vertices $v$ and $w$ of $\widetilde{Z}$, there is a unique graph isomorphism $\widetilde{f}_{vw} \colon \widetilde{Z} \to \widetilde{Z}$ with $f_{vw}(v) = w$ such that $f_{vw}$ preserves the colors and orientations of the edges. These form the deck group. $\square$

Example 2.7.8. Consider the cover




This is an infinite degree irregular cover with trivial deck group. To see this, note that just like in Example 2.7.3 any deck transformation of $\widetilde{Z}$ must fix the unique orange loop in $\widetilde{Z}$, and thus must be the identity (see Lemma 2.2.1). $\square$

## 2.8. Covering space actions

Let $p \colon \widetilde{X} \to X$ be a cover with deck group $G$. The group $G$ acts on $\widetilde{X}$. If $\widetilde{X}$ is connected, then Lemma 2.2.1 says that action is *free*, i.e., that for all $z \in \widetilde{X}$ the stabilizer subgroup $G_z$ is trivial. In fact, even more is true, and this section is devoted to studying this action.

Remark 2.8.1. In this book, a group action is always assumed to preserve any structure a set has. In particular, an action of a group $G$ on a topological space $Z$ is assumed to be by homeomorphisms. In other words, for all $g \in G$ the map $Z \to Z$ that multiplies points by $g$ is assumed to be a homeomorphism. $\square$

The following lemma isolates the key property of the action of the deck group of a connected cover:

Lemma 2.8.2. *Let $p \colon \widetilde{X} \to X$ be a covering space with $\widetilde{X}$ connected, let $G = \mathrm{Deck}(p \colon \widetilde{X} \to X)$, and let $z \in \widetilde{X}$. Then there is an open neighborhood $V$ of $z$ whose translates $\{g \cdot V \mid g \in G\}$ are all disjoint.*

Proof. Let $U$ be a trivialized neighborhood of $p(z)$ and let $\widetilde{U}$ be the sheet lying above $U$ with $z \in \widetilde{U}$. We claim that $V = \widetilde{U}$ has the indicated property. Indeed, let $g_1, g_2 \in G$ satisfy $(g_1 \cdot \widetilde{U}) \cap (g_2 \cdot \widetilde{U}) \neq \emptyset$. We must prove that $g_1 = g_2$. Pick $z_1, z_2 \in \widetilde{U}$ with $g_1 \cdot z_1 = g_2 \cdot z_2$. Since the action of $G$ preserves the fibers of $p \colon \widetilde{X} \to X$, the points $z_1, z_2 \in \widetilde{U}$ must lie in the same fiber. Since the restriction of $p \colon \widetilde{X} \to X$ to $\widetilde{U}$ is injective, this implies that $z_1 = z_2$. Letting $w = z_1 = z_2$ be this common value, we have $g_1 \cdot w = g_2 \cdot w$. Lemma 2.2.1 now implies that $g_1 = g_2$, as desired. $\square$

Actions satisfying the conclusions of this lemma are important, so we give them a special name:

[6]A tree is a nonempty graph with no cycles, that is, no embedded circles.

Definition 2.8.3. A *covering space action* is an action of a group $G$ on a space $Z$ such that for all $z \in Z$, there exists an open neighborhood $V$ of $z$ such that the translates $\{g \cdot V \mid g \in G\}$ are all disjoint. □

Remark 2.8.4. All covering space actions are free. If $G$ is finite and $Z$ is Hausdorff, then the converse is true: all free actions of $G$ on $Z$ are covering space actions (see Exercise 2.6). □

## 2.9. Quotients by covering space actions

Let $G$ be a group and let $Z$ be a space equipped with a left action of $G$. Endow the quotient[7] $Z/G$ with the quotient topology. In other words, if $q\colon Z \to Z/G$ is the projection then a set $U \subset Z/G$ is open if and only if $q^{-1}(U)$ is open. If the action of $G$ on $Z$ is a covering space action, then the quotient map $q\colon Z \to Z/G$ is a regular covering space:

Lemma 2.9.1. *Let $G$ be a group acting on a space $Z$ by a covering space action. Then the quotient map $q\colon Z \to Z/G$ is a regular covering space. Moreover, if $Z$ is connected then $G = \mathrm{Deck}(q\colon Z \to Z/G)$.*

Proof. Consider $x \in Z/G$. Write $x = q(z)$ with $z \in Z$. Let $V$ be an open neighborhood of $z$ such that the sets in the $G$-orbit of $V$ are disjoint. Set $U = q(V)$. We have

$$q^{-1}(U) = \bigcup_{g \in G} g \cdot V.$$

Since each $g \cdot V$ is open, it follows that $q^{-1}(U)$ is open and thus by definition $U$ is open. The $g \cdot V$ are disjoint open subsets of $Z$ and each projects homeomorphically onto $U$. We conclude that $U$ is a trivialized neighborhood of $x$ and the $g \cdot V$ are the sheets lying above $U$. This implies that $q\colon Z \to Z/G$ is a covering space. By construction, the action of $G$ on $Z$ is by deck transformations of $q\colon Z \to Z/G$, so $G < \mathrm{Deck}(q\colon Z \to Z/G)$. This action is transitive on fibers, so if $Z$ is connected then Lemma 2.2.1 implies that $G = \mathrm{Deck}(q\colon Z \to Z/G)$. □

All of our examples of regular covering spaces could have been constructed using Lemma 2.9.1. For instance, $C_2$ acts on $\mathbb{S}^n$ via the antipodal map $z \mapsto -z$. This is a free action, so since $C_2$ is finite it is a covering space action (cf. Remark 2.8.4). We could have defined $\mathbb{RP}^n = \mathbb{S}^n/C_2$ and identified the covering space $p\colon \mathbb{S}^n \to \mathbb{RP}^n$ with the quotient projection. This would be a little artificial, but here is an example where this point of view is essential:

Example 2.9.2 (Configuration space). Let $X$ be a Hausdorff space. The *ordered configuration space* of $n$ points on $X$ is the space[8]

$$\mathrm{PConf}_n(X) = \left\{(x_1, \ldots, x_n) \in X^{\times n} \mid x_i \neq x_j \text{ for all distinct } 1 \leq i, j \leq n\right\}.$$

Topologize this as a subspace of $X^{\times n}$. The symmetric group $\mathfrak{S}_n$ on $n$ letters acts on $\mathrm{PConf}_n(X)$ via the formula

$$\sigma \cdot (x_1, \ldots, x_n) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}) \quad \text{for } \sigma \in \mathfrak{S}_n \text{ and } (x_1, \ldots, x_n) \in \mathrm{PConf}_n(X).$$

The inverses are there to make this a left action.[9] This is a free action since the $x_i$ are all distinct, and thus since $\mathfrak{S}_n$ is finite it is a covering space action. The *configuration space* of $n$ points on $X$ is the quotient $\mathrm{Conf}_n(X) = \mathrm{PConf}_n(X)/\mathfrak{S}_n$. Points of $\mathrm{Conf}_n(X)$ can be viewed as unordered sets $\{x_1, \ldots, x_n\}$ of $n$ distinct points in $X$. The projection $p\colon \mathrm{PConf}_n(X) \to \mathrm{Conf}_n(X)$ is a regular covering space. □

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[7] This is potentially confusing notation since $G$ is acting on the left. A purist would insist that $Z/G$ is the quotient of $Z$ by an action of $G$ on the right, and denote the quotient of $Z$ by an action of $G$ on the left by $G\backslash Z$. However, our notation is common and traditional, and we will follow it. There will be a few situations where we will have both left and right actions, and we will work hard to be clear about what our notation means in those cases.

[8] This is sometimes also called the *pure configuration space*, which is why it is written $\mathrm{PConf}_n(X)$.

[9] This is the same reason that inverses appear in the action of $\mathrm{GL}(V)$ on the dual of a vector space $V$.

## 2.10. Exercises

EXERCISE 2.1. Prove the following:

(a) Let $G$ be a group acting on a space $X$ by a covering space action. Let $Y$ be a subspace of $X$ that is preserved by the action of $G$, so $G$ acts on $Y$. Prove that the action of $G$ on $Y$ is a covering space action.
(b) For $i = 1, 2$, let $X_i$ be a space and let $G_i$ be a group acting on $X_i$ by a covering space action. Prove that the action of $G_1 \times G_2$ on $X_1 \times X_2$ is a covering space action. □

EXERCISE 2.2. Let $p\colon \widetilde{X} \to X$ be a covering space with $X$ connected and locally connected. Let $f, g\colon \widetilde{X} \to \widetilde{X}$ be two deck transformations such that there exists some $x_0 \in X$ with $f(z) = g(z)$ for all $z \in p^{-1}(x_0)$. Prove that $f = g$. Hint: this is a variant of Lemma 2.2.1, and can be proved in a similar way. The key thing that must be shown is that each component of $\widetilde{X}$ contains a point of $p^{-1}(x_0)$. For this, Exercise 1.10 might be useful. □

EXERCISE 2.3. Construct a covering space action of $C_2 \times C_2$ on a compact surface $\Sigma$. □

EXERCISE 2.4. Let $G$ be a finite group. Construct a covering space action of $G$ on a path connected open subset of some Euclidean space. Hint: Embed $G$ into the unitary group $\mathrm{U}(n)$ for some $n \gg 0$. This gives an action of $G$ on $\mathbb{C}^n$. For each $g \in G$, let $X_g = \{v \in \mathbb{C}^n \mid gv = v\}$. Set

$$U = \mathbb{C}^n \setminus \bigcup_{g \in G \setminus 1} X_g.$$

Prove that $U$ is path connected and that $G$ acts on $U$ by a covering space action. □

EXERCISE 2.5. Let $\alpha \in \mathbb{R}$ be an irrational number. Let $G \cong \mathbb{Z}$ be an infinite cyclic group generated by $s \in G$. Let $G$ act on $\mathbb{S}^1$ via the formula

$$t^n \cdot z = e^{2\pi i n\alpha} z \quad \text{for } z \in \mathbb{S}^1 \text{ and } n \in \mathbb{Z}.$$

Let $p\colon \mathbb{S}^1 \to \mathbb{S}^1/G$ be the quotient map.

(a) Prove directly that $p$ is not a covering space. Hint: First prove that every orbit of $G$ on $\mathbb{S}^1$ is dense. The key to this is as follows. For $x \in \mathbb{R}$, let $\{x\} \in [0, 1)$ be the fractional part of $x$, so $x = d + \{x\}$ for some $d \in \mathbb{Z}$. You then must prove that for all $\epsilon > 0$ there exist $n \in \mathbb{Z}$ such that $\{n\alpha\} \in (0, \epsilon)$. This can be proved using the pigeonhole principle.
(b) Prove that $\mathbb{S}^1/G$ is not Hausdorff. □

EXERCISE 2.6. Let $X$ be a Hausdorff space and let $G$ be a group acting freely on $X$. For all $x \in X$, assume that there is an open neighborhood $U$ of $x$ such that the set

$$\{g \in G \mid g \cdot U \cap U \neq \emptyset\}$$

is finite. Prove that the action of $G$ on $X$ is a covering space action. The above condition is automatically satisfied if $G$ is finite, so this shows that a free action of a finite group on a Hausdorff space is a covering space action. □

EXERCISE 2.7. Let $Z$ be a Hausdorff topological space and let $G$ be a group acting on $Z$ by a covering space action. Assume that $Z/G$ is Hausdorff (cf. Exercise 2.13), and let $x, y \in Z$ be two points that project to different points of $Z/G$. Prove that there exist open neighborhoods $U$ of $x$ and $V$ of $y$ such that the translates $\{g \cdot U \mid g \in G\}$ and $\{g \cdot V \mid g \in G\}$ are all disjoint from each other. □

EXERCISE 2.8. Let $Z$ be a first countable topological space. Let $G$ be a group acting on $Z$ by a covering space action. Assume that $Z/G$ is Hausdorff (cf. Exercise 2.13), and let $K \subset Z$ be compact. Prove that the set $\{g \in G \mid g \cdot K \cap K \neq \emptyset\}$ is finite. Hint: the previous exercise will be useful. □

EXERCISE 2.9. Let $p\colon \widetilde{X} \to X$ be a degree 2 cover. Prove that $p\colon \widetilde{X} \to X$ is a regular cover. □

EXERCISE 2.10. Let $X$ be a space, $\mathcal{I}$ be a discrete set, and $p\colon X \times \mathcal{I} \to X$ be the trivial cover.

(a) If $X$ is connected, prove that all elements of the deck group of $p\colon X \times \mathcal{I} \to X$ are of the form $f_\sigma(x, i) = (x, \sigma(i))$ for some bijection $\sigma\colon \mathcal{I} \to \mathcal{I}$.
(b) If $X$ is not connected, construct elements of the deck group that are not of this form. □

Exercise 2.11. Let $Z$ be a graph with one vertex and two edges. Construct degree 4 covers $p\colon \widetilde{Z} \to Z$ and $q\colon \widetilde{Z}' \to Z$ with $\widetilde{Z}$ and $\widetilde{Z}'$ path connected such that $p\colon \widetilde{Z} \to Z$ is regular and $q\colon \widetilde{Z}' \to Z$ is irregular. □

Exercise 2.12. Let $p\colon \widetilde{X} \to X$ be a cover with $\widetilde{X}$ connected. Let $Z$ be a nonempty space and $f\colon Z \to X$ be a map. As in Exercise 1.14, let $q\colon \widetilde{Z} \to Z$ be the pullback of $p$ along $f$, so

$$\widetilde{Z} = \left\{ (\widetilde{x}, z) \in \widetilde{X} \times Z \mid p(\widetilde{x}) = f(z) \right\}.$$

Do the following:

(a) Assume that $\widetilde{Z} \neq \emptyset$. Prove that there is an injection $\mathrm{Deck}(\widetilde{X}) \hookrightarrow \mathrm{Deck}(\widetilde{Z})$.
(b) Give an example where $\widetilde{Z}$ is nonempty and connected but the injection $\mathrm{Deck}(\widetilde{X}) \hookrightarrow \mathrm{Deck}(\widetilde{Z})$ constructed in (a) is not surjective.
(c) Assume that $p\colon \widetilde{X} \to X$ is a regular cover and $\widetilde{Z}$ is nonempty and connected. Prove that the injection $\mathrm{Deck}(\widetilde{X}) \hookrightarrow \mathrm{Deck}(\widetilde{Z})$ constructed in (a) is an isomorphism.
(d) Assume that $p\colon \widetilde{X} \to X$ is a regular cover. Prove that $q\colon \widetilde{Z} \to Z$ is a regular cover. □

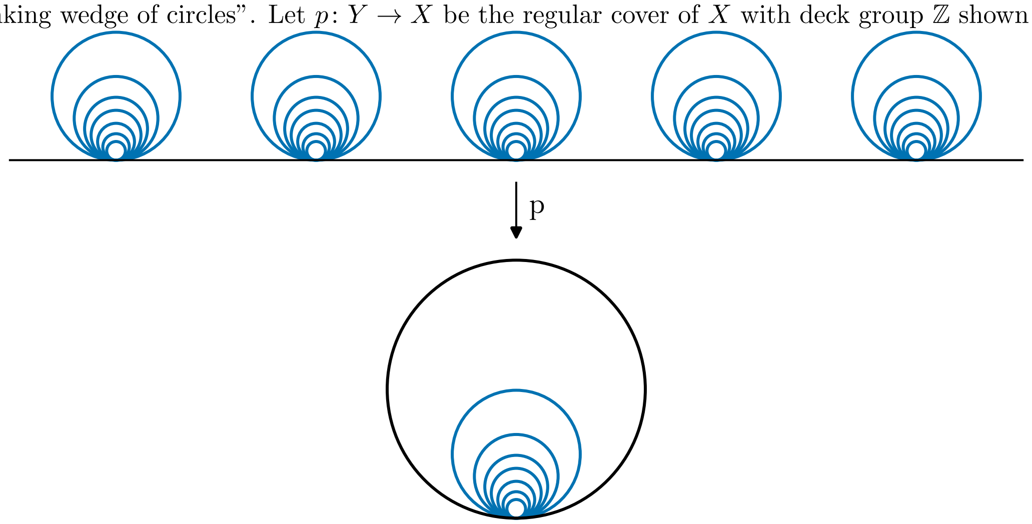
Exercise 2.13. In this exercise you will see that non-Hausdorff spaces can have Hausdorff covers. Set $X = \mathbb{R}^2 \setminus 0$. Let $G = \mathbb{Z}$ with generator $t = 1$. Define an action of $G$ on $X$ by letting

$$t^n \cdot (x, y) = (2^n x, 2^{-n} y) \quad \text{for all } (x, y) \in X \text{ and } n \in \mathbb{Z}.$$

Prove the following:

(a) The action of $G$ on $X$ is a covering space action.
(b) The quotient $X/G$ is not Hausdorff. □

Exercise 2.14**. In this exercise, you will see that the composition of two covering maps is not necessarily a covering map (though later we will see that this does hold for reasonable spaces; see Exercises 4.10 and 10.7). For $n \geq 1$, let $C_n \subset \mathbb{R}^2$ be the circle of radius $1/n$ with center $(0, 1/n)$. Let $X = \cup_{n=1}^{\infty} C_n$, topologized as a subspace of $\mathbb{R}^2$. This is sometimes called the "earring space" or the "shrinking wedge of circles". Let $p\colon Y \to X$ be the regular cover of $X$ with deck group $\mathbb{Z}$ shown here:




Construct a degree 2 cover $q\colon Z \to Y$ such that the composition $p \circ q\colon Z \to X$ is *not* a covering space. Hint: Enumerate the copies of the earring space in $Y$ by $(E_k)_{k \in \mathbb{Z}}$. Construct a 2-fold cover of $Y$ that is trivial on $E_k$ for $k \leq 0$. For $k \geq 1$, the cover should be nontrivial on the loop $C_k \cong \mathbb{S}^1$ of $E_k$ and trivial on all the other loops in $E_k$. □

CHAPTER 3

# Lifting paths and homotopies

This chapter studies lifting problems, which play a key role in both the classification of covering spaces and their applications. As an application, we develop the theory of winding numbers and degrees of maps from $\mathbb{S}^1$ to iteself.

## 3.1. Lifting problems in general

Let $p\colon \widetilde{X} \to X$ be a covering space and let $f\colon Y \to X$ be a map. A *lift* of $f$ through $p$ is a map $\widetilde{f}\colon Y \to \widetilde{X}$ such that the diagram

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\widetilde{f}"] \arrow[r, "f"] & X \end{tikzcd}$$

commutes, i.e., such that $f = p \circ \widetilde{f}$.

Example 3.1.1. A deck transformation $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ is a lift of the covering space map $p\colon \widetilde{X} \to X$ itself:

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ \widetilde{X} \arrow[ur, "\widetilde{f}"] \arrow[r, "p"] & X \end{tikzcd}$$

Of course, it is possible that a lift of $p\colon \widetilde{X} \to X$ to a map $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ exists such that $\widetilde{f}$ is not a homeomorphism, so not all such lifts are deck transformations. $\square$

A lift might or might not exist. However, just like a deck transformation if a lift exists then under favorable hypotheses it is determined by what it does to a single point:

Lemma 3.1.2. *Let $p\colon \widetilde{X} \to X$ be a covering space and let $f\colon Y \to X$ be a map. Assume that $Y$ is connected. Let $\widetilde{f}_1, \widetilde{f}_2\colon Y \to \widetilde{X}$ be two lifts of $f$ through $p$ such that there is some $y_0 \in Y$ with $\widetilde{f}_1(y_0) = \widetilde{f}_2(y_0)$. Then $\widetilde{f}_1 = \widetilde{f}_2$.*

Proof. The proof is identical to that of Lemma 2.2.1, which is the analgous result for deck transformations. $\square$

## 3.2. Sections

We now discuss a special kind of lifting problem. Let $p\colon \widetilde{X} \to X$ be a covering space. A *section* of $p$ is a lift $\sigma\colon X \to \widetilde{X}$ of the identity map $\mathbb{1}_X\colon X \to X$ through $p$. In other words, $\sigma\colon X \to \widetilde{X}$ is a map such that $p(\sigma(x)) = x$ for all $x \in X$.

Example 3.2.1. Let $X$ be a space and $\mathcal{I}$ be a discrete set. Consider the trivial cover $p\colon X \times \mathcal{I} \to X$. Let $i_0 \in \mathcal{I}$, and define $\sigma\colon X \to X \times \mathcal{I}$ via the formula $\sigma(x) = (x, i_0)$. Then $\sigma$ is a section. $\square$

The above example might be unsatisfying; however, covers typically have no sections:

Lemma 3.2.2. *Let $p\colon \widetilde{X} \to X$ be a covering space with $\widetilde{X}$ connected. Assume that there exists a section $\sigma\colon X \to \widetilde{X}$. Then $p\colon \widetilde{X} \to X$ is a homeomorphism, and in particular has degree 1.*

Proof. It is enough to prove that $p\colon \widetilde{X} \to X$ has degree 1; indeed, this will imply that $p$ is a bijection, and since covering space maps are open maps we will be able to conclude that $p$ is a homeomorphism.[1] To prove that $p$ has degree 1, it is enough to prove that $\sigma$ is surjective. Since $\widetilde{X}$ is connected, this will follow if we prove that the image of $\sigma$ is both open and closed.

Consider $x \in X$. It is enough to prove that $\sigma(x)$ lies in the interior of $\sigma(X)$ and that all points of $p^{-1}(x)$ other than $\sigma(x)$ lie in the interior of $\widetilde{X} \setminus \sigma(X)$. Let $U$ be a trivialized neighborhood of $x$ and let $\{\widetilde{U}_i\}_{i \in \mathcal{I}}$ be the sheets lying above $U$. Let $i_0 \in \mathcal{I}$ be such that $\sigma(x) \in \widetilde{U}_{i_0}$. Naively, one might expect that $\sigma(U) = \widetilde{U}_{i_0}$; however, without further assumptions (like $U$ being connected, which could only be ensured if $X$ is locally connected) this need not hold.

However, let $V = \sigma^{-1}(\widetilde{U}_{i_0})$. The set $V$ is also a trivialized neighborhood of $x$. Let $\{\widetilde{V}_i\}_{i \in \mathcal{I}}$ be the sheets lying above $V$, enumerated such that $\widetilde{V}_i \subset \widetilde{U}_i$ for all $i \in \mathcal{I}$. Then $\widetilde{V}_{i_0} = \sigma(V)$ is an open neighborhood of $\sigma(x)$ lying in $\sigma(X)$. Also, the union of the $\widetilde{V}_i$ with $i \neq i_0$ is an open neighborhood of $p^{-1}(x) \setminus \sigma(x)$ lying in $\widetilde{X} \setminus \sigma(X)$. The lemma follows. $\square$

## 3.3. Formulas for roots of polynomials

We explain an interesting application of Lemma 3.2.2. As discussed in Example 1.6*.4, let $\mathrm{Poly}_n^{\mathrm{sf}}$ be the space of monic degree-$n$ polynomials without repeated roots, let $\mathrm{RPoly}_n^{\mathrm{sf}}$ be the space of pairs $(f, x)$ with $f \in \mathrm{Poly}_n^{\mathrm{sf}}$ and $f(x) = 0$, and let $p\colon \mathrm{RPoly}_n^{\mathrm{sf}} \to \mathrm{Poly}_n^{\mathrm{sf}}$ be the map $p(f, x) = f$. The fundamental theorem of algebra inplies that $p\colon \mathrm{RPoly}_n^{\mathrm{sf}} \to \mathrm{Poly}_n^{\mathrm{sf}}$ has degree $n$. We now prove that $\mathrm{RPoly}_n^{\mathrm{sf}}$ is path connected:

Lemma 3.3.1. *For $n \geq 1$, the space $\mathrm{RPoly}_n^{\mathrm{sf}}$ is path connected.*

Proof. Let $(f_1, x_1)$ and $(f_2, x_2)$ be two points of $\mathrm{RPoly}_n^{\mathrm{sf}}$. We want to find a path from $(f_1, x_1)$ to $(f_2, x_2)$. Since the polynomial $f_i(z)$ has no repeated roots, we can factor it as

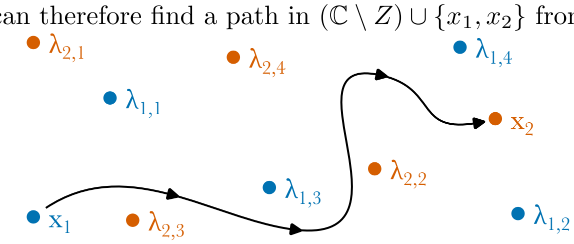
$$f_i(z) = (z - x_i)(z - \lambda_{i,1}) \cdots (z - \lambda_{i,n-1}).$$

Here the $\lambda_{i,j}$ are distinct complex numbers that are different from $x_i$. We remark that the ordering on $\{\lambda_{i,1}, \ldots, \lambda_{i,n-1}\}$ is not canonical. We can move $(f_i, x_i)$ in $\mathrm{RPoly}_n^{\mathrm{sf}}$ by moving $x_i$ and the $\lambda_{i,j}$ while keeping them distinct. Moving $x_1$ and the $\lambda_{1,j}$ slightly, we can ensure that the numbers

$$Z = \{x_1, \lambda_{1,1}, \ldots, \lambda_{1,n-1}, x_2, \lambda_{2,1}, \ldots, \lambda_{2,n-1}\}$$

are all distinct. We will now move the points $x_1, \lambda_{1,1}, \ldots, \lambda_{1,n-1}$ to $x_2, \lambda_{2,1}, \ldots, \lambda_{2,n-1}$ one at a time, starting with $x_1$.

Since removing finitely many points from $\mathbb{C}$ does not disconnect it, the space $(\mathbb{C} \setminus Z) \cup \{x_1, x_2\}$ is path connected. We can therefore find a path in $(\mathbb{C} \setminus Z) \cup \{x_1, x_2\}$ from $x_1$ to $x_2$:




By moving $x_1$ along this path, we move $(f_1, x_1)$ and reduce ourselves to the case where $x_1 = x_2$. Next, the space $(\mathbb{C} \setminus Z) \cup \{\lambda_{1,1}, \lambda_{2,1}\}$ is path connected, so we can find a path in it from $\lambda_{1,1}$ to $\lambda_{2,1}$. By moving $\lambda_{1,1}$ along this path, we move $(f_1, x_1)$ and reduce ourselves to the case where $x_1 = x_2$ and $\lambda_{1,1} = \lambda_{2,1}$. Repeating this process, we move $(f_1, x_1)$ to $(f_2, x_2)$. $\square$

Combining this with Lemma 3.2.2, we deduce the following:

[1] We remark that there is a simpler proof that $p$ has degree 1 if $\widetilde{X}$ is path connected. Consider $x \in X$. Set $z_1 = \sigma(x)$, so $z_1 \in p^{-1}(x)$. Consider $z_2 \in p^{-1}(x)$. We must prove that $z_1 = z_2$. Let $\widetilde{\gamma}\colon [0,1] \to \widetilde{X}$ be a path with $\widetilde{\gamma}(0) = z_1$ and $\widetilde{\gamma}(1) = z_2$. Set $\gamma = p \circ \gamma$, so $\gamma\colon [0,1] \to X$ is a path in $X$ from $x = p(z_1)$ to $x = p(z_2)$. Define $\widetilde{\gamma}' = \sigma \circ \gamma$. Both $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ are lifts of $\gamma\colon [0,1] \to X$ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{\gamma}'(0) = z_1$, so by Lemma 3.1.2 we have $\widetilde{\gamma} = \widetilde{\gamma}'$. We conclude that $z_1 = \widetilde{\gamma}(1) = \widetilde{\gamma}'(1) = z_2$, as desired.

COROLLARY 3.3.2. *For $n \geq 2$, the covering space $p\colon \mathrm{RPoly}_n^{\mathrm{sf}} \to \mathrm{Poly}_n^{\mathrm{sf}}$ has no section.*

Why is this interesting? Recall that $\mathrm{Poly}_n^{\mathrm{sf}} \subset \mathbb{C}^n$, where

$$f(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n \in \mathrm{Poly}_n^{\mathrm{sf}}$$

is identified with $(a_1, \ldots, a_n) \in \mathbb{C}^n$. A section of $p\colon \mathrm{RPoly}_n^{\mathrm{sf}} \to \mathrm{Poly}_n^{\mathrm{sf}}$ is thus a function that takes as input the coefficients of a polynomial $f(z)$ with no repeated roots and returns $(f, x)$, where $x \in \mathbb{C}$ is a root of $f(z)$. In other words, it is a continuous "formula" for the roots of a degree-$n$ polynomial.

The fact that these do not exist for $n \geq 2$ might seem to contradict that we do in fact have such formulas in low degrees. For instance, we have the quadratic formula: for a quadratic polynomial $f(z) = z^2 + bz^1 + c$, its roots are

$$\frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

The point here is that this is not really a well-defined function because of the $\pm$, and indeed there is no way to choose a canonical square root of a complex number in a continuous way. In Essay YYY, we will see that this forms the germ of a beautiful proof of Arnold of the classical fact (usually proved with Galois theory) that there is no elementary formula for the roots of a degree-$n$ polynomial for $n \geq 5$, even if you allow multivalued $k^{\text{th}}$ roots like in the quadratic formula.

## 3.4. Lifting paths

Once the basic theory of the fundamental group is in place, we will be able to give a satisfying necessary and sufficient condition for a lift to exist, at least for reasonable spaces (see Chapter 9). Before we can do this, we need to solve some important special cases. As notation, let $I = [0, 1]$. A *path* in a space $X$ is a map $\gamma\colon I \to X$. The *initial point* of $\gamma$ is $\gamma(0)$ and the *terminal point* is $\gamma(1)$, and we say that $\gamma$ goes from $\gamma(0)$ to $\gamma(1)$. Paths can always be lifted:

LEMMA 3.4.1. *Let $p\colon \widetilde{X} \to X$ be a covering space and let $\gamma\colon I \to X$ be a path. For all $\widetilde{x}_0 \in \widetilde{X}$ with $p(\widetilde{x}_0) = \gamma(0)$, there exists a unique lift $\widetilde{\gamma}\colon I \to \widetilde{X}$ of $\gamma$ through $p$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$.*

PROOF. Uniqueness follows from Lemma 3.1.2, so we must only prove existence. Using the Lebesgue number lemma,[2] we can partition $I$ into subintervals

$$0 = \epsilon_1 < \epsilon_2 < \cdots < \epsilon_n = 1$$

such that for all $1 \leq k < n$ the image $\gamma([\epsilon_k, \epsilon_{k+1}])$ is contained in a trivialized open set in $X$. We construct our lift $\widetilde{\gamma}$ inductively as follows.

First, define $\widetilde{\gamma}(0) = \widetilde{x}_0$. Next, assume that for some $1 \leq k < n$ we have constructed a lift $\widetilde{\gamma}\colon [0, \epsilon_k] \to \widetilde{X}$ of $\gamma|_{[0,\epsilon_k]}\colon [0, \epsilon_k] \to X$. We extend $\widetilde{\gamma}$ to $[0, \epsilon_{k+1}]$ as follows. Let $U$ be a trivialized open set in $X$ such that $\gamma([\epsilon_k, \epsilon_{k+1}]) \subset U$. Let $\widetilde{U}$ be the sheet lying above $U$ with $\widetilde{\gamma}(\epsilon_k) \in \widetilde{U}$. The restriction $p|_{\widetilde{U}}\colon \widetilde{U} \to U$ is a homeomorphism, and on the interval $[\epsilon_k, \epsilon_{k+1}]$ we define $\widetilde{\gamma}$ to be the composition

$$[\epsilon_k, \epsilon_{k+1}] \xrightarrow{\ \gamma\ } U \xrightarrow{(p|_{\widetilde{U}})^{-1}} \widetilde{U} \hookrightarrow \widetilde{X}.$$

By construction, this agrees with our already-constructed partial lift $\widetilde{\gamma}\colon [0, \epsilon_k] \to \widetilde{X}$ at $\epsilon_k$. $\square$

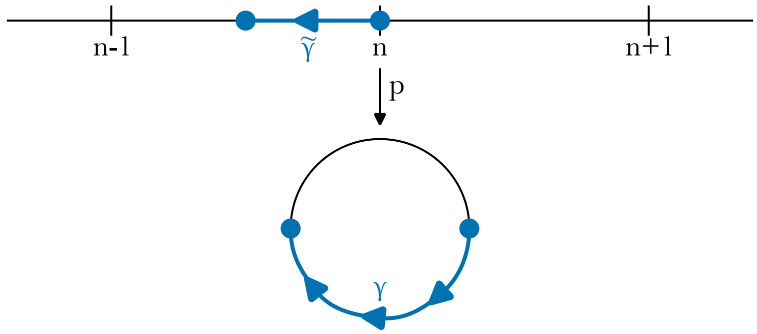
To help the reader understand the content of this lemma, we give several examples.

EXAMPLE 3.4.2 (Circle). Let $p\colon \mathbb{R} \to \mathbb{S}^1$ be the universal cover of $\mathbb{S}^1$, so $p(\theta) = e^{2\pi i\theta}$. Let $\gamma\colon [0, 1] \to \mathbb{S}^1$ be the path that starts at $1 \in \mathbb{S}^1 \subset \mathbb{C}$ and travels clockwise half-way around the circle:

$$\gamma(t) = e^{-\pi i t} \quad \text{for } 0 \leq t \leq 1.$$

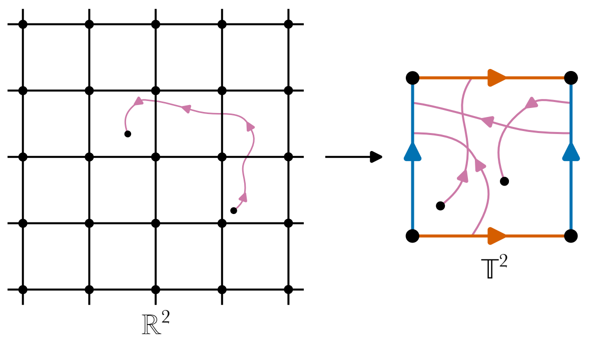
The points of $\mathbb{R}$ that project to $\gamma(0) = 1$ are precisely the integers. For $n \in \mathbb{Z}$, the lift $\widetilde{\gamma}\colon [0, 1] \to \mathbb{R}$ of $\gamma$ with $\widetilde{\gamma}(0) = n$ is the map that looks like this:

[2] Recall that the Lebesgue number lemma says that if $Z$ is a compact metric space and $\{W_j\}_{j \in J}$ is an open cover of $Z$, then we can find some $\epsilon > 0$ such that for all $z \in Z$ the $\epsilon$-ball $B_\epsilon(z)$ is contained in some $W_j$. To find the indicated partition of $I$, apply this to the cover of $I$ by preimages of trivialized open subsets of $X$ and choose the partition such that each segment $[\epsilon_k, \epsilon_{k+1}]$ has diameter at most the $\epsilon > 0$ given by the lemma.



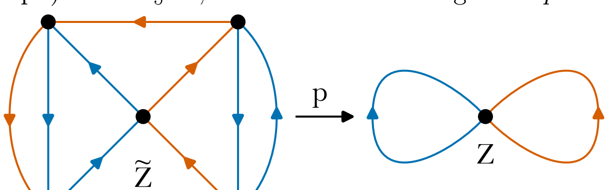

In coordinates, $\widetilde{\gamma}(t) = n - t/2$ for $0 \leq t \leq 1$. □

EXAMPLE 3.4.3 (Torus). As in Example 1.3.3, identify $\mathbb{R}^2/\mathbb{Z}^2$ with the 2-torus $\mathbb{T}^2$ and let $p\colon \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$ be the associated cover. Here is an example of a path $\gamma\colon [0,1] \to \mathbb{T}^2$ and one choice of lift $\widetilde{\gamma}\colon [0,1] \to \mathbb{T}^2$:

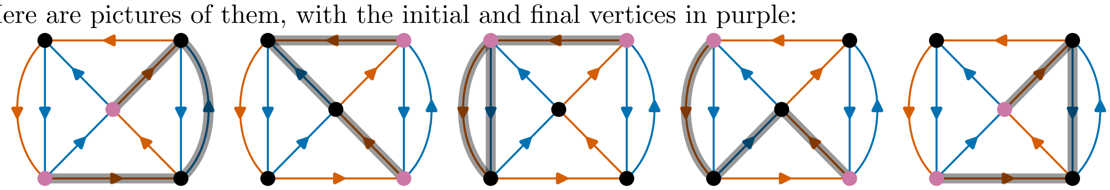



The torus on the right is obtained by gluing the sides of the square together as indicated. Because of this gluing, a path can e.g. pass through the top edge of the square and come out of the bottom edge. The other possible lifts are obtained by varying the initial point, which results in translating the entire lift by some element of $\mathbb{Z}^2$. □

EXAMPLE 3.4.4 (Graph). As in §2.5, consider the following cover $p\colon \widetilde{Z} \to Z$:




Let $\gamma\colon [0,1] \to Z$ be the path that starts at the vertex, goes around the orange circle in the positive direction, then goes around the blue circle in the positive direction, and finally goes around the orange circle in the negative direction. There are five possible lifts, one starting at each vertex of $\widetilde{Z}$. Here are pictures of them, with the initial and final vertices in purple:



Constructing these illustrates the necessity that each vertex of $\widetilde{Z}$ has one incoming edge of each color and one outgoing edge of each color. □

EXAMPLE 3.4.5 (Polar coordinates). Points of $\mathbb{R}^2 \setminus 0$ can be expressed using polar coordinates $(r, \theta)$ with $r > 0$ and $\theta \in \mathbb{R}$:

$$(x, y) = (r\cos(\theta), r\sin(\theta)).$$

While $r = \sqrt{x^2+y^2}$ is unambiguous, the $\theta$-coordinate is ambiguous since $(r,\theta) = (r, \theta + 2\pi n)$ for all $n \in \mathbb{Z}$. Letting $p\colon \mathbb{R} \to \mathbb{S}^1$ be the universal cover, a choice of polar coordinates for $(x,y) \in \mathbb{R}^2 \setminus 0$ is the same as a choice of lift $\theta \in \mathbb{R}$ for the point

$$(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}) \in \mathbb{S}^1.$$

Lemma 3.4.1 explains why maps $f\colon I \to \mathbb{R}^2 \setminus 0$ can always be continuously expressed using polar coordinates. This is depends on the topology of $I$, and $I$ cannot be replaced by an arbitrary space.

For instance, the inclusion $\iota\colon \mathbb{S}^1 \hookrightarrow \mathbb{R}^2 \setminus 0$ cannot be continuously described using polar coordinates as $\iota(x) = (r(x), \theta(x))$ for some $r\colon \mathbb{S}^1 \to \mathbb{R}_{>0}$ and $\theta\colon \mathbb{S}^1 \to \mathbb{R}$. Indeed, in such an expression the function $r$ would be identically 1, but the function $\theta$ would be a section of the cover $p\colon \mathbb{R} \to \mathbb{S}^1$, and Lemma 3.2.2 implies that such a section does not exist. $\square$

## 3.5. Homotopies

In algebraic topology, spaces are modeled by algebra. Spaces can vary continuously, while algebraic objects are typically discrete. In this section, we introduce a formalism called homotopy for studying deformations of maps. The algebraic invariants we later study will be insensitive to these deformations.

Consider two maps $f, g\colon X \to Y$. We say that $f$ and $g$ are *homotopic* if there exists a continuous map $H\colon X \times I \to Y$ such that $f(x) = H(x,0)$ and $g(x) = H(x,1)$ for all $x \in X$. For $t \in I$, let $h_t\colon X \to Y$ be the map $h_t(x) = H(x,t)$. We thus have $f = h_0$ and $g = h_1$, and we view the $h_t$ as a continuous family of maps witnessing $f$ being deformed to $g$. Typically we will demonstrate that $f$ and $g$ are homotopic by describing the $h_t$ rather than $H$, and will call $h_t$ a *homotopy from $f$ to $g$*. This is an equivalence relations on the set of maps from $X$ to $Y$ (see Exercise 3.3).

Example 3.5.1. Let $X$ be a space. Any two maps $f, g\colon X \to \mathbb{R}^n$ are homotopic via the straight-line homotopy $h_t\colon X \to \mathbb{R}^n$ defined via the formula $h_t(x) = (1-t)f(x) + tg(x)$ for all $x \in X$ and $t \in I$. In this, we have $h_0 = f$ and $h_1 = g$. $\square$

We say that a map $f\colon X \to Y$ is *null-homotopic* if $f$ is homotopic to a constant map. As Example 3.5.1 shows, any map $f\colon X \to \mathbb{R}^n$ is null-homotopic. Here is another example:

Example 3.5.2. Let $X$ be a space. Then any map $f\colon \mathbb{R}^n \to X$ is null-homotopic via the homotopy $h_t\colon \mathbb{R}^n \to X$ defined via the formula $h_t(x) = f((1-t)x)$ for all $x \in X$ and $t \in I$. In this, we have $h_0 = f$ and $h_1$ is the constant map taking all points of $\mathbb{R}^n$ to $f(0)$. $\square$

To show that two maps are homotopic, one typically exhibits an explicit homotopy. It is harder to show that two maps are *not* homotopic. This requires invariants of maps. For instance, the identity map $\mathbb{S}^1 \to \mathbb{S}^1$ is not null-homotopic, but this is not so easy to prove directly. In §3.7 below we will prove this by developing the theory of degrees and winding numbers. In fact, using this we will completely describe *all* homotopy classes of maps $\mathbb{S}^1 \to \mathbb{S}^1$. Doing this requires studying the interaction between homotopies and lifting problems.

## 3.6. Lifting homotopies

Roughly speaking, our goal in this section is to prove that lifting problems are insensitive to homotopies. To make this precise, consider a covering space $p\colon \widetilde{X} \to X$. Let $f, g\colon Y \to X$ be two homotopic maps. One thing we would like to prove is that a lift $\widetilde{f}\colon Y \to \widetilde{X}$ of $f$ exists if and only if a lift $\widetilde{g}\colon Y \to \widetilde{X}$ exists. We would also like to prove that if these lifts exist then we can choose lifts $\widetilde{f}\colon Y \to \widetilde{X}$ and $\widetilde{g}\colon Y \to \widetilde{X}$ such that $\widetilde{f}$ and $\widetilde{g}$ are themselves homotopic. The following result implies both of these claims.

Lemma 3.6.1. *Let $p\colon \widetilde{X} \to X$ be a covering space. Let $f\colon Y \to X$ be a map and let $\widetilde{f}\colon Y \to \widetilde{X}$ be a lift of $f$ through $p$. Let $h_t\colon Y \to X$ be a homotopy with $h_0 = f$. There is then a unique lift of $h_t$ through $p$ to a homotopy $\widetilde{h}_t\colon Y \to \widetilde{X}$ such that $\widetilde{h}_0 = \widetilde{f}$.*

Proof. Let $H\colon Y \times I \to X$ be the map with $H(y,t) = h_t(y)$ for all $y \in Y$ and $t \in I$. Our goal is to prove that there is a unique lift $\widetilde{H}\colon Y \times I \to \widetilde{X}$ of $H$ through $p$ such that $\widetilde{H}(y,0) = \widetilde{f}(y)$ for all $y \in Y$. Uniqueness follows from Lemma 3.1.2, so we must only prove existence. In fact, even more is true. For $y \in Y$, let $\gamma_y\colon I \to X$ be the path $\gamma_y(t) = H(y,t)$. By path-lifting (Lemma 3.4.1), we can lift $\gamma_y$ to a path $\widetilde{\gamma}_y\colon I \to \widetilde{X}$ such that $\widetilde{\gamma}_y(0) = \widetilde{f}(0)$. Define $\widetilde{H}\colon Y \times I \to \widetilde{X}$ via the formula

$$\widetilde{H}(y,t) = \widetilde{\gamma}_y(t) \quad \text{for all } y \in Y \text{ and } t \in I.$$

By construction, $\widetilde{H}$ is a lift of $H$ with $\widetilde{H}(y,0) = \widetilde{f}(y)$ for all $y \in Y$.

There is only one problem: it is not obvious that this $H$ is continuous. To see that it is, fix some $y_0 \in Y$. We will prove that $\widetilde{H}$ is continuous at all points of the form $(y_0,t)$ by imitating our proof of the path-lifting lemma (Lemma 3.4.1). Just like in that proof, using the Lebesgue number lemma we can partition $I$ into subintervals

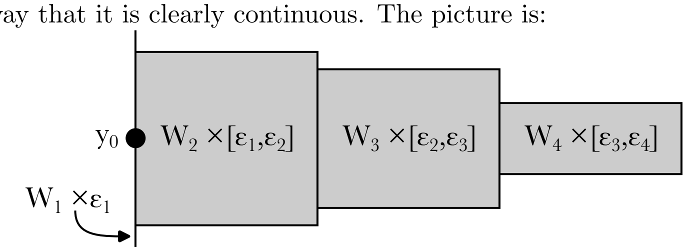
$$0 = \epsilon_1 < \epsilon_2 < \cdots < \epsilon_n = 1$$

such that for all $1 \leq k < n$ the image $H(y_0 \times [\epsilon_k, \epsilon_{k+1}])$ is contained in a trivialized open set in $X$. In fact, we can even find some open neighborhood $V_k$ of $y_0$ such that the image $H(V_k \times [\epsilon_k, \epsilon_{k+1}])$ is contained in a trivialized open set in $X$.

Define $W_1 = V_1 \cap \cdots \cap V_{n-1}$, so $W_1$ is an open neighborhood of $y_0$. We will find a nested sequence

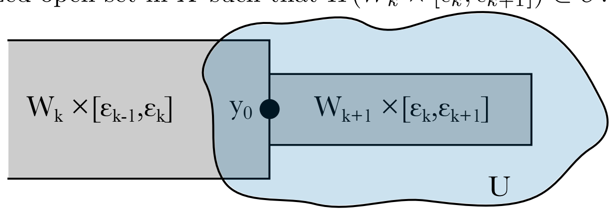
$$W_1 \supset W_2 \supset \cdots \supset W_n$$

of open neighborhoods of $y_0$ such that $\widetilde{H}$ is continuous on each $W_k \times [0, \epsilon_k]$ by constructing $\widetilde{H}$ on this set in such a way that it is clearly continuous. The picture is:




The construction will be inductive. First, define $\widetilde{H}\colon W_1 \times 0 \to \widetilde{X}$ via the formula $\widetilde{H}(y,0) = \widetilde{f}(y)$. Next, assume that for some $1 \leq k < n$ we have constructed a continuous lift $\widetilde{H}\colon W_k \times [0,\epsilon_k] \to \widetilde{X}$ of $H|_{W_k \times [0,\epsilon_k]}\colon W_k \times [0,\epsilon_k] \to X$. We find an open neighborhood $W_{k+1}$ of $y_0$ with $W_{k+1} \subset W_k$ and an extension of $\widetilde{H}$ to $W_{k+1} \times [0, \epsilon_{k+1}]$ as follows.

Let $U$ be a trivialized open set in $X$ such that $H(W_k \times [\epsilon_k, \epsilon_{k+1}]) \subset U$:




Let $\widetilde{U}$ be the sheet lying above $U$ with $H(y_0, \epsilon_k) \in \widetilde{U}$. Let $W_{k+1}$ be the preimage of $\widetilde{U}$ under the map[3]

$$W_k = W_k \times \epsilon_k \xrightarrow{\widetilde{H}(-,\epsilon_k)} \widetilde{X}.$$

The set $W_{k+1}$ is an open neighborhood of $y_0$ and $\widetilde{H}$ takes $W_{k+1} \times \epsilon_k$ to $\widetilde{U}$. The restriction $p|_{\widetilde{U}}\colon \widetilde{U} \to U$ is a homeomorphism, and on $W_{k+1} \times [\epsilon_k, \epsilon_{k+1}]$ we define $\widetilde{H}$ to be the composition

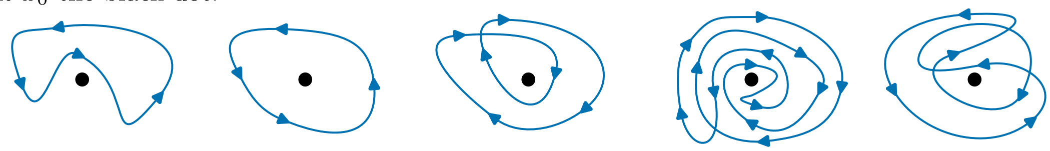
$$W_{k+1} \times [\epsilon_k, \epsilon_{k+1}] \xrightarrow{H} U \xrightarrow{(p|_{\widetilde{U}})^{-1}} \widetilde{U} \hookrightarrow \widetilde{X}.$$

[3]If we could ensure that $\widetilde{H}(W_k \times \epsilon_k) \subset \widetilde{U}$ (which would hold, for instance, if $W_k$ and $\widetilde{U}$ were connected), then there would be no need to pass to the nested sequence $W_1 \supset W_2 \supset \cdots$. Since we are not assuming that our spaces are locally connected, this is unfortunately necessary.

By construction, this agrees with our already-constructed partial lift $\widetilde{H}\colon W_{k+1} \times [0, \epsilon_k] \to \widetilde{X}$ on $W_{k+1} \times \epsilon_k$. $\square$

## 3.7. Winding numbers

To illustrate the meaning of all this machinery, we study the classical subject of winding numbers. Fix a point $x_0 \in \mathbb{C}$. We start with an intuitive discussion. Consider a map $f\colon \mathbb{S}^1 \to \mathbb{C} \setminus x_0$. Roughly speaking, the winding number $\mathrm{wind}_{x_0}(f)$ of $f$ around $x_0$ measures the number of times the vector $f(z) - x_0$ rotates as $z$ moves around $\mathbb{S}^1$. The "number" here includes a sign: a counterclockwise rotation counts as $+1$, while a clockwise rotation counts as $-1$. Here are several examples, with the point $x_0$ the black dot:



$\mathrm{wind}_{x_0}(f)$=0 $\quad$ $\mathrm{wind}_{x_0}(f)$=1 $\quad$ $\mathrm{wind}_{x_0}(f)$=-2 $\quad$ $\mathrm{wind}_{x_0}(f)$=-2 $\quad$ $\mathrm{wind}_{x_0}(f)$=0

One feature of the winding number is that it is invariant under homotopies of $f$ through maps that avoid $x_0$. In fact, it is a *complete* invariant of such maps (see Exercise 3.1). It is enlightening to verify that the different $f$ above with the same winding number are homotopic.

We now give a precise definition. Let $p\colon \mathbb{R} \to \mathbb{S}^1$ be the universal cover. Consider some $f\colon \mathbb{S}^1 \to \mathbb{C} \setminus x_0$. Define a map $F\colon I \to \mathbb{S}^1$ via the formula

$$F(t) = \frac{f(e^{2\pi i t}) - x_0}{\|f(e^{2\pi i t}) - x_0\|} \quad \text{for } t \in I.$$

This definition makes sense since $f(z) \neq x_0$ for all $z \in \mathbb{S}^1$. Pick some $\theta_0 \in \mathbb{R}$ such that $p(\theta_0) = F(0)$, and use path lifting (Lemma 3.4.1) to lift $F$ through $p$ to $\widetilde{F}\colon I \to \mathbb{R}$ with $\widetilde{F}(0) = \theta_0$. Since $F(0) = F(1)$, the lifts $\widetilde{F}(0) = \theta_0$ and $\widetilde{F}(1)$ differ by an integer. We define

$$\mathrm{wind}_{x_0}(f) = \widetilde{F}(1) - \widetilde{F}(0) \in \mathbb{Z}.$$

The only arbitrary choice we made was the lift $\theta_0$. Any other choice of $\theta_0$ is of the form $\theta_0 + m$ for some $m \in \mathbb{Z}$, and using $\theta_0 + m$ as our initial lift would change $\widetilde{F}$ to $\widetilde{F} + m$. Since

$$(\widetilde{F}(1) + m) - (\widetilde{F}(0) + m) = \widetilde{F}(1) - \widetilde{F}(0),$$

this would not change $\mathrm{wind}_{x_0}(f)$. In other words, $\mathrm{wind}_{x_0}(f) \in \mathbb{Z}$ is well-defined.

EXAMPLE 3.7.1. Fix $k \in \mathbb{Z}$, and define $f\colon \mathbb{S}^1 \to \mathbb{C} \setminus x_0$ via the formula

$$f(z) = x_0 + z^k \quad \text{for } z \in \mathbb{S}^1 \subset \mathbb{C}.$$

In the above recipe, we then have

$$F(t) = \frac{f(e^{2\pi i t}) - x_0}{\|f(e^{2\pi i t}) - x_0\|} = e^{2\pi i k t} \quad \text{for } t \in I.$$

We can take $\theta_0 = 0$, and then

$$\widetilde{F}(\theta) = k\theta \quad \text{for } \theta \in \mathbb{R}.$$

It follows that $\mathrm{wind}_{x_0}(f) = k$. We thus see that all integers can be winding numbers. $\square$

One of the main properties of the winding number is that it is unchanged under homotopies:

LEMMA 3.7.2. *Let $x_0 \in \mathbb{C}$, and let $f, g\colon \mathbb{S}^1 \to \mathbb{C} \setminus x_0$ be homotopic maps. Then* $\mathrm{wind}_{x_0}(f) = \mathrm{wind}_{x_0}(g)$.

PROOF. Let $p\colon \mathbb{R} \to \mathbb{S}^1$ be the universal cover. As in the definition of the winding number, define $F\colon I \to \mathbb{S}^1$ and $G\colon I \to \mathbb{S}^1$ via the formulas[4]

$$F(s) = \frac{f(e^{2\pi i s}) - x_0}{\|f(e^{2\pi i s}) - x_0\|} \quad \text{and} \quad G(s) = \frac{g(e^{2\pi i s}) - x_0}{\|g(e^{2\pi i s}) - x_0\|} \quad \text{for } s \in I.$$

[4]We use $s$ instead of $t$ since we will later use $t$ when we talk about homotopies.

Pick some $\theta_0 \in \mathbb{R}$ such that $p(\theta_0) = F(0)$, and use path lifting (Lemma 3.4.1) to lift $F$ through $p\colon \mathbb{R} \to \mathbb{S}^1$ to $\widetilde{F}\colon I \to \mathbb{R}$ with $\widetilde{F}(0) = \theta_0$. We then have $\text{wind}_{x_0}(f) = \widetilde{F}(1) - \widetilde{F}(0)$. We will hold off on constructing the lift of $G$ that would determine $\text{wind}_{x_0}(g)$.

Now let $h_t\colon \mathbb{S}^1 \to \mathbb{C} \setminus x_0$ be a homotopy from $f$ to $g$. Define $H_t\colon I \to \mathbb{S}^1$ via the formula

$$H_t(s) = \frac{h_t(e^{2\pi i s}) - x_0}{\|h_t(e^{2\pi i s}) - x_0\|} \quad \text{for } s \in I,$$

so $H_t$ is a homotopy from $H_0 = F$ to $H_1 = G$. Use the homotopy lifting lemma (Lemma 3.6.1) to lift $H_t$ to a homotopy $\widetilde{H}_t\colon I \to \mathbb{R}$ with $\widetilde{H}_0 = \widetilde{F}$. It follows that $\widetilde{H}_1$ is a lift of $G$, so $\text{wind}_{x_0}(g) = \widetilde{H}_1(1) - \widetilde{H}_1(0)$. More generally, we have $\text{wind}_{x_0}(h_t) = \widetilde{H}_t(1) - \widetilde{H}_t(0)$ for all $t \in I$. This implies that the map

$$t \mapsto \widetilde{H}_t(1) - \widetilde{H}_t(0) \quad \text{for } t \in I$$

is a continuous integer-valued function. It is thus constant, so

$$\text{wind}_{x_0}(f) = \widetilde{H}_0(1) - \widetilde{H}_0(0) = \widetilde{H}_1(1) - \widetilde{H}_1(0) = \text{wind}_{x_0}(g). \qquad \square$$

## 3.8. Degree of map of circle

Consider a map $f\colon \mathbb{S}^1 \to \mathbb{S}^1$. We can regard $f$ as a map to $\mathbb{C} \setminus 0$, giving an integer $\text{wind}_0(f)$ that we will call the *degree*[5] of $f$. Denote this by $\deg(f)$. Lemma 3.7.2 implies that $\deg(f)$ is invariant under homotopy, and just like in Example 3.7.1 we have $\deg(z^n) = n$ for all $n \in \mathbb{Z}$. In particular, the degree of the identity is 1 and the degree of a constant map is 0. Since these are different, we deduce the following, which was promised at the end of §3.5:

LEMMA 3.8.1. *The identity map $\mathbb{1}\colon \mathbb{S}^1 \to \mathbb{S}^1$ is not nullhomotopic.*

The following basic result says that the degree is a *complete* invariant of homotopy classes of maps from $\mathbb{S}^1$ to itself:

LEMMA 3.8.2. *Let $f, g\colon \mathbb{S}^1 \to \mathbb{S}^1$ be maps with $\deg(f) = \deg(g)$. Then $f$ is homotopic to $g$.*

PROOF. By postcomposing $f$ and $g$ with paths of rotations of $\mathbb{S}^1$, we can homotope them to maps with $f(1) = g(1) = 1$. Define $F\colon I \to \mathbb{S}^1$ and $G\colon I \to \mathbb{S}^1$ via the formulas

$$F(s) = f(e^{2\pi i s}) \quad \text{and} \quad G(s) = g(e^{2\pi i s}) \quad \text{for } s \in I.$$

We thus have $F(0) = G(0) = 1$. Letting $p\colon \mathbb{R} \to \mathbb{S}^1$ be the universal cover, by the path lifting lemma (Lemma 3.4.1) we can lift $F$ and $G$ through $p$ to maps $\widetilde{F}, \widetilde{G}\colon I \to \mathbb{R}$ with $\widetilde{F}(0) = \widetilde{G}(0) = 0$. We have $\widetilde{F}(1) = \deg(f)$ and $\widetilde{G}(1) = \deg(g)$, which are equal by assumption. Define $H_t\colon I \to \mathbb{S}^1$ via the formula

$$H_t(s) = p((1-t)\widetilde{F}(s) + t\widetilde{G}(s)) \quad \text{for } s \in I.$$

The maps $H_t$ are a homotopy from $F$ to $G$, and since $\widetilde{F}(0) = \widetilde{G}(0) = 0$ and $\widetilde{F}(1) = \widetilde{G}(1) \in \mathbb{Z}$ we have $H_t(0) = 1$ and $H_t(1) = 1$ for all $t \in I$. This implies that there exists some $h_t\colon \mathbb{S}^1 \to \mathbb{S}^1$ with

$$H_t(s) = h_t(e^{2\pi i s}) \quad \text{for } s \in I.$$

This $h_t$ is a homotopy from $f$ to $g$. $\square$

REMARK 3.8.3. In later volumes when we develop some basic results about homology, we will generalize the notion of degree to an integer-valued invariant of maps $f\colon M^n \to N^n$ with $M^n$ and $N^n$ compact oriented $n$-manifolds. Lemma 3.8.2 generalizes to a deep theorem of Hopf saying that this degree is a complete invariant for maps $f\colon M^n \to \mathbb{S}^n$. $\square$

[5] If $f$ is a covering space, this is different from the degree of $f$ as a covering space. For instance, it can be negative.

## 3.9. Exercises

EXERCISE 3.1. Let $x_0 \in \mathbb{C}$ and let $f, g\colon \mathbb{S}^1 \to \mathbb{C} \setminus x_0$ be maps with $\operatorname{wind}_{x_0}(f) = \operatorname{wind}_{x_0}(g)$. Prove that $f$ is homotopic to $g$. Hint: first homotope $f$ and $g$ such that their images lie in $\mathbb{S}^1$, then appeal to Lemma 3.8.2. □

EXERCISE 3.2. Let $p\colon \widetilde{X} \to X$ be a covering space and let $f\colon \mathbb{D}^n \to X$ be a map. Set $x_0 = f(0)$. For each $\widetilde{x}_0 \in p^{-1}(x_0)$, prove that there is a unique lift $\widetilde{f}\colon \mathbb{D}^n \to \widetilde{X}$ with $\widetilde{f}(0) = \widetilde{x}_0$. □

EXERCISE 3.3. Let $X$ and $Y$ be spaces. Prove that the relation of being homotopic is an equivalence relation on maps from $X$ to $Y$. □

EXERCISE 3.4. Let $X$ and $Y$ and $Z$ be spaces. For $i = 0, 1$, let $f_i\colon X \to Y$ and $g_i\colon Y \to Z$ be maps. Assume that $f_0$ is homotopic to $f_1$ and that $g_0$ is homotopic to $g_1$. Prove that $g_0 \circ f_0\colon X \to Z$ is homotopic to $g_1 \circ f_1\colon X \to Z$. □

EXERCISE 3.5. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be maps. Prove that $g \circ f\colon X \to Z$ is nullhomotopic if either $f$ or $g$ is nullhomotopic. □

EXERCISE 3.6. Let $X$ be a space and let $f\colon \mathbb{S}^n \to X$ be a map. Prove that $f$ is null-homotopic if and only if $f$ extends to a map $F\colon \mathbb{D}^{n+1} \to X$. □

EXERCISE 3.7. A space $Y$ is *contractible* if the identity map $\mathbb{1}_Y\colon Y \to Y$ is null-homotopic (we will say more about this concept in Chapter 6). Let $p\colon \widetilde{X} \to X$ be a covering space with $\widetilde{X} \neq \emptyset$ and let $f\colon Y \to X$ be a continuous map with $Y$ contractible. Prove that $f$ can be lifted to $\widetilde{f}\colon Y \to \widetilde{X}$. Hint: First prove that $f$ is null-homotopic. □

EXERCISE 3.8. Let $p\colon \mathbb{R} \to \mathbb{S}^1$ be the universal cover. For a space $X$, prove that a map $f\colon X \to \mathbb{S}^1$ can be lifted to a map $\widetilde{f}\colon X \to \mathbb{R}$ if and only if $f$ is null-homotopic. □

EXERCISE 3.9. Let $p\colon \widetilde{X} \to X$ be a cover. Let $Z$ be a nonempty space and $f\colon Z \to X$ be a map. As in Exercise 1.14, let $q\colon \widetilde{Z} \to Z$ be the pullback of $p$ along $f$, so

$$\widetilde{Z} = \left\{ (\widetilde{x}, z) \in \widetilde{X} \times Z \mid p(\widetilde{x}) = f(z) \right\}.$$

Finally, let $W$ be a space and $g\colon W \to Z$ be a map. Prove that $g$ can be lifted to $\widetilde{g}\colon W \to \widetilde{Z}$ if and only if $g \circ f\colon W \to X$ can be lifted to $\widetilde{g \circ f}\colon W \to \widetilde{X}$. □

EXERCISE 3.10. Let $X$ be a topological space and let $f\colon X \to \mathbb{C}$ be a continuous function such that $f(x) \neq 0$ for all $x \in X$.

(a) Construct a degree 2 cover $p\colon \widetilde{X} \to X$ such that $f \circ p\colon \widetilde{X} \to \mathbb{C}$ has a continuous square root, i.e., there exists a continuous function $g\colon \widetilde{X} \to \mathbb{C}$ such that $f(p(\widetilde{x})) = g(\widetilde{x})^2$ for all $\widetilde{x} \in \widetilde{X}$.
(b) Prove that $p\colon \widetilde{X} \to X$ is a trivial cover if and only if $f\colon X \to \mathbb{C}$ has a continuous square root. □

EXERCISE 3.11. Define a map $f\colon \mathbb{S}^1 \times I \to \mathbb{S}^1 \times I$ via the formula

$$f(z, s) = (e^{2\pi i s} z, s) \quad \text{for all } z \in \mathbb{S}^1 \subset \mathbb{C} \text{ and } s \in I.$$

Prove the following:

(a) There is a homotopy $f_t\colon \mathbb{S}^1 \times I \to \mathbb{S}^1 \times I$ with $f_0 = f$ and $f_1 = \text{unit}$ such that $f_t(z, 0) = (z, 0)$ for all $z \in \mathbb{S}^1$ and $t \in I$.
(b) There does not exist a homotopy $f_t\colon \mathbb{S}^1 \times I \to \mathbb{S}^1 \times I$ with $f_0 = f$ and $f_1 = \text{unit}$ such that $f_t(z, 0) = (z, 0)$ and $f_t(z, 1) = (z, 1)$ for all $z \in \mathbb{S}^1$ and $t \in I$. □

EXERCISE 3.12. Let $f\colon \mathbb{S}^1 \to \mathbb{C} \setminus \{x_0\}$ be a smooth map. Prove that you can compute the winding number of $f$ around $x_0$ using the following formula from complex analysis:

$$\operatorname{wind}_{x_0}(f) = \frac{1}{2\pi i} \int_f \frac{1}{z - x_0} \, \mathrm{d}z\,.$$

Here the integral is the usual path integral. □

EXERCISE 3.13. Let $f, g\colon \mathbb{S}^1 \to \mathbb{S}^1$ be maps. Prove that $\deg(f \circ g) = \deg(f)\deg(g)$. $\square$

EXERCISE 3.14. Let $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ be a map with $\deg(f) \neq 1$. Prove that there exists some $x \in \mathbb{S}^1$ such that $f(x) = x$. $\square$

EXERCISE 3.15. Let $A \in \mathrm{GL}_2(\mathbb{R})$. Identify $\mathbb{C}$ with $\mathbb{R}^2$, so for $z \in \mathbb{C}$ we have $Az \in \mathbb{C}$. Define $f\colon \mathbb{S}^1 \to \mathbb{C} \setminus 0$ via the formula $f(z) = Az$. Prove that

$$\mathrm{wind}_0(f) = \begin{cases} 1 & \text{if } \det(A) > 0, \\ -1 & \text{if } \det(A) < 0. \end{cases}$$

In other words, $\mathrm{wind}_0(f)$ is the sign of $\det(A)$. $\square$

EXERCISE 3.16. Let $f\colon \mathbb{S}^1 \to \mathbb{C}$ be a smooth map. We say that $f$ is an *immersion* if for all $x \in \mathbb{S}^1$ the derivative map $D_x f\colon T_x\mathbb{S}^1 \to T_{f(x)}\mathbb{C}$ is an injective map from the 1-dimensional tangent space $T_x\mathbb{S}^1$ to the 2-dimensional tangent space $T_{f(x)}\mathbb{C} = \mathbb{C}$. A homotopy $f_t\colon \mathbb{S}^1 \to \mathbb{C}$ is a *regular homotopy* if each $f_t$ is an immersion. Define $g\colon \mathbb{S}^1 \to \mathbb{C}$ and $h\colon \mathbb{S}^1 \to \mathbb{C}$ via the formulas

$$g(z) = z \quad \text{and} \quad h(z) = z^{-1} \quad \text{for } z \in \mathbb{S}^1.$$

Prove that there does not exist a regular homotopy from $g$ to $h$. Hint: for $x \in \mathbb{S}^1$, we can identify $T_x\mathbb{S}^1$ with a subspace of $T_x\mathbb{C} = \mathbb{C}$. Using this identification, we have $ix \in T_x\mathbb{S}^1$. For an immersion $f\colon \mathbb{S}^1 \to \mathbb{C}$, consider the winding number around 0 of the map $\tau_f\colon \mathbb{S}^1 \to \mathbb{C} \setminus 0$ defined by $\tau_f(x) = (D_x f)(ix)$. $\square$

EXERCISE 3.17. For $a < b$, a map $\gamma\colon [a, b] \to \mathbb{R}^n$ is *linear* if for some $x, y \in \mathbb{R}^n$ it can be written in the form

$$\gamma(t) = x + ty \quad \text{for } t \in [a, b].$$

A map $\gamma\colon I \to \mathbb{R}^n$ is *piecewise linear* if there exist

$$0 = \epsilon_1 < \epsilon_2 < \cdots < \gamma_n = 1$$

such that $\gamma|_{[\epsilon_k, \epsilon_{k+1}]}$ is linear for all $1 \leq k < n$. For an open set $U \subset \mathbb{R}^n$ and a map $\gamma\colon I \to U$, prove that there is a homotopy $\gamma_t\colon I \to U$ such that:

- $\gamma_0 = \gamma$ and $\gamma_1$ is piecewise-linear; and
- $\gamma_t(0) = \gamma(0)$ and $\gamma_t(1) = \gamma(1)$ for all $t \in I$. $\square$

CHAPTER 4

# Homotopy classes of paths

Setting the stage for introducing the fundamental group in the next chapter, this chapter discusses homotopy classes of paths and how they lift to covers. As an application, for a broad class of spaces $X$ we give a criterion that implies that all covers $p\colon \widetilde{X} \to X$ are trivial.

## 4.1. Homotopies of paths

Let $X$ be a space and let $x, y \in X$. Recall from §3.4 that a *path* in $X$ from $x$ to $y$ is a map $\gamma\colon I \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. We wish to study paths up to homotopy. This would be uninteresting if we allowed the endpoints of $\gamma$ to move during the homotopy since then all paths would be homotopic if $X$ is path connected (see Exercise 4.1). We therefore make the following definition:

Definition 4.1.1. Let $X$ be a space, let $x, y \in X$, and let $\gamma_0, \gamma_1\colon I \to X$ be two paths from $x$ to $y$. We say that $\gamma_0$ and $\gamma_1$ are *homotopic* paths from $x$ to $y$ if there exists a homotopy $\gamma_t\colon I \to X$ from $\gamma_0$ to $\gamma_1$ that fixes the endpoints in the sense that

$$\gamma_t(0) = x \quad \text{and} \quad \gamma_t(1) = y \quad \text{for all } t \in I.$$

The relation of being homotopic is an equivalence relation[1] on paths, and we will call the equivalence classes of paths under this equivalence relation *homotopy classes*. For a path $\gamma$, we will write $[\gamma]$ for its homotopy class. □

Here are two examples:

Example 4.1.2. For all $x, y \in \mathbb{R}^n$, there is a unique homotopy class of path from $x$ to $y$. Indeed, let $\gamma_0\colon I \to \mathbb{R}^n$ be the straight line path

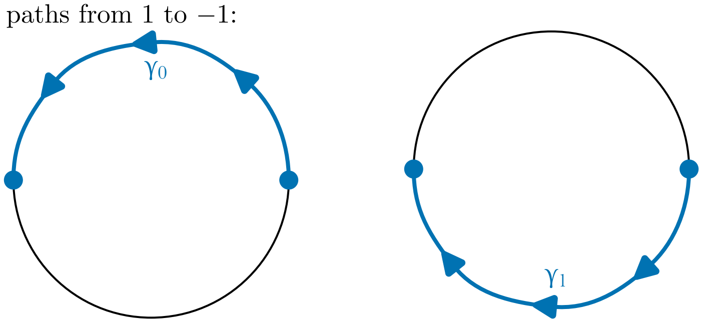
$$\gamma_0(s) = (1 - s)x + sy \quad \text{for } s \in I.$$

If $\gamma\colon I \to \mathbb{R}^n$ is any other path from $x$ to $y$, then $\gamma_0$ is homotopic to $\gamma$ via the homotopy $\gamma_t\colon I \to \mathbb{R}^n$ defined by $\gamma_t(s) = (1 - t)\gamma_0(s) + t\gamma(s)$ for all $s \in I$ and $t \in I$. □

Example 4.1.3. View $\mathbb{S}^1$ as a subspace of $\mathbb{C}$, and let $\gamma_0\colon I \to \mathbb{S}^1$ and $\gamma_1\colon I \to \mathbb{S}^1$ be the paths defined by the formulas

$$\gamma_0(s) = e^{\pi i s} \quad \text{and} \quad \gamma_1(s) = e^{-\pi i s} \quad \text{for } s \in I. \tag{4.1.1}$$

Both $\gamma_0$ and $\gamma_1$ are paths from 1 to $-1$:




We claim that $\gamma_0$ and $\gamma_1$ are not homotopic. To see this, assume that they are homotopic and that

[1] The proof is the same as the one needed for Exercise 3.3.

$\gamma_t\colon I \to \mathbb{S}^1$ is a homotopy. Let $p\colon \mathbb{R} \to \mathbb{S}^1$ be the universal cover, so $p(\theta) = e^{2\pi i\theta}$ for all $\theta \in \mathbb{R}$. The lift of $\gamma_0$ to $\mathbb{R}$ starting at 0 is the map $\widetilde{\gamma}_0\colon I \to \mathbb{R}$ defined by

$$\widetilde{\gamma}_0(s) = s/2 \quad \text{for } s \in I. \tag{4.1.2}$$

By Lemma 3.6.1, we can lift the homotopy $\gamma_t$ to a homotopy $\widetilde{\gamma}_t\colon I \to \mathbb{R}$ with $\widetilde{\gamma}_0$ the map (4.1.2). Since $\gamma_t(0) = 1$ and $\gamma_t(1) = -1$ for all $t \in I$, we have that

$$\widetilde{\gamma}_t(0) \in p^{-1}(1) = 2\pi\mathbb{Z} \quad \text{and} \quad \widetilde{\gamma}_t(1) \in p^{-1}(-1) = 2\pi\mathbb{Z} + \pi \quad \text{for } t \in I.$$

Since $2\pi\mathbb{Z}$ is discrete, it follows that both $\widetilde{\gamma}_t(0)$ and $\widetilde{\gamma}_t(1)$ are constant functions of $t$, i.e., that $\widetilde{\gamma}_t(0) = 0$ and $\widetilde{\gamma}_t(1) = 1/2$ for all $t \in I$. This implies in particular that $\widetilde{\gamma}_1$ is the lift of $\gamma_1$ to $\mathbb{R}$ starting at 0. From (4.1.1), we see that

$$\widetilde{\gamma}_1(s) = -s/2 \quad \text{for } s \in I.$$

In particular, $\widetilde{\gamma}_1(1) = -1/2$, contradicting the fact that $\widetilde{\gamma}_1(1) = 1/2$. $\square$

## 4.2. Lifting homotopies of paths

Let $p\colon \widetilde{X} \to X$ be a cover and let $\gamma\colon I \to X$ be a path in $X$ from $x \in X$ to $y \in X$. For $\widetilde{x} \in p^{-1}(x)$, Lemma 3.4.1 implies that there exists a unique lift $\widetilde{\gamma}\colon I \to \widetilde{X}$ of $\gamma$ with $\widetilde{\gamma}(0) = \widetilde{x}$. Generalizing the reasoning from Example 4.1.3, we prove that this lift only depends on the homotopy class $[\gamma]$:

Lemma 4.2.1. *Let $p\colon \widetilde{X} \to X$ be a cover and let $\gamma_0, \gamma_1\colon I \to X$ be two homotopic paths in $X$ from $x \in X$ to $y \in X$. Pick $\widetilde{x} \in p^{-1}(x)$, and for $i = 0, 1$ let $\widetilde{\gamma}_i\colon I \to \widetilde{X}$ be the lift of $\gamma_i$ to $\widetilde{X}$ with $\widetilde{\gamma}_i(0) = \widetilde{x}$. Then $\widetilde{\gamma}_0$ and $\widetilde{\gamma}_1$ are homotopic, i.e., $[\widetilde{\gamma}_0] = [\widetilde{\gamma}_1]$ In particular, $\widetilde{\gamma}_0(1) = \widetilde{\gamma}_1(1)$.*

Proof. Let $\gamma_t\colon I \to X$ be a homotopy of paths from $\gamma_0$ to $\gamma_1$. We thus have

$$\gamma_t(0) = x \quad \text{and} \quad \gamma_t(1) = y \quad \text{for all } t \in I.$$

By Lemma 3.6.1, we can lift the homotopy $\gamma_t$ to a homotopy[2] $\widetilde{\gamma}'_t\colon I \to X$ with $\widetilde{\gamma}'_0 = \widetilde{\gamma}_0$. We claim that $\gamma'_t$ is a homotopy of paths from $\widetilde{\gamma}_0$ to $\widetilde{\gamma}_1$. Indeed, since $\gamma_t(0) = x$ for all $t \in I$ it follows that the map $t \mapsto \widetilde{\gamma}'_t(0)$ is a path in the discrete space $p^{-1}(x)$. This path must be constant, so

$$\widetilde{\gamma}'_t(0) = \widetilde{\gamma}'_0(0) = \widetilde{\gamma}_0(0) = \widetilde{x} \quad \text{for all } t \in I.$$

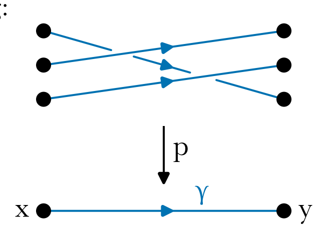
Similarly, $\widetilde{\gamma}'_t(1) = \widetilde{\gamma}_0(1)$ for all $t \in I$, as claimed. It follows that $\widetilde{\gamma}'_0 = \widetilde{\gamma}_0$ is homotopic to $\widetilde{\gamma}'_1$. Moreover, $\widetilde{\gamma}'_1$ is a lift of $\gamma_1$ with $\widetilde{\gamma}'_1(0) = \widetilde{x}$, so by the uniqueness of lifts of paths we have $\widetilde{\gamma}'_1 = \widetilde{\gamma}_1$. The lemma follows. $\square$

## 4.3. Connecting fibers

Let $p\colon \widetilde{X} \to X$ be a cover and let $\gamma\colon I \to X$ be a path in $X$ from $x \in X$ to $y \in X$. Define a map $\tau_\gamma\colon p^{-1}(x) \to p^{-1}(y)$ as follows:

- For $\widetilde{x} \in p^{-1}(x)$, let $\widetilde{\gamma}$ be the lift of $\gamma$ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}$. We then set $\tau_\gamma(\widetilde{x}) = \widetilde{\gamma}(1) \in p^{-1}(y)$.

If $\gamma$ is an embedding, then the lifts $\widetilde{\gamma}$ used to construct $\tau_\gamma$ are all disjoint (see Exercise 4.2). The picture thus looks like the following:




By Lemma 4.2.1, the map $\tau_\gamma$ only depends on the homotopy class $[\gamma]$ of $\gamma$. The following lemma says that it is a bijection:

Lemma 4.3.1. *Let $p\colon \widetilde{X} \to X$ be a cover and let $\gamma\colon I \to X$ be a path in $X$ from $x \in X$ to $y \in X$. Then the map $\tau_\gamma\colon p^{-1}(x) \to p^{-1}(y)$ is a bijection.*

[2] We call this $\widetilde{\gamma}'_t$ since it is not obvious that $\widetilde{\gamma}'_1 = \widetilde{\gamma}_1$, though we will soon prove this.

PROOF. Let $\overline{\gamma}\colon I \to X$ be the path obtained by reversing $\gamma$, i.e.,

$$\overline{\gamma}(s) = \gamma(1-s) \quad \text{for } s \in I.$$

The path $\overline{\gamma}$ goes from $y$ to $x$. To prove the lemma, it is enough to prove that $\tau_{\overline{\gamma}}\colon p^{-1}(y) \to p^{-1}(x)$ is an inverse to $\tau_\gamma$. To see this, consider $\widetilde{x} \in p^{-1}(x)$. Let $\widetilde{\gamma}$ be the lift of $\gamma$ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}$, so $\tau_\gamma(\widetilde{x}) = \widetilde{\gamma}(1)$. The lift $\widetilde{\overline{\gamma}}$ of $\overline{\gamma}$ to $\widetilde{X}$ with $\widetilde{\overline{\gamma}}(0) = \widetilde{\gamma}(1)$ is exactly the path obtained by reversing $\widetilde{\gamma}$, so

$$\tau_{\overline{\gamma}}(\tau_\gamma(\widetilde{x})) = \tau_{\overline{\gamma}}(\widetilde{\gamma}(1)) = \widetilde{\gamma}(0) = \widetilde{x}.$$

Similarly, $\tau_\gamma(\tau_{\overline{\gamma}}(\widetilde{y})) = \widetilde{y}$ for all $\widetilde{y} \in p^{-1}(y)$. The lemma follows. □

REMARK 4.3.2. For a cover $p\colon \widetilde{X} \to X$ with $X$ connected, Lemma 1.2.1 implies that the fibers $p^{-1}(x)$ all have the same cardinality. If $X$ is path connected, then, Lemma 4.3.1 also implies this, giving an alternate proof in this special case. □

## 4.4. Regularity from action on single fiber

Let $p\colon \widetilde{X} \to X$ be a cover with deck group $G$. Recall that $p\colon \widetilde{X} \to X$ is regular if for all $x_0 \in X$ the group $G$ acts transitively on the fiber $p^{-1}(x_0)$. One application of Lemma 4.3.1 is that if $X$ is path connected it is enough to check this on a single fiber:

LEMMA 4.4.1. *Let $p\colon \widetilde{X} \to X$ be a cover with deck group $G$. Assume that $X$ is path connected and that there exists some $x_0 \in X$ such that $G$ acts transitively on $p^{-1}(x_0)$. Then $p\colon \widetilde{X} \to X$ is regular.*

PROOF. Let $x_1 \in X$. We must prove that $G$ acts transitively on $p^{-1}(x_1)$. Let $\gamma$ be a path in $X$ from $x_0$ to $x_1$. Lemma 4.3.1 says that the the map $\tau_\gamma\colon p^{-1}(x_0) \to p^{-1}(x_1)$ is a bijection. Since the action of $G$ takes lifts of $\gamma$ to lifts of $\gamma$, it commutes with $\tau_\gamma$ in the sense that

$$\tau_\gamma(g\widetilde{x}_0) = g\tau_\gamma(\widetilde{x}_0) \quad \text{for all } g \in G \text{ and } \widetilde{x}_0 \in p^{-1}(x_0).$$

Since $\tau_\gamma$ is a bijection and $G$ acts transitively on $p^{-1}(x_0)$, it follows that $G$ acts transitively on $p^{-1}(x_1)$, as desired. □

## 4.5. 1-connectivity, spheres, and general position

We say that a space $X$ is 0-*connected* if it is nonempty and path connected.[3] For each $x, y \in X$, there thus exists a path from $x$ to $y$. We say that $X$ is 1-*connected* if $X$ is 0-connected and for all $x, y \in X$ there is a *unique* homotopy class of paths from $x$ to $y$. It is also common to say that a 1-connected space is *simply connected*.

EXAMPLE 4.5.1. We showed in Example 4.1.2 that $\mathbb{R}^n$ is 1-connected. We also showed in Example 4.1.3 that $\mathbb{S}^1$ is not 1-connected. □

Spheres of dimension at least 2 are important examples of 1-connected spaces:

LEMMA 4.5.2. *Let $n \geq 2$. Then $\mathbb{S}^n$ is 1-connected.*

PROOF. Consider $x, y \in \mathbb{S}^n$. We must prove that there is a unique homotopy class of paths from $x$ to $y$. Let $z \in \mathbb{S}^n$ be a point with $z \neq x, y$. Since $\mathbb{S}^n \setminus z \cong \mathbb{R}^n$, it follows from Example 4.1.2 that there exists a unique homotopy class of path from $x$ to $y$ in $\mathbb{S}^n \setminus r$. Letting $\gamma$ be a path from $x$ to $y$ in $\mathbb{S}^n$, to prove the claim it is enough to prove that $\gamma$ can be homotoped into $\mathbb{S}^n \setminus z$. This is nontrivial since there do exist space-filling curves in $\mathbb{S}^n$.

One way to do this is to use smooth manifold techniques. Indeed, it follows from standard results that $\gamma$ can be homotoped to a smooth map that is transverse to $z$. The point $z$ is 0-dimensional, and thus is a codimension $n$ submanifold of $\mathbb{S}^n$. It follows that $\gamma^{-1}(z)$ is a codimension $n \geq 2$ submanifold of $I$, and thus that $\gamma^{-1}(z) = \emptyset$.

Here is another approach that avoids using any technology. As in the following figure, let $V \cong \mathbb{R}^n$ be a small open neighborhood of $z$ with $x, y \notin V$:

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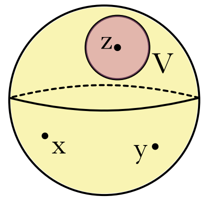
[3] See Exercise 4.7 for the origin of this terminology.




The subspace $V \cong \mathbb{R}^n$ is 1-connected (Example 4.1.2), and $V \setminus z$ is path connected. Intuitively, we should be able to make a small homotopy to the portions of $\gamma$ that pass through $V$ to make them miss $z$. Indeed, this is what the smooth manifold approach in the previous paragraph did. Lemma 4.5.3 below shows that this is in fact possible even in more general settings where smooth manifold techniques are unavailable. $\square$

The above proof used the following result:

Lemma 4.5.3 (General position). *Let $X$ be a space, let $x, y \in X$, and let $\gamma$ be a path in $X$ from $x$ to $y$. For some $z \in X$ with $z \neq x, y$, assume there is an open neighborhood $V$ of $z$ such that:*

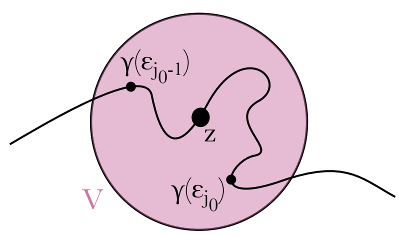
- *$V$ is 1-connected; and*
- *$V \setminus z$ is path connected.*

*Then $\gamma$ can be homotoped such that its image does not contain $z$.*

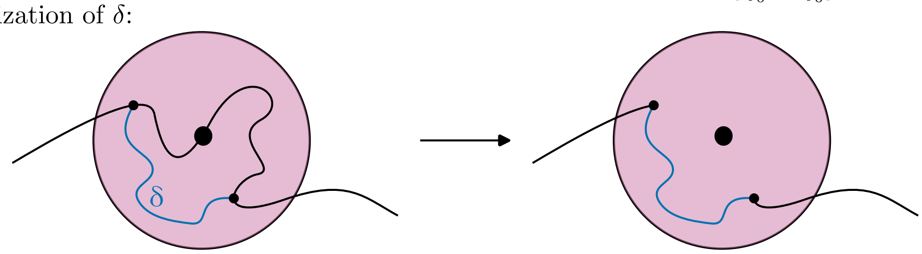
Proof. Set $U = X \setminus z$. The set $\{U, V\}$ is an open cover of $X$, so by the Lebesgue number lemma (cf. the proof of Lemma 3.4.1) we can find

$$0 = \epsilon_0 < \epsilon_1 < \cdots < \epsilon_k = 1$$

such that $\gamma([\epsilon_{j-1}, \epsilon_j])$ is contained in either $U$ or $V$. After possibly deleting some $\epsilon_j$ whose adjacent intervals are mapped to the same open set we can also assume that for all $1 \leq j \leq k-1$ we have $\gamma(\epsilon_j) \in U \cap V$. In particular, $\gamma(\epsilon_j) \neq z$ for all $0 \leq j \leq k$. Consider some $j_0$ such that $\gamma([\epsilon_{j_0-1}, \epsilon_{j_0}]) \subset V$:




Since $V \setminus z$ is path connected, there is some path $\delta$ in $V \setminus z$ from $\gamma(\epsilon_{j_0-1})$ to $\gamma(\epsilon_{j_0})$. Since $V$ is 1-connected, the path obtained by re-parameterizing $\gamma|_{[\epsilon_{j_0-1}, \epsilon_{j_0}]}$ to make its domain $I = [0, 1]$ is homotopic to $\delta$. It follows that we can homotope $\gamma$ to change $\gamma|_{[\epsilon_{j_0-1}, \epsilon_{j_0}]}$ to a suitable re-parametrization of $\delta$:




This ensures that the image of $\gamma|_{[\epsilon_{i_0-1}, \epsilon_{i_0}]}$ does not contain $z$. Doing this repeatedly homotopes $\gamma$ to a path that avoids $z$, as desired. $\square$

## 4.6. 1-connectivity and covers

Recall that a trivial cover of a space $X$ is a cover that is isomorphic to a cover of the form $X \times \mathcal{I} \to X$ for some discrete set $\mathcal{I}$. We will prove below that if $X$ is 1-connected and has reasonable local properties, then all covers of $X$ are trivial. For instance, this is why we have not seen any nontrivial covers of $\mathbb{S}^n$ for $n \geq 2$.

A space $X$ is *locally path connected* if for all $x \in X$ and all open neighborhoods $U$ of $x$, there is a path connected open neighborhood $V$ of $x$ with $V \subset U$. Most geometrically natural spaces are locally path connected; for instance, all manifolds have this property. We have:

THEOREM 4.6.1. *Let $X$ be a 1-connected space that is locally path connected and let $p\colon \widetilde{X} \to X$ be a cover. Then $p\colon \widetilde{X} \to X$ is trivial.*

REMARK 4.6.2. This would be false without the assumption that $X$ is locally path connected. See Exercise 4.11. □

PROOF OF THEOREM 4.6.1. Fix a point $x_0 \in X$. Consider some $x \in X$. Let $\gamma$ be a path in $X$ from $x_0$ to $x$ and let

$$\tau_\gamma\colon p^{-1}(x_0) \longrightarrow p^{-1}(x)$$

be the map constructed in §4.3 by lifting $\gamma$ to $\widetilde{X}$ starting at different points of $p^{-1}(x_0)$. As we observed in §4.3, the map $\tau_\gamma$ only depends on the homotopy class of $\gamma$. Since $X$ is 1-connected, all choices of $\gamma$ are homotopic, so $\tau_\gamma$ only depends on $x$. We therefore write $\tau_x = \tau_\gamma$.

We proved in Lemma 4.3.1 that $\tau_x$ is a bijection from $p^{-1}(x_0)$ to $p^{-1}(x)$. Every point of $\widetilde{X}$ lies in $p^{-1}(x)$ for some unique $x \in X$. Letting $\mathcal{I} = p^{-1}(x_0)$, we can therefore define a bijective set map $\phi\colon X \times \mathcal{I} \to \widetilde{X}$ as follows:

$$\phi(x, \widetilde{x}_0) = \tau_x(\widetilde{x}_0) \quad \text{for } x \in X \text{ and } \widetilde{x}_0 \in \mathcal{I} = p^{-1}(x_0).$$

Let $q\colon X \times \mathcal{I} \to X$ be the projection onto the first factor. By construction, $\phi$ fits into a commutative diagram

$$\begin{tikzcd} X \times \mathcal{I} \arrow[rr, "\phi"] \arrow[dr, "q"'] & & \widetilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

To prove that $\phi$ is an isomorphism from the trivial cover $q\colon X \times \mathcal{I} \to X$ to $p\colon \widetilde{X} \to X$, we must prove that $\phi$ is a homeomorphism. At this point, we remark that we do not even know that $\phi$ is continuous.

Since $\phi$ is a bijection, to prove that it is continuous and a homeomorphism it is enough to prove that it is a local homeomorphism. Consider some $x \in X$ and $\widetilde{x}_0 \in \mathcal{I} = p^{-1}(x_0)$. Set $\widetilde{x} = \phi(x, \widetilde{x}_0) = \tau_x(\widetilde{x}_0)$, so $\widetilde{x} \in p^{-1}(x)$. Let $U \subset X$ be a trivialized neighborhood of $x$ and let $\widetilde{U} \subset \widetilde{X}$ be the sheet lying above $U$ with $\widetilde{x} \in \widetilde{U}$. Since $X$ is locally path connected, we can shrink $U$ and assume that it is path connected. We claim that the restriction of $\phi$ to $U \times \widetilde{x}_0$ is the composition

$$U \times \widetilde{x}_0 \xrightarrow{q} U \xrightarrow{(p|_{\widetilde{U}})^{-1}} \widetilde{U}.$$

Since this is a homeomorphism between the open sets $U \times \widetilde{x}_0$ and $\widetilde{U}$, this will give the theorem.

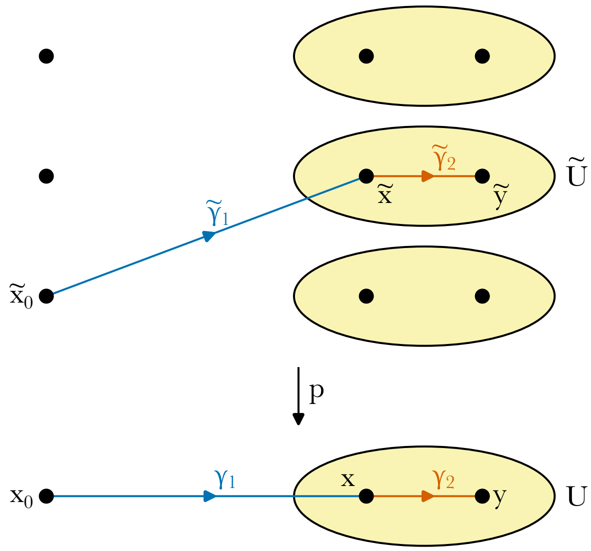
To see this claim, consider some $y \in U$. Let $\gamma$ be a path in $X$ from $x_0$ to $y$, and let $\widetilde{\gamma}$ be the lift of $\gamma$ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$. Set $\widetilde{y} = \widetilde{\gamma}(1)$, so $\widetilde{y} \in p^{-1}(y)$. What we must prove is that $\widetilde{y} \in \widetilde{U}$. In fact, since $X$ is 1-connected any two paths from $x_0$ to $y$ are homotopic, so we can choose $\gamma$ to be any such path we like.

Let $\gamma_1$ be a path in $X$ from $x_0$ to $x$ and let $\gamma_2$ be a path in the path connected subspace $U$ from $x$ to $y$. Let $\gamma\colon I \to X$ be the path[4]

$$\gamma(s) = \begin{cases} \gamma_1(2s) & \text{if } 0 \le s \le 1/2, \\ \gamma_2(2s-1) & \text{if } 1/2 \le s \le 1. \end{cases}$$

The path $\gamma$ thus first goes along $\gamma_1$ at $2\times$ speed and then goes along $\gamma_2$ at $2x$ speed. Let $\widetilde{\gamma}_1$ be the lift of $\gamma_1$ to $\widetilde{X}$ with $\widetilde{\gamma}_1(0) = \widetilde{x}_0$. By assumption, $\widetilde{\gamma}_1(1) = \phi(x, \widetilde{x}_0) = \widetilde{x} \in \widetilde{U}$. Let $\widetilde{\gamma}_2$ be the lift of $\gamma_2$ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}$:

[4]In the next chapter, we will systemize this kind of operation. In the notation of that chapter, we have $\gamma = \gamma_1 \cdot \gamma_2$.




As this figure shows, the lift $\widetilde{\gamma}$ of $\gamma$ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$ is

$$\widetilde{\gamma}(s) = \begin{cases} \widetilde{\gamma}_1(2s) & \text{if } 0 \leq s \leq 1/2, \\ \widetilde{\gamma}_2(2s-1) & \text{if } 1/2 \leq s \leq 1. \end{cases}$$

Since the image of $\gamma_2$ lies in $U$ and $\widetilde{\gamma}_2(0) \in \widetilde{U}$, it follows that $\widetilde{\gamma}_2$ is the composition

$$I \xrightarrow{\gamma_2} U \xrightarrow{(p|_{\widetilde{U}})^{-1}} \widetilde{U}.$$

In particular, $\widetilde{y} = \widetilde{\gamma}_2(1) \in \widetilde{U}$, as desired. □

### 4.7. Exercises

Exercise 4.1. Let $X$ be a path connected space and let $\gamma_0, \gamma_1 \colon I \to X$ be two paths in $X$. Prove that $\gamma_0$ is homotopic to $\gamma_1$ if we do not require the homotopy to fix the endpoints of the path. □

Exercise 4.2. Let $p \colon \widetilde{X} \to X$ be a cover and let $\gamma \colon I \to X$ be a path in $X$. Assume that $\gamma$ is an embedding. Let $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ be two lifts of $\gamma$ to $\widetilde{X}$ such that $\widetilde{\gamma}_1(0) \neq \widetilde{\gamma}_2(0)$. Prove that the images of $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ are disjoint. □

Exercise 4.3. Let $n_1, \ldots, n_m \geq 2$. Fix basepoints $x_i \in \mathbb{S}^{n_i}$ for all $1 \leq i \leq n_m$. Define $\mathbb{S}^{n_1} \vee \cdots \vee \mathbb{S}^{n_m}$ to be the space obtained from the disjoint union $\mathbb{S}^{n_1} \sqcup \cdots \sqcup \mathbb{S}^{n_m}$ by identifying all the $x_i$ to a single point $p_0$ (this is called the *wedge sum* of the $\mathbb{S}^{n_i}$; see §8.6). Prove that $\mathbb{S}^{n_1} \vee \cdots \vee \mathbb{S}^{n_m}$ is 1-connected. □

Exercise 4.4. Prove that a nonempty space $X$ is 1-connected if and only if any two maps $f, g \colon \mathbb{S}^1 \to X$ are homotopic. □

Exercise 4.5. Calculate a complete set of homotopy classes of paths from $1 \in \mathbb{S}^1 \subset \mathbb{C}$ to $-1 \in \mathbb{S}^1 \in \mathbb{C}$. Hint: the ideas from Example 4.1.3 will be helpful. □

Exercise 4.6. As discussed in Example 1.6*.4, let $\mathrm{Poly}_n^{\mathrm{sf}}$ be the space of monic degree-$n$ polynomials without repeated roots, let $\mathrm{RPoly}_n^{\mathrm{sf}}$ be the space of pairs $(f, x)$ with $f \in \mathrm{Poly}_n^{\mathrm{sf}}$ and $f(x) = 0$, and let $p \colon \mathrm{RPoly}_n^{\mathrm{sf}} \to \mathrm{Poly}_n^{\mathrm{sf}}$ be the map $p(f, x) = f$, so $p$ is a degree $n$ covering space. Fix points $f, g \in \mathrm{Poly}_n^{\mathrm{sf}}$. Letting $x_1, \ldots, x_n \in \mathbb{C}$ be the roots of $f$ and $y_1, \ldots, y_n \in \mathbb{C}$ be the roots of $g$, we have

$$p^{-1}(f) = \{(f, x_1), \ldots, (f, x_n)\},$$
$$p^{-1}(g) = \{(g, y_1), \ldots, (g, y_n)\}.$$

Let $\mathfrak{S}_n$ be the symmetric group on $n$ elements and let $\sigma \in \mathfrak{S}_n$. Prove that there is a path $\gamma$ in $\mathrm{Poly}_n^{\mathrm{sf}}$ from $f$ to $g$ such that the map $\phi_\gamma \colon p^{-1}(f) \to p^{-1}(g)$ is given by

$$\phi_\gamma(f, x_i) = (g, y_{\sigma(i)}) \quad \text{for all } 1 \leq i \leq n.$$

Hint: the ideas from the proof of Lemma 3.3.1 might be helpful. □

Exercise 4.7. This exercise explains why we defined 0- and 1-connectivity the way we did. The sphere $\mathbb{S}^d$ and the disk $\mathbb{D}^d$ are usually defined for $d \geq 0$. As a convention, for any $d < 0$ define $\mathbb{S}^d = \emptyset$ and $\mathbb{D}^d = \emptyset$. For all $d \in \mathbb{Z}$, the space $\mathbb{D}^{d+1}$ contains $\mathbb{S}^d$ as a subspace. Say that a space $X$ is *$n$-connected* if the following holds for all $d \leq n$:

- All continuous maps $\mathbb{S}^d \to X$ extend to continuous maps $\mathbb{D}^{d+1} \to X$.

Prove the following using the above definition for 0- and 1-connectivity rather than the one we gave in the text:

(a) All spaces are $n$-connected for $n \leq -2$.
(b) A space $X$ is $-1$-connected if and only if it is nonempty.
(c) A space $X$ is 0-connected if and only if it is nonempty and path connected.
(d) A space $X$ is 1-connected if and only if it is nonempty, path connected, for all $x, y \in X$ there is a unique homotopy class of paths from $x$ to $y$. □

Exercise 4.8. Let $X$ be a graph and let $p \colon \widetilde{X} \to X$ be a cover. Prove that $\widetilde{X}$ is a graph. Hint: Start by adding vertices to the interiors of edges as necessary to ensure that $X$ has no loops (make sure this does not change the truth of the exercise!). Next, use Theorem 4.6.1 to prove that the restriction of $p \colon \widetilde{X} \to X$ to each edge of $X$ is trivial. □

Exercise 4.9. Fix a space $Z$. Do the following:

(a) Let $r \colon \widetilde{W} \to Z \times I$ be a cover. Prove that there is a cover $q \colon \widetilde{Z} \to Z$ such that $r \colon \widetilde{W} \to Z \times I$ is isomorphic to the cover $q \times \mathbb{1}_I \colon \widetilde{Z} \times I \to Z \times I$. Hint: For each $z \in Z$ and $t \in I$, let $\gamma_{z,t} \colon I \to Z \times I$ be the path

$$\gamma_{z,t}(s) = (z, st) \quad \text{for all } s \in I$$

from $(z, 0)$ to $(z, t)$. Use the bijections between the fibers $r^{-1}(z, 0)$ and $r^{-1}(z, t)$ given by $\tau_{\gamma_{z,t}}$. Make sure to check that all the maps in your covers and covering space isomorphisms are continuous.

(b) Let $p \colon \widetilde{X} \to X$ be a cover. For $i = 0, 1$, let $f_i \colon Z \to X$ be a map and let $q_i \colon \widetilde{Z}_i \to Z$ be the pullback of $p \colon \widetilde{X} \to X$ along $f_i \colon Z \to X$ as in Exercise 1.14, so

$$\widetilde{Z}_i = \left\{ (\widetilde{x}, z) \in \widetilde{X} \times Z \mid p(\widetilde{x}) = f_i(z) \right\}.$$

Assume that $f_0$ is homotopic to $f_1$. Prove that the covers $q_0 \colon \widetilde{Z}_0 \to Z$ and $q_1 \colon \widetilde{Z}_1 \to Z$ are isomorphic. □

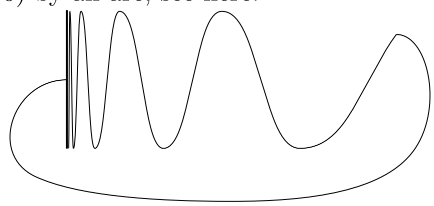
Exercise 4.10. Let $X$ be a locally path connected space. Do the following:

(a) Let $p \colon \widetilde{X} \to X$ be a cover and let $U \subset X$ be a 1-connected open set. Prove that $U$ is a trivialized neighborhood for $p \colon \widetilde{X} \to X$.
(b) Say that $X$ is *locally 1-connected* if for all $x \in X$ and all open neighborhoods $V$ of $x$, there exists a 1-connected open neighborhood $U$ of $x$ with $U \subset V$. This strengthens $X$ being locally path connected. If $p \colon \widetilde{X} \to X$ is a cover and $X$ is locally 1-connected, then prove that $\widetilde{X}$ is locally 1-connected.
(c) Let $p \colon Y \to X$ and $q \colon Z \to Y$ be covers. Assume that $X$ is locally 1-connected. Prove that $p \circ q \colon Z \to X$ is a cover. We remark that the composition of two covering space maps is not always a covering space map (see Exercise 2.14**). □

Exercise 4.11. The *quasi-circle* is the space $Y$ obtained from the topologist's sine curve

$$X = \left\{ (x, \sin(1/x)) \in \mathbb{R}^2 \mid 0 < x \leq 1 \right\} \cup \{(0, y) \mid -1 \leq y \leq 1\}$$

by connecting $(1, \sin(1))$ to $(0, 0)$ by an arc; see here:



Prove the following:

(a) The space $Y$ is 1-connected.

(b) There exists a nontrivial cover $p\colon \widetilde{Y} \to Y$. $\square$

# Fundamental group

CHAPTER 5

# Definition and basic properties of the fundamental group

As the last chapter showed, there is a close connection between covers of a space $X$ and the collection of homotopy classes of paths in $X$. In this section, we organize the collection of homotopy classes of paths into algebraic objects called the fundamental group and groupoid.

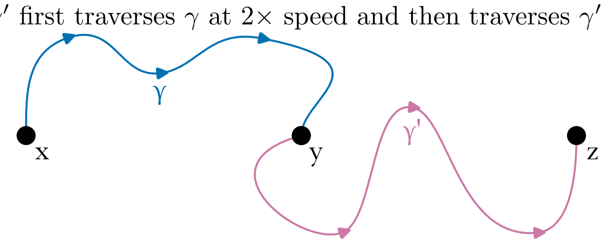
## 5.1. Multiplying homotopy classes of paths

Let $X$ be a space. Our goal is to endow the set of homotopy classes of paths between points of $X$ with an algebraic structure. In this structure, only some paths can be "multiplied". The definition is as follows:

Definition 5.1.1. Let $\gamma\colon I \to X$ and $\gamma'\colon I \to X$ be paths between points of $X$. We say that $\gamma$ and $\gamma'$ are *composable* if the terminal point of $\gamma$ equals the initial point of $\gamma'$. If $\gamma$ and $\gamma'$ are composable, then $\gamma\cdot\gamma'\colon I \to X$ is the path defined by the formula

$$(\gamma\cdot\gamma')(s) = \begin{cases} \gamma(2s) & \text{if } 0 \leq s \leq 1/2, \\ \gamma'(2s-1) & \text{if } 1/2 \leq s \leq 1. \end{cases} \quad \text{for } s \in I. \qquad \square$$

In other words, $\gamma\cdot\gamma'$ first traverses $\gamma$ at $2\times$ speed and then traverses $\gamma'$ at $2\times$ speed:




If $\gamma$ goes from $x$ to $y$ and $\gamma'$ goes from $y$ to $z$, then $\gamma\cdot\gamma'$ goes from $x$ to $z$. This only makes sense if $\gamma$ and $\gamma'$ are composable, and we do not define $\gamma\cdot\gamma'$ if they are not.

Remark 5.1.2. Being composable is *not* symmetric: if $\gamma\cdot\gamma'$ is defined, then it need not be the case that $\gamma'\cdot\gamma$ is defined. $\square$

For a path $\gamma\colon I \to X$, recall that $[\gamma]$ denotes its homotopy class. The following lemma says that our "multiplication" descends to a multiplication on homotopy classes:

Lemma 5.1.3. *Let $X$ be a space. Let $\gamma_0$ and $\gamma'_0$ be composable paths in $X$. Let $\gamma_1$ be a path that is homotopic to $\gamma_0$ and let $\gamma'_1$ be a path that is homotopic to $\gamma'_0$, so $[\gamma_0] = [\gamma_1]$ and $[\gamma'_0] = [\gamma'_1]$. Then $[\gamma_0\cdot\gamma'_0] = [\gamma'_0\cdot\gamma'_1]$.*

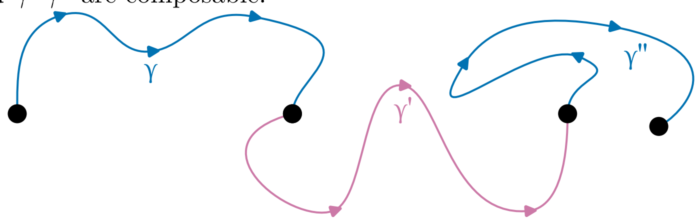
Proof. Assume that $\gamma_0$ goes from $x$ to $y$ and that $\gamma'_0$ goes from $y$ to $z$. Let $\gamma_t$ be a homotopy from $\gamma_0$ to $\gamma_1$ and let $\gamma'_t$ be a homotopy from $\gamma'_0$ to $\gamma'_1$. For each $t \in I$, we have

$$\gamma_t(0) = x \quad \text{and} \quad \gamma_t(1) = \gamma'_t(0) = y \quad \text{and} \quad \gamma'_t(1) = z,$$

so $\gamma_t$ and $\gamma'_t$ are composable and $\gamma_t\cdot\gamma'_t$ is a well-defined path from $x$ to $z$. As $t$ varies over $I$, the paths $\gamma_t\cdot\gamma'_t$ form a homotopy from $\gamma_0\cdot\gamma'_0$ to $\gamma_1\cdot\gamma'_1$. $\square$

## 5.2. Properties of multiplication

Let $X$ be a space. This section explores properties of our multiplication that resemble the properties of a group. We start with associativity. Let $\gamma$ and $\gamma'$ and $\gamma''$ be paths in $X$ such that $\gamma$ and $\gamma'$ are composable and $\gamma'$ and $\gamma''$ are composable. It follows that $\gamma\cdot\gamma'$ and $\gamma''$ are composable, and also that $\gamma$ and $\gamma'{\cdot}\gamma''$ are composable:



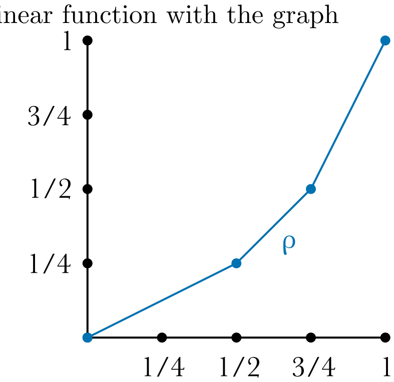

Both $(\gamma{\cdot}\gamma'){\cdot}\gamma''$ and $\gamma{\cdot}(\gamma'{\cdot}\gamma'')$ thus make sense; however except in degenerate cases we have $(\gamma{\cdot}\gamma'){\cdot}\gamma'' \neq \gamma{\cdot}(\gamma'{\cdot}\gamma'')$. The following lemma shows that passing to homotopy fixes this:

LEMMA 5.2.1 (Associativity). *Let $X$ be a space. Let $\gamma$ and $\gamma'$ and $\gamma''$ be paths in $X$ such that $\gamma$ and $\gamma'$ are composable and $\gamma'$ and $\gamma''$ are composable. Then $[(\gamma{\cdot}\gamma'){\cdot}\gamma''] = [\gamma{\cdot}(\gamma'{\cdot}\gamma'')]$.*

PROOF. The paths $f_1 = (\gamma{\cdot}\gamma'){\cdot}\gamma''$ and $f_2 = \gamma{\cdot}(\gamma'{\cdot}\gamma'')$ are almost the same. They both traverse $\gamma$ and then $\gamma'$ and then $\gamma''$; however, they do this at different speeds. As functions on $I = [0,1]$, we have the following:

- The path $f_1$ traverses $\gamma$ at $4\times$ speed on the interval $[0,1/4]$, then $\gamma'$ at $4\times$ speed on the interval $[1/4,1/2]$, and then $\gamma''$ at $2\times$ speed on the interval $[1/2,1]$.
- The path $f_2$ traverses $\gamma$ at $2\times$ speed on the interval $[0,1/2]$, then $\gamma'$ at $4\times$ speed on the interval $[1/2,3/4]$, and then $\gamma''$ at $4\times$ speed on the interval $[3/4,1]$.

Let $\rho\colon I \to I$ be the piecewise linear function with the graph




We then have $f_2 = f_1 \circ \rho$. The lemma now follows from Lemma 5.2.2 below. $\square$

LEMMA 5.2.2 (Reparameterization lemma). *Let $X$ be a space and $\gamma\colon I \to X$ be a path. Let $\rho\colon I \to I$ be a function such that $f(0) = 0$ and $f(1) = 1$. Then $[\gamma \circ \rho] = [\gamma]$.*

PROOF. The desired homotopy from $\gamma \circ \rho$ to $\gamma$ is given by

$$\gamma_t(s) = \gamma((1-t)\rho(s) + ts) \quad \text{for } t, s \in I.$$

Here we use the fact that $f(0) = 0$ and $f(1) = 1$ to ensure that the endpoints of $\gamma_t$ do not move:

$$\gamma_t(0) = \gamma((1-t)\rho(0) + 0) = \gamma(0) \quad \text{and} \quad \gamma_t(1) = \gamma((1-t)\rho(1) + t) = \gamma(1-t+t) = \gamma(1). \qquad \square$$

We now turn to multiplicative identities. For a point $x \in X$, let $\mathfrak{c}_x\colon I \to X$ be the constant path

$$\mathfrak{c}_x(s) = x \quad \text{for } s \in I.$$

This serves as an identity for our multiplication. However, since we can only multiply composable paths an appropriate $\mathfrak{c}_x$ must be chosen for the left- and right-identities of any given path:

LEMMA 5.2.3 (Multiplicative identities). *Let $X$ be a space and let $\gamma$ be a path in $X$ from $x$ to $y$. Then $[\gamma{\cdot}\mathfrak{c}_x] = [\gamma]$ and $[\mathfrak{c}_y{\cdot}\gamma] = [\gamma]$.*

PROOF. The path $\gamma{\cdot}\mathfrak{c}_x$ stays at $x$ on the interval $[0,1/2]$ and then traverses $\gamma$ at $2\times$ speed:

$$(\gamma{\cdot}\mathfrak{c}_x)(s) = \begin{cases} x & \text{if } s \in [0,1/2], \\ \gamma(2s-1) & \text{if } s \in [1/2,1]. \end{cases} \quad \text{for } s \in I.$$

Letting $\rho\colon I \to I$ be the map

$$\rho(s) = \begin{cases} 0 & \text{if } s \in [0,1/2], \\ 2s-1 & \text{if } s \in [1/2,1] \end{cases} \quad \text{for } s \in I,$$

we thus have $\gamma\cdot\mathfrak{c}_x = \gamma \circ \rho$. Applying Lemma 5.2.2, we see that $[\gamma\cdot\mathfrak{c}_x] = [\gamma \circ \rho] = [\gamma]$, as desired. The proof that $[\mathfrak{c}_y\cdot\gamma] = [\gamma]$ is similar. □

Having found identities, our final goal is to find inverses. Let $\gamma$ be a path in $X$ from $x$ to $y$. Define $\overline{\gamma}\colon I \to X$ to be the path that traverses $\gamma$ in the reverse order:

$$\overline{\gamma}(s) = \gamma(1-s) \quad \text{for } s \in I.$$

The path $\overline{\gamma}$ goes from $y$ to $x$, and serves as a sort of "inverse" to our multiplication:

LEMMA 5.2.4 (Inverses). *Let $X$ be a space and let $\gamma$ be a path in $X$ from $x$ to $y$. Then $[\gamma\cdot\overline{\gamma}] = [\mathfrak{c}_x]$ and $[\overline{\gamma}\cdot\gamma] = [\mathfrak{c}_y]$.*

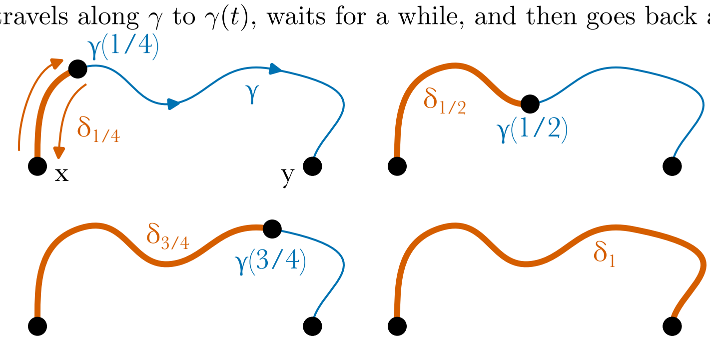
PROOF. The path $\gamma\cdot\overline{\gamma}$ goes from $x$ to $x$. For $t \in I$, define $\delta_t\colon I \to X$ to be the path

$$\delta_t(s) = \begin{cases} \gamma(2s) & \text{if } s \in [0,t/2], \\ \gamma(t) & \text{if } s \in [t/2, 1-t/2], \\ \gamma(2(1-s)) & \text{if } s \in [1-t/2, 1]. \end{cases} \quad \text{for } s \in I.$$

This makes sense since

$$\gamma(2(t/2)) = \gamma(t) = \gamma(2(1-(1-t/2))).$$

Geometrically, $\delta_t$ travels along $\gamma$ to $\gamma(t)$, waits for a while, and then goes back along $\overline{\gamma}$:




Since $\delta_t$ is a homotopy from $\mathfrak{c}_x$ to $\gamma\cdot\overline{\gamma}$, we deduce that $[\mathfrak{c}_x] = [\gamma\cdot\overline{\gamma}]$, as desired. The proof that $[\overline{\gamma}\cdot\gamma] = [\mathfrak{c}_y]$ is similar. □

## 5.3. Fundamental group

Let $X$ be a space. In the previous sections, we showed that the set of homotopy classes of paths between points of $X$ has a partially-defined "multiplication" that is associative, has identities, and has inverses. What kind of algebraic structure could this be? As we will see later in this chapter, this is an example of what is called a "groupoid". The definition of a groupoid is a little abstract, and in the applications this full structure is rarely needed. We therefore start by extracting from it a group called the fundamental group.

Fix a basepoint $x_0 \in X$. A *loop* in $X$ based at $x_0$ is a path $\gamma$ from $x_0$ to itself. The *fundamental group* of $X$ with basepoint $x_0$, denoted $\pi_1(X, x_0)$ is

$$\pi_1(X, x_0) = \{[\gamma] \mid \gamma \text{ is a loop in } X \text{ based at } x_0\}.$$

If $\gamma$ and $\gamma'$ are loops in $X$ based at $x_0$, then $\gamma$ and $\gamma'$ are composable and $\gamma\cdot\gamma'$ is also a loop based at $x_0$. It follows that $\pi_1(X, x_0)$ can be endowed with the multiplication

$$[\gamma][\gamma'] = [\gamma\cdot\gamma'] \quad \text{for } [\gamma], [\gamma'] \in \pi_1(X, x_0).$$

The results of §5.2 show that this makes $\pi_1(X, x_0)$ into a group. The unit is $[\mathfrak{c}_{x_0}]$, and the inverse of $[\gamma] \in \pi_1(X, x_0)$ is the reversed path $[\overline{\gamma}] \in \pi_1(X, x_0)$.

## 5.4. Dependence on basepoint

Consider $x_0, x_0' \in X$. How are the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(X, x_0')$ related? If $\alpha$ is a path from $x_0$ to $x_0'$, then we get a set map

$$\alpha_*\colon \pi_1(X, x_0') \to \pi_1(X, x_0)$$

defined by

$$\alpha_*([\gamma]) = [\alpha\cdot\gamma\cdot\overline{\alpha}] \quad \text{for all } [\gamma] \in \pi_1(X, x_0').$$

This is a group homomorphism: for $[\gamma], [\gamma'] \in \pi_1(X, x_0')$, we have

$$\alpha_*([\gamma\cdot\gamma']) = [\alpha\cdot\gamma\cdot\gamma'\cdot\overline{\alpha}] = [\alpha\cdot\gamma\cdot\overline{\alpha}\cdot\alpha\cdot\gamma'\cdot\overline{\alpha}] = [\alpha\cdot\gamma\cdot\overline{\alpha}][\alpha\cdot\gamma'\cdot\overline{\alpha}] = \alpha_*([\gamma])\alpha_*([\gamma']).$$

We claim that $\alpha_*$ is an isomorphism with inverse $\overline{\alpha}_*\colon \pi_1(X, x_0) \to \pi_1(X, x_0')$. Indeed, for $[\gamma] \in \pi_1(X, x_0')$ we have

$$\overline{\alpha}_*(\alpha_*([\gamma])) = \overline{\alpha}_*([\alpha\cdot\gamma\cdot\overline{\alpha}]) = [\overline{\alpha}\cdot\alpha\cdot\gamma\cdot\overline{\alpha}\cdot\alpha] = [\gamma],$$

so $\overline{\alpha}_* \circ \alpha_* = \mathbb{1}_{\pi_1(X, x_0')}$. Similarly, $\alpha_* \circ \overline{\alpha}_* = \mathbb{1}_{\pi_1(X, x_0)}$. We call $\alpha_*$ the *change of basepoint isomorphism* associated to $\alpha$. Using these isomorphisms, we deduce the following:

LEMMA 5.4.1. *Let $X$ be a path connected space. Then for all $x_0, x_0' \in X$ we have $\pi_1(X, x_0) \cong \pi_1(X, x_0')$.*

PROOF. Use the change of basepoint isomorphism associated to a path $\alpha$ from $x_0$ to $x_0'$. □

We will give many computations of $\pi_1(X, x_0)$ over the next few chapters. For $X$ path connected, Lemma 5.4.1 says that the isomorphism type of $\pi_1(X, x_0)$ is independent of the basepoint $x_0$. The isomorphism type of $\pi_1(X, x_0)$ is thus a useful invariant of path connected spaces, i.e., if two path connected spaces have different fundamental groups, then they are not homeomorphic.

We remark that serious applications of $\pi_1(X, x_0)$ often require a careful treatment of the basepoint $x_0$. Simply identifying the fundamental group at different basepoints will quickly lead you astray. This is analogous to the fact that while all finite-dimensional vector spaces over a field $\mathbf{k}$ are isomorphic to $\mathbf{k}^n$ for some $n \geq 0$, one cannot simply identify vector spaces with $\mathbf{k}^n$. Such an identification requires a choice of basis, and often there is no natural choice. Much of linear algebra focuses on carefully choosing bases adapted to different situations and studying how all these different bases are related.

## 5.5. Categorical language

To a space $X$ equipped with a basepoint $x_0$, we have associated a group $\pi_1(X, x_0)$. Our next goal is to provide a language to discuss the way in with $\pi_1(X, x_0)$ depends on $X$ and $x_0$. Recall that a category $\mathbf{C}$ consists of the following data:

- A collection of objects. We will write $A \in \mathbf{C}$ to indicate that $A$ is an object of $\mathbf{C}$.
- For all objects $A, B \in \mathbf{C}$, a set $\mathbf{C}(A, B)$ of morphisms. We will often write $f\colon A \to B$ to indicate that $f$ is a morphism from $A$ to $B$.
- For all objects $A \in \mathbf{C}$, an identity morphism $\mathbb{1}_A\colon A \to A$.

These morphisms can be composed: if $f\colon A \to B$ and $g\colon B \to C$ are morphisms, then we have a morphism $g \circ f\colon A \to C$. This composition should be associative in the sense that if $f\colon A \to B$ and $g\colon B \to C$ and $h\colon C \to D$ are morphisms, then

$$(f \circ g) \circ h = f \circ (g \circ h).$$

Because of this, there is no need to insert parentheses when composing morphisms. Under this composition, the identity morphisms should be units: if $f\colon A \to B$ is a morphism, then $f \circ \mathbb{1}_A = f$ and $\mathbb{1}_B \circ f = f$.

EXAMPLE 5.5.1. The collection of all sets and set maps forms a category $\mathtt{Set}$. □

EXAMPLE 5.5.2. The collection of all topological spaces and continuous maps forms a category $\mathtt{Top}$. □

EXAMPLE 5.5.3. The collection of all groups and homomorphisms forms a category $\mathtt{Group}$. □

EXAMPLE 5.5.4. For a group $G$, there is a category $\mathbf{G}$ with one object $\bullet$ and with $\mathbf{G}(\bullet, \bullet) = G$. □

REMARK 5.5.5. The language of category theorem might seem overly abstract, but it turns out to be very useful and clarifying. Fundamentally, it is just a way of organizing information. Typically you cannot prove interesting new theorems by just defining a category, but the language of category theory often suggests useful constructions. □

## 5.6. Functoriality: fundamental group

We now return to the fundamental group. Let $X$ and $Y$ be spaces. Consider a map $f\colon X \to Y$ and $x_0 \in X$. If $\gamma$ is a loop in $X$ based at $x_0$, then $f \circ \gamma$ is a loop in $Y$ based at $f(x_0)$. The loop $[f \circ \gamma]$ only depends on the homotopy class of $\gamma$, and the map $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ defined by

$$f_*([\gamma]) = [f \circ \gamma] \quad \text{for all } [\gamma] \in \pi_1(X, x_0)$$

is a homomorphism called the homomorphism *induced* by $f$.

Since the fundamental group is the group of homotopy classes of loops, one expects the homomorphism induced by a map to only depend on the homotopy class of the map. However, this is not quite right since we have to be careful about the basepoint. To state things properly, we introduce the following terminology:

DEFINITION 5.6.1. A *pointed space* is a pair $(X, x_0)$ with $X$ a space and $x_0 \in X$. A map between pointed space $(X, x_0)$ and $(Y, y_0)$ is a map $f\colon X \to Y$ such that $f(x_0) = y_0$. We will denote such a map by $f\colon (X, x_0) \to (Y, y_0)$. A homotopy of maps from $(X, x_0)$ to $(Y, y_0)$ is a homotopy $f_t\colon X \to Y$ such that $f_t(x_0) = y_0$ for all $t \in I$. Just like for maps, we will denote this by $f_t\colon (X, x_0) \to (Y, y_0)$, and if such an $f_t$ exists we will say that $f_0\colon (X, x_0) \to (Y, y_0)$ and $f_1\colon (X, x_0) \to (Y, y_0)$ are homotopic. □

With this setup, a map $f\colon (X, x_0) \to (Y, y_0)$ between pointed spaces induces a homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$, and if $f\colon (X, x_0) \to (Y, y_0)$ and $g\colon (X, x_0) \to (Y, Y_0)$ are homotopic maps between pointed spaces then $f_* = g_*$. The homomorphisms induced by maps of pointed spaces have the following two simple properties:

- for maps of pointed space $f\colon (X, x_0) \to (Y, y_0)$ and $g\colon (Y, y_0) \to (Z, z_0)$, we have $(g \circ f)_* = g_* \circ f_*$; and
- the identity map $\mathbb{1}\colon (X, x_0) \to (X, x_0)$ induces the identity homomorphism, i.e., $\mathbb{1}_* = \mathbb{1}$.

All of this can be summarized in categorical language as follows. Recall that if $\mathbf{C}$ and $\mathbf{D}$ are categories, then a *functor* $F\colon \mathbf{C} \to \mathbf{D}$ consists of the following data:

- For all objects $C \in \mathbf{C}$, an object $F(D) \in \mathbf{D}$.
- For all morphisms $f\colon C_1 \to C_2$ between objects of $\mathbf{C}$, a morphism $F(f)\colon F(C_1) \to F(C_2)$.

These are required to satisfy:

- for all morphisms $f\colon C_1 \to C_2$ and $g\colon C_2 \to C_3$ between objects of $\mathbf{C}$, we have $F(g \circ f) = F(g) \circ F(f)$; and
- for all identity morphisms $\mathbb{1}_C\colon C \to C$ in $\mathbf{C}$, we have $F(\mathbb{1}_C) = \mathbb{1}_{F(C)}$.

EXAMPLE 5.6.2. There is a functor $F\colon \mathtt{Top} \to \mathtt{Set}$ that takes a topological space and returns its underlying set. There is also a functor $G\colon \mathtt{Set} \to \mathtt{Top}$ that takes a set and returns the corresponding discrete topological space. □

EXAMPLE 5.6.3. There is a functor $Z\colon \mathtt{Set} \to \mathtt{Group}$ defined as follows:

- For $S \in \mathtt{Set}$, define $Z(S)$ to be the abelian group of finite formal $\mathbb{Z}$-linear combinations of elements of $S$. In other words, each element of $Z(S)$ can for some $n \geq 0$ be written in the form $\lambda_1 s_1 + \cdots + \lambda_n s_n$ with $\lambda_i \in \mathbb{Z}$ and $s_i \in S$ for all $1 \leq i \leq n$.
- For $S, T \in \mathtt{Set}$ and a morphism $f\colon S \to T$ in $\mathtt{Set}$, define $Z(f)\colon Z(S) \to Z(T)$ to be the group homomorphism induced by $f$ in the following way:

$$Z(f)(\lambda_1 s_1 + \cdots + \lambda_n s_n) = \lambda_1 f(s_1) + \cdots + \lambda_n f(s_n) \quad \text{for all } \lambda_1 s_1 + \cdots + \lambda_n s_n \in Z(S). \qquad \square$$

To fit the fundamental group into this, let $\mathtt{Top}_*$ be the category of pointed spaces, so the objects of $\mathtt{Top}_*$ are pointed spaces $(X, x_0)$ and the morphisms in $\mathtt{Top}_*$ are the maps $f\colon (X, x_0) \to (Y, y_0)$ between pointed spaces. We can then summarize our discussion by:

LEMMA 5.6.4. *The fundamental group is a functor* $\pi_1\colon \mathtt{Top}_* \to \mathtt{Group}$.

REMARK 5.6.5. We can fit homotopies into this framework as follows. Let $\mathtt{h}\mathrm{Top}_*$ be the category whose objects are pointed spaces $(X, x_0)$ and whose morphisms are homotopy classes of pointed maps $f\colon (X, x_0) \to (Y, y_0)$. The fundamental group can then be viewed as a functor from $\mathtt{h}\mathrm{Top}_*$ to $\mathtt{Group}$. There is a functor $\mathtt{Top}_* \to \mathtt{h}\mathrm{Top}_*$ that is the identity on objects and takes a morphism $f\colon (X, x_0) \to (Y, y_0)$ to its homotopy class, and the functor $\pi_1\colon \mathtt{Top}_* \to \mathtt{Group}$ factors as

$$\mathtt{Top}_* \to \mathtt{h}\mathrm{Top}_* \to \mathtt{Group}\,. \qquad \square$$

## 5.7. Homotopies that move the basepoint

Let $(X, x_0)$ be a pointed space and let $f_0, f_1\colon X \to Y$ be two homotopic maps. Set $y_0 = f_0(x_0)$ and $y_1 = f_1(x_0)$. We therefore have maps $f_0\colon (X, x_0) \to (Y, y_0)$ and $f_1\colon (X, x_0) \to (Y, y_1)$. Since their targets are different, it does not make sense to say that the induced maps

$$\begin{aligned}(f_0)_*\colon \pi_1(X, x_0) &\to \pi_1(Y, y_0),\\ (f_1)_*\colon \pi_1(X, x_0) &\to \pi_1(Y, y_1)\end{aligned}$$

are equal. Instead, they are related as follows:

LEMMA 5.7.1. *Let $X$ and $Y$ be spaces and let $f_t\colon X \to Y$ be a homotopy of maps from $X$ to $Y$. Let $x_0 \in X$ be a basepoint, and set $y_0 = f_0(x_0)$ and $y_1 = f_1(x_0)$. Let $\delta\colon I \to Y$ be the following path from $y_0$ to $y_1$:*

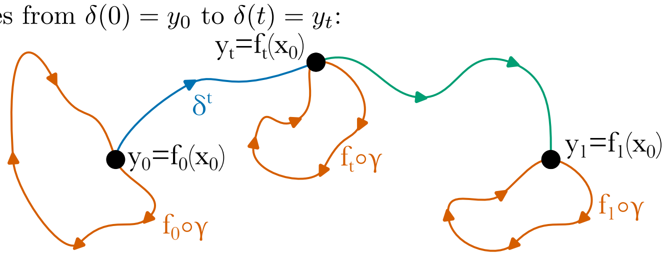
$$\delta(s) = f_s(x_0) \quad \textit{for } s \in I.$$

*Then for all $[\gamma] \in \pi_1(X, x_0)$ we have $(f_0)_*([\gamma]) = [\delta](f_1)_*([\gamma])[\overline{\delta}] \in \pi_1(Y, y_0)$.*

PROOF. For $t \in I$, let $y_t = \delta(t)$ and let $\delta^t\colon I \to Y$ be the path

$$\delta^t(s) = f_{ts}(x_0) = \delta(ts) \quad \text{for } s \in I.$$

The path $\delta^t$ thus goes from $\delta(0) = y_0$ to $\delta(t) = y_t$:




Since $f_t \circ \gamma$ is a loop based at $f_t(x_0) = \delta(t) = y_t$, it follows that the map $t \mapsto \delta^t{\cdot}(f_t \circ \gamma){\cdot}\overline{\delta}^t$ is a homotopy of paths. Since $\delta^0$ is the constant path based at $y_0$, we conclude that

$$(f_0)_*([\gamma]) = [f_0 \circ \gamma] = [\delta^0{\cdot}(f_0 \circ \gamma){\cdot}\overline{\delta}^0] = [\delta^1{\cdot}(f_1 \circ \gamma){\cdot}\overline{\delta}^1] = [\delta](f_1)_*([\gamma])[\overline{\delta}]. \qquad \square$$

## 5.8*. Fundamental groupoid

For a pointed space $(X, x_0)$, the fundamental group $\pi_1(X, x_0)$ consists of homotopy classes of loops in $X$ based at $x_0$. Our final goal in this chapter is to encode *all* homotopy classes of paths in $X$ into an algebraic structure called the fundamental groupoid. Since this structure will only play a small role in this book, we mark these sections with an $*$ to indicate that a reader might want to skip them when first learning the subject.

The fundamental groupoid will be a category. Its objects will be the points of $X$. For points $x, y \in X$, the morphisms from $x$ to $y$ will be the homotopy classes of paths from $x$ to $y$. There is one annoying technical point: in a category, composition goes from right to left like functions. However, we multiply paths from left to right: if $\gamma$ is a path from $x$ to $y$ and $\gamma'$ is a path from $y$ to $z$, then $\gamma{\cdot}\gamma'$ is a path from $x$ to $z$. To fix this, we introduce the following notation:

NOTATION 5.8*.1. Let $X$ be a space. For points $x, y, z \in X$, let $\gamma$ be a path in $X$ from $x$ to $y$ and let $\delta$ be a path in $X$ from $y$ to $z$. We then define $\gamma' * \gamma = \gamma{\cdot}\gamma'$. This descends to homotopy classes of paths, and we also write $[\gamma'] * [\gamma] = [\gamma' * \gamma]$. $\square$

We now define the following:

DEFINITION 5.8*.2. Let $X$ be a space. The *fundamental groupoid* of $X$, denoted $\Pi(X)$, is the following category:

- The objects of $\Pi(X)$ are the points of $X$.
- For points $x$ and $y$, the $\Pi(X)$-morphisms from $x$ to $y$ are the set of all homotopy classes of paths from $x$ to $y$. For a path $\gamma$ from $x$ to $y$, we will write $[\gamma]\colon x \to y$ for the corresponding morphism from $x$ to $y$.
- If $\gamma$ is a path from $x$ to $y$ and $\gamma'$ is a path from $y$ to $z$, then the composition of the morphisms $[\gamma]\colon x \to y$ and $[\gamma']\colon y \to z$ is the morphism $[\gamma'] * [\gamma]\colon x \to z$.
- For a point $x \in X$, the identity morphism of $x$ is the constant path $[\mathfrak{c}_x]\colon x \to x$. □

Lemma 5.2.4 says that all the morphisms in the fundamental groupoid $\Pi(X)$ are invertible. This is the defining property of a groupoid:

DEFINITION 5.8*.3. A *groupoid* is a category $\mathbf{G}$ in which all morphisms are invertible, i.e., such that for all morphisms $\phi\colon A \to B$, there is a morphism $\overline{\phi}\colon B \to A$ with $\overline{\phi} \circ \phi = \mathbb{1}_A$ and $\phi \circ \overline{\phi} = \mathbb{1}_B$. □

EXAMPLE 5.8*.4. As we discussed in Example 5.5.4, a group can be identified with a groupoid containing a single object. □

Consider a groupoid $\mathbf{G}$. For $x_0 \in \mathbf{G}$, write

$$\mathrm{Aut}_{\mathbf{G}}(x_0) = \{g \mid g\colon x_0 \to x_0 \text{ is a morphism in } \mathbf{G}\}.$$

Since all morphisms in $\mathbf{G}$ are invertible, this is a group. The one-object category corresponding to $\mathrm{Aut}_{\mathbf{G}}(x_0)$ is a subcategory of $\mathbf{G}$ with one object $x_0$.

The following relates the fundamental groupoid and group:

LEMMA 5.8*.5. *Let $X$ be a space. For $x_0 \in X$, we have $\mathrm{Aut}_{\Pi(X)}(x_0) \cong \pi_1(X, x_0)$.*

PROOF. By definition, $\mathrm{Aut}_{\Pi(X)}(x_0)$ consists of all homotopy classes of loops $[\gamma] \in \pi_1(X, x_0)$ with the multiplication

$$[\gamma] * [\gamma'] = [\gamma' \cdot \gamma] \quad \text{for } [\gamma], [\gamma'] \in \pi_1(X, x_0).$$

This is similar to $\pi_1(X, x_0)$, but the order of multiplication is reversed. This is something that can be done to any group. For a group $G$ with multiplication $\cdot$, the *opposite group* of $G$, denoted $G^{\mathrm{op}}$ is the group with the same elements as $G$ but with the multiplication

$$g * g' = g' \cdot g \quad \text{for } g, g' \in G.$$

We therefore have $\mathrm{Aut}_{\Pi(X)}(x_0) = \pi_1(X, x_0)^{\mathrm{op}}$. For any group $G$, it turns out that $G \cong G^{\mathrm{op}}$; see Exercise 5.11. The lemma follows. □

For a groupoid $\mathbf{G}$, a morphism $\psi\colon x_0 \to x_1$ in $\mathbf{G}$ induces an isomorphism $\psi_*\colon \mathrm{Aut}_{\mathbf{G}}(x_0) \to \mathrm{Aut}_{\mathbf{G}}(x_1)$ defined by

$$\psi_*(g) = \psi \circ g \circ \overline{\psi} \quad \text{for all } g\colon x_0 \to x_0 \text{ in } \mathrm{Aut}_{\mathbf{G}}(x_0).$$

In this way, a groupoid packages together a collection of groups along with certain isomorphisms between them. For the fundamental groupoid, these induced isomorphisms are the change of basepoint isomorphisms from §5.4.

Unlike the fundamental group, the fundamental groupoid is not so useful as an invariant of a space since it knows far too much about the space; for instance, its objects are literally the points of the space. You might wonder why we bothered to introduce the fundamental groupoid at all. There are two reasons:

- While for a path connected space the isomorphism type of the fundamental group does not depend on the basepoint, the isomorphisms between the fundamental groups at different basepoints are not canonical. The fundamental groupoid packages them all together, and is present at least implictly in all serious treatements of the fundamental group. It seems perverse to refuse to give a name to a structure you use.
- There are many constructions in topology that are most naturally phrased in terms of the fundamental groupoid. For instance, the most general form of the classification of covering spaces uses the fundamental groupoid (see Chapter 13*). Later volumes of this book will contain other examples.

## 5.9*. Functoriality: fundamental groupoid

For completeness, we now explain how to think about the fundamental groupoid as a functor. Recall that a groupoid is a category in which all morphisms are invertible. For groupoids $\mathbf{G}_1$ and $\mathbf{G}_2$, a groupoid homomorphism from $\mathbf{G}_1$ to $\mathbf{G}_2$ is a functor $F\colon \mathbf{G}_1 \to \mathbf{G}_2$. Unpacking this, $F$ consists of the following data:

- For each object $x \in \mathbf{G}_1$, an object $F(x) \in \mathbf{G}_2$.
- For each morphism $\phi\colon x \to y$ in $\mathbf{G}_1$, a morphism $F(\phi)\colon F(x) \to F(y)$ in $\mathbf{G}_2$.

The morphisms $F(\phi)$ must respect composition in the obvious sense. For each $x \in \mathbf{G}_1$, we have the group $\mathrm{Aut}_{\mathbf{G}_1}(x)$, and $F\colon \mathbf{G}_1 \to \mathbf{G}_2$ induces a group homomorphism $F_*\colon \mathrm{Aut}_{\mathbf{G}_1}(x) \to \mathrm{Aut}_{\mathbf{G}_2}(f(x))$. If we think of a groupoid as a collection of groups connected by isomorphisms, the homomorphism $F\colon \mathbf{G}_1 \to \mathbf{G}_2$ can be regarded as a collection of group homomorphisms that respect the given isomorphisms.

Let $\mathtt{Groupoid}$ be the category whose objects are groupoids and whose objects are groupoid homomorphisms. The fundamental groupoid can then be regarded as a functor $\Pi\colon \mathtt{Top} \to \mathtt{Groupoid}$:

- For a space $X$, we have the groupoid $\Pi(X)$.
- For a map of space $f\colon X \to Y$, we have the groupoid homomorphism $f_*\colon \Pi(X) \to \Pi(Y)$ defined as follows:
  - An object of $\Pi(X)$ is a point $x \in X$, and $f_*(x) = f(x) \in Y$.
  - A morphism in $\Pi(X)$ from $x \in X$ to $y \in X$ is the homotopy class of a path $\gamma$ from $x$ to $y$, and $f_*([\gamma]) = [f \circ \gamma]$.

We remark that unlike for the fundamental group, the groupoid homomorphisms $f_*\colon \Pi(X) \to \Pi(Y)$ are not homotopy invariant, at least not in a naive sense. See Exercise 5.12 for one way to think about this.

## 5.10. Exercises

EXERCISE 5.1. Let $(X, x_0)$ and $(Y, y_0)$ be pointed spaces. Prove that

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0). \qquad \square$$

EXERCISE 5.2. Let $X$ be a space and let $x_0, x_0' \in X$. For each path $\alpha$ from $x_0$ to $x_0'$, we defined a change of basepoint isomorphism $\alpha_*\colon \pi_1(X, x_0') \to \pi_1(X, x_0)$ in §5.4:

$$\alpha_*([\gamma]) = [\alpha \cdot \gamma \cdot \overline{\alpha}] \quad \text{for all } [\gamma] \in \pi_1(X, x_0').$$

Prove that $\alpha_*$ is independent of the path $\alpha$ if and only if $\pi_1(X, x_0)$ is abelian. This exercise shows that when the fundamental group is abelian there is a canonical isomorphism between the fundamental groups at different basepoints. $\square$

EXERCISE 5.3. Prove that $\pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z}$. Hint: though this does not follow directly from the fact that the degree is a complete invariant of homotopy classes of maps $\mathbb{S}^1 \to \mathbb{S}^1$ (Lemma 3.8.2), it can be proved by carefully examining the construction of the degree and the proof of Lemma 3.8.2. $\square$

EXERCISE 5.4. Let $(X, x_0)$ be a pointed space. Do the following:

(a) Prove that there is a bijection between elements of $\pi_1(X, x_0)$ and homotopy classes of maps of pointed spaces $f\colon (\mathbb{S}^1, 1) \to (X, x_0)$.

(b) Let $f\colon (\mathbb{S}^1, 1) \to (X, x_0)$ be a map of pointed spaces. Prove that $f$ represents the trivial element of $\pi_1(X, x_0)$ if and only if $f$ extends to a map $F\colon (\mathbb{D}^2, 1) \to (X, x_0)$.

(c) Let $Z$ be a graph with a single vertex $*$ and two loops $\ell_1$ and $\ell_2$ based at $*$. Regarding the $\ell_i$ as circles, for maps $f_1, f_2\colon (\mathbb{S}^1, 1) \to (X, x_0)$ let $f_1 \vee f_2\colon (Z, *) \to (X, x_0)$ be the map that equals $f_1$ on $\ell_1$ and $f_2$ on $\ell_2$. **Problem**: Construct a map $m\colon (\mathbb{S}^1, 1) \to (Z, *)$ with the following property:
   - Consider elements $[\gamma_1], [\gamma_2] \in \pi_1(X, x_0)$. Represent $[\gamma_i]$ by a map $f_i\colon (\mathbb{S}^1, 1) \to (X, x_0)$. Then $[\gamma_1 \cdot \gamma_2]$ is represented by the map $(f_1 \vee f_2) \circ m\colon (\mathbb{S}^1, 1) \to (X, x_0)$. $\square$

EXERCISE 5.5. Let $X$ be a space and let $f_t\colon X \to X$ be a homotopy from $f_0 = \mathbb{1}_X$ to $f_1 = \mathbb{1}_X$. For a basepoint $x_0 \in X$, let $\gamma\colon I \to X$ be the path $\gamma(s) = f_s(x_0)$. Prove that $[\gamma] \in \pi_1(X, x_0)$ is a central element, that is, that $[\gamma]$ commutes with all elements of $\pi_1(X, x_0)$. $\square$

Exercise 5.6. Let $\mathbf{G}$ be a topological group, that is, a space $\mathbf{G}$ that is also a group such that the multiplication map $\mathbf{G} \times \mathbf{G} \to \mathbf{G}$ and the inversion map $\mathbf{G} \to \mathbf{G}$ are continuous. Letting $1 \in \mathbf{G}$ be the identity, prove the following:

(a) Define an alternate multiplication on $\pi_1(\mathbf{G}, 1)$ using the multiplication on $\mathbf{G}$ as follows: for $[\gamma_1], [\gamma_2] \in \pi_1(\mathbf{G}, 1)$, define $[\gamma_1] \bullet [\gamma_2] = [\gamma_3]$ where $\gamma_3 \colon I \to \mathbf{G}$ is the loop
$$\gamma_3(s) = \gamma_1(s)\gamma_2(s) \quad \text{for } s \in I.$$
Prove that $\bullet$ is the same as the usual multiplication on $\pi_1(\mathbf{G}, 1)$.

(b) Prove that $\pi_1(G, 1)$ is abelian. $\square$

Exercise 5.7. Let $X$ be a 0-connected space and let $x, y \in X$. Set
$$\mathcal{P}(x, y) = \{[\delta] \mid \delta \text{ is a path in } X \text{ from } x \text{ to } y\}.$$
Define an action of $\pi_1(X, x)$ on $\mathcal{P}(x, y)$ as follows:
$$[\gamma][\delta] = [\gamma{\cdot}\delta] \quad \text{for all } [\gamma] \in \pi_1(X, x) \text{ and } [\delta] \in \mathcal{P}(x, y).$$
Do the following:

(a) Prove that this action is faithful, i.e., that for all $[\delta] \in \mathcal{P}(x, y)$ the only $[\gamma] \in \pi_1(X, x)$ with $[\gamma][\delta] = [\delta]$ is $[\gamma] = 1$.

(b) Prove that this action is transitive, i.e., that for all $[\delta_1], [\delta_2] \in \mathcal{P}(x, y)$ there exists some $[\gamma] \in \pi_1(X, x)$ with $[\gamma][\delta_1] = [\delta_2]$.

For a group $G$, a *torsor* for $G$ is a set $X$ equipped with a faithful and transitive action of $G$. In this case, for $x_0 \in X$ we have a bijection $G \to X$ taking $g \in G$ to $gx_0 \in X$. The above shows that $\mathcal{P}(x, y)$ is a torsor for $\pi_1(X, x)$. $\square$

Exercise 5.8. Let $p \colon \widetilde{X} \to X$ be a cover. Fix a basepoint $x_0 \in X$, and set $S = p^{-1}(x_0)$. Prove that the following defines an action of $\pi_1(X, x_0)$ on $S$:
$$[\gamma]s = \tau_\gamma^{-1}(s) \quad \text{for } s \in S \text{ and } [\gamma] \in \pi_1(X, x_0).$$
Here $\tau_\gamma$ is the map defined in §4.3. For path connected pointed spaces $(X, x_0)$ with reasonable local properties and sets $S$, we will later prove this construction gives a bijection between actions of $\pi_1(X, x_0)$ on $S$ and isomorphism classes of covers $p \colon \widetilde{X} \to X$ with $p^{-1}(x_0) = S$. See Chapter 12*. $\square$

Exercise 5.9. Let $p \colon \widetilde{X} \to X$ be a cover. Fix a basepoint $x_0 \in X$ and some $\widetilde{x}_0 \in p^{-1}(x_0)$. Prove that the induced map $p_* \colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective. We will later prove that if $(X, x_0)$ is a path connected space with reasonable local properties, then there is a bijection between subgroups $G < \pi_1(X, x_0)$ and pointed maps $p \colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that $p \colon \widetilde{X} \to X$ is a cover and $\widetilde{X}$ is path connected (up to an appropriate notion of isomorphism of covers). This bijection takes $p \colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ to the image of $p_* \colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$. See Chapter 11. $\square$

Exercise 5.10. Let $\mathbf{G}$ be a groupoid and $\phi \colon A \to B$ be a morphism in $\mathbf{G}$. Consider $\overline{\phi}, \overline{\phi}' \colon B \to A$. Prove that $\phi = \overline{\phi}'$ if any of the following conditions are satisfied:

- $\overline{\phi} \circ \phi = \overline{\phi}' \circ \phi = \mathbb{1}_A$; or
- $\phi \circ \overline{\phi} = \phi \circ \overline{\phi}' = \mathbb{1}_B$; or
- $\overline{\phi} \circ \phi = \mathbb{1}_A$ and $\phi \circ \overline{\phi}' = \mathbb{1}_B$. $\square$

Exercise 5.11. Let $G$ be a group with multiplication $*$. The *opposite group*, denoted $G^{\mathrm{op}}$, has the same elements as $G$. However, the multiplication $\cdot$ in $G^{\mathrm{op}}$ is the "opposite" multiplication to $G$:
$$x{\cdot}y = y * x \quad \text{for all } x, y \in G^{\mathrm{op}}.$$
Prove the following:

(a) The opposite group $G^{\mathrm{op}}$ is a group.

(b) The group $G^{\mathrm{op}}$ is isomorphic to $G$. $\square$

EXERCISE 5.12. Let $f_t\colon X \to Y$ be a homotopy of maps between spaces. Prove that $f_t$ induces a natural isomorphism between the functors $f_0\colon \Pi_1(X) \to \Pi_1(Y)$ and $f_1\colon \Pi_1(X) \to \Pi_1(Y)$ giving the induced maps between fundamental groupoids. Here recall that if $F, G\colon \mathbf{C} \to \mathbf{D}$ are functors between categories $\mathbf{C}$ and $\mathbf{D}$, then a *natural isomorphism* $\Psi\colon F \to G$ consists of the following data:

- For all objects $A$ of $\mathbf{C}$, a $\mathbf{D}$-isomorphism $\Psi(A)\colon F(A) \to G(A)$.

These must satisfy the following:

- For all morphisms $\lambda\colon A \to B$ between objects of $\mathbf{C}$, the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(\lambda)"] \arrow[d, "\Psi(A)"] & F(B) \arrow[d, "\Psi B"] \\ G(A) \arrow[r, "G(\lambda)"] & G(B) \end{tikzcd}$$

  must commute. □

CHAPTER 6

# Triviality of certain fundamental groups

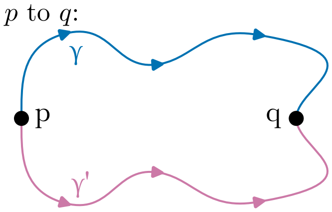
In this chapter and the next, we calculate the fundamental groups of many spaces. This chapter focuses on spaces with trivial fundamental groups. The results we prove will be needed in the next chapter to handle spaces with nontrivial fundamental groups.

## 6.1. 1-connectivity and the fundamental group

Recall that a space $X$ is 1-connected if it is nonempty and for all $x, y \in X$ there is a unique homotopy class of paths from $x$ to $y$. If $X$ is 1-connected and locally path connected, then Theorem 4.6.1 says that all covers of $X$ are trivial. We can relate this to the fundamental group as follows:

LEMMA 6.1.1. *Let $(X, x_0)$ be a path connected pointed space.*[1] *Then $X$ is 1-connected if and only if $\pi_1(X, x_0) = 1$.*

PROOF. If $X$ is 1-connected, then in particular there is only one homotopy class of paths from $x_0$ to itself, so $\pi_1(X, x_0) = 1$. Conversely, assume that $\pi_1(X, x_0) = 1$. Let $p$ and $q$ be two points of $X$, and let $\gamma$ and $\gamma'$ be paths from $p$ to $q$:




Since $X$ is path connected, Lemma 5.4.1 implies that $\pi_1(X, p) = 0$. The path $\gamma{\cdot}\overline{\gamma}'$ is a path from $p$ to $p$, so $[\gamma{\cdot}\overline{\gamma}'] \in \pi_1(X, p)$ must be trivial. We therefore have

$$[\gamma'] = 1[\gamma'] = [\gamma{\cdot}\overline{\gamma}'][\gamma'] = [\gamma][\overline{\gamma}'][\gamma'] = [\gamma],$$

as desired. □

Lemma 4.5.2 says that $\mathbb{S}^n$ is 1-connected for $n \geq 2$, so we deduce the following:

LEMMA 6.1.2. *For $n \geq 2$, we have $\pi_1(\mathbb{S}^n, x_0) = 1$ for all $x_0 \in \mathbb{S}^n$.*

## 6.2. Retracts and deformation retracts

Let $X$ be a space and let $A \subset X$ be a subspace. A *retract* of $X$ to $A$ is a map $r\colon X \to A$ such that $r(a) = a$ for all $a \in A$, i.e., such that $r|_A = \mathbb{1}$. A *deformation retraction* of $X$ to $A$ is a homotopy $r_t\colon X \to X$ from the identity $\mathbb{1}\colon X \to X$ to a map $r_1\colon X \to X$ such that:

- the map $r_1$ is a retraction of $X$ to $A$; and
- for all $t \in I$ and $a \in A$, we have $r_t(a) = a$.

If there exists a deformation retraction of $X$ to $A$, then we say that $A$ is a *deformation retract* of $X$ and that $X$ *deformation retracts* to $A$. Here are several examples:

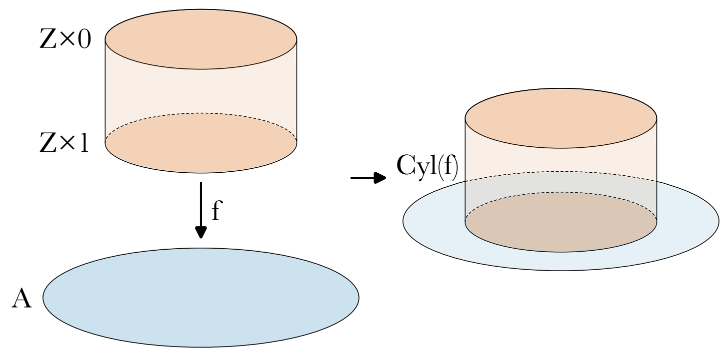
EXAMPLE 6.2.1. Let $U \subset \mathbb{R}^n$ be a set that is *star-shaped*, i.e., such that there exists a point $p_0 \in U$ such that for all $x \in U$ the line segment from $p_0$ to $x$ is contained in $U$. For instance, $U$ might be convex. We claim that $U$ deformation retracts to $p_0$. Indeed, the maps $r_t\colon U \to U$ defined by

$$r_t(x) = (1 - t)x + tp_0 \quad \text{for } x \in U \text{ and } t \in I$$

form a deformation retraction. □

[1] Since $x_0 \in X$, the space $X$ is nonempty and thus 0-connected.

Example 6.2.2. Let $f\colon Z \to A$ be a map between spaces. The *mapping cylinder* of $f$, denoted $\mathrm{Cyl}(f)$, is the quotient of the disjont union $(Z \times I) \sqcup A$ that identifies $(z,1) \in Z \times I$ with $f(z) \in A$ for all $z \in Z$:

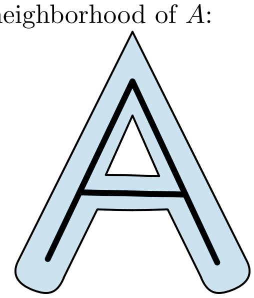



For $z \in Z$ and $s \in I$, let $\overline{(z,s)}$ be the image of $(z,s) \in Z \times I$ in $\mathrm{Cyl}(f)$. Both $Z$ and $A$ are subspaces of $\mathrm{Cyl}(f)$: the space $Z$ can be identified with $\left\{\overline{(z,0)} \mid z \in Z\right\}$, and the copy of $A$ in $(Z \times I) \sqcup A$ maps homeomorphically to a copy of $A$ in $\mathrm{Cyl}(f)$. The space $\mathrm{Cyl}(f)$ deformation retracts to $A$ via the deformation retract $r_t\colon \mathrm{Cyl}(f) \to A$ defined by

$$\begin{cases} r_t\overline{(z,s)} = \overline{(z,(1-t)s)} & \text{for } (z,s) \in Z \times [0,1], \\ r_t(a) = a & \text{for } a \in A. \end{cases}$$

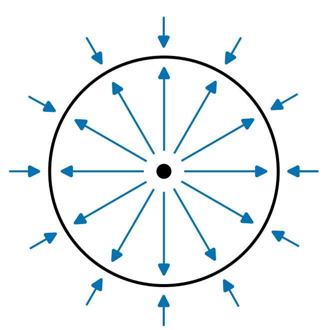
The reader can easily check that this makes sense and is continuous. □

Example 6.2.3. As in the following figure, let $A$ be the letter $A$ embedded in the plane and for some small $\epsilon > 0$ let $X$ be a closed $\epsilon$-neighborhood of $A$:



Then $X$ deformation retracts to $A$ via a deformation retraction during which points travel along straight line segments to $A$. In fact, this is a special case of the previous example: the boundary of $X$ consists of two circles $\mathbb{S}^1 \sqcup \mathbb{S}^1$, and $X$ is homeomorphic to the mapping cylinder of a map $f\colon \mathbb{S}^1 \sqcup \mathbb{S}^1 \to A$. □

Example 6.2.4. We claim that $\mathbb{S}^{n-1}$ is a deformation retract of $\mathbb{R}^n \setminus 0$. Geometrically, the picture is as follows, where the blue arrows show the paths points of $\mathbb{R}^n \setminus 0$ travel during the deformation retraction:



In formulas, this deformation retraction is given by the maps $r_t\colon \mathbb{R}^n \setminus 0 \to \mathbb{R}^n \setminus 0$ defined by

$$r_t(x) = \left((1-t) + \frac{t}{\|x\|}\right) x \quad \text{for } x \in \mathbb{R}^n \setminus 0 \text{ and } t \in I.$$ □

Recall from §5.6 that the fundamental group is functorial under maps of pointed spaces. Using this, we have:

LEMMA 6.2.5. *Let $X$ be a space, let $A \subset X$ be a subspace, and let $a_0 \in A$. Let $\iota\colon (A, a_0) \to (X, a_0)$ be the inclusion. Then:*

*(i) If $A$ is a retract of $X$, then the map $\iota_*\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$ is injective.*
*(ii) If $A$ is a deformation retraction of $X$, then the map $\iota_*\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$ is an isomorphism.*

PROOF. We start with (i). Let $r\colon X \to A$ be a retraction. Since $r \circ \iota = \mathbb{1}_A$, it follows that $r_* \circ \iota_* = \mathbb{1}_{\pi_1(A,a_0)}$, i.e., that the following composition is the identity:

$$\pi_1(A, a_0) \xrightarrow{\iota_*} \pi_1(X, a_0) \xrightarrow{r_*} \pi_1(A, a_0).$$

This implies that $\ker(\iota_*)$ is trivial, so $\iota_*$ is injective.

We now prove (ii). Let $r_t\colon X \to X$ be a deformation retraction. In light of (i), it is enough to prove that the map $\iota_*\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$ is surjective. Consider a loop $\delta\colon I \to X$ based at $p$. We must prove that $\delta$ can be homotoped to a loop lying in $A$. Since $r_t(a_0) = a_0$ for all $t \in I$, we have a homotopy of paths $r_t \circ \delta$. Since $r_1(X) \subset A$, the image of the endpoint $r_1 \circ \delta$ of this homotopy lies in $A$, as desired. $\square$

Here is one consequence:

LEMMA 6.2.6. *Let $n \geq 3$. For $x_0 \in \mathbb{R}^n \setminus 0$, we have $\pi_1(\mathbb{R}^n \setminus 0, x_0) = 0$.*

PROOF. Since $\mathbb{R}^n \setminus 0$ is path connected, we can change $x_0$ without changing the fundamental group. Choose $x_0$ such that $x_0 \in \mathbb{S}^{n-1} \subset \mathbb{R}^n \setminus 0$. Since $\mathbb{R}^n \setminus 0$ deformation retracts to $\mathbb{S}^{n-1}$, Lemma 6.2.5 implies that

$$\pi_1(\mathbb{R}^n \setminus 0, x_0) \cong \pi_1(\mathbb{S}^{n-1}, x_0).$$

Since $n \geq 3$, this vanishes by Lemma 4.5.2. $\square$

## 6.3. Contractibility

A nonempty space $X$ is said to be *contractible* if the identity map $\mathbb{1}\colon X \to X$ is homotopic to a constant map. This holds, for instance, if $X$ deformation retracts to any one-point subspace $x_0$. Star-shaped or convex subspaces of $\mathbb{R}^n$ are therefore contractible. However, being contractible is more general than this since none of the points of $X$ need to be fixed during the contraction. See Exercise 6.12 for an example where these are genuinely different notions.

If a space $X$ deformation retracts to a point $x_0 \in X$, then it follows from Lemma 6.2.5 that

$$\pi_1(X, x_0) \cong \pi_1(x_0, x_0) = 1.$$

The following shows that this vanishing holds more generally if $X$ is merely contractible.

LEMMA 6.3.1. *Let $X$ be a contractible space and let $x_0 \in X$. Then $\pi_1(X, x_0) = 1$.*

PROOF. Let $f_t\colon X \to X$ be a homotopy from the identity $\mathbb{1}\colon X \to X$ to a constant map. Since $X$ is path connected, its fundamental groups at different basepoints are all isomorphic. We can therefore assume without loss of generality that $x_0$ is the constant value of $f_1$. Let $\delta\colon I \to X$ be the following path from $x_0$ to $x_0$:

$$\delta(s) = f_s(x_0) \quad \text{for } s \in I.$$

Consider $[\gamma] \in \pi_1(X, x_0)$. Since $f_0 = \mathbb{1}$ we have $(f_0)_*([\gamma]) = [\gamma]$, and since $f_1$ is the constant map $x_0$ we have $(f_1)_*([\gamma]) = 1$. By Lemma 5.7.1, we therefore have

$$[\gamma] = (f_0)_*([\gamma]) = [\delta](f_1)_*([\gamma])[\overline{\delta}] = [\delta][\overline{\delta}] = 1. \qquad \square$$

## 6.4. Trees

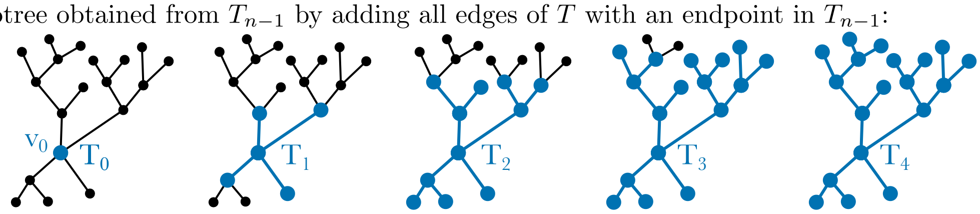
Here is an important example. Recall that we discussed graphs in §2.5. A tree is a nonempty connected graph with no cycles. We will prove:

Lemma 6.4.1. *Let $T$ be a tree and let $v_0$ be a vertex of $T$. Then $T$ deformation retracts to $v_0$, and in particular $T$ is contractible.*

Proof. We will omit some of the point-set details, and invite the reader in Exercise 6.9 to verify that all the maps we construct are continuous. Inductively define subtrees

$$T_0 \subset T_1 \subset T_2 \subset \cdots$$

of $T$ in the following way. Start by letting $T_0 = v_0$. Next, if $T_{n-1}$ has been constructed, let $T_n$ be the subtree obtained from $T_{n-1}$ by adding all edges of $T$ with an endpoint in $T_{n-1}$:




Since $T$ is a tree, each new edge $e$ added to $T_{n-1}$ to form $T_n$ has the property that exactly one endpoint of $e$ lies in $T_{n-1}$; otherwise, $e$ would form part of a cycle in $T_n$. This implies that $T_n$ deformation retracts to $T_{n-1}$ via a deformation retract where the points of these new edges $e$ move along $e$ to the vertex lying in $T_{n-1}$. Let $r_t^n \colon T_n \to T_n$ be this deformation retract. Since $T$ is connected, we have

$$T = \bigcup_{n=0}^{\infty} T_n.$$

For each $n \geq 1$ and $m \geq 0$, consider the retractions

$$R_m^n = r_1^n \circ \cdots \circ r_1^{n+m} \colon T_{n+m} \to T_{n-1}.$$

For $m_1 \geq m_2 \geq 0$, the retractions $R_{m_1}^n$ and $R_{m_2}^n$ agree where they both are defined, namely on $T_{n+m_2}$. It follows that for a fixed $n \geq 1$ the different $R_m^n$ glue together to give a retraction $R^n \colon T \to T_{n-1}$.

Assume first that $T = T_{n_1}$ for some $n_1 \gg 0$ (which holds, for instance, if $T$ is a finite tree). In this case, we can deformation retract $T = T_{n_1}$ to $T_0 = v_0$ by first using $r_t^{n_1}$ to deformation retract $T_{n_1}$ to $T_{n_1-1}$, then using $r_t^{n_1-1}$ to deformation retract $T_{n_1-1}$ to $T_{n_1-2}$, etc. For the general case, we have to be a bit more careful. Write

$$I = \{0\} \cup \bigcup_{n=1}^{\infty} I_n \quad \text{with } I_n = [1/2^n, 1/2^{n-1}],$$

so $I_n$ has length $1/2^n$. Define $r_t \colon T \to T$ in the following way:

- For $t \in I_n$ and $x \in T$, let $r_t(x) = r^n_{2^n(t-1/2^n)}(R^{n+1}(x))$.
- For $t = 0$ and $x \in T$, define $r_0(x) = x$.

The reader will check in Exercise 6.9 that this definition makes sense and is continuous. By definition we have $r_0 = \mathbb{1}$, and since $1 \in I_1$ we have

$$r_1(x) = r_1^1(R^2(x)) = v_0 \quad \text{for } x \in T,$$

where we recall that $T_0$ is the vertex $v_0$. It follows that $r_t$ is a deformation retraction of $T$ to $v_0$, as desired. □

## 6.5. Projective spaces

Recall that real projective space $\mathbb{RP}^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$. This has the degree 2 cover $p \colon \mathbb{S}^n \to \mathbb{RP}^n$ taking $x \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$ to the line through $x$. This reflects the fact that $\mathbb{RP}^n$ is not 1-connected, as we will see rigorously in Chapter 7.

An important relative of $\mathbb{RP}^n$ is *complex projective space*, that is, the space $\mathbb{CP}^n$ of lines through the origin in $\mathbb{C}^{n+1}$. This can be topologized just like $\mathbb{RP}^n$: letting $\mathbb{C}^{n+1} \setminus 0 \to \mathbb{CP}^n$ be the map taking $z \in \mathbb{C}^{n+1}$ to the line through $z$, we give $\mathbb{CP}^n$ the quotient topology:

- A set $U \subset \mathbb{CP}^n$ is open if and only if its preimage in $\mathbb{C}^{n+1} \setminus 0$ is open.

The space $\mathbb{CP}^n$ will play an important role in subsequent volumes when we discuss homology and cohomology. However, it is less important in this volume since it turns out to be 1-connected. We outline a proof of this in Exercise 6.3.

## 6.6. Exercises

Exercise 6.1. Let $X$ be a contractible space and let $A \subset X$ be a subspace such that there is a retract $r\colon X \to A$. Prove that $A$ is contractible. □

Exercise 6.2. Let $X$ be a space. Letting $p$ be a one-point space, the *cone* on $X$, denoted $\mathrm{Cone}(X)$, is the mapping cylinder of the constant map $X \to p$. The space $X$ is a subspace of $\mathrm{Cone}(X)$. Also, the image of $p$ in $\mathrm{Cone}(X)$ is called the *cone point*. The *suspension* of $X$, denoted $\Sigma X$, is the quotient of the disjoint union $\mathrm{Cone}(X) \sqcup \mathrm{Cone}(X)$ that identifies the copies of $X$ in the two cones. Do the following:

(a) Prove that $\mathrm{Cone}(X)$ deformation retracts to the cone point, and in particular is contractible.
(b) Prove that $\Sigma \mathbb{S}^n \cong \mathbb{S}^{n+1}$.
(c) If $X$ is 0-connected, then prove that $\Sigma X$ is 1-connected. Note that by (b) this generalizes the fact that $\mathbb{S}^n$ is 1-connected for $n \geq 2$. Hint: apply Lemma 4.5.3 (general position). □

Exercise 6.3. This exercise outlines a proof that $\mathbb{CP}^n$ is 1-connected. As notation, for $(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \setminus 0$ let $[z_1, \ldots, z_{n+1}]$ be the corresponding point in $\mathbb{CP}^n$, i.e., the line in $\mathbb{C}^{n+1}$ though the origin and $(z_1, \ldots, z_{n+1})$. The proof will be by induction.

(a) For the base case, prove that $\mathbb{CP}^1 \cong \mathbb{S}^2$, so $\mathbb{CP}^1$ is 1-connected.
(b) Now assume that $n \geq 2$ and that $\mathbb{CP}^{n-1}$ is 1-connected. Fix a basepoint $x_0 \in \mathbb{CP}^{n-1}$. Embed $\mathbb{CP}^{n-1}$ into $\mathbb{CP}^n$ via the map taking $[z_1, \ldots, z_n] \in \mathbb{CP}^{n-1}$ to $[z_1, \ldots, z_n, 0] \in \mathbb{CP}^n$. Using this, we identify $x_0$ with a point in $\mathbb{CP}^n$. Finally, set $r = [0, \ldots, 0, 1] \in \mathbb{CP}^n$. **Problem**: for $[\gamma] \in \pi_1(\mathbb{CP}^n, x_0)$, prove that $\gamma$ can be homotoped such that its image does not contain $r$. Hint: use Lemma 4.5.3 (general position).
(c) Prove that $\mathbb{CP}^n \setminus r$ deformation retracts to $\mathbb{CP}^{n-1}$. Since our inductive hypothesis implies that $\pi_1(\mathbb{CP}^{n-1}, x_0) = 1$, this will imply that $\pi_1(\mathbb{CP}^n \setminus r, x_0) = 0$ and thus by (b) that $\pi_1(\mathbb{CP}^n, x_0) = 1$. We conclude that $\mathbb{CP}^n$ is 1-connected. □

Exercise 6.4. Let $X$ be a space. Prove the following:

(a) The space $X$ is contractible if and only if for all spaces $Y$, every map $f\colon X \to Y$ is nullhomotopic.
(b) The space $X$ is contractible if and only if for all spaces $Z$, every map $g\colon Z \to X$ is nullhomotopic. □

Exercise 6.5. Let $X$ be a space. Let $\mathrm{Cone}(X)$ be the cone on $X$ from Exercise 6.2. Prove that $X$ is contractible if and only if $X$ is a retract of $\mathrm{Cone}(X)$. □

Exercise 6.6. Let $f\colon X \to Y$ be a map. Prove that there exists a map $g\colon Y \to X$ such that $g \circ f\colon X \to X$ is homotopic to the identity if and only if $X$ is a retract of the mapping cylinder $\mathrm{Cyl}(f)$. □

Exercise 6.7. Do the following:

(a) Let $X$ be a space. Assume that $X = U \cup V$ where $U$ and $V$ are 1-connected open sets such that $U \cap V$ is 0-connected. For $x_0 \in U \cap V$, prove that $\pi_1(X, x_0) = 1$. Hint: Start with some $[\gamma] \in \pi_1(X, x_0)$ and try to write $[\gamma] = [\gamma_1] \cdots [\gamma_n]$ where each $[\gamma_i] \in \pi_1(X, x_0)$ is such that the image of $\gamma_i$ lies in either $U$ or $V$. The Lebesgue number lemma will be useful.
(b) Use part (a) to give an alternate proof that $\mathbb{S}^n$ is 1-connected for $n \geq 2$.
(c) Generalize part (a) as follows. Let $X$ be a space and let $X = \cup_{i \in I} U_i$ with each $U_i$ open. Let $x_0 \in X$ be a basepoint such that $x_0 \in U_i$ for all $i \in I$. Assume the following:
  - The open set $U_i$ is 0-connected for all $i \in I$.
  - The open set $U_i \cap U_j$ is 0-connected for all $i, j \in I$.
  - The map $\pi_1(U_i, x_0) \to \pi_1(X, x_0)$ is trivial for all $i \in I$.

Prove that $\pi_1(X, x_0) = 1$. □

EXERCISE 6.8. Let $X$ and $Y$ be spaces. The *join* of $X$ and $Y$, denoted $X * Y$, is

$$X * Y = X \sqcup Y \sqcup (X \times Y \times I)/\sim,$$

where $\sim$ makes the following identifications for all $x \in X$ and $y \in Y$:

$$(x, y, 0) \sim x \quad \text{and} \quad (x, y, 1) \sim y.$$

Identify $X$ and $Y$ with their images in $X * Y$. The space $X * Y$ can be viewed as the space of all lines connecting points of $X$ to points of $Y$. In analogy with the usual way of writing a line segment between points of $\mathbb{R}^n$ using barycentric coordinates, it is useful to denote the image in $X * Y$ of $(x, y, t) \in X \times Y \times I$ by the formal sum $(1 - t)x + ty$. Do the following:

(a) For $n, m \geq 0$, we have $\mathbb{S}^n * \mathbb{S}^m \cong \mathbb{S}^{n+m+1}$.
(b) For $n, m \geq 0$, we have $\mathbb{D}^n * \mathbb{D}^m \cong \mathbb{D}^{n+m+1}$.
(c) Prove that $(X * Y) \setminus X$ deformation retracts to $Y$.
(d) Assume that $Y$ is nonempty. For $x_0 \in X$, prove that the map $\pi_1(X, x_0) \to \pi_1(X * Y, x_0)$ is trivial.
(e) Assume that $Y$ is 0-connected and that $Y$ is nonempty and locally path connected. Prove that $X * Y$ is 1-connected. Hint: Part (c) of Exercise 6.7 might be useful. It also might be easier to first prove this when $Y$ is 0-connected. For the general case, since $Y$ is locally path connected, we can decompose it into clopen path components $Y = \sqcup_{i \in I} Y_i$. □

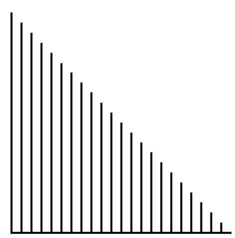
EXERCISE 6.9. Verify that the maps constructed in the proof of Lemma 6.4.1 are well-defined and continuous. □

EXERCISE 6.10. Let $\mathrm{O}(n)$ be the $n$-dimensional orthogonal group. Prove that $\mathrm{GL}_n(\mathbb{R})$ deformation retracts to $\mathrm{O}(n)$. Hint: analyze the Gram–Schmidt orthogonalization process. □

EXERCISE 6.11. Let $X \subset \mathbb{R}^2$ be the union of the following subspaces:

- the horizontal segment $[0, 1] \times 0$; and
- for each $r \in \mathbb{Q}$, the vertical segment $r \times [0, 1 - r]$.

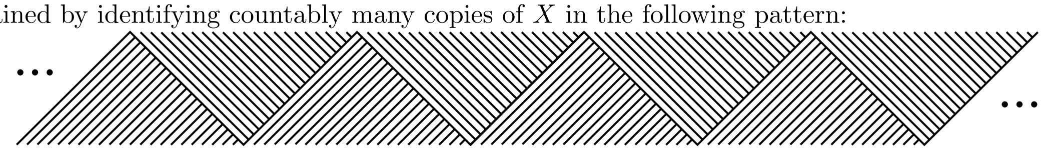
See here:



Prove the following:

(a) The space $X$ deformation retracts to any point on the horizontal segment $[0, 1] \times 0$.
(b) The space $X$ does not deformation retract to any point that does not lie on $[0, 1] \times 0$. □

EXERCISE 6.12. Let $X \subset \mathbb{R}^2$ be the subspace from Exercise 6.11. Let $Y \subset \mathbb{R}^2$ be the space obtained by identifying countably many copies of $X$ in the following pattern:



Prove the following:

(a) The space $Y$ is contractible.
(b) The space $Y$ does not deformation retract to any point. □

CHAPTER 7

# Basic calculations of the fundamental group

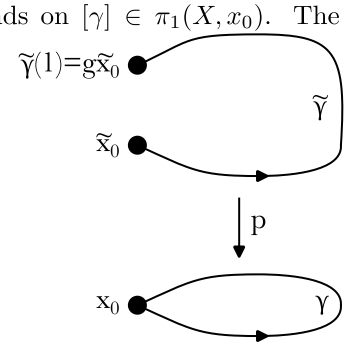
We now calculate a number of nontrivial fundamental groups. One of the main places areas to which the fundamental group can be applied is group theory, where it allows geometric arguments that would be difficult to express purely algebraically. We give a first example of this at the end of this chapter, where we use the fundamental group to construct free groups.

## 7.1. Calculating the fundamental group using covering spaces

The following is our main tool for calculating fundamental groups:

THEOREM 7.1.1. *Let $p\colon \widetilde{X} \to X$ be a regular cover such that $\widetilde{X}$ is 1-connected. Set $G = \mathrm{Deck}(\widetilde{X})$. Then for $x_0 \in X$ we have $\pi_1(X, x_0) \cong G$.*

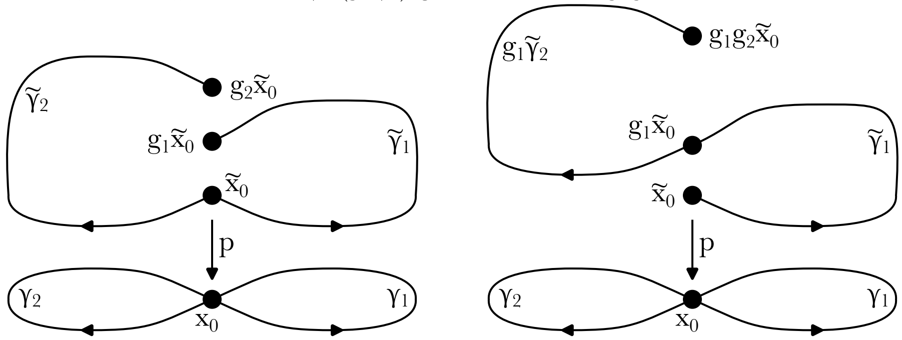
PROOF. Pick $\widetilde{x}_0 \in \widetilde{X}$ with $p(\widetilde{x}_0) = x_0$. Define a set map $f\colon \pi_1(X, x_0) \to G$ as follows. Consider $[\gamma] \in \pi_1(X, x_0)$. By path lifting (Lemma 3.4.1), we can lift $\gamma$ to a path $\widetilde{\gamma}$ in $\widetilde{X}$ starting at $\widetilde{x}_0$. Lemma 4.2.1 implies that the homotopy class of $\widetilde{\gamma}$ only depends on the homotopy class of $\gamma$. In particular, $\widetilde{\gamma}(1) \in \widetilde{X}$ only depends on $[\gamma] \in \pi_1(X, x_0)$. The point $\widetilde{\gamma}(1)$ projects to $\gamma(1) = x_0$:




Since $\widetilde{X}$ is a path connected regular cover of $X$, there exists a unique $g \in G$ with $g\widetilde{x}_0 = \widetilde{\gamma}(1)$; see Lemma 2.2.1. Define $f([\gamma]) = g$. To prove the theorem, it is enough to prove that $f$ is a group homomorphism that is injective and surjective. We do this in the following three claims:

CLAIM 1. *The set map $f$ is a group homomorphism.*

Consider $[\gamma_1], [\gamma_2] \in \pi_1(X, x_0)$. For $i = 1, 2$, let $\widetilde{\gamma}_i$ be the lift of $\gamma_i$ to $\widetilde{X}$ with $\widetilde{\gamma}_i(0) = \widetilde{x}_0$. Letting $g_i = f([\gamma_i])$, the path $\widetilde{\gamma}_i$ thus goes from $\widetilde{x}_0$ to $g_i\widetilde{x}_0$. The deck group $G$ acts not only on $\widetilde{X}$, but also on paths in $\widetilde{X}$. Under this group action, the path $g_1\widetilde{\gamma}_2$ goes from $g_1\widetilde{x}_0$ to $g_1g_2\widetilde{x}_0$. It follows that $\widetilde{\gamma}_1$ and $g_1\widetilde{\gamma}_2$ are composable paths, and $\widetilde{\gamma}_1 \cdot (g_1\widetilde{\gamma}_2)$ goes from $\widetilde{x}_0$ to $g_1g_2\widetilde{x}_0$:




The path $\widetilde{\gamma}_1 \cdot (g_1\widetilde{\gamma}_2)$ is the lift of $\gamma_1 \cdot \gamma_2$, so by definition this implies that $f([\gamma_1 \cdot \gamma_2]) = g_1g_2$, as desired.

CLAIM 2. *The homomorphism $f$ is surjective.*

Consider $g \in G$. Since $\widetilde{X}$ is path connected, we can find a path $\widetilde{\gamma}$ in $\widetilde{X}$ from $\widetilde{x}_0$ to $g\widetilde{x}_0$. The path $\widetilde{\gamma}$ projects to a path $\gamma$ in $X$ from $x_0$ to $x_0$, so $[\gamma] \in \pi_1(X, x_0)$. By definition, $f([\gamma]) = g$.

CLAIM 3. *The homomorphism $f$ is injective.*

Consider $[\gamma] \in \pi_1(X, x_0)$ such that $f([\gamma]) = 1$. Let $\widetilde{\gamma}$ be the lift of $\gamma$ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$. Since $f([\gamma]) = 1$, we must have $\widetilde{\gamma}(1) = \widetilde{x}_0$, so $\widetilde{\gamma}$ is a loop based at $\widetilde{x}_0$. Since $\widetilde{X}$ is 1-connected, the loop $\widetilde{\gamma}$ is homotopic to a constant loop. Composing this homotopy with the map $p \colon \widetilde{X} \to X$, we obtain a homotopy from $\gamma$ to a constant loop, so $[\gamma] = 1$, as desired. $\square$

## 7.2. The lifting map

Before giving examples of Theorem 7.1.1, we give a name to the isomorphism underlying its conclusion. Let $p \colon \widetilde{X} \to X$ be a regular cover such that $\widetilde{X}$ is 1-connected. Let $\widetilde{x}_0 \in \widetilde{X}$, and set $x_0 = p(\widetilde{x}_0)$ and $G = \text{Deck}(\widetilde{X})$. Theorem 7.1.1 says that $\pi_1(X, x_0) \cong G$. In the proof of that theorem, this isomorphism is given by the following set map $f \colon \pi_1(X, x_0) \to G$:

- Consider $[\gamma] \in \pi_1(X, x_0)$. Let $\widetilde{\gamma}$ be the lift of $\gamma$ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$. There exists a unique $g \in G$ with $\widetilde{\gamma}(1) = g\widetilde{x}_0$. We then have $f([\gamma]) = g$.

We will call this isomorphism $f \colon \pi_1(X, x_0) \to G$ the *lifting map*. The proof that it is a well-defined surjective homomorphism only depends on the fact that $\widetilde{X}$ is path connected. The 1-connectedness of $\widetilde{X}$ is only used in the proof that $f$ is injective. We record these observations in the following lemma:

LEMMA 7.2.1. *Let $p \colon \widetilde{X} \to X$ be a regular cover with $\widetilde{X}$ path connected. Let $\widetilde{x}_0 \in \widetilde{X}$, and set $x_0 = p(\widetilde{x}_0)$ and $G = \text{Deck}(\widetilde{X})$. Then the lifting map $f \colon \pi_1(X, x_0) \to G$ is a surjective homomorphism.*

## 7.3. Circle and torus

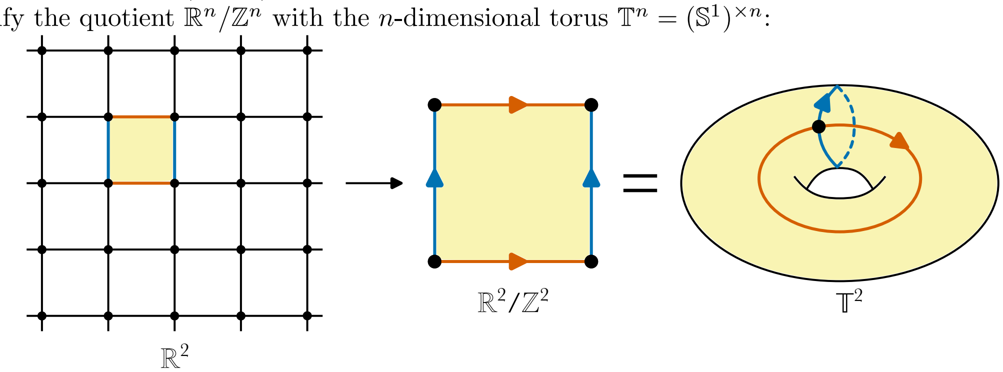
We now give some calculations using Theorem 7.1.1. Our first is important enough that we single it out as a lemma. Recall that we identify $\mathbb{S}^1$ with a subset of $\mathbb{C}$, so $1 \in \mathbb{S}^1$.

LEMMA 7.3.1 (Circle). *We have $\pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z}$, where $n \in \mathbb{Z}$ corresponds to the loop $\gamma_n \colon I \to \mathbb{S}^1$ defined by $\gamma_n(s) = e^{2\pi i n s}$ for $s \in I$.*

PROOF. Consider the universal cover $p \colon \mathbb{R} \to \mathbb{S}^1$ of $\mathbb{S}^1$, so $p(\theta) = e^{2\pi i \theta}$. As we observed in Example 2.2.2, this is a regular cover with deck group $\mathbb{Z}$, which acts on $\mathbb{R}$ by integer translations. Since $\mathbb{R}$ is contractible, it is 1-connected. We can therefore apply Theorem 7.1.1 to see that $\pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z}$. This isomorphism is given by the lifting map, and under the lifting map $n \in \mathbb{Z}$ corresponds to the loop $\gamma_n$. $\square$

Our next example generalizes this:

EXAMPLE 7.3.2 (Torus). As in Example 1.3.3 let $\mathbb{Z}^n$ act on $\mathbb{R}^n$ by integer translations and identify the quotient $\mathbb{R}^n/\mathbb{Z}^n$ with the $n$-dimensional torus $\mathbb{T}^n = (\mathbb{S}^1)^{\times n}$:




This figure shows the case $n = 2$. The projection $p \colon \mathbb{R}^n \to \mathbb{Z}^n$ is a regular cover with deck group $\mathbb{Z}^n$. Set $x_0 = p(0)$. Since $\mathbb{R}^n$ is contractible, it is 1-connected. We can thus apply Theorem 7.1.1 to see that $\pi_1(\mathbb{T}^n, x_0) = \pi_1((\mathbb{S}^1)^{\times n}, x_0) \cong \mathbb{Z}^n$. $\square$

Remark 7.3.3. More generally, by Exercise 5.1 if $X$ and $Y$ are spaces with basepoints $x_0 \in X$ and $y_0 \in Y$, then $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$. □

## 7.4. Brouwer fixed point theorem

Since $\pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z}$ but $\mathbb{S}^n$ is 1-connected for all $n \geq 2$, it follows that $\mathbb{S}^1$ is not homeomorphic to $\mathbb{S}^n$ for any $n \geq 2$. This is not a particularly profound result, and can be proved in many other ways.[1] In the next chapter we will talk about homotopy equivalences, and we will be able to deduce the more interesting result that $\mathbb{S}^1$ and $\mathbb{S}^n$ are not homotopy equivalent for any $n \geq 2$.

Here, however, we give an interesting application of a different flavor that also illustrates how the functorality of the fundamental group can be used to obstruct the existence of maps. Regard $\mathbb{S}^1$ as the boundary of $\mathbb{D}^2$.

Lemma 7.4.1. *There does not exist a retraction* $r\colon \mathbb{D}^2 \to \mathbb{S}^1$.

Proof. Assume for the sake of contradiction that a retraction $r\colon \mathbb{D}^2 \to \mathbb{S}^1$ exists. Let $\iota\colon \mathbb{S}^1 \to \mathbb{D}^2$ be the inclusion, and fix some $x_0 \in \mathbb{S}^1$. Since $r \circ \iota = \mathbb{1}_{\mathbb{S}^1}$, the following composition is the identity:

$$\begin{array}{ccccc} \pi_1(\mathbb{S}^1, x_0) & \xrightarrow{\iota_*} & \pi_1(\mathbb{D}^2, x_0) & \xrightarrow{r_*} & \pi_1(\mathbb{S}^1, x_0). \\ \| & & \| & & \| \\ \mathbb{Z} & & 0 & & \mathbb{Z} \end{array}$$

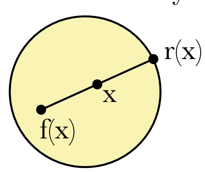
However, since the middle group is 0 this composition is also the 0 map, a contradiction. □

This has the following consequence:

Theorem 7.4.2 (Brouwer fixed point theorem). *Let* $f\colon \mathbb{D}^2 \to \mathbb{D}^2$ *be a continuous map. Then* $f$ *has a fixed point, i.e., there exists some* $x \in \mathbb{D}^2$ *with* $f(x) = x$.

Proof. Assume for the sake of contradiction that $f(x) \neq x$ for all $x \in \mathbb{D}^2$. Define $r\colon \mathbb{D}^2 \to \mathbb{S}^1$ to be following map:

- For $x \in \mathbb{D}^2$, let $r(x)$ be the point where the ray from $f(x)$ to $x$ intersects $\mathbb{S}^1$:




By construction, $f$ is a continuous retraction, contradicting Lemma 7.4.1. □

Remark 7.4.3. The Brouwer fixed point theorem (Theorem 7.4.2) actually holds for maps $f\colon \mathbb{D}^n \to \mathbb{D}^n$ with $n \geq 1$ arbitrary. See Exercise 7.6 for the (easy) proof when $n = 1$. Similarly, Lemma 7.4.1 holds in all dimension. For general $n \geq 1$, we gave an elementary proof of both of these results in Volume 1 using a combinatorial result called Sperner's Lemma. Once we develop the theory of homology, we will be able to give a proof of the general case that is very similar to the proof we gave above for $n = 2$. □

## 7.5. Real projective space

We now turn to real projective space:

Example 7.5.1 (Real projective space). Let $n \geq 2$. Recall that $\mathbb{RP}^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$. As we described in Example 1.3.4, there is a 2-fold cover $p\colon \mathbb{S}^n \to \mathbb{RP}^n$ taking $x \in \mathbb{S}^n$ to the line through $x$. This is a regular cover with deck group the cyclic group $C_2$ of order 2, which acts on $\mathbb{S}^n$ by the antipodal map $x \mapsto -x$. Fix some $x_0 \in \mathbb{S}^n$, and let $\ell_0 = p(x_0) \in \mathbb{RP}^n$. Since $n \geq 2$, we have $\pi_1(\mathbb{S}^n, x_0) = 1$ (Lemma 4.5.2). We can therefore apply Theorem 7.1.1 and see that

$$\pi_1(\mathbb{RP}^n, \ell_0) \cong C_2.$$

[1] For instance, by observing that $\mathbb{S}^1 \setminus \{p, q\}$ has two path components for any distinct $p, q \in \mathbb{S}^1$, while removing two points from $\mathbb{S}^n$ cannot disconnect $\mathbb{S}^n$ for any $n \geq 2$.

This isomorphism is given by the lifting map, and under the lifting map the generator of $C_2$ corresponds to the loop in $\pi_1(\mathbb{RP}^n, \ell_0)$ that rotates the line $\ell_0$ around an axis by an angle of $\pi$, coming back to itself but with the reversed orientation. □

REMARK 7.5.2. We have $\mathbb{RP}^1 \cong \mathbb{S}^1$ (see Exercise 7.8), so $\pi_1(\mathbb{RP}^1, \ell_0) \cong \mathbb{Z}$. □

## 7.6. Free groups: definition

Before we can give our next example, we need some background about free groups. Let $S$ be a set. Roughly speaking, a free group on $S$ is a group that is easy to map out of. One need only say where the elements of $S$ must go. Here is the formal definition:

DEFINITION 7.6.1. Let $S$ be a set. A *free group* on $S$ is a group $F(S)$ equipped with a map of sets $\iota\colon S \to F(S)$ such that the following holds:

(†) Let $G$ be a group and $h\colon S \to G$ be a map of sets. Then there is a unique homomorphism $H\colon F(S) \to G$ such that $h = H \circ \iota$.

The set $S$ is called a *free basis* for $F(S)$. □

REMARK 7.6.2. The condition (†) is an example of a *universal mapping property.* □

The map $\iota$ in the definition of a free group is necessarily injective:

LEMMA 7.6.3. *Let $S$ be a set and let $F(S)$ be a free group on $S$. Then the associated map $\iota\colon S \to F(S)$ is injective.*

PROOF. Let $G$ be a group of cardinality at least $|S|$ and let $h\colon S \to G$ be an injection. By the universal property (†), there is a homomorphism $H\colon F(S) \to G$ such that $h = H \circ \iota$. Since $h$ is injective, it follows that $\iota$ is injective. □

Let $F(S)$ be a free group on a set $S$. By Lemma 7.6.3, we can identify $S$ with a subset of $F(S)$ via the corresponding map $\iota$. Having done this, we can now rephrase (†) as follows:

(†′) Let $G$ be a group and $h\colon S \to G$ be a map of sets. Then $h$ extends uniquely to a homomorphism $H\colon F(S) \to G$.

Whenever we work with free groups in this book, we will identify $S$ with a subset of $F(S)$ and use (†′) as the defining property of $F(S)$. The subset $S$ generates $F(S)$:

LEMMA 7.6.4. *Let $S$ be a set and let $F(S)$ be a free group on $S$. Then $S$ generates $F(S)$.*

PROOF. Let $G < F(S)$ be the subgroup generated by $S$. By (†′), the inclusion $h\colon S \hookrightarrow G$ extends uniquely to a homomorphism $H\colon F(S) \to G$. The composition

$$F(S) \xrightarrow{H} G \hookrightarrow F(S)$$

and the identity map $\mathbb{1}_{F(S)}\colon F(S) \to F(S)$ both extend $\mathbb{1}_S\colon S \to S$, and thus must be equal. We conclude that $G = F(S)$, as desired. □

It is not obvious that a free group on a set $S$ exists. We will soon construct one, but first we prove that they are unique in the following sense:

LEMMA 7.6.5. *Let $S$ be a set and let $F(S)$ and $F'(S)$ be free groups on $S$. There is then a unique isomorphism $\phi\colon F(S) \to F'(S)$ with $\phi|_S = \mathbb{1}_S$.*

PROOF. Applying (†′) to the identity maps between $S \subset F(S)$ and $S \subset F'(S)$, we get:

- a homomorphism $\phi\colon F(S) \to F'(S)$ such that $\phi|_S = \mathbb{1}_S$; and
- a homomorphism $\phi'\colon F'(S) \to F(S)$ such that $\phi'|_S = \mathbb{1}_S$.

The composition $\phi' \circ \phi\colon F(S) \to F(S)$ fixes each element of the generating set $S$, so $\phi' \circ \phi = \mathbb{1}_{F(S)}$. Similarly, $\phi \circ \phi' = \mathbb{1}_{F'(S)}$. We conclude that $\phi$ and $\phi'$ are inverse isomorphisms. □

REMARK 7.6.6. A similar argument shows uniqueness for other mathematical objects defined by universal mapping properties. □

## 7.7. Free groups: reduced words

Let $S$ be a set. A *word* in $S$ is a formal expression $w = s_1^{\epsilon_1} \cdots s_n^{\epsilon_n}$ with $s_i \in S$ and $\epsilon_i \in \{\pm 1\}$ for all $1 \leq i \leq n$. This word is *reduced* if for all $1 \leq i < n$ we do *not* have

$$s_i^{\epsilon_i} s_{i+1}^{\epsilon_{i+1}} \in \left\{ ss^{-1},\, s^{-1}s \mid s \in S \right\}.$$

By cancelling terms of the form $ss^{-1}$ and $s^{-1}s$ with $s \in S$ we can reduce any word to a reduced word. If $S$ is a subset of a group $\Gamma$, then we can regard words in $S$ as elements of $\Gamma$. Cancelling terms of the form $ss^{-1}$ and $ss^{-1}$ does not change the associated element of $\Gamma$, so every element of $\Gamma$ that can be written as a word in $S$ can be written as a reduced word in $S$.

The main existence theorem for free groups is:

Theorem 7.7.1. *Let $S$ be a set. There then exists a group $F(S)$ with $S \subset F(S)$ such that:*

*(i) the group $F(S)$ is a free group on $S$; and*
*(ii) every element of $F(S)$ can be represented by a unique reduced word in $S$.*

The standard proof of this is algebraic, and proves the two parts separately:

- First, free groups are proved to exist, establishing (i).
- Next, (ii) is proved using the universal property of the free group.

See Exercises 7.15 and 7.16 for an outline of this proof. We will prove Theorem 7.7.1 geometrically in the next section. Our argument will reverse the logic of the algebraic proof:

- First, we will use geometry to construct a group $F(S)$ containing a set $S$ such that every element of $F(S)$ can be represented by a unique reduced word in $S$. This establishes part (ii) of Theorem 7.7.1.
- Next, we will prove that $F(S)$ has the desired universal property, establishing part (i).

In fact, the second step is quite formal:

Lemma 7.7.2. *Let $S$ be a set and let $\Gamma$ be a group containing $S$ such that every element of $\Gamma$ can be represented by a unique reduced word in $S$. Then $\Gamma$ is a free group on $S$.*

Proof. Let $G$ be a group and let $h\colon S \to G$ be a set map. We extend $h$ to $H\colon \Gamma \to G$ as follows. Consider $g \in \Gamma$. We can uniquely write $g$ as a reduced word in $S$:

$$g = s_1^{\epsilon_1} \cdots s_n^{\epsilon_n} \quad \text{with } s_1, \ldots, s_n \in S \text{ and } \epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}.$$

We then define

$$H(g) = h(s_1)^{\epsilon_1} \cdots h(s_n)^{\epsilon_n}.$$

This is a homomorphism. Indeed, consider $g, g' \in \Gamma$. Write them as reduced words in $S$:

$$\begin{aligned} g =& s_1^{\epsilon_1} \cdots s_n^{\epsilon_n} \quad \text{with } s_1, \ldots, s_n \in S \text{ and } \epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}, \\ g' =& t_1^{e_1} \cdots t_m^{e_m} \quad \text{with } t_1, \ldots, t_m \in S \text{ and } e_1, \ldots, e_m \in \{\pm 1\}. \end{aligned}$$

Then the reduced word representing $gg'$ is obtained from

$$s_1^{\epsilon_1} \cdots s_n^{\epsilon_n} t_1^{e_1} \cdots t_m^{e_m}$$

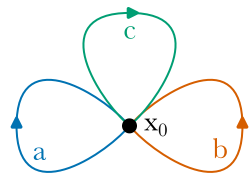
by cancelling terms. The images of those terms under $H$ also cancel, so

$$H(gg') = h(s_1)^{\epsilon_1} \cdots h(s_n)^{\epsilon_n} h(t_1)^{e_1} \cdots h(t_m)^{e_m} = H(g)H(g').$$

It follows that $H$ is a well-defined homomorphism. Since $S$ generates $\Gamma$, it is the only possible homomorphism extending $h$. $\square$

## 7.8. Constructing free groups using graphs

We now prove Theorem 7.7.1. In fact, we prove something better. Recall our convention from §2.5 that all graphs are oriented. Let $S$ be a set. Denote by $X_S$ the graph with one vertex $x_0$ and with $|S|$ oriented edges, each labeled with an element of $S$. For instance, if $S = \{a, b, c\}$ then $X_S$ is




Each $s \in S$ corresponds to a loop in $X_S$ based at $x_0$, and we will identify $s$ with the corresponding element of $\pi_1(X_S, x_0)$. The elements of $S$ correspond to distinct elements of $\pi_1(X_S, x_0)$. Indeed, for $s \in S$ let $r_s\colon (X_S, x_0) \to (\mathbb{S}^1, 1)$ be the retraction onto the loop labeled $s$. The induced map

$$(r_s)_*\colon \pi_1(X_S, x_0) \to \pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z}$$

takes $s$ to $1 \in \mathbb{Z}$ and each $s' \in S \setminus \{s\}$ to $0 \in \mathbb{Z}$, so $s \neq s'$ for all $s' \in S \setminus \{s\}$. We can therefore identify $S$ with a subset of $\pi_1(X_S, x_0)$. We then have:

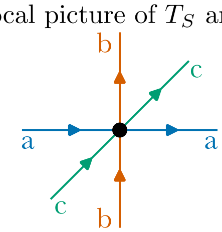
Theorem 7.8.1. *Let $S$ be a set. The following hold:*

*(i) the group $\pi_1(X_S, x_0)$ is a free group on $S$; and*
*(ii) every element of $\pi_1(X_S, x_0)$ can be represented by a unique reduced word in $S$.*

Proof. By Lemma 7.7.2, it is enough to prove (ii). Define $T_S$ to be an infinite tree each of whose vertices has valence $2|S|$. Label the oriented edges of $T_S$ by elements of $S$ such that for each vertex $v$ of $T_S$ there are:

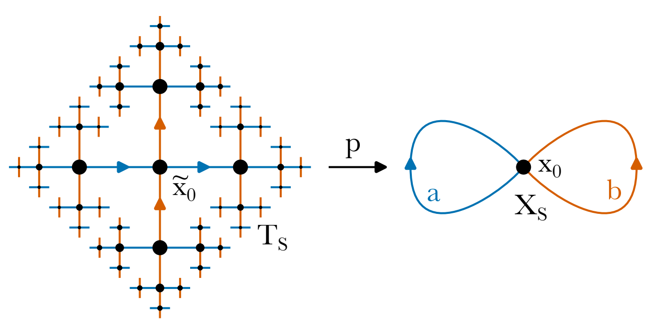
- $|S|$ edges coming out of $v$ labeled by elements of $S$; and
- $|S|$ edges going into $v$ labeled by elements of $S$.

For instance, if $S = \{a, b, c\}$ then the local picture of $T_S$ around $v$ looks like




Fix a vertex $\widetilde{x}_0$ of $T_S$.

There is a covering space $p\colon T_S \to X_S$ taking each vertex of $T_S$ to $x_0$ and each oriented edge of $T_S$ labeled by $s \in S$ to the corresponding loop in $X_S$ labeled by $s$. For instance, if $S = \{a, b\}$ this is the cover




Just like in the $S = \{a, b\}$ case, the cover $p\colon T_S \to X_S$ is regular.[2] Since $T_S$ is a tree, it is contractible and hence 1-connected (see Lemma 6.4.1).

Letting $G$ be the deck group of $p\colon T_S \to X$, we can apply Theorem 7.1.1 to see that

$$\pi_1(X_S, x_0) \cong G. \tag{7.8.1}$$

The group $G$ acts freely and transitively on the vertices of $T$, so each vertex is of the form $g\widetilde{x}_0$ for some unique $g \in G$. The isomorphism (7.8.1) is given by the lifting map, so the element of $\pi_1(X, x_0)$ corresponding to $g \in G$ is the homotopy class of the loop in $X$ based at $x_0$ obtained by taking a path in $T$ from $\widetilde{x}_0$ to $g\widetilde{x}_0$ and projecting it to $X$.

[2] The point here is that $T$ is a tree each of whose vertices has the same local picture, so there are edge-label preserving graph automorphisms of $T$ taking any vertex to any other vertex. These are deck transformations.

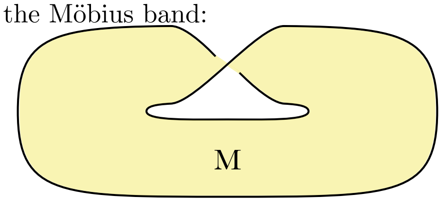
For each vertex $\widetilde{x}_1$ of $T$, there is a unique sequence of edges connecting $\widetilde{x}_0$ to $\widetilde{x}_1$ that does not backtrack, that is, traverse an edge in one direction and then go backwards along the same edge. This non-backtracking condition is exactly what is needed to ensure that this edge-path corresponds to an element of $\pi_1(X_S, x_0)$ represented by a reduced word in $S$. In this way, we see that every element of $\pi_1(X_S, x_0)$ is represented by a unique reduced word in $S$, as desired. $\square$

## 7.9. Exercises

Exercise 7.1. For $d \geq 2$, construct a pointed space $(X, x_0)$ with $\pi_1(X, x_0) \cong C_d$. Hint: generalize the construction of $\mathbb{RP}^n$ as $\mathbb{S}^n/C_2$. It might be helpful to view $\mathbb{S}^{2n-1}$ as a subspace of $\mathbb{C}^n \cong \mathbb{R}^{2n}$. $\square$

Exercise 7.2. Let $A$ be a finitely generated abelian group. Construct a pointed space $(X, x_0)$ with $\pi_1(X, x_0) \cong A$. Hint: the previous exercise might be helpful here. $\square$

Exercise 7.3. Let $M$ be the Möbius band:



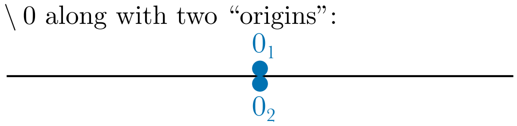

Fixing a basepoint $x_0 \in M$, prove that $\pi_1(M, x_0) \cong \mathbb{Z}$ in two ways:

(a) By identifying $M$ as $Z/\mathbb{Z}$ for an explicit 1-connected space $Z$ equipped with a covering space action of $\mathbb{Z}$.

(b) By showing that $M$ deformation retracts to a subspace $A \subset M$ with $A \cong \mathbb{S}^1$. $\square$

Exercise 7.4. As in Exercise 7.3, let $M$ be the Möbius band. Let $\partial M$ be its boundary, so $\partial M \cong \mathbb{S}^1$. Prove that there does not exist a retraction $r\colon M \to \partial M$. Hint: fixing a basepoint $x_0 \in \partial M$, first calculate the image of $\pi_1(\partial M, x_0) \cong \mathbb{Z}$ in $\pi_1(M, x_0) \cong \mathbb{Z}$. $\square$

Exercise 7.5. Let $X$ be the "line with two origins", i.e., the quotient of $\mathbb{R} \sqcup \mathbb{R}$ that for $x \in \mathbb{R}$ nonzero identifies the points $x$ in the two copies of $\mathbb{R}$ to a single point. This is a non-Hausdorff space composed of an open set $\mathbb{R} \setminus 0$ along with two "origins":




Letting $x_0 \in X$, prove that $\pi_1(X, x_0) \cong \mathbb{Z}$. Hint: Exercise 6.7 might be useful for proving that the cover you produce is 1-connected. $\square$

Exercise 7.6. Prove the 1-dimensional Brouwer fixed point theorem: every continuous map $f\colon \mathbb{D}^1 \to \mathbb{D}^1$ has a fixed point. $\square$

Exercise 7.7. Say that a space $X$ has the *fixed point property* if every continuous map $f\colon X \to X$ has a fixed point.

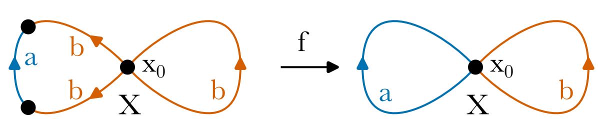
(a) Prove that if $X$ has the fixed point property and $Y \subset X$ is a retract of $X$, then $Y$ has the fixed point property.

(b) Prove that every finite tree $T$ has the fixed point property. Hint: embed $T$ into $\mathbb{D}^2$ such that $\mathbb{D}^2$ retracts to $T$, and apply the Brouwer fixed point theorem.

(c) A challenging problem: give a direct proof of part (b) that does not use the Brouwer fixed point theorem. $\square$

Exercise 7.8. Prove that $\mathbb{RP}^1 \cong \mathbb{S}^1$, and thus that for all $x_0 \in \mathbb{RP}^1$ we have $\pi_1(\mathbb{RP}^1, x_0) \cong \mathbb{Z}$. $\square$

Exercise 7.9. Let $X$ be a graph with one vertex $x_0$ and two oriented edges labeled $a$ and $b$, so $\pi_1(X, x_0)$ is the free group $F(a, b)$ on $a$ and $b$. Define a map $f\colon (X, x_0) \to (X, x_0)$ as in the following figure:



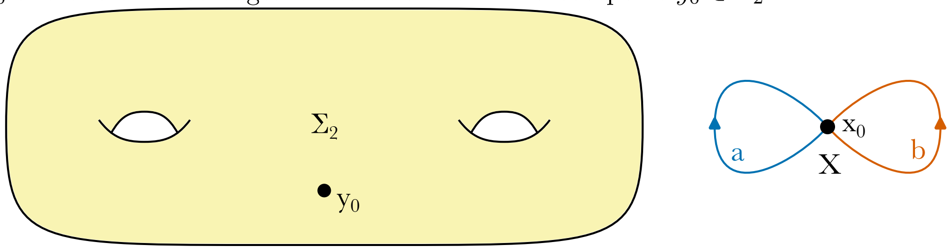
Here the three vertices on the left hand side all map to the single vertex $x_0$ on the right hand side, and the edges connecting those three vertices map to the edges on the right hand side as indicated. Do the following:

(a) Calculate the induced map $f_*\colon \pi_1(X, x_0) \to \pi_1(X, x_0)$ by determining the images of the two generators $a, b \in F(a, b) = \pi_1(X, x_0)$ in $\pi_1(X, x_0) = F(a, b)$.

(b) Prove that the map $f_*$ you computed in the previous part is not the identity, and conclude that $f\colon (X, x_0) \to (X, x_0)$ is not homotopic to the identity as a map of pointed spaces.

(c) Prove $f\colon X \to X$ is homotopic to the identity as a map of unpointed spaces. □

Exercise 7.10. Let $\Sigma_2$ be a closed oriented genus 2 surface and let $X$ be a graph with one vertex $x_0$ and two oriented edges labeled $a$ and $b$. Fix a basepoint $y_0 \in \Sigma_2$:




We do not yet know how to calculate $\pi_1(\Sigma_2, y_0)$, but we can still say various things about it using the calculations we have done. Construct a map $f\colon (\Sigma_2, y_0) \to (X, x_0)$ such that the induced map $f_*\colon \pi_1(\Sigma_2, y_0) \to \pi_1(X, x_0)$ is surjective. Hint: To prove it is surjective, draw explicit elements of $\pi_1(\Sigma_2, y_0)$ mapping to $a$ and to $b$. □

Exercise 7.11. Let $G$ be a group. The goal of this exercise is to construct a pointed space $(X, x_0)$ with $\pi_1(X, x_0) \cong G$. This exercise uses the notion of the join of spaces from Exercise 6.8. Do the following:

(a) Let $A$ and $B$ be spaces equipped with left $G$-actions. Prove that there is an induced action of $G$ on the join $A * B$.

(b) Let $X = G$ and $Y = G$, both viewed as discrete spaces. Let $J$ be the join $J = X * Y$. Prove that $J$ is 0-connnected.

(c) Let $Z = G$, again viewed as a discrete space. Prove that $K = Z * J$ is 1-connected. Hint: Exercise 6.8 will be useful.

(d) The group $G$ acts on itself on the left. By part (a), we get induced actions of $G$ on $J$ and $K$. Prove that the action of $G$ on $K$ is a covering space action. Letting $X = G/K$ and fixing a basepoint $x_0 \in X$, deduce that $\pi_1(X, x_0) \cong G$. □

Exercise 7.12. Let $F(a_1, \ldots, a_n)$ be the free group on elements $a_1, \ldots, a_n$. Recall that the *abelianization* of a group $G$ is the abelian group $G/[G, G]$, where $[G, G]$ is the normal subgroup generated by elements of the form $[g, h] = ghg^{-1}h^{-1}$ with $g, h \in G$. Prove that the abelianization of $F(a_1, \ldots, a_n)$ is isomorphic to $\mathbb{Z}^n$. Hint: start by constructing a map $F(a_1, \ldots, a_n) \to \mathbb{Z}^n$ using the universal property of the free group. □

Exercise 7.13. Let $F(S)$ be the free group on a set $S$. Prove that $F(S)$ is torsion-free. □

Exercise 7.14. Let $F(S)$ be the free group on a set $S$ and let $x, y \in F(S)$. Prove that $xy = yx$ if and only if there exists some $z \in F(S)$ and $n, m \in \mathbb{Z}$ with $x = z^n$ and $y = z^m$. □

Exercise 7.15. In this exercise, we outline one of the classical constructions of a free group on a set $S$.

(a) Let $M$ be the set of all words $w = s_1^{\epsilon_1} \cdots s_n^{\epsilon_n}$, including the empty word. Prove that $M$ is an associative monoid under concatanation. Here an associative monoid is a set equipped with an associative multiplication with an identity, but unlike in a group there need not be inverses.

(b) Let $\sim$ be the equivalence relation on $M$ where $w \sim w'$ if $w'$ can be obtained from $w$ by a sequence of the following two moves:
- For some $s \in S$, insert either $ss^{-1}$ or $s^{-1}s$ somewhere in $w$.

- For some $s \in S$, delete a subword of the form $ss^{-1}$ or $s^{-1}s$ from $w$.

Set $F = M/\sim$. Prove that the multiplication on $M$ descends to an associative multiplication on $M$, and that $F$ is a group.

(c) Prove that $F$ along with the evident map $\iota\colon S \to F$ satisfies the universal property of a free group. Hint: first prove that $M$ satisfies an appropriate universal property to be a "free associative monoid" on $S$. $\square$

EXERCISE 7.16. Let $S$ be a set and let $F(S)$ be a free group on $S$. This exercise outlines the classical algebraic proof that every element of $F(S)$ is represented by a unique reduced word in $S$.

(a) Let $\mathcal{W}$ be the set of reduced words in $S$. Use the universal property of the free group to construct a left action of $F(S)$ on $\mathcal{W}$ that for $s \in S$ multiplies a word $w \in \mathcal{W}$ by $s$ to get $sw$ and then reduces appropriately to get a reduced word. Hint: an action is the same as a homomorphism $F(S) \to \mathrm{Sym}(\mathcal{W})$ where $\mathrm{Sym}(\mathcal{W})$ is the symmetric group consisting of all bijections of $\mathcal{W}$. You can construct such homomorphisms using the universal property of $F(S)$.

(b) Prove that every element of $F(S)$ is represented by a unique reduced word. Hint: consider $w \in F(S)$, and think about where $w$ takes the trivial word $1 \in \mathcal{W}$. $\square$

EXERCISE 7.17. For a group $G$ generated by a set $S$, we have the following two classic decision problems:

- The *word problem*: decide if two words in $S$ represent the same element of $G$.
- The *conjugacy problem*: decide if two words in $S$ represent conjugate elements of $G$, that is, elements $g_1, g_2 \in G$ such that there is some $h \in G$ with $g_2 = hg_1h^{-1}$.

For a free group $F(S)$ on a set $S$, the word problem is easy since two words in $S$ represent the same element of $G$ if and only if they become the same reduced word after repeatedly cancelling terms of the form $ss^{-1}$ and $s^{-1}s$ with $s \in S$. In this exercise, you will solve the conjugacy problem in $F(S)$.

(a) Consider a word $w = s_1^{\epsilon_1} \cdots s_n^{\epsilon_n}$ in $S$. For some $k \geq 1$, let

$$w' = s_{1+k}^{\epsilon_{1+k}} \cdots s_{n+k}^{\epsilon_{n+k}},$$

where the subscripts are interpreted modulo $n$. Prove that $w$ and $w'$ are conjugate elements of $F(S)$. We say that they are *cyclically conjugate*.

(b) Say that a word $w = s_1^{\epsilon_1} \cdots s_n^{\epsilon_n}$ is *cyclically reduced* if $w$ is reduced and $s_1^{\epsilon_1} \neq s_n^{-\epsilon_n}$. Prove that every element of $F(S)$ is conjugate to a cyclically reduced word.

(c) Prove that if $w$ and $w'$ are cyclically reduced words, then $w$ and $w$ are conjugate elements of $F(S)$ if and only if they are cyclically conjugate. Since we can check this by just enumerating all the cyclic conjugates of $w$, this gives an algorithm for solving the conjugacy problem. $\square$

CHAPTER 8

# Homotopy equivalences

In this chapter, we discuss a weakening of a homeomorphism called a homotopy equivalences that plays an important role in algebraic topology.

## 8.1. Pointed homotopy equivalences

A map $f\colon (X, x_0) \to (Y, y_0)$ between pointed spaces is a *homotopy equivalence* if there exists a map $g\colon (Y, y_0) \to (X, x_0)$ such that $g \circ f\colon (X, x_0) \to (X, x_0)$ and $f \circ g\colon (Y, y_0) \to (Y, y_0)$ are both homotopic to the identity. We call $g$ a *homotopy inverse* to $f$, and if there is a homotopy equivalence between $(X, x_0)$ and $(Y, y_0)$ then we will say that $(X, x_0)$ is *homotopy equivalent* to $(Y, y_0)$ and write $(X, x_0) \simeq (Y, y_0)$. Here is an example:

Example 8.1.1. Let $X$ be a space, let $A \subset X$ be a subspace, and let $r_t\colon X \to X$ be a deformation retract to $A$. We can therefore regard $r_1$ as a retraction $r_1\colon X \to A$. Pick a basepoint $a_0 \in A$, and let $\iota\colon (A, a_0) \to (X, a_0)$ be the inclusion. Then $\iota$ is a homotopy equivalence with homotopy inverse $r_1\colon (X, a_0) \to (A, a_0)$. Indeed, $r_1 \circ \iota\colon (A, a_0) \to (A, a_0)$ is literally the identity, and $r_t$ is a homotopy from the identity $r_0 = \mathbb{1}_X\colon (X, x_0) \to (X, x_0)$ to $r_1 = \iota \circ r_1$. □

It will become more and more clear as we delve deeper into algebraic topology that homotopy equivalent pointed spaces are in many ways the "same" from the perspective of the tools of the subject. Here is one easy way in which this is true, which generalizes the corresponding fact for deformation retracts (Lemma 6.2.5):

Lemma 8.1.2. *Let $f\colon (X, x_0) \to (Y, y_0)$ be a homotopy equivalence between pointed spaces. Then $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.*

Proof. Let $g\colon (Y, y_0) \to (X, x_0)$ be a homotopy inverse to $f$. Since $g \circ f\colon (X, x_0) \to (X, x_0)$ and $f \circ g\colon (Y, y_0) \to (Y, y_0)$ are homotopic to the identity, the induced maps $(g \circ f)_*\colon \pi_1(X, x_0) \to \pi_1(X, x_0)$ and $(f \circ g)_*\colon \pi_1(Y, y_0) \to \pi_1(Y, y_0)$ are the identity. Functorality implies that

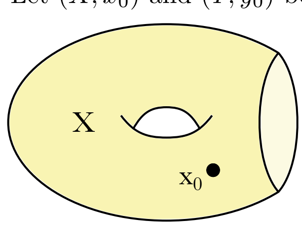
$$\begin{aligned} \mathbb{1}_{\pi_1(X, x_0)} &= (g \circ f)_* = g_* \circ f_* \quad \text{and} \\ \mathbb{1}_{\pi_1(Y, y_0)} &= (f \circ g)_* = f_* \circ g_*, \end{aligned}$$

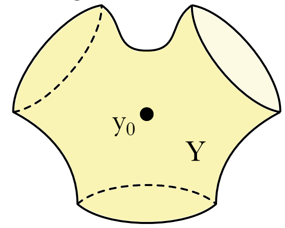
so $f_*$ and $g_*$ are inverses to each other. This implies that that $f_*$ and $g_*$ are isomorphisms. □

## 8.2. Composing deformation retractions

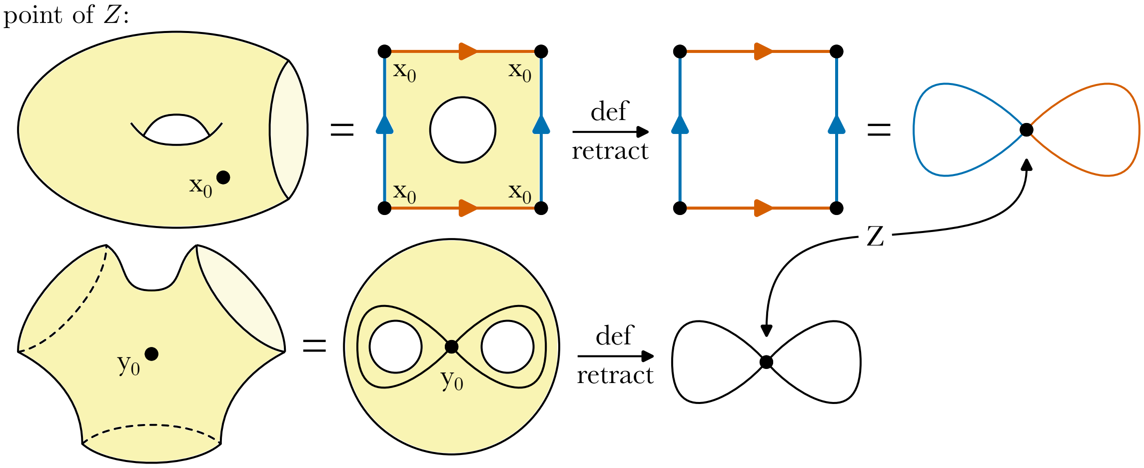
As we already noted, if $X$ deformation retracts to a subspace $Y$ and $y_0 \in Y$, then $(Y, y_0) \simeq (X, y_0)$. Being homotopy equivalent is an equivalence relation (see Exercise 8.1), so by applying this multiple times we can get interesting homotopy equivalences. For instance, if $Y$ is a subspace of both $X$ and $Z$ and both $X$ and $Z$ deformation retract to $Y$, then $(X, y_0) \simeq (Z, y_0)$ even though neither $X$ or $Z$ is contained in the other. Here is an example:

Example 8.2.1. Let $(X, x_0)$ and $(Y, y_0)$ be the following surfaces with boundary:



Both $X$ and $Y$ deformation retract to the same space $Z$, with $x_0$ and $y_0$ corresponding to the same point of $Z$:




It follows that $(X, x_0) \simeq (Y, y_0)$. Moreover, since the fundamental group of $Z$ is a free group on two generators (Theorem 7.8.1), it follows that $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$ are free groups on two generators. $\square$

## 8.3. Unpointed homotopy equivalences

We now explain how this works without basepoints. A map $f\colon X \to Y$ between spaces is a *homotopy equivalence* if there exists a map $g\colon Y \to X$ such that $g \circ f\colon X \to X$ and $f \circ g\colon Y \to Y$ are homotopic to the identity. The difference between this and the pointed case is that these homotopies need not fix a basepoint. We call $g$ a *homotopy inverse* to $f$, and if a homotopy equivalence from $X$ to $Y$ exists we say that $X$ and $Y$ are *homotopy equivalent* and write $X \simeq Y$.

In Lemma 6.3.1, we proved that even though contractions need not fix a basepoint, it is still true that contractible spaces have trivial fundamental groups. By being similarly careful with the basepoint, we prove the following:

LEMMA 8.3.1. *Let $f\colon X \to Y$ be a homotopy equivalence and let $x_0 \in X$. Set $y_0 = f(x_0)$. Then $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.*

REMARK 8.3.2. Since $\pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z}$ but $\mathbb{S}^n$ is 1-connected for $n \geq 2$, it follows from this lemma that $\mathbb{S}^1$ and $\mathbb{S}^n$ are not homotopy equivalent for $n \geq 2$. Similarly, since $\pi_1(\mathbb{RP}^n, 1) \cong C_2$ for $n \geq 2$ it follows that $\mathbb{RP}^n$ and $\mathbb{S}^n$ are not homotopy equivalent for $n \geq 2$. We remark that $\mathbb{RP}^1$ and $\mathbb{S}^1$ are homeomorphic (see Exercise 7.8). $\square$

PROOF OF LEMMA 8.3.1. Let $g\colon Y \to X$ be a homotopy inverse to $f$. Set $x_1 = g(y_0)$ and $y_1 = f(x_1)$. The naive thing to do would be to prove that the maps $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $g_*\colon \pi_1(Y, y_0) \to \pi_1(X, x_1)$ were inverses to each other. However, this does not make sense since the domain $\pi_1(X, x_0)$ of $f_*$ is not the same as the codomain $\pi_1(X, x_1)$ of $g_*$.

Instead, what we will prove is that

$$(g \circ f)_*\colon \pi_1(X, x_0) \to \pi_1(X, x_1) \quad \text{and} \quad (f \circ g)_*\colon \pi_1(Y, y_0) \to \pi_1(Y, y_1) \tag{8.3.1}$$

are both isomorphisms. This will imply that $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $g_*\colon \pi_1(Y, y_0) \to \pi_1(X, x_1)$ are both injections. We claim that this implies that $f_*$ is a surjection and hence an isomorphism. Indeed, consider $\zeta \in \pi_1(Y, y_0)$. We want to find some $\eta \in \pi_1(X, x_0)$ with $f_*(\eta) = \zeta$. Since $g_* \circ f_*$ is an isomorphism by (8.3.1), there exists some $\eta \in \pi_1(X, x_0)$ such that $g_*(f_*(\eta)) = g_*(\zeta)$. Since $g_*$ is an injection, we must have $f_*(\eta) = \zeta$, as desired.

It remains to prove that the two maps in (8.3.1) are isomorphisms. The proofs are the same, so we will give the details for $(g \circ f)_*\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$. Since $g$ is a homotopy inverse to $f$, the map $g \circ f\colon X \to X$ is homotopic to the identity. Let $h_t\colon X \to X$ be a homotopy from $g \circ f$ to $\mathbb{1}_X$. Let $\delta\colon I \to X$ be the following path from $x_1 = (g \circ f)(x_0) = h_0(x_0)$ to $x_0 = h_1(x_0)$:

$$\delta(s) = h_s(x_0) \quad \text{for } s \in I.$$

For $[\gamma] \in \pi_1(X, x_0)$, Lemma 5.7.1 implies that

$$(g \circ f)_*([\gamma]) = (h_0)_*([\gamma]) = [\delta](h_1)_*([\gamma])[\overline{\delta}] = [\delta][\gamma][\overline{\delta}].$$

In other words, $(g \circ f)_*$ equals the change of basepoint isomorphism from $\pi_1(X, x_0)$ to $\pi_1(X, x_1)$ given by the path $\delta$. In particular, $(g \circ f)_*$ is an isomorphism. □

## 8.4. Mapping cylinder neighborhoods

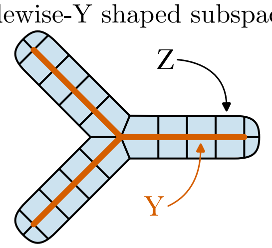
Let $X$ be a space and $Y$ be a contractible subspace of $X$. Consider the quotient map $q\colon X \to X/Y$. It turns out that in many cases $q$ is a homotopy equivalence. Roughly speaking, this holds as long as $Y$ is embedded into $X$ with reasonable local properties. There are a number of conditions that ensure this. Our next goal is to give one that is fairly easy to state and prove. This requires some preliminary definitions.

Recall from Example 6.2.2 that for a map $f\colon Z \to Y$ between spaces, the mapping cylinder of $f$ is the space $\mathrm{Cyl}(f)$ obtained by quotienting the disjoint union $(Z \times I) \sqcup Y$ to identify $(z, 1) \in Z \times I$ with $f(z) \in Y$ for all $z \in Z$. For $z \in Z$ and $s \in I$, let $\overline{(z,s)}$ be the image of $(z, s) \in Z \times I$ in $\mathrm{Cyl}(f)$. We now define:

Definition 8.4.1. Let $X$ be a space and let $Y \subset X$ be a subspace. A *mapping cylinder neighborhood* of $Y$ is a closed subset $N$ of $X$ containing $Y$ along with a closed subset $Z \subset N$ such that:

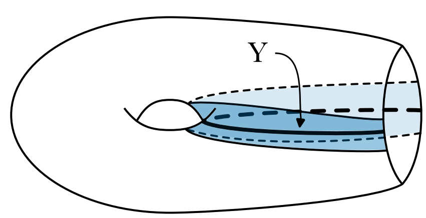
- $N \setminus Z$ is an open neighborhood of $Y$ in $X$; and
- there exists a map $f\colon Z \to Y$ and a homeomorphism $\phi\colon \mathrm{Cyl}(f) \to N$ such that $f(\overline{(z,0)}) = z$ and $f(y) = y$ for all $z \in Z$ and $y \in Y$. □

Example 8.4.2. Let $Y$ be the sidewise-Y shaped subspace of $\mathbb{R}^2$ shown here:



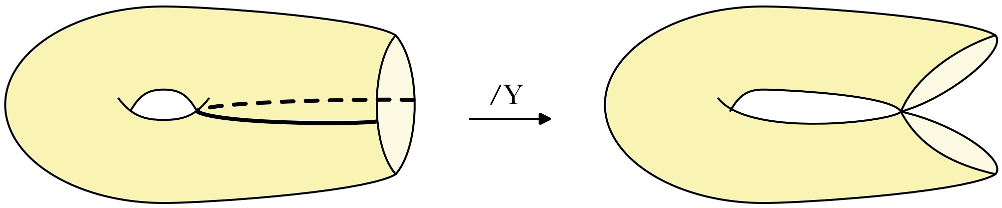

The subspace $Y$ of $\mathbb{R}^2$ has a mapping cylinder neighborhood $N$ indicated in blue. This blue subspace is the mapping cylinder of a map $f\colon Z \to Y$ with $Z \cong \mathbb{S}^1$ the indicated subspace. The lines connect points $z \in Z$ with their images $f(z) \in Y$. Since $Y$ is contractible and has a mapping cylinder neighborhood, it will follow from Theorem 8.5.1 below that the quotient map $q\colon \mathbb{R}^2 \to \mathbb{R}^2/Y$ is a homotopy equivalence.[1] □

Example 8.4.3. Let $X$ be the following surface with boundary and let $Y \cong I$ be the indicated arc in $X$:




A mapping cylinder neighborhood $N$ of $Y$ is drawn in blue. Here $N \cong Y \times I$, and $N$ is homeomorphic to the mapping cylinder of the projection $f\colon Y \sqcup Y \to Y$. Since $Y$ is contractible and has a mapping cylinder neighborhood, it will follow from Theorem 8.5.1 below that the quotient map $q\colon X \to X/Y$ is a homotopy equivalence. Here is a picture of this map:

[1] In fact, it is not hard to see that $\mathbb{R}^2/Y \cong \mathbb{R}^2$. However, letting $\phi\colon \mathbb{R}^2/Y \to \mathbb{R}^2$ be a homeomorphism, the composition $\phi \circ q\colon \mathbb{R}^2 \to \mathbb{R}^2$ is not a homeomorphism, so knowing that $\mathbb{R}^2/Y \cong \mathbb{R}^2$ is not directly useful for seeing that $q$ is a homotopy equivalence.

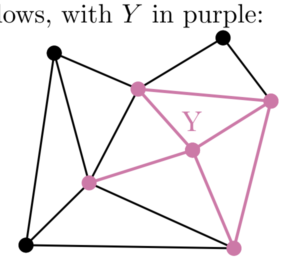



In this case, $X$ and $X/Y$ are not homeomorphic; indeed, $X/Y$ is not a manifold around the point that is the image of $Y$. We showed in Example 8.2.1 that the fundamental group of $X$ is the free group on two generators, so the same is true for $X/Y$. $\square$

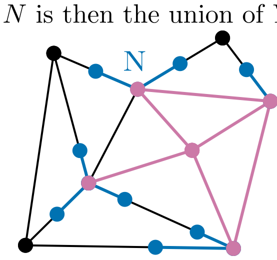
Our final example will be important later, so we separate it out as a lemma:

Lemma 8.4.4. *Let $X$ be a graph and let $Y \subset X$ be a subgraph. Then $Y$ has a mapping cylinder neighborhood.*

Proof. Once an example is understood this lemma will be clear, so we give one in lieu of a formal proof. Let $X$ and $Y$ be as follows, with $Y$ in purple:




The mapping cylinder neighborhood $N$ is then the union of $Y$ with the blue region here:



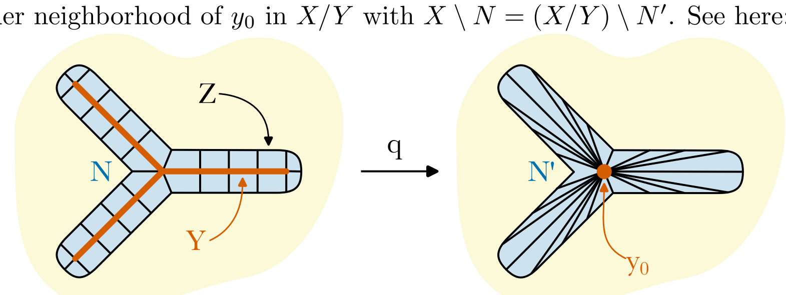

Here $Z$ consists of eight points, and $N$ is the mapping cylinder of a map $f\colon Z \to Y$ taking each of those eight points to a vertex of $Y$. $\square$

## 8.5. Collapsing subspaces with mapping cylinder neighborhoods

We now prove:

Theorem 8.5.1. *Let $X$ be a space and let $Y \subset X$ be a contractible subspace with a mapping cylinder neighborhood. Then the quotient map $q\colon X \to X/Y$ is a homotopy equivalence.*

Proof. We must use the hypotheses to construct a homotopy inverse $g\colon X/Y \to X$ to $q$. Let $N$ be a mapping cylinder neighborhood of $Y$. Identify $N$ with $\mathrm{Cyl}(f)$ for some $Z \subset N$ and some map $f\colon Z \to Y$. Let $y_0$ be the point of $X/Y$ corresponding to $Y$, let $f'\colon Z \to y_0$ be the projection, and let $N' = \mathrm{Cyl}(f')$. We can identify $N'$ with $N/Y$, and after making this identification $N'$ is a mapping cylinder neighborhood of $y_0$ in $X/Y$ with $X \setminus N = (X/Y) \setminus N'$. See here:




To construct a continuous map $X/Y \to X$, it is enough to construct a continuous map $N' \to N$ that is the identity on $Z$ and then extend $N' \to N$ to $X/Y$ by the identity.[2] In a similar way, we can construct continuous maps $X \to X$ (resp. $X/Y \to X/Y$) by constructing continuous maps $N \to N$ (resp. $N' \to N$) and extending by the identity.

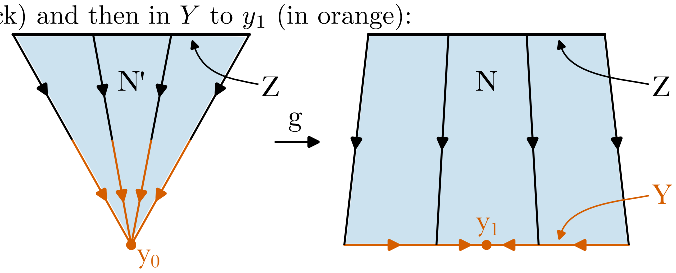
For $z \in Z$ and $s \in I$, let $\overline{(z,s)} \in N$ and $\overline{(z,s)}' \in N'$ be the corresponding points. We now divide the proof into three steps:

Step 1. *We construct the purported homotopy inverse $g\colon X/Y \to X$.*

Since $Y$ is contractible, there is a homotopy $h_t\colon Y \to Y$ from $\mathbb{1}_Y$ to a constant map. Let $y_1 \in Y$ be the constant value of $h_1$. Define $g\colon X/Y \to X$ via the formulas

$$\begin{cases} g(\overline{(z,s)}') = \overline{(z,2s)} & \text{for } z \in Z \text{ and } s \in [0,1/2], \\ g(\overline{(z,s)}') = h_{2s-1}(f(z)) & \text{for } z \in Z \text{ and } s \in [1/2,1], \\ g(y_0) = y_1 & \\ g(x) = x & \text{for } x \in (X/Y) \setminus N' = X \setminus N. \end{cases}$$

See here, where the indicated map takes each line segment from $z \in Z$ to $y_0$ to the path that first goes to $f(z)$ (in black) and then in $Y$ to $y_1$ (in orange):




By what we said above, this map $g\colon X/Y \to X$ is continuous.

Step 2. *We prove that the composition $g \circ q\colon X \to X$ is homotopic to the identity.*

The map $g \circ q\colon X \to X$ is given by the formulas

$$\begin{cases} g \circ q(\overline{(z,s)}) = \overline{(z,2s)} & \text{for } z \in Z \text{ and } s \in [0,1/2], \\ g \circ q(\overline{(z,s)}) = h_{2s-1}(f(z)) & \text{for } z \in Z \text{ and } s \in [1/2,1], \\ g \circ q(y) = y_1 & \text{for } y \in Y, \\ g \circ q(x) = x & \text{for } x \in X \setminus N. \end{cases}$$

This is homotopic to the identity via the homotopy $\phi_t\colon X \to X$ given by the formulas

$$\begin{cases} \phi_t(\overline{(z,s)}) = \overline{(z,(2-t)s)} & \text{for } z \in Z \text{ and } s \in [0,1/(2-t)], \\ \phi_t(\overline{(z,s)}) = h_{(2-t)s-1}(f(z)) & \text{for } z \in Z \text{ and } s \in [1/(2-t),1], \\ \phi_t(y) = h_{1-t}(y) & \text{for } y \in Y, \\ \phi_t(x) = x & \text{for } x \in X \setminus N. \end{cases}$$

By the discussion at the beginning of the proof this is continuous.

Step 3. *We prove that the composition $q \circ g\colon X/Y \to X/Y$ is homotopic to the identity.*

The map $q \circ g\colon X/Y \to X/Y$ is given by the formulas

$$\begin{cases} q \circ g(\overline{(z,s)}') = \overline{(z,2s)}' & \text{for } z \in Z \text{ and } s \in [0,1/2], \\ q \circ g(\overline{(z,s)}') = y_0 & \text{for } z \in Z \text{ and } s \in [1/2,1], \\ q \circ g(y_0) = y_0 & \\ g \circ q(x) = x & \text{for } x \in (X/Y) \setminus N'. \end{cases}$$

---

[2]To see that this is continuous, note that the map $X/Y \to X$ is continuous on the closed sets $N'$ and $(X/Y)\setminus(N'\setminus Z)$; indeed, on the latter set it is the identity. These cover $X/Y$. Now apply the fact that if $\psi\colon A \to B$ is a map of sets between spaces and $\{C_1,\ldots,C_n\}$ is a cover of $A$ by closed sets such that each $\psi|_{C_i}$ is continuous, then $\psi$ is continuous. Note that this would be false if our cover had infinitely many closed sets in it.

This is homotopic to the identity via the homotopy $\psi_t\colon X/Y \to X/Y$ given by the formulas

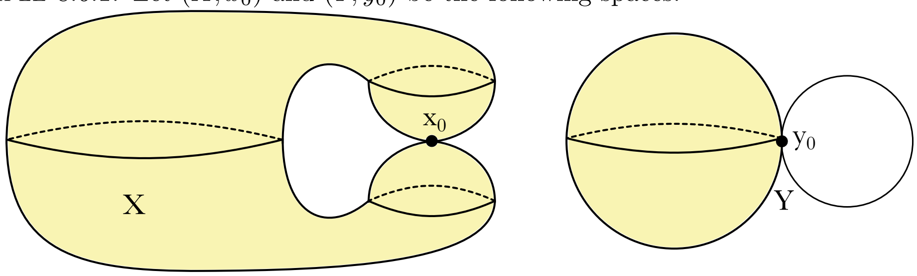
$$\begin{cases} \psi_t(\overline{(z,s)}') = \overline{(z,(2-t)s)}' & \text{for } z \in Z \text{ and } s \in [0, 1/(2-t)], \\ \psi_t(\overline{(z,s)}') = y_0 & \text{for } z \in Z \text{ and } s \in [1/(2-t), 1], \\ \psi_t(y_0) = y_0 & \\ \psi_t(x) = x & \text{for } x \in X \setminus N. \end{cases}$$

By the discussion at the beginning of the proof this is continuous. □

## 8.6. Example of collapsing

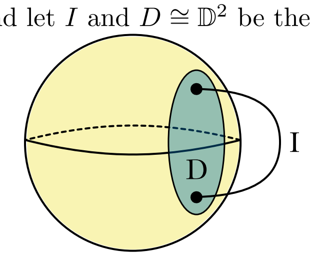
We now give an example of how to this to analyze an interesting example.

EXAMPLE 8.6.1. Let $(X, x_0)$ and $(Y, y_0)$ be the following spaces:




The space $X$ is obtained by quotienting $\mathbb{S}^2$ to identify two points together,[3] and the space $Y$ is obtained by gluing $\mathbb{S}^2$ and $\mathbb{S}^1$ together at a single point. We will prove that $X \simeq Y$ and that $\pi_1(X, x_0) \cong \pi_1(Y, y_0) \cong \mathbb{Z}$.

Let $Z$ be the following space and let $I$ and $D \cong \mathbb{D}^2$ be the indicated subspaces of $Z$:




We have $Z/I \cong X$ and $Z/D \cong Y$. Since $I$ is contractible and has a mapping cylinder neighborhood in $Z$, it follows that $Z \simeq X$. Similarly, since $D$ is contractible and has a mapping cylinder neighborhood in $Z$, it follows that $Z \simeq Y$. We conclude that $X \simeq Y$.

In particular, $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$. It remains to prove that $\pi_1(Y, y_0) \cong \mathbb{Z}$. The space $Y$ is the union of $\mathbb{S}^2$ and $\mathbb{S}^1$. Identifying $\mathbb{S}^2$ and $\mathbb{S}^1$ with their images in $Y$, we have $\mathbb{S}^2 \cap \mathbb{S}^1 = y_0$. There is a retraction $Y \to \mathbb{S}^1$ that collapses $\mathbb{S}^2$ to $y_0$. It follows that the inclusion $(\mathbb{S}^1, y_0) \to (Y, y_0)$ induces an injection from $\pi_1(\mathbb{S}^1, y_0) \cong \mathbb{Z}$ to $\pi_1(Y, y_0)$.

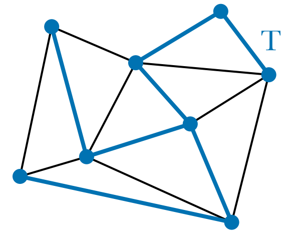
We must prove that this injection is surjective. Consider some $[\gamma] \in \pi_1(Y, y_0)$. Let $r \in \mathbb{S}^2$ be a point other than $y_0$. Just like in our proof that $\mathbb{S}^n$ has a trivial fundamental group (Lemma 4.5.2), we can use Lemma 4.5.3 (general position) to homotope $\gamma$ such that its image does not contain $r$. Since $\mathbb{S}^2 \setminus r \cong \mathbb{R}^2$ deformation retracts to $y_0$, it follows that $\gamma$ can be homotoped to a loop in $\mathbb{S}^1$, as desired. □

We close this section by introducing some terminology. For pointed spaces $\{(Z_i, z_i)\}_{i \in I}$, the *wedge product* of the $Z_i$ is the space $W = \vee_{i \in I}(Z_i, z_i)$ obtained from the disjoint union of the $Z_i$ by identifying all the basepoints $z_i$ together to a single point $w_0$. The point $w_0$ serves as a basepoint, so $(W, w_0)$ is a pointed space. We will often omit explicit mention of the basepoints $z_i$ and just write $W = \vee_{i \in I} Z_i$, and if $I$ is finite we will use the $\vee$ like a sum and e.g. write $Z_1 \vee Z_2$. For instance, the pointed space $(Y, y_0)$ in Example 8.6.1 is $\mathbb{S}^2 \vee \mathbb{S}^1$.

[3] It does not matter which two points are identified. Indeed, any two points "look the same" in the sense that they differ by a homeomorphism of $\mathbb{S}^2$. More generally, if $M^n$ is a connected $n$-manifold with $n \geq 2$ and $\{p_1, \ldots, p_k\}$ and $\{q_1, \ldots, q_k\}$ are two sets of $k$ distinct points on $M^n$, then there exists a homeomorphism $f\colon M^n \to M^n$ with $f(p_i) = q_i$ for all $1 \leq i \leq k$.

## 8.7. Maximal trees

Our final goal in this chapter is to use these tools to calculate the fundamental group of an arbitrary connected graph. This requires some preliminaries. Recall that a tree is a nonempty connected graph with no cycles. Each tree is contractible (Lemma 6.4.1). For a graph $X$, a *maximal tree* in $X$ is a subtree $T$ of $X$ that contains every vertex of $X$. For instance:




These always exist:

LEMMA 8.7.1. *Let $X$ be a nonempty connected graph. Then $X$ contains a maximal tree.*

PROOF. Inductively define subtrees

$$T_0 \subset T_1 \subset T_2 \subset \cdots$$

of $X$ in the following way. Start by choosing a vertex $v_0$ of $X$ and letting $T_0 = v_0$. Next, if $T_{n-1}$ has been constructed, let $T_n$ be the subtree obtained from $T_{n-1}$ as follows:

- For each vertex $v$ of $X$ that does not lie in $T_{n-1}$ but is connected by an edge to $T_{n-1}$, choose an edge connecting $T_{n-1}$ to $v$ and add it to $T_n$.

Now define

$$T = \bigcup_{n=0}^{\infty} T_n.$$

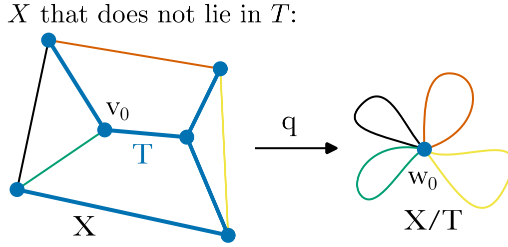
This is a subgraph of $G$. Since a cycle in $T$ only involves finitely many edges, a cycle of $T$ must be contained in some $T_n$. Since each $T_n$ is a tree, it follows that $T$ has no cycles, so $T$ is a tree. Since $X$ is connected, each vertex of $X$ must lie in $T$, so $T$ is a maximal tree. □

## 8.8. Fundamental groups of graphs

Using maximal trees, we will prove:

THEOREM 8.8.1. *Let $X$ be a connected graph and let $v_0$ be a vertex of $X$. Then $\pi_1(X, v_0)$ is a free group.*

PROOF. Let $T$ be a maximal tree in $X$. The quotient graph $X/T$ contains a single vertex $w_0$ and a loop for each edge of $X$ that does not lie in $T$:




Note that though our graphs are always oriented, this picture does not indicate the orientations of the edges. We saw in the proof of Theorem 7.7.1 that $\pi_1(X, w_0)$ is a free group. Since $T$ is a contractible subspace of $X$ with a mapping cylinder neighborhood (Lemma 8.4.4), the quotient map $q\colon X \to X/T$ is a homotopy equivalence. The induced map $q_*\colon \pi_1(X, v_0) \to \pi_1(W, w_0)$ is therefore an isomorphism, so $\pi_1(X, v_0)$ is also a free group. □

If $X$ is a finite connected graph with vertices $\mathcal{V}(X)$ and edges $\mathcal{E}(X)$, then we can give a formula for the rank of the free group $\pi_1(X, v_0)$ as follows. The *Euler characteristic* of $X$ is

$$\chi(X) = |\mathcal{V}(X)| - |\mathcal{E}(X)| \in \mathbb{Z}.$$

We then have:

THEOREM 8.8.2. *Let $X$ be a finite connected graph and let $v_0$ be a vertex of $X$. Then $\pi_1(X, v_0)$ is a free group of rank $n$, where $\chi(X) = 1 - n$.*

PROOF. Let $T$ be a maximal tree of $X$. The proof of Theorem 8.8.1 shows that $\pi_1(X, v_0)$ is a free group of rank $n$ where $n$ is the number of edges of $X/T$. Collapsing a single non-loop edge to a point does not change $\chi(X)$ since it causes the number of vertices and edges to both go down by 1. Iterating this, we see that $\chi(X) = \chi(X/T) = 1 - n$, as desired. □

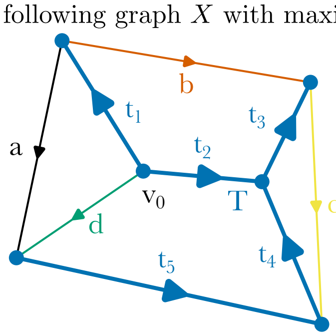
## 8.9. Free bases for fundamental groups of graphs

Let $X$ be a connected graph and let $v_0$ be a vertex of $X$. By analyzing the proof of Theorem 8.8.1, we can construct a free basis for the free group $\pi_1(X, v_0)$. Begin by choosing a maximal tree $T$ of $X$. Let $w_0$ be the single vertex of $X/T$. As in the proof of Theorem 8.8.1, the quotient map $q\colon X \to X/T$ induces an isomorphism $q_*\colon \pi_1(X, v_0) \to \pi_1(X/T, w_0)$.

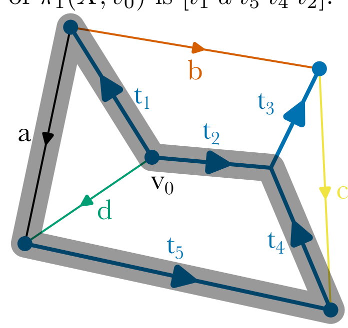
Let $\{e_i \mid i \in I\}$ be the edges of $X$ that do *not* lie in $T$. The map $q$ maps each $e_i$ to a loop $\overline{e}_i$ in $X/T$ that is based at $w_0$. Recall our convention that each edge of a graph is oriented (cf. §2.5). Using the orientation on $\overline{e}_i$, we get an element $[\overline{e}_i] \in \pi_1(X/T, w_0)$. By Theorem 7.8.1, the set $\{[\overline{e}_i] \mid i \in I\}$ is a basis for the free group $\pi_1(X/T, w_0)$.

For each $i \in I$, we must lift $\overline{e}_i$ to a loop in $X$ that is based at $v_0$. To do this, let $t_i$ be a path in $T$ from $v_0$ to the initial vertex of the edge $e_i$ and let $t_i'$ be a path in $T$ from the terminal vertex of $e_i$ back to $v_0$. We then have a loop $t_i \cdot e_i \cdot t_i'$ in $X$ based at $v_0$, and $q_*([t_i \cdot e_i \cdot t_i']) = [\overline{e}_i]$. We deduce that $\{[t_i \cdot e_i \cdot t_i'] \mid i \in I\}$ is a basis for the free group $\pi_1(X, v_0)$.

EXAMPLE 8.9.1. Consider the following graph $X$ with maximal tree $T$:




We have labeled and shown the orientation on each edge of $X$. Following the above algorithm, we obtain a free basis for $\pi_1(X, v_0)$. There is one element of this basis for each edge $\{a, b, c, d\}$. For the edge $a$, the corresponding element of $\pi_1(X, v_0)$ is $[t_1 \cdot a \cdot t_5 \cdot t_4 \cdot \overline{t}_2]$:




We similarly get basis elements corresponding to $b$ and $c$ and $d$. In summary, the following is a free basis for $\pi_1(X_S, v_0)$:

$$\{[t_1 \cdot a \cdot t_5 \cdot t_4 \cdot \overline{t}_2], [t_1 \cdot b \cdot \overline{t}_3 \cdot \overline{t}_2], [t_2 \cdot t_3 \cdot c \cdot \overline{t}_4 \cdot \overline{t}_2], [d \cdot t_5 \cdot t_4 \cdot \overline{t}_2]\}. \qquad \square$$

## 8.10. Exercises

EXERCISE 8.1. Prove that being homotopy equivalent is an equivalence relation on pointed spaces and on spaces. □

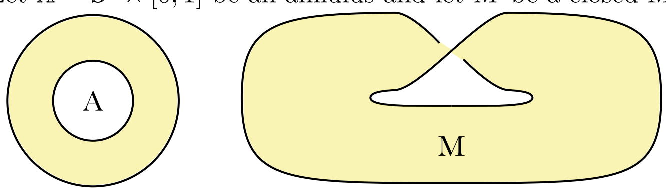
EXERCISE 8.2. Prove that a space $X$ is contractible if and only if it is homotopy equivalent to a one-point space $p_0$. $\square$

EXERCISE 8.3. Let $(X, x_0)$ and $(Y, y_0)$ be pointed spaces. Let $[(X, x_0), (Y, y_0)]_*$ be the set of homotopy classes of maps $f\colon (X, x_0) \to (Y, y_0)$. We will often omit the basepoints and just write $[X, Y]_*$. We remark that set of homotopy classes of maps $f\colon X \to Y$ that do not necessarily preserve the basepoints is written $[X, Y]$. Prove the following:

(a) Precomposition with a pointed homotopy equivalence $h\colon (Z, z_0) \to (X, x_0)$ induces a bijection $h_*\colon [X, Y]_* \to [Z, Y]_*$.
(b) Postcomposition with a pointed homotopy equivalence $h\colon (Y, y_0) \to (Z, z_0)$ induces a bijection $h_*\colon [X, Y]_* \to [X, Z]_*$. $\square$

EXERCISE 8.4. For some $n \geq 1$, let $X = \{(x, y) \in \mathbb{S}^n \times \mathbb{S}^n \mid x \neq -y\}$. Define a map $f\colon \mathbb{S}^n \to X$ via the formula $f(x) = (x, x)$. Prove that $f$ is a homotopy equivalence. $\square$

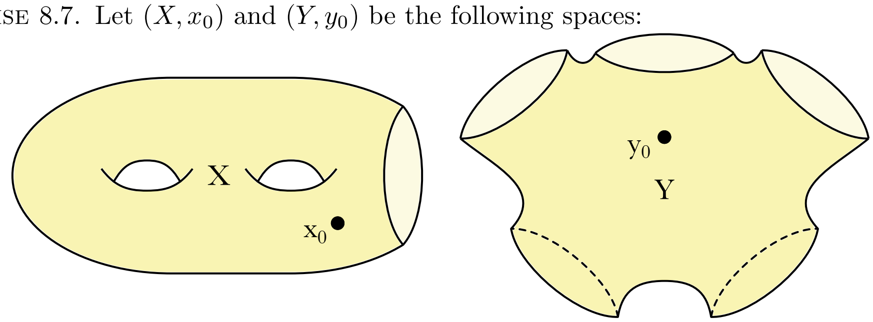
EXERCISE 8.5. Let $\mathbb{A} = \mathbb{S}^1 \times [0, 1]$ be an annulus and let $M$ be a closed Möbius band:




Prove that $\mathbb{A}$ and $M$ are homotopy equivalent. $\square$

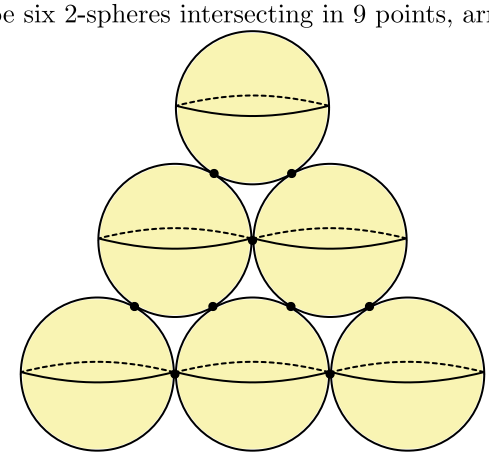
EXERCISE 8.6. Let $\mathbb{A} = \mathbb{S}^1 \times [0, 1]$ be an annulus and let $\mathrm{Int}(\mathbb{A}) = \mathbb{S}^1 \times (0, 1)$ be an open annulus. Prove that $\mathbb{A}$ and $\mathrm{Int}(\mathbb{A})$ are homotopy equivalent. $\square$

EXERCISE 8.7. Let $(X, x_0)$ and $(Y, y_0)$ be the following spaces:




Prove that $X \simeq Y$, and calculate $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$. $\square$

EXERCISE 8.8. Let $X$ be six 2-spheres intersecting in 9 points, arranged as follows:

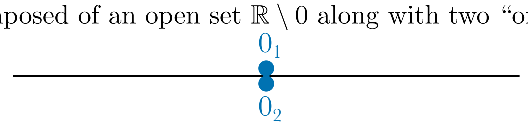


Do the following:

(a) Prove that $X$ is homotopy equivalent to $(\vee_{i=1}^{6} \mathbb{S}^2) \vee (\vee_{j=1}^{4} \mathbb{S}^1)$.
(b) Letting $x_0 \in X$ be a basepoint, calculate $\pi_1(X, x_0)$. The result will be a free group, and you should also draw loops on $X$ corresponding to generators of this free group. $\square$

Exercise 8.9. For a space $X$, let $\text{Covers}(X)$ be the set of isomorphism classes of covers $p\colon \widetilde{X} \to X$. For a map $f\colon Y \to X$, let $f^*\colon \text{Covers}(X) \to \text{Covers}(Y)$ take $p\colon \widetilde{X} \to X$ to the pullback $q\colon \widetilde{Y} \to Y$ of $p\colon \widetilde{X} \to X$ along $f$ as in Exercise 1.14. If $f\colon Y \to X$ is a homotopy equivalence, prove that the induced map $f^*\colon \text{Covers}(X) \to \text{Covers}(Y)$ is a bijection. Hint: Exercise 4.9 says that $f^*$ only depends on the homotopy class of $f$. $\square$

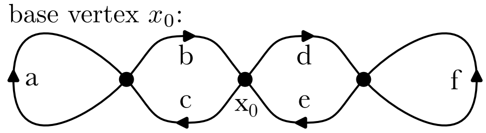
Exercise 8.10. As in Exercise 7.5, let $X$ be the "line with two origins", i.e., the quotient of $\mathbb{R} \sqcup \mathbb{R}$ that for $x \in \mathbb{R}$ nonzero identifies the points $x$ in the two copies of $\mathbb{R}$ to a single point. This is a non-Hausdorff space composed of an open set $\mathbb{R} \setminus 0$ along with two "origins":



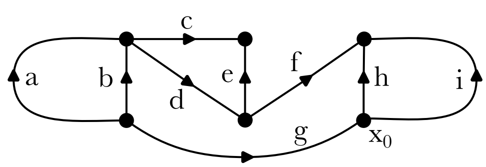

Prove that $X$ is not homotopy equivalent to any Hausdorff space. Hint: you proved in Exercise 7.5 that the fundamental group of $X$ is $\mathbb{Z}$. Prove that any map from $X$ to a Hausdorff space induces the trivial map on the fundamental group. $\square$

Exercise 8.11. Calculate free generating sets for $\pi_1(X, x_0)$ where $X$ is one of the following three graphs with the indicated base vertex $x_0$:

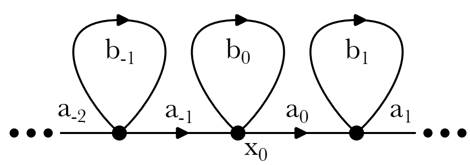
(i)




(ii)

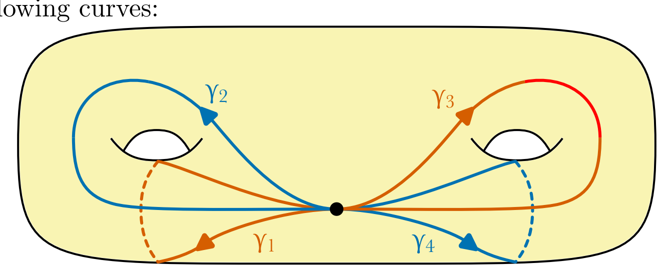



(iii)




The edges are labeled to make it easy to describe loops based at $x_0$. Hint: first find a maximal tree. $\square$

Exercise 8.12. Let $\Sigma_2$ be a closed oriented genus 2 surface with a basepoint $y_0$. Let $[\gamma_1], \ldots, [\gamma_4] \in \pi_1(\Sigma_2, y_0)$ be the following curves:




In this exercise, you will prove that the abelianization $\pi_1(\Sigma_2, y_0)^{\text{ab}}$ is isomorphic to $\mathbb{Z}^4$.

(a) Choose a point $z_0 \in \Sigma_2$ with $z_0 \neq y_0$. Prove that the map $\pi_1(\Sigma_2 \setminus z_0, y_0) \to \pi_1(\Sigma_2, y_0)$ is surjective. Hint: use Lemma 4.5.3 (general position).

(b) Prove that $\pi_1(\Sigma_2 \setminus z_0, y_0)$ is a free group on the $[\gamma_i]$, considered as elements of $\pi_1(\Sigma_2 \setminus z_0, y_0)$. Hint: view $\Sigma_2$ as an octagon $P$ with sides identified in pairs and $z_0$ as the central point $c \in P$. Deformation retract $P \setminus c$ to its boundary, and use this to deformation retract $\Sigma_2 \setminus z_0$ to a graph with one vertex $y_0$ and 4 edges corresponding to the $[\gamma_i]$.

(c) Deduce from (b) that there is a surjection from $\pi_1(\Sigma_2 \setminus z_0, y_0)^{\text{ab}} \cong \mathbb{Z}^4$ to $\pi_1(\Sigma_2, y_0)$.

(d) Let $\mathbb{T}^4 = (\mathbb{S}^1)^{\times 4}$ be the 4-dimensional torus. Construct a map $f\colon \Sigma_2 \to \mathbb{T}^4$ such that the map $f_*\colon \pi_1(\Sigma_2, y_0) \to \pi_1(\mathbb{T}^4, f(y_0)) \cong \mathbb{Z}^4$ takes $[\gamma_i]$ to the $i^{\text{th}}$ basis vector of $\mathbb{Z}^4$. Hint: It might be easier to construct maps $f_1, f_2\colon \Sigma_2 \to \mathbb{T}^2$ such that:
- $(f_1)_*$ takes $[\gamma_1]$ and $[\gamma_2]$ to the basis elements of $\pi_1(\mathbb{T}^2, f_1(y_0)) \cong \mathbb{Z}^2$ and has $[\gamma_3], [\gamma_4] \in \ker((f_1)_*)$; and
- $(f_2)_*$ takes $[\gamma_3]$ and $[\gamma_4]$ to the basis elements of $\pi_1(\mathbb{T}^2, f_1(y_0)) \cong \mathbb{Z}^2$ and has $[\gamma_1], [\gamma_2] \in \ker((f_2)_*)$.

You can then take $f = f_1 \times f_2$. To construct the $f_i$, think about collapsing subsurfaces of $\Sigma_2$ to points.

(e) Deduce from (d) that there is a surjection from $\pi_1(\Sigma_2, y_0)^{\text{ab}}$ to $\mathbb{Z}^4$.

(f) Combine (c) and (e) and prove that $\pi_1(\Sigma_2, y_0)^{\text{ab}} \cong \mathbb{Z}^4$. □

EXERCISE 8.13. As a silly exercise in pure point-set topology, prove that for all $n \geq 1$ there exist topological spaces $X$ with exactly $n$ points that are contractible. □

# Classifying covers

CHAPTER 9

# The lifting criterion for covers

In this chapter, we use the fundamental group to characterize when a map can be lifted to a cover. Using this, we will prove that in favorable situations covering spaces can be understood using fundamental group information. We will elaborate on this more in subsequent chapters when we describe the classification of covering spaces.

## 9.1. Fundamental group of cover

Before giving the lifting criterion, we introduce some terminology and prove a preliminary result. Let $p\colon \widetilde{X} \to X$ be a cover and let $\gamma\colon I \to X$ be a path. For $\widetilde{x} \in \widetilde{X}$ such that $p(\widetilde{x}) = \gamma(0)$, Lemma 3.4.1 (path lifting) says that there is a unique lift $\widetilde{\gamma}\colon I \to \widetilde{X}$ of $\gamma$ such that $\widetilde{\gamma}(0) = \widetilde{x}$. We will simply call this the *lift of* $\gamma$ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}$.

We will also need to be careful with basepoints. A *pointed cover* is a pointed map $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that the map $p\colon \widetilde{X} \to X$ is a cover. A pointed cover $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ induces a map $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$. This map is injective; in fact, the following holds:

Theorem 9.1.1. *Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a pointed cover. Then the following hold:*

*(i) The map $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective.*
*(ii) The image of $p_*$ consists of all $[\gamma] \in \pi_1(X, x_0)$ such that the following holds:*
  *(†) Let $\widetilde{\gamma}$ be the lift of $\gamma$ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$. Then $\widetilde{\gamma}$ is a loop, i.e., $\widetilde{\gamma}(1) = \widetilde{x}_0$.*
*(iii) If $\widetilde{X}$ is path connected, then the index of $\mathrm{Im}(p_*)$ in $\pi_1(X, x_0)$ equals the degree of the cover.*

Proof. We prove each part separately:

Step 1. *The map $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective.*

Consider $[\widetilde{\gamma}] \in \pi_1(\widetilde{X}, \widetilde{x}_0)$ in the kernel of $p_*$. Our goal is to prove that $[\widetilde{\gamma}] = 1$. Letting $\gamma = p \circ \gamma$, the element $[\gamma] \in \pi_1(X, x_0)$ is trivial. Lemma 4.2.1 says that the homotopy class of the lift $\widetilde{\gamma}$ of $\gamma$ to $\widetilde{X}$ only depends on the homotopy class of $\gamma$. Since $[\gamma] = 1$, it follows that $[\widetilde{\gamma}] = 1$.

Step 2. *The image of $p_*$ consists of all $[\gamma] \in \pi_1(X, x_0)$ such that the following holds:*

(†) *Let $\widetilde{\gamma}$ be the lift of $\gamma$ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$. Then $\widetilde{\gamma}$ is a loop, i.e., $\widetilde{\gamma}(1) = \widetilde{x}_0$.*

It is immediate from the definitions that the image of $p_*$ consists of all $[\gamma] \in \pi_1(X, x_0)$ such that $\gamma$ is homotopic as a path to a loop $\gamma'$ based at $x_0$ such that $\gamma'$ satisfies (†). Just like in Step 1, we can lift a homotopy of paths from $\gamma$ to $\gamma'$ to a homotopy of paths in the cover. It follows that $\gamma$ also satisfies (†). The step follows.

Step 3. *If $\widetilde{X}$ is path connected, then the index of $\mathrm{Im}(p_*)$ in $\pi_1(X, x_0)$ equals the degree of the cover.*

We have $\widetilde{x}_0 \in p^{-1}(x_0)$. Enumerate the entire fiber $p^{-1}(x_0)$ as $\{\widetilde{x}_i \mid i \in I\}$ for some indexing set $I$ with $0 \in I$. The cardinality $|I|$ is thus the degree of the cover. For each $i \in I$, pick a path $\widetilde{\delta}_i$ in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}_i$. Let $\delta_i$ be the image of $\widetilde{\delta}_i$ in $X$, so $[\delta_i] \in \pi_1(X, x_0)$. We claim that $\{[\delta_i] \mid i \in I\}$ is a set of left coset representatives for $\mathrm{Im}(p_*)$.

To see this, consider $[\gamma] \in \pi_1(X, x_0)$. Lift $\gamma$ to a path $\widetilde{\gamma}$ in $\widetilde{X}$ starting at $\widetilde{x}_0$. Let $\widetilde{x}_{i_0}$ be the endpoint of $\widetilde{\gamma}$. As in the previous steps, $\widetilde{x}_{i_0}$ only depends on $[\gamma]$. We have

$$[\gamma] = [\gamma \cdot \overline{\delta}_{i_0}][\delta_{i_0}]. \tag{9.1.1}$$

Since $\gamma \cdot \overline{\delta}_{i_0}$ lifts to the loop $\widetilde{\gamma} \cdot \widetilde{\overline{\delta}}_{i_0}$ based at $\widetilde{x}_0$, Step 2 implies that $[\gamma \cdot \overline{\delta}_{i_0}] \in \mathrm{Im}(p_*)$. The equation (9.1.1) exhibits $[\gamma]$ as an element of the left $[\delta_{i_0}]$-coset of $\mathrm{Im}(p_*)$. Since this expression is clearly unique, this proves the claim. $\square$

## 9.2. Lifting criterion

Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a pointed cover. Consider a pointed map $f\colon (Y, y_0) \to (X, x_0)$. Our goal is to understand when $f$ can be lifted to $\widetilde{X}$, i.e., when there exists a pointed map $\widetilde{f}\colon (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ such that $f = p \circ \widetilde{f}$. In other words, we want the following diagram to commute:

$$\begin{tikzcd} & (\widetilde{X}, \widetilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, "\widetilde{f}"] \arrow[r, "f"] & (X, x_0). \end{tikzcd}$$

There is one obvious necessary condition. Assume that $\widetilde{f}$ exists, and pass to the fundamental group. We get a commutative diagram

$$\begin{tikzcd} & \pi_1(\widetilde{X}, \widetilde{x}_0) \arrow[d, "p_*"] \\ \pi_1(Y, y_0) \arrow[ur, "\widetilde{f}_*"] \arrow[r, "f_*"] & \pi_1(X, x_0). \end{tikzcd}$$

Since $p_* \circ \widetilde{f}_* = f_*$, it follows immediately that the image of $f_*$ is contained in the image of the injective map $p_*$. It turns out that for reasonable spaces $Y$, this necessary condition is sufficient. Since the fundamental group only depends on the path component containing the basepoint, we clearly need to assume that $Y$ is path connected.

To avoid pathological local behavior, we also need to assume that $Y$ is locally path connected, i.e., that for all $y \in Y$ and all open neighborhoods $U$ of $y$ there exists a path connected open neighborhood $V$ of $y$ with $V \subset U$. This property passes to covers in the sense that if $Y$ is locally path connected and $q\colon \widetilde{Y} \to Y$ is a cover, then $\widetilde{Y}$ is locally path connected (see Exercise 9.1). Our lifting criterion is as follows:

Theorem 9.2.1 (Lifting criterion). *Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a pointed cover. Let $f\colon (Y, y_0) \to (X, x_0)$ be a pointed map such that the image of $f_*\colon \pi_1(Y, y_0) \to \pi_1(X, x_0)$ is contained in the image of $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$. Assume that $Y$ is path connected and locally path connected. Then $f$ can be uniquely lifted to a map $\widetilde{f}\colon (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$.*

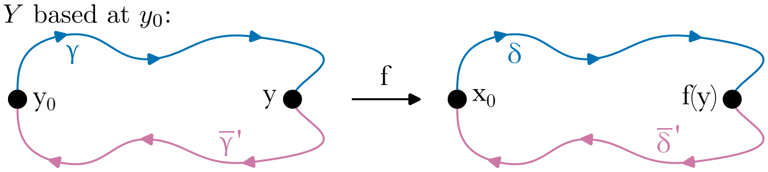
Remark 9.2.2. See Exercise 9.12 for an example showing that it is necessary to assume that $Y$ is locally path connected. $\square$

Proof of Theorem 9.2.1. We proved in Lemma 3.1.2 that for connected spaces lifts are determined by what they do to a single point. Since we are assuming that our lift $\widetilde{f}$ takes $y_0$ to $\widetilde{x}_0$, this implies that the lift is unique if it exists. We must prove existence.

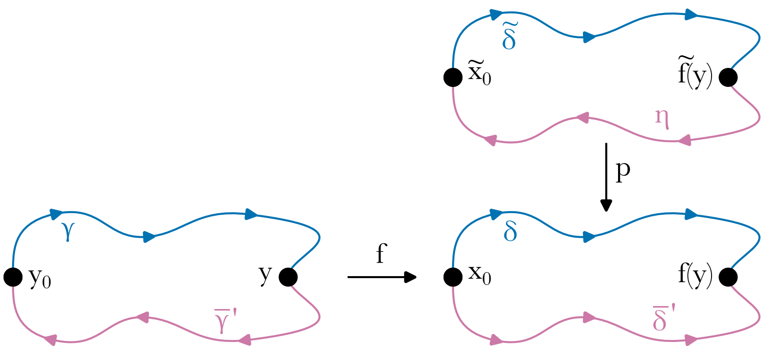
Consider $y \in Y$. We define $\widetilde{f}(y)$ as follows. Since $Y$ is path connected, we can find a path $\gamma$ from $y_0$ to $y$. Its image $\delta = f \circ \gamma$ is a path from $f(y_0) = x_0$ to $f(y)$. Let $\widetilde{\delta}\colon I \to \widetilde{X}$ be the lift of $\delta$ to $\widetilde{X}$ with $\widetilde{\delta}(0) = \widetilde{x}_0$. We would like to define $\widetilde{f}(y) = \widetilde{\delta}(1)$. To do this, we must prove that this does not depend on the choice of $\gamma$:

Claim. *The above definition of $\widetilde{f}(y) \in \widetilde{X}$ does not depend on the choice of $\gamma$.*

Proof of claim. Let $\gamma'$ be another path from $y_0$ to $y$. Letting $\delta' = f \circ \gamma'$ be its image in $X$ and $\widetilde{\delta}'\colon I \to \widetilde{X}$ be the lift of $\delta'$ with $\widetilde{\delta}'(0) = \widetilde{x}_0$, our goal is to prove that $\widetilde{\delta}(1) = \widetilde{\delta}'(1)$. Observe that $\gamma \cdot \overline{\gamma}'$ is a loop in $Y$ based at $y_0$:



As is shown here, the image of $[\gamma \cdot \overline{\gamma}'] \in \pi_1(Y, y_0)$ under $f_*$ is the loop $[\delta \cdot \overline{\delta}'] \in \pi_1(X, x_0)$. By assumption, this lies in the image of $p_*$. Theorem 9.1.1 therefore implies that $\delta \cdot \overline{\delta}' : I \to X$ lifts to a loop in $\widetilde{X}$ starting at $\widetilde{x}_0$:




As is shown here, this lifted loop equals $\widetilde{\delta} \cdot \eta$, where:

- as introduced above, the path $\widetilde{\delta}$ is the lift of $\delta$ with $\widetilde{\delta}(0) = \widetilde{x}_0$; and
- the path $\eta$ is the lift of $\overline{\delta}'$ with $\eta(0) = \widetilde{\delta}(1) = \widetilde{f}(y)$.

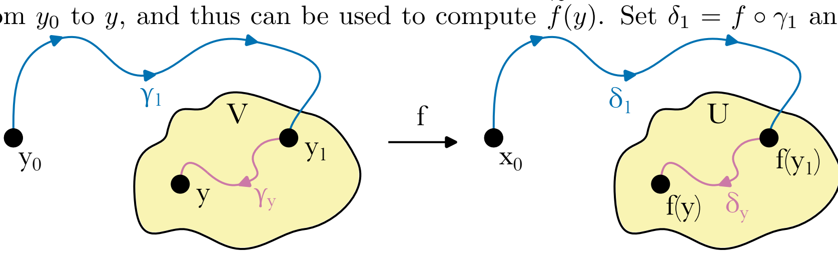
Since $\eta(1) = \widetilde{x}_0$, it follows that $\overline{\eta}$ is the path $\widetilde{\delta}'$ introduced above that we obtained by lifting $\delta'$ to a path with $\widetilde{\delta}'(0) = \widetilde{x}_0$. We conclude that $\widetilde{\delta}(1) = \eta(0) = \widetilde{\delta}'(1)$, as desired. $\square$

We have now defined $\widetilde{f} \colon Y \to \widetilde{X}$. This map satisfies $\widetilde{f}(y_0) = \widetilde{x}_0$ since in its definition when calculating $\widetilde{f}(y_0)$ we can take $\gamma$ to be the constant path at $y_0$. All that remains to prove is that $\widetilde{f}$ is continuous. Consider some $y_1 \in Y$. Let $U \subset X$ be a trivialized open neighborhood of $f(y_1) \in X$ and let $\widetilde{U} \subset X$ be the sheet above $U$ containing $\widetilde{f}(y_1)$. Since $Y$ is locally path connected, we can find a path connected open neighborhood $V$ of $y_1$ such that $V \subset f^{-1}(U)$. To prove that $\widetilde{f}$ is continuous at $y_1$, it is enough to prove that on $V$ the map $\widetilde{f}$ is the composition

$$V \xrightarrow{f} U \xrightarrow{(p|_{\widetilde{u}})^{-1}} \widetilde{U}.$$

To see this, consider some $y \in V$. The image $\widetilde{f}(y) \in \widetilde{X}$ lies in the fiber $p^{-1}(f(y))$, and we must prove that it is the point of this fiber lying in $\widetilde{U}$. In other words, we must prove that $\widetilde{f}(y) \in \widetilde{U}$.

Let $\gamma_1$ be a path in $Y$ from $y_0$ to $y_1$ and let $\gamma_y$ be a path in $V$ from $y_1$ to $y$. The path $\gamma_1 \cdot \gamma_y$ in $Y$ goes from $y_0$ to $y$, and thus can be used to compute $\widetilde{f}(y)$. Set $\delta_1 = f \circ \gamma_1$ and $\delta_y = f \circ \gamma_y$:




By definition, $\widetilde{f}(y)$ is the endpoint of the lift of $\delta_1 \cdot \delta_y$ to $\widetilde{X}$ starting at $\widetilde{x}_0$. We construct this lift in two steps:

- Let $\widetilde{\delta}_1$ be the lift of $\delta_1$ to $\widetilde{X}$ with $\widetilde{\delta}_1(0) = \widetilde{x}_0$. By definition, $\widetilde{f}(y_1) = \widetilde{\delta}_1(1) \in \widetilde{U}$.
- Let $\widetilde{\delta}_y$ be the lift of $\delta_y$ to $\widetilde{X}$ with $\widetilde{\delta}_y(0) = \widetilde{\delta}_1(1) \in \widetilde{U}$. By definition, $\widetilde{f}(y) = \widetilde{\delta}_y(1)$.

Since $\gamma_y$ is a path in $V$ and $f(V) \subset U$, it follows that $\delta_y$ is a path in $U$. Since $\widetilde{\delta}_y(0) \in \widetilde{U}$, we have that $\widetilde{\delta}_y$ is a path in $\widetilde{U}$; in fact, $\widetilde{\delta}_y = (p|_{\widetilde{u}})^{-1} \circ \delta_y$. We conclude that $\widetilde{f}(y) = \widetilde{\delta}_y(1) \in \widetilde{U}$, as desired. $\square$

## 9.3. Pointed isomorphisms of covers

Our next goal is to apply Theorem 9.2.1 (lifting criterion) to help us classify pointed covers and construct deck transformations using fundamental group information. We defined what it means for two covers to be isomorphic in §1.4. Adding basepoints, we make the following definition:

DEFINITION 9.3.1. Let $(X, x_0)$ be a pointed space and $p_1\colon (\widetilde{X}_1, \widetilde{x}_1) \to (X, x_0)$ and $p_2\colon (\widetilde{X}_2, \widetilde{x}_2) \to (X, x_0)$ be two pointed covers of $(X, x_0)$. A *pointed covering space isomorphism* from $(\widetilde{X}_1, \widetilde{x}_1)$ to $(\widetilde{X}_2, \widetilde{x}_2)$ is a homeomorphism $\widetilde{f}\colon (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ such that the diagram

$$\begin{tikzcd}
(\widetilde{X}_1, \widetilde{x}_1) \arrow[rr, "\widetilde{f}"] \arrow[dr, "p_1"'] & & (\widetilde{X}_2, \widetilde{x}_2) \arrow[dl, "p_2"] \\
& (X, x_0) &
\end{tikzcd}$$

commutes, i.e., such that $p_2 \circ \widetilde{f} = p_1$. If a pointed covering space isomorphism from $(\widetilde{X}_1, \widetilde{x}_1)$ to $(\widetilde{X}_2, \widetilde{x}_2)$ exists, we say that $(\widetilde{X}_1, \widetilde{x}_1)$ and $(\widetilde{X}_2, \widetilde{x}_2)$ are *isomorphic* pointed covers of $X$. This is clearly an equivalence relation. □

Using Theorem 9.2.1 (lifting criterion), we will prove that in favorable situations pointed covers are determined up to isomorphism by the images of their fundamental groups:

THEOREM 9.3.2. *Let $p_1\colon (\widetilde{X}_1, \widetilde{x}_1) \to (X, x_0)$ and $p_2\colon (\widetilde{X}_2, \widetilde{x}_2) \to (X, x_0)$ be two pointed covers such that:*

- *the images of $(p_1)_*\colon \pi_1(\widetilde{X}_1, \widetilde{x}_1) \to \pi_1(X, x_0)$ and $(p_2)_*\colon \pi_1(\widetilde{X}_2, \widetilde{x}_2) \to \pi_1(X, x_0)$ are the same; and*
- *both $\widetilde{X}_1$ and $\widetilde{X}_2$ are path connected.*

*Assume that $X$ is path connected and locally path connected. Then $(\widetilde{X}_1, \widetilde{x}_1)$ and $(\widetilde{X}_2, \widetilde{x}_2)$ are isomorphic pointed covers of $(X, x_0)$.*

PROOF. Since $X$ is locally path connected, so are $\widetilde{X}_1$ and $\widetilde{X}_2$. Applying Theorem 9.2.1 twice, we get pointed maps $\widetilde{f}\colon (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ and $\widetilde{g}\colon (\widetilde{X}_2, \widetilde{x}_2) \to (\widetilde{X}_1, \widetilde{x}_1)$ such that the following diagrams commute:

$$\begin{tikzcd}
& (\widetilde{X}_2, \widetilde{x}_2) \arrow[d, "p_2"] \\
(\widetilde{X}_1, \widetilde{x}_1) \arrow[ur, "\widetilde{f}"] \arrow[r, "p_1"] & (X, x_0)
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
& (\widetilde{X}_1, \widetilde{x}_1) \arrow[d, "p_1"] \\
(\widetilde{X}_2, \widetilde{x}_2) \arrow[ur, "\widetilde{g}"] \arrow[r, "p_2"] & (X, x_0)
\end{tikzcd}$$

commute. To prove that $\widetilde{f}$ is the desired isomorphism of pointed covers, it is enough to prove that it is a homeomorphism. In fact, we will prove that $\widetilde{f}$ and $\widetilde{g}$ are inverse homeomorphisms, i.e., $\widetilde{g} \circ \widetilde{f}\colon (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_1, \widetilde{x}_1)$ and $\widetilde{f} \circ \widetilde{g}\colon (\widetilde{X}_2, \widetilde{x}_2) \to (\widetilde{X}_2, \widetilde{x}_2)$ are both the identity. The two proofs are similar, so we will prove the first. By construction, the diagrams

$$\begin{tikzcd}
& (\widetilde{X}_1, \widetilde{x}_1) \arrow[d, "p_1"] \\
(\widetilde{X}_1, \widetilde{x}_1) \arrow[ur, "\widetilde{g} \circ \widetilde{f}"] \arrow[r, "p_1"] & (X, x_0)
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
& (\widetilde{X}_1, \widetilde{x}_1) \arrow[d, "p_1"] \\
(\widetilde{X}_1, \widetilde{x}_1) \arrow[ur, "\mathbb{1}"] \arrow[r, "p_1"] & (X, x_0)
\end{tikzcd}$$

both commute. Since $\widetilde{X}_1$ is path connected and $\widetilde{g} \circ \widetilde{f}$ and $\mathbb{1}$ are both lifts of $p_1\colon (\widetilde{X}_1, \widetilde{x}_1) \to (X, x_0)$ that are equal at the point $\widetilde{x}_1$, it follows from Lemma 3.1.2 that they are equal, as desired. □

## 9.4. Deck transformations

Let $p\colon \widetilde{X} \to X$ be a cover with deck group $G$. For $x_0 \in X$, the group $G$ acts on the fiber $p^{-1}(x_0)$. If $\widetilde{X}$ is connected, then Lemma 2.2.1 says that if $g, g' \in G$ are elements such that there is some $\widetilde{x}_0 \in p^{-1}(x_0)$ with $g\widetilde{x}_0 = g'\widetilde{x}_0$, then $g = g'$. In other words, the group $G$ is not only determined by its action on $p^{-1}(x_0)$, but even by its action on any single point of $p^{-1}(x_0)$. In favorable situations, the following theorem therefore completely describes $G$:

THEOREM 9.4.1. *Let $p\colon \widetilde{X} \to X$ be a cover with deck group $G$. Let $x_0 \in X$ and let $\widetilde{x}_0, \widetilde{x}_0' \in p^{-1}(X)$. Set $\Gamma = \pi_1(X, x_0)$, and let $K$ and $K'$ be the following subgroups of $\Gamma$:*

$$K = \operatorname{Im}(p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)),$$
$$K' = \operatorname{Im}(p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0') \to \pi_1(X, x_0)).$$

*Assume that $\widetilde{X}$ is path connected and that $X$ is locally path connected. Then there exists $g \in G$ with $g\widetilde{x}_0 = \widetilde{x}_0'$ if and only if $K = K'$.*

PROOF. If such a $g$ exists, then the corresponding deck transformation $\widetilde{f}_g$ and its inverse $\widetilde{f}_{g^{-1}}$ fit into commutative diagrams

$$\begin{tikzcd} & (\widetilde{X}, \widetilde{x}_0') \arrow[d, "p"] \\ (\widetilde{X}, \widetilde{x}_0) \arrow[ur, "\widetilde{f}_g"] \arrow[r, "p"] & (X, x_0) \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} & (\widetilde{X}, \widetilde{x}_0) \arrow[d, "p"] \\ (\widetilde{X}, \widetilde{x}_0') \arrow[ur, "\widetilde{f}_{g^{-1}}"] \arrow[r, "p"] & (X, x_0). \end{tikzcd}$$

Applying the fundamental group, we get commutative diagrams

$$\begin{tikzcd} & \pi_1(\widetilde{X}, \widetilde{x}_0') \arrow[d, "p"] \\ \pi_1(\widetilde{X}, \widetilde{x}_0) \arrow[ur, "\widetilde{f}_g"] \arrow[r, "p"] & \pi_1(X, x_0) \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} & \pi_1(\widetilde{X}, \widetilde{x}_0) \arrow[d, "p"] \\ \pi_1(\widetilde{X}, \widetilde{x}_0') \arrow[ur, "\widetilde{f}_{g^{-1}}"] \arrow[r, "p"] & \pi_1(X, x_0). \end{tikzcd}$$

From this, we see that $K = K'$. Conversely, if $K = K'$ then Theorem 9.2.1 (lifting criterion) says that there exists $\widetilde{f}_g$ and $\widetilde{f}_{g^{-1}}$ making the above diagrams commute. Just like in the proof of Theorem 9.3.2, the compositions $\widetilde{f}_g \circ \widetilde{f}_{g^{-1}}$ and $\widetilde{f}_{g^{-1}} \circ \widetilde{f}_g$ are both the identity, so they are homeomorphisms and thus the desired deck transformations. $\square$

## 9.5. Regular covers

Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a pointed cover with deck group $G$. Recall that the cover is *regular* if for all $x \in X$ the group $G$ acts transitively on the fiber $p^{-1}(x)$. In fact, if $X$ is path connected then Lemma 4.4.1 says that it is enough to check this transitivity on a single fiber, for instance $p^{-1}(x_0)$. Also, recall from §7.2 that if $\widetilde{X}$ is path connected the *lifting map* is the set map $f\colon \pi_1(X, x_0) \to G$ defined as follows:

- Consider $[\gamma] \in \Gamma$. Let $\widetilde{\gamma}$ be the lift of $\gamma$ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$. Letting $f([\gamma]) = g$, we then have $\widetilde{\gamma}(1) = g\widetilde{x}_0$.

The lifting map is a surjective group homomorphism (see Lemma 7.2.1). Moreover, when we proved Theorem 7.1.1 we showed that the lifting map is an isomorphism if $\widetilde{X}$ is 1-connected. Our final result in this chapter generalizes this as follows. See the next section for some examples of it.

THEOREM 9.5.1. *Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a pointed cover. Let $\Gamma = \pi_1(X, x_0)$, and let $K$ be the following subgroup of $\Gamma$:*

$$K = \operatorname{Im}(p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0) = \Gamma).$$

*Assume that $\widetilde{X}$ is path connected and that $X$ is locally path connected. The following hold:*

*(i) The cover $p\colon \widetilde{X} \to X$ is a regular cover if and only if $K$ is a normal subgroup of $\Gamma$.*
*(ii) Assume that $p\colon \widetilde{X} \to X$ is a regular cover, and let $G$ be its deck group. Then the lifting map $f\colon \Gamma \to G$ is a surjective homomorphism with kernel $K$.*

PROOF. We prove (i) and (ii) separately:

STEP 1. *The cover $p\colon \widetilde{X} \to X$ is a regular cover if and only if $K$ is a normal subgroup of $\Gamma$.*

By Lemma 4.4.1, the cover is regular if and only if the deck group $G$ acts transitively on $p^{-1}(x_0)$. For $\widetilde{x}_1 \in p^{-1}(x_0)$, set

$$K_{\widetilde{x}_1} = \mathrm{Im}(p_* \colon \pi_1(\widetilde{X}, \widetilde{x}_1) \to \pi_1(X, x_0)).$$

By Theorem 9.4.1, there exists some $g \in G$ with $g\widetilde{x}_0 = \widetilde{x}_1$ if and only if $K_{\widetilde{x}_1} = K$. We deduce that $p \colon \widetilde{X} \to X$ is regular if and only if $K_{\widetilde{x}_1} = K$ for all $\widetilde{x}_1 \in p^{-1}(x_0)$.

For $\widetilde{x}_1 \in p^{-1}(x_0)$, let $\widetilde{\gamma}_{\widetilde{x}_1}$ be a path in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}_1$. The change of basepoint isomorphism from $\pi_1(\widetilde{X}, \widetilde{x}_1)$ to $\pi_1(\widetilde{X}, \widetilde{x}_0)$ takes $[\eta] \in \pi_1(\widetilde{X}, \widetilde{x}_1)$ to $[\widetilde{\gamma}_{\widetilde{x}_1} \cdot \eta \cdot \overline{\widetilde{\gamma}_{\widetilde{x}_1}}] \in \pi_1(\widetilde{X}, \widetilde{x}_0)$. Let $\gamma_{\widetilde{x}_1}$ be the image of $\widetilde{\gamma}_{\widetilde{x}_1}$ in $X$, so $\gamma_{\widetilde{x}_1}$ is a path from $x_0$ to $x_0$. We thus have $[\widetilde{\gamma}_{\widetilde{x}_1}] \in \pi_1(X, x_0) = \Gamma$. It follows that

$$[\gamma_{\widetilde{x}_1}] K [\gamma_{\widetilde{x}_1}]^{-1} = K_{\widetilde{x}_1}.$$

Moreover, every $g \in \Gamma = \pi_1(X, x_0)$ can be lifted to a path connecting $\widetilde{x}_0$ to $\widetilde{x}_1$ for some choice of $\widetilde{x}_1 \in p^{-1}(x_0)$. We conclude that our cover is regular if and only if $gKg^{-1} = K$ for all $g \in \Gamma$, i.e., if and only if $K$ is a normal subgroup.

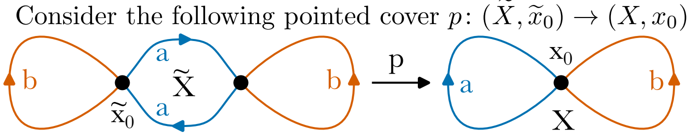
Step 2. *Assume that $p \colon \widetilde{X} \to X$ is a regular cover, and let $G$ be its deck group. Then the lifting map $f \colon \Gamma \to G$ is a surjective homomorphism with kernel $K$.*

Lemma 7.2.1 gives all of this except for the fact that the kernel of the lifting map is $K$, so this is what we must prove. Consider $[\gamma] \in \pi_1(X, x_0)$ such that $f([\gamma]) = 1$. Let $\widetilde{\gamma}$ be the lift of $\gamma$ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$. Since $f([\gamma]) = 1$, we must have $\widetilde{\gamma}(1) = \widetilde{x}_0$, so $\widetilde{\gamma}$ is a loop based at $\widetilde{x}_0$. It follows that $[\gamma] = p_*([\widetilde{\gamma}]) \in K$. Conversely, reversing the above logic we see that elements of $K$ lie in the kernel of $f$, as desired. □

## 9.6. Examples of regular and irregular covers

We now give two examples of Theorem 9.5.1.

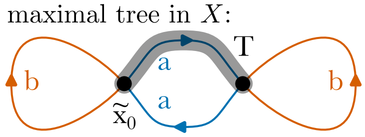
Example 9.6.1. Consider the following pointed cover $p \colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$:




This is a regular cover with deck group $C_2$. Letting $t$ be the generator of $C_2$, the generator $t$ acts on $\widetilde{X}$ by flipping the two vertices, the two oriented edges labeled $a$, and the two oriented edges labeled $b$. We explain how we could see this using Theorem 9.5.1. To do this, we must prove:

- The image of $p_* \colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is a normal subgroup of $\pi_1(X, x_0)$.
- The quotient of $\pi_1(X, x_0)$ by the image of $p_*$ is $C_2$.

The group $\pi_1(X, x_0)$ is isomorphic to the free group $F(a, b)$ on generators $a$ and $b$. We have labeled the edges of $\widetilde{X}$ with the loops in $X$ they map to. We calculate the image of $p_* \colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ as follows. Let $T$ be the following maximal tree in $\widetilde{X}$:

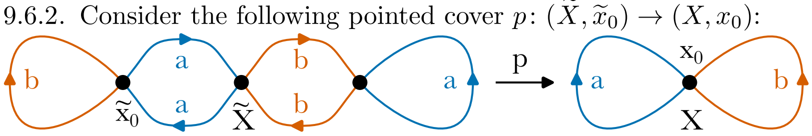



Using the algorithm described in §8.9, we can use $T$ to calculate a free basis $S$ for $\pi_1(\widetilde{X}, \widetilde{x}_0)$. Since $p_*$ is injective, we might as well describe the elements of this free basis by giving their images in $\pi_1(X, x_0) = F(a, b)$:

$$S = \{b, a^2, aba^{-1}\}.$$

The subgroup of $F(a, b)$ generated by $S$ is normal; indeed, it is the kernel of the map $f \colon F(a, b) \to C_2$ taking $a$ to $t$ and $b$ to 1. □

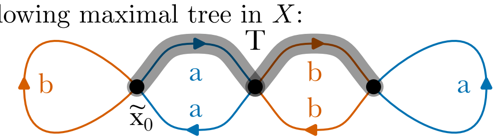
Example 9.6.2. Consider the following pointed cover $p \colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$:




As we noted in §2.5, this is an irregular cover. We explain how we could see this using Theorem 9.5.1. To do this, we must prove:

- The image of $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is a non-normal subgroup of $\pi_1(X, x_0)$.

The group $\pi_1(X, x_0)$ is isomorphic to the free group $F(a, b)$ on generators $a$ and $b$. We have labeled the edges of $\widetilde{X}$ with the loops in $X$ they map to. We calculate the image of $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ as follows. Let $T$ be the following maximal tree in $\widetilde{X}$:




Using the algorithm described in §8.9, we can use $T$ to calculate a free basis $S$ for $\pi_1(\widetilde{X}, \widetilde{x}_0)$. Since $p_*$ is injective, we might as well describe the elements of this free basis by giving their images in $\pi_1(X, x_0) = F(a, b)$:

$$S = \{b, a^2, ab^2a^{-1}, abab^{-1}a^{-1}\}.$$

The subgroup of $F(a, b)$ generated by $S$ is not normal. Indeed, it contains $b$, but we claim it does not contain $aba^{-1}$. To see this, let $S' = S \cup \{aba^{-1}\}$. Since

$$abab^{-1}a^{-1} = (aba^{-1})(a^2)(b^{-1})a^{-1},$$

the subgroup generated by $S'$ contains $a^{-1}$ and hence $a$. But this implies that this subgroup contains both $a$ and $b$, and hence $S'$ generates all of $F(a, b)$. Since $S$ generates a proper subgroup[1] of $F(a, b)$, this implies that $aba^{-1}$ is not in the subgroup generated by $S$. □

## 9.7. Exercises

Exercise 9.1. Let $X$ be a locally path connected space and let $p\colon \widetilde{X} \to X$ be a cover. Prove that $\widetilde{X}$ is locally path connected. □

Exercise 9.2. Let $(X, x_0)$ be a pointed space that is path connected and locally path connected. Assume that $\pi_1(X, x_0)$ is a finite group. Let $f\colon (X, x_0) \to (\mathbb{S}^1, 1)$ be a map. Prove that $f$ can be lifted to the universal cover $p\colon (\mathbb{R}, 0) \to (\mathbb{S}^1, 1)$. □

Exercise 9.3. Let $n \geq 2$ and $m \geq 1$. Let $\mathbb{T}^m = (\mathbb{S}^1)^{\times m}$ be the $m$-torus. Prove that every continuous map $f\colon \mathbb{S}^n \to \mathbb{T}^m$ is null homotopic. □

Exercise 9.4. Let $f\colon X \to Y$ be a continuous map with $X$ and $Y$ both path connected and locally path connected. Let $p\colon \widetilde{X} \to X$ and $q\colon \widetilde{Y} \to Y$ be covers such that $\widetilde{X}$ and $\widetilde{Y}$ are both 1-connected (we will prove in the next couple of chapters that such covers exist if $X$ and $Y$ are reasonable; they are called *universal covers*). Prove that there exists a map $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ such that the diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{f}"] \arrow[d, "p"] & \widetilde{Y} \arrow[d, "q"] \\ X \arrow[r, "f"] & X \end{tikzcd}$$

commutes. □

Exercise 9.5. Let $X$ and $Y$ be path connected and locally path connected space. As in the previous exercise, let $p\colon \widetilde{X} \to X$ and $q\colon \widetilde{Y} \to Y$ be covers such that $\widetilde{X}$ and $\widetilde{Y}$ are both 1-connected. Assume that $X$ and $Y$ are homotopy equivalent. Prove that $\widetilde{X}$ and $\widetilde{Y}$ are homotopy equivalent. □

Exercise 9.6. Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a pointed cover. Let $\Gamma = \pi_1(X, x_0)$, and let $K$ be the following subgroup of $\Gamma$:

$$K = \operatorname{Im}(p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)).$$

Assume that $\widetilde{X}$ is path connected and that $X$ is locally path connected. Let $N(K)$ be the normalizer of $K$ in $\Gamma$, i.e., the subgroup of $\Gamma$ consisting of $g \in \Gamma$ with $gKg^{-1} = K$. Prove that the deck group of $p\colon \widetilde{X} \to X$ is isomorphic to $N(K)/K$. □

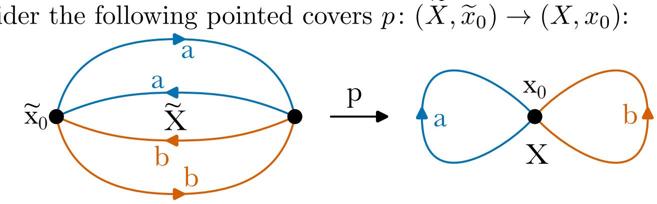
[1] For instance, we know that it generates a rank-4 free group, which has abelianization $\mathbb{Z}^4$ (see Exercise 7.12). On the other hand, $F(a, b)$ has abelianization $\mathbb{Z}^2$.

EXERCISE 9.7. Let $\mathbf{G}$ be a topological group, i.e., a space that is also a group such that the multiplication map $m\colon \mathbf{G} \times \mathbf{G} \to \mathbf{G}$ and the inversion map $i\colon \mathbf{G} \to \mathbf{G}$ are continuous. Assume that $\mathbf{G}$ is path connected and locally path connected. Prove the following:

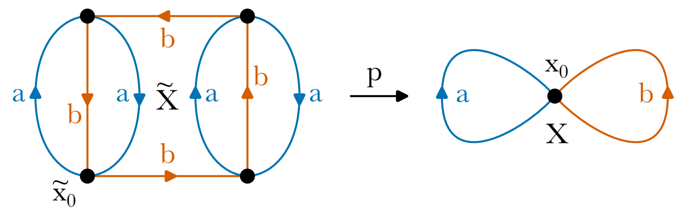
(a) Let $p\colon \widetilde{\mathbf{G}} \to \mathbf{G}$ be a cover. Prove that $\widetilde{\mathbf{G}}$ can be given the structure of a topological group such that $p$ is a group homomorphism.
(b) If $\mathbf{G}$ is abelian, then prove that $\widetilde{\mathbf{G}}$ is abelian.
(c) Prove that $\ker(p)$ is a central subgroup of $\widetilde{\mathbf{G}}$, i.e., that each $g \in \ker(p)$ commutes with all elements of $\widetilde{\mathbf{G}}$. □

EXERCISE 9.8. Consider the following pointed covers $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$:

(a)




(b)




For both, give a free basis for image of $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$. Also, determine if these images are normal. If they are, describe them as the kernels of maps to the deck group of the cover. If they are not, prove it. □

EXERCISE 9.9. As in Example 1.6*.4, let $\mathrm{Poly}_n$ be the space of degree-$n$ monic polynomials over $\mathbb{C}$. Such an $f \in \mathrm{Poly}_n$ can be written as

$$f(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n \quad \text{with } a_1, \ldots, a_n \in \mathbb{C},$$

so $\mathrm{Poly}_n \cong \mathbb{C}^n$. Let

$$\mathrm{Poly}_n^{\mathrm{sf}} = \{f \in \mathrm{Poly}_n \mid f \text{ has } n \text{ distinct roots}\},$$
$$\mathrm{RPoly}_n^{\mathrm{sf}} = \left\{(f, x) \in \mathrm{Poly}_n^{\mathrm{sf}} \times \mathbb{C} \mid f(x) = 0\right\}.$$

We showed in Example 1.6*.4 that the projection $p\colon \mathrm{RPoly}_n^{\mathrm{sf}} \to \mathrm{Poly}_n^{\mathrm{sf}}$ is a degree $n$ cover. In this exercise, you will show that it is an irregular cover for $n \geq 3$. We remark that another proof of this can be found in Exercise 12*.3. Do the following:

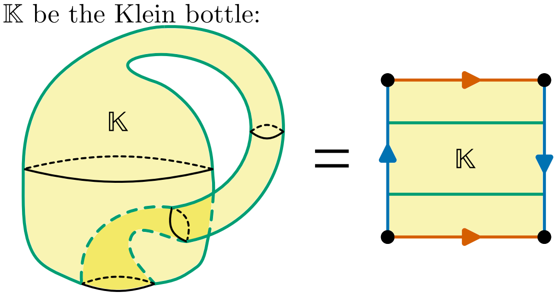
(a) Let
$$C_n = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \mid \lambda_i \neq \lambda_j \text{ for all } 1 \leq i < j \leq n\}.$$
Let $m\colon C_n \to \mathrm{Poly}_n^{\mathrm{sf}}$ be the map
$$m(\lambda_1, \ldots, \lambda_n) = (x - \lambda_1) \cdots (x - \lambda_n) \quad \text{for all } (\lambda_1, \ldots, \lambda_n) \in C_n.$$
Prove that $m\colon C_n \to \mathrm{Poly}_n^{\mathrm{sf}}$ is a regular cover with deck group the symmetric group $\mathfrak{S}_n$ on $n$ generators.
(b) Let $q\colon C_n \to \mathrm{RPoly}_n^{\mathrm{sf}}$ be the map
$$q(\lambda_1, \ldots, \lambda_n) = (m(\lambda_1, \ldots, \lambda_n), \lambda_n) \quad \text{for all } (\lambda_1, \ldots, \lambda_n) \in C_n.$$
Prove that $q\colon C_n \to \mathrm{RPoly}_n^{\mathrm{sf}}$ is a regular cover with deck group the symmetric group $\mathfrak{S}_{n-1}$ on $(n-1)$ generators.
(c) Use the previous two parts to prove that $p\colon \mathrm{RPoly}_n^{\mathrm{sf}} \to \mathrm{Poly}_n^{\mathrm{sf}}$ is an irregular cover for $n \geq 3$. Hint: Fix a basepoint $f_0 \in \mathrm{Poly}_n^{\mathrm{sf}}$. Let $x_1, \ldots, x_n \in \mathbb{C}$ be the roots of $f_0$. We will then use $r_0 = (x_1, \ldots, x_n)$ for our basepoint of $C_n$ and $(f_0, x_n)$ as our basepoint for $\mathrm{RPoly}_n^{\mathrm{sf}}$. Applying

Theorem 9.5.1 to the cover from (a), we get a homomorphism $\phi\colon \pi_1(\mathrm{Poly}_n^{\mathrm{sf}}, f_0) \to \mathfrak{S}_n$ whose kernel is the image of $m_*\colon \pi_1(C_n, r_0) \to \pi_1(\mathrm{Poly}_n^{\mathrm{sf}}, f_0)$. Argue using (b) that the homomorphism $\phi$ takes the image of $p_*\colon \pi_1(\mathrm{RPoly}_n^{\mathrm{sf}}, (f_0, x_n)) \to \pi_1(\mathrm{Poly}_n^{\mathrm{sf}}, f_0)$ to the subgroup $\mathfrak{S}_{n-1}$ of $\mathfrak{S}_n$. Conclude by using the fact that $\mathfrak{S}_{n-1}$ is a non-normal subgroup of $\mathfrak{S}_n$ for $n \geq 3$. □

Exercise 9.10. Let $\mathbb{T}^n = (\mathbb{S}^1)^{\times n}$ be the $n$-torus. Using the group structure on $\mathbb{S}^1 \subset \mathbb{C}$ coming from the multiplication on $\mathbb{C}$, endow $\mathbb{T}^n$ with the structure of a topological group. Let $\mathrm{Aut}(\mathbb{T}^n)$ be the group of continuous group homomorphisms. Prove that $\mathrm{Aut}(\mathbb{T}^n) \cong \mathrm{GL}_n(\mathbb{Z})$.

Hint: Let $0 \in \mathbb{T}^n$ be the identity, and let $f\colon (\mathbb{T}^n, 0) \to (\mathbb{T}^n, 0)$ be a continuous group homomorphism. Let $p\colon (\mathbb{R}^n, 0) \to (\mathbb{T}^n, 0)$ be the pointed covering space obtained by taking the product of the universal cover $\mathbb{R} \to \mathbb{S}^1$. The space $\mathbb{R}^n$ is a topological group under addition, and $p$ is a group homomorphism. Prove that you can lift $f$ to a map $\widetilde{f}\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, then prove that $\widetilde{f}$ is a linear map, and then finally prove that $\widetilde{f} \in \mathrm{GL}_n(\mathbb{Z})$. The uniqueness of lifts will be important here! □

Exercise 9.11. Let $\mathbb{K}$ be the Klein bottle:

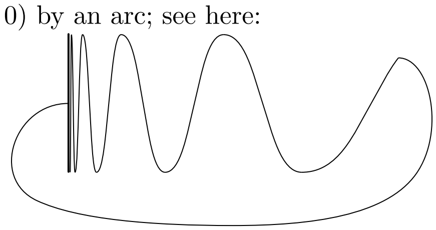



In this figure, the green loop on the left corresponds to the two green lines on the right, whose ends match up to form a circle. Prove that the fundamental group of $\mathbb{K}$ is the following group $\Gamma$:

- Let $\mathbb{Z}$ act on $\mathbb{Z}$ via the homomorphism $\phi\colon \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z})$ defined by
$$\phi(n)(m) = (-1)^n m \quad \text{for all } n \in \mathbb{Z} \text{ and } m \in \mathbb{Z}.$$
Then $\Gamma = \mathbb{Z} \rtimes \mathbb{Z}$, i.e., the semidirect product[2] of $\mathbb{Z}$ and $\mathbb{Z}$.

Hint: First prove that there is a homomorphism $f\colon \mathbb{Z} \rtimes \mathbb{Z} \to C_2$ with $\ker(f) = \mathbb{Z} \times \mathbb{Z}$. Then construct a regular degree 2 cover $p\colon \mathbb{T}^2 \to \mathbb{K}$ with deck group $C_2$ and use Theorem 9.5.1 to analyze the fundamental group of $\mathbb{K}$ using this cover. □

Exercise 9.12. The *quasi-circle* is the space $Y$ obtained from the topologist's sine curve
$$X = \big\{(x, \sin(1/x)) \in \mathbb{R}^2 \mid 0 < x \leq 1\big\} \cup \{(0, y) \mid -1 \leq y \leq 1\}$$
by connecting $(1, \sin(1))$ to $(0,0)$ by an arc; see here:



We saw in Exercise 4.11 that $X$ is 1-connected but has nontrivial covers. Note that this would be impossible if $Y$ were locally path connected (cf. Theorem 4.6.1). Prove the following:

- Let $f\colon Y \to \mathbb{S}^1$ be the map that collapses $\{(0, y) \mid -1 \leq y \leq 1\}$ to a point and identifies the resulting space with $\mathbb{S}^1$. Prove that $f$ cannot be lifted to the universal cover $p\colon \mathbb{R} \to \mathbb{S}^1$. Note that $f$ could be lifted if $Y$ were locally path connected (cf. Theorem 9.2.1). □

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[2]Let $G$ and $H$ be groups such that $G$ acts on $H$ on the left. For $g \in G$ and $h \in H$, we will write ${}^g h$ for the image of $h$ under the action of $g$. The corresponding semidirect product $H \rtimes G$ is the following group:
- The elements of $H \rtimes G$ are pairs $(h, g)$ with $h \in H$ and $g \in G$.
- For $(h_1, g_1), (h_2, g_2) \in H \rtimes G$, their product is $(h_1, g_1)(h_2, g_2) = (h_1\, {}^{g_1} h_2, g_1 g_2)$.

It is enlightening to prove that this is a group. If the $G$-action on $H$ is trivial, this is just the usual direct product $H \times G$.

CHAPTER 10

# Universal covers

Our next major goal is to classify covers of spaces in terms of fundamental group information. The first step is to construct what is called the universal cover.

## 10.1. Definition and examples of universal covers

Let $(X, x_0)$ be a pointed space that is path connected and locally path connected. Assume that there exists a pointed cover $p_u \colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ such that $\widetilde{X}_u$ is 1-connected. If such a cover exists, then Theorem 9.3.2 implies that it is unique up to isomorphism. It turns out that such a cover exists for most reasonable spaces. We will give a precise statement of this soon. The cover $p_u \colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ is called the *universal cover* of $(X, x_0)$. The following lemma explains this terminology: it covers all other covers.

LEMMA 10.1.1. *Let $(X, x_0)$ be a pointed space that is path connected and locally path connected. Let $p_u \colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ be a pointed cover such that $\widetilde{X}_u$ is 1-connected. Then for all covers $p \colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$, there exists a unique pointed cover $q \colon (\widetilde{X}_u, \widetilde{x}_u) \to (\widetilde{X}, \widetilde{x}_0)$ such that $p_u = p \circ q$, i.e., such that the following diagram commutes:*

$$\begin{tikzcd} (\widetilde{X}_u, \widetilde{x}_u) \arrow[r, "q"] \arrow[rr, bend right, "p_u"'] & (\widetilde{X}, \widetilde{x}_0) \arrow[r, "p"] & (X, x_0). \end{tikzcd}$$

PROOF. Since $\pi_1(\widetilde{X}_u, \widetilde{x}_u) = 1$, we can apply the lifting criterion (Theorem 9.2.1) to lift the map $p_u \colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ to a map $q \colon (\widetilde{X}_u, \widetilde{x}_u) \to (\widetilde{X}, \widetilde{x}_0)$ such that $p_u = p \circ q$. That $q$ is a covering space map follows from Lemma 10.1.2 below. $\square$

LEMMA 10.1.2. *Let $q \colon (Z, z_0) \to (Y, y_0)$ and $p \colon (Y, y_0) \to (X, x_0)$ be maps such that $p$ and $p \circ q$ are covering space maps. Assume that $X$ is locally connected. Then $q$ is a covering space map.*

PROOF. Consider $y \in Y$. Our goal is to find a trivialized open neighborhood $V$ of $y$ for $q$. Since $p$ and $p \circ q$ are covering space maps there exists:

- a trivialized open neighborhood $U_1 \subset X$ of $p(y)$ for $p$; and
- a trivialized open neighborhood $U_2 \subset X$ of $p(y)$ for $p \circ q$.

Since $X$ is locally connected, we can find a connected open neighborhood $U$ of $p(y)$ such that $U \subset U_1 \cap U_2$. The connected open set $U$ is a trivialized open set for both $p$ and $p \circ q$. Let the sheets above $U$ be

$$p^{-1}(U) = \bigsqcup_{i \in I} V_i \quad \text{and} \quad (p \circ q)^{-1}(U) = \bigsqcup_{j \in J} W_j.$$

Each $V_i$ and $W_j$ is connected, and they satisfy the following key property:

CLAIM. *For all $j \in J$, there exists some $i(j) \in I$ such that $q$ takes $W_j$ homeomorphically to $V_{i(j)}$.*

PROOF OF CLAIM. We have $q(W_j) \subset \sqcup_{i \in I} V_i$. Since $q(W_j)$ and each $V_i$ is connected, there must exist some $i(j) \in I$ such that $q(W_j) \subset V_{i(j)}$. The map $q|_{W_j} \colon W_j \to V_{i(j)}$ is then a homeomorphism with inverse the composition

$$V_{i(j)} \xrightarrow[\cong]{p|_{V_{i(j)}}} U \xrightarrow[\cong]{((p \circ q)|_{W_j})^{-1}} W_j. \qquad \square$$

Letting $i_0 \in I$ be such that $y \in V_{i_0}$, it follows from the above claim that $V = V_{i_0}$ is a trivialized open neighborhood of $y$ for $q$. The sheets of $q$ lying above $V_{i_0}$ are

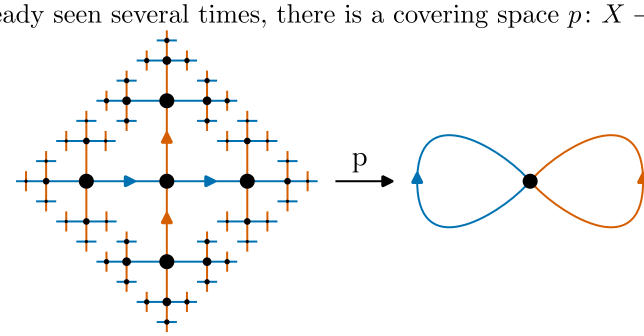
$$q^{-1}(V_{i_0}) = \bigsqcup_{\substack{j \in J \\ i(j)=i_0}} W_j. \qquad \square$$

We will also talk about the unpointed version of a universal cover: if $X$ is path connected and locally path connected and $p_u\colon \widetilde{X}_u \to X$ is a cover with $\widetilde{X}_u$ a 1-connected space, then $\widetilde{X}_u$ will be called the *universal cover* of $X$.

Example 10.1.3. If $X$ is a 1-connected space that is locally path connected, then $X$ is its own universal cover with covering space map the identity map $\mathbb{1}\colon X \to X$. $\square$

Example 10.1.4. Since $\mathbb{R}$ is 1-connected, the universal cover $p\colon \mathbb{R} \to \mathbb{S}^1$ we have discussed since the beginning of this book exhibits $\mathbb{R}$ as the universal cover of $\mathbb{S}^1$. $\square$

Example 10.1.5. Let $X$ be a graph with one vertex and two edges. Let $\widetilde{X}$ be a regular 4-valent tree. As we have already seen several times, there is a covering space $p\colon \widetilde{X} \to X$; see here:




Since $\widetilde{X}$ is a tree, it is contractible and hence 1-connected (see Lemma 6.4.1). It follows that $\widetilde{X}$ is the universal cover of $X$. $\square$

Example 10.1.6. For $n \geq 2$, the degree 2 cover $p\colon \mathbb{S}^n \to \mathbb{RP}^n$ taking $x \in \mathbb{S}^n$ to the line through $x$ exhibits $\mathbb{S}^n$ as the universal cover of $\mathbb{RP}^n$. $\square$

Remark 10.1.7. We emphasize that in the above we only defined the universal cover of a pointed space $(X, x_0)$ such that $X$ is path connected, locally path connected, and has a 1-connected cover. For general connected pointed spaces $(X, x_0)$, it is natural define a universal cover of $(X, x_0)$ to be a pointed cover $p_u\colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ satisfying the conclusion of Lemma 10.1.1:

- for all covers $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$, there exists a unique pointed cover $q\colon (\widetilde{X}_u, \widetilde{x}_u) \to (\widetilde{X}, \widetilde{x}_0)$ such that $p_u = p \circ q$, i.e., such that the following diagram commutes:

$$\begin{tikzcd} (\widetilde{X}_u, \widetilde{x}_u) \arrow[r, "q"] \arrow[rr, bend right, "p_u"'] & (\widetilde{X}, \widetilde{x}_0) \arrow[r, "p"] & (X, x_0). \end{tikzcd}$$

These might or might not exist. Even for spaces $X$ that are path connected and locally path connected, these general universal covers can behave in unexpected ways; for instance, they do not need to be 1-connected (see Exercise 10.12). We remark that pathological spaces like this rarely show up in algebraic topology. $\square$

## 10.2. Semilocal 1-connectedness

Let $X$ be a space that is path connected and locally path connected. As we said in §10.1, there might not exist a 1-connected cover of $X$. The following lemma gives one property of spaces with 1-connected covers:

Lemma 10.2.1. *Let $X$ be a space. Assume there exists a cover $p\colon \widetilde{X} \to X$ such that $\widetilde{X}$ is 1-connected. Then for all $x \in X$, there exists an open neighborhood $U$ of $x$ such that the map $\pi_1(U, x) \to \pi_1(X, x)$ is the trivial map.*

PROOF. Let $x \in X$, let $U \subset X$ be a trivialized open neighborhood of $X$ for $p\colon \widetilde{X} \to X$, and let $\widetilde{U} \subset \widetilde{X}$ be any sheet lying above $U$. Let $\widetilde{x} \in \widetilde{U}$ be the point lying in the fiber over $x$. We can factor the inclusion $(U, x) \to (X, x)$ as

$$(U, x) \xrightarrow{(p|_{\widetilde{U}})^{-1}} (\widetilde{U}, \widetilde{x}) \longrightarrow (\widetilde{X}, x) \xrightarrow{p} (X, x),$$

where $(\widetilde{U}, \widetilde{x}) \to (\widetilde{X}, x)$ is the inclusion. Passing to fundamental groups, the map $\pi_1(U, x) \to \pi_1(X, x)$ factors as

$$\begin{array}{ccccccc} \pi_1(U, x) & \xrightarrow{(p|_{\widetilde{U}})^{-1}_*} & \pi_1(\widetilde{U}, \widetilde{x}) & \longrightarrow & \pi_1(\widetilde{X}, x) & \xrightarrow{p_*} & \pi_1(X, x). \\ & & & & \| & & \\ & & & & 1 & & \end{array}$$

It follows that the map $\pi_1(U, x) \to \pi_1(X, x)$ is the trivial map, as desired. □

A space $X$ is *semilocally* 1*-connected* if for all $x \in X$, there exists an open neighborhood $U$ of $x$ such that the map $\pi_1(U, x) \to \pi_1(X, x)$ is the trivial map. Lemma 10.2.1 implies that spaces with 1-connected are semilocally 1-connected. Many spaces have this property:

EXAMPLE 10.2.2. Let $X$ be a space that is locally 1-connected, that is, for all $x \in X$ and all open neighborhoods $V$ of $x$, there is a 1-connected open neighborhood $U$ of $x$ with $U \subset V$. For this $U$, since $\pi_1(U, x) = 1$ it follows that the map $\pi_1(U, x) \to \pi_1(X, x)$ is trivial. We conclude that $X$ is semilocally 1-connected. Many spaces are locally 1-connected. Indeed, many satisfy even stronger local conditions like being locally contractible. For instance, manifolds are locally contractible. □

## 10.3. Existence of universal covers

The main result about semilocally 1-connected spaces is the following:

THEOREM 10.3.1 (Existence of universal covers). *Let $(X, x_0)$ be a pointed space. Assume that $X$ is path connected, locally path connected, and semilocally 1-connected. Then there exists a universal cover of $(X, x_0)$, i.e., a pointed cover $p_u\colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ such that $\widetilde{X}_u$ is 1-connected. This cover is a regular cover with deck group $\pi_1(X, x_0)$.*

REMARK 10.3.2. Let $(X, x_0)$ be path connected and locally path connected. Theorem 9.5.1 implies that a cover $p_u\colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ such that $\widetilde{X}_u$ is 1-connected is regular with deck group $\pi_1(X, x_0)$. The final sentence of Theorem 10.3.1 is therefore automatic. We include it for emphasis. □

The rest of this chapter is devoted to proving Theorem 10.3.1. We remark that while the details of the proof are beautiful, they are not important for applying it in concrete situations. Indeed, when trying to understand the fundamental group of a concrete space the first step is usually to identify its universal cover explictly like we did in the examples in the previous section. We therefore mark the rest of the sections of this chapter with $*$ to indicate that a reader might want to skip them when first learning the subject.

## 10.4*. Basis for topology of semilocally 1-connected spaces

Let $X$ be a space that is locally path connected and semilocally 1-connected. If $p\colon \widetilde{X} \to X$ is a cover, then all 1-connected open subsets $U \subset X$ are trivialized open sets for $p\colon \widetilde{X} \to X$ (see Exercise 4.10). Unfortunately, our assumptions do not imply that $X$ has any 1-connected open subsets. Our goal in this section is to use the fact that $X$ is semilocally 1-connected to identify a larger collection of open sets that are trivialized open sets for all covers of $X$.

Call a set $U \subset X$ an *sl1c subset* if $U$ is open and path connected, and for all $x \in U$ the map $\pi_1(U, x) \to \pi_1(X, x)$ is the trivial map. We have:

LEMMA 10.4*.1. *Let $X$ be a space that is locally path connected and semilocally 1-connected. The following hold:*

(i) *If $p\colon \widetilde{X} \to X$ is a cover of $X$ and $U \subset X$ is an sl1c subset, then $U$ is a trivialized open set for $X$.*

*(ii) If $U \subset X$ is an sl1c subset and if $U'$ is a subset of $U$ that is open and path connected, then $U'$ is an sl1c subset.*

*(iii) The sl1c subsets of $X$ form a basis for the topology on $X$.*

Proof. We will not use (i) in our proofs, so we leave it as an exercise (see Exercise 10.7). For (ii), let $U \subset X$ be an sl1c subset and let $U' \subset U$ be open and path connected. To prove that $U'$ is an sl1c subset, we must prove that for $x \in U'$ the map $\pi_1(U', x) \to \pi_1(X, x)$ is trivial. To see this, note that this map factors through the map $\pi_1(U, x) \to \pi_1(X, x)$, which is trivial by assumption.

For (iii), let $y \in X$ and let $W$ be an open neighborhood of $y$. We must find an sl1c subset $V$ of $y$ with $V \subset W$. Since $X$ is semilocally 1-connected, there is an open neighborhood $V'$ of $y$ such that the map $\pi_1(V', y) \to \pi_1(X, y)$ is trivial. Since $X$ is locally path connected, there is a path connected open neighborhood $V$ of $y$ with $V \subset W \cap V'$. The map $\pi_1(V, y) \to \pi_1(X, y)$ factors through the map $\pi_1(V', y) \to \pi_1(X, y)$, so $\pi_1(V, y) \to \pi_1(X, y)$ is trivial.

We claim that $V$ is an sl1c subset. To see this, consider $z \in V$. We must prove that the map $\pi_1(V, z) \to \pi_1(X, z)$ is trivial. Since $V$ is path connected, there is a path $\alpha$ in $V$ from $y$ to $z$. Letting $\alpha_*$ denote the change of basepoint isomorphism (see §5.4), we have a commutative diagram

$$\begin{tikzcd}
\pi_1(V,z) \arrow[r] \arrow[d, "\cong"', "\alpha_*"] & \pi_1(X,z) \arrow[d, "\cong"', "\alpha_*"] \\
\pi_1(V,y) \arrow[r] & \pi_1(X,y).
\end{tikzcd}$$

Since the map $\pi_1(V, y) \to \pi_1(X, y)$ is trivial, it follows that the map $\pi_1(V, z) \to \pi_1(X, z)$ is trivial. □

### 10.5*. Motivation for construction of universal cover

Let $(X, x_0)$ be a pointed space that is path connected, locally path connected, and semilocally 1-connected. Our goal is to construct the universal cover $\widetilde{X}_u$ of $X$. This should be a 1-connected cover of $X$. We start with some motivation. Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a pointed cover such that $\widetilde{X}$ is 1-connected. Consider $\widetilde{x} \in \widetilde{X}$. Since $\widetilde{X}$ is 1-connected, there is a unique homotopy class of paths $\widetilde{\gamma}$ in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}$. Projecting $\widetilde{\gamma}$ to $X$, we get a path $\gamma$ in $X$ from $x_0$ to $p(\widetilde{x})$. We can recover $\widetilde{\gamma}$ from $\gamma$ by lifting $\gamma$ to $\widetilde{X}$. We conclude that there is a bijection between the following two sets:

- The set of homotopy classes of paths $\gamma$ in $X$ with $\gamma(0) = x_0$.
- The set of points of $\widetilde{X}$.

This bijection takes a path $\gamma$ in $X$ with $\gamma(0) = x_0$ to the point $\widetilde{\gamma}(1) \in \widetilde{X}$, where $\widetilde{\gamma}$ is the lift of $\gamma$ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$.

Inspired by this, we will *define* the universal cover $\widetilde{X}_u$ to be the set of homotopy classes of paths in $X$ that start at $x_0$. This comes equipped with a set map $p_u\colon \widetilde{X}_u \to X$ defined by $p_u([\gamma]) = \gamma(1)$. Our main task will be to put a topology on $\widetilde{X}_u$ such that $p_u\colon \widetilde{X}_u \to X$ is a cover. The topology will be constructed from the sl1c sets in $X$, which by Lemma 10.4*.1 form a basis for the topology on $X$ and will all be trivialized open sets for $p_u\colon \widetilde{X}_u \to X$.

### 10.6*. Construction of universal cover

We now turn to the construction of the universal cover. Let $(X, x_0)$ be a pointed space. Assume that $X$ is path connected, locally path connected, and semilocally 1-connected. As a set, define

$$\widetilde{X}_u = \{[\gamma] \mid \gamma \text{ a path in } X \text{ with } \gamma(0) = x_0\}.$$

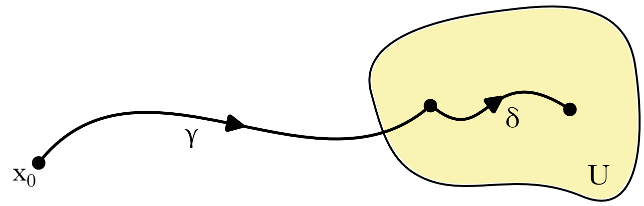
Let $p_u\colon \widetilde{X}_u \to X$ be the map

$$p_u([\gamma]) = \gamma(1) \quad \text{for } [\gamma] \in \widetilde{X}_u$$

and let $\widetilde{x}_u \in \widetilde{X}_u$ be the homotopy class of the constant path $\mathfrak{c}_{x_0}$. In this section, we endow $\widetilde{X}_u$ with a topology. The next section will prove that $p_u\colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ is a covering space, and the section after that will prove that $\widetilde{X}_u$ is 1-connected.

Let $[\gamma] \in \widetilde{X}_u$ and let $U \subset X$ be an sl1c subset such that $\gamma(1) \in U$. Define

$$\widetilde{U}_{[\gamma]} = \{[\gamma \cdot \delta] \mid \delta \text{ a path in } U \text{ with } \delta(0) = \gamma(1)\} \subset \widetilde{X}_u.$$




Set

$$\mathfrak{B} = \left\{ \widetilde{U}_{[\gamma]} \mid [\gamma] \in \widetilde{X}_u \text{ and } U \subset X \text{ is an sl1c subset with } \gamma(1) \in U \right\}.$$

Our goal is to prove that $\mathfrak{B}$ is the basis for a topology on $\widetilde{X}_u$. We need the following lemma:

LEMMA 10.6*.1. *Let the notation be as above. Let $U \subset X$ be an sl1c subset. The following hold:*

*(i) Let $[\gamma] \in \widetilde{X}_u$ be such that $\gamma(1) \in U$. Then the map $p_u$ takes $\widetilde{U}_{[\gamma]}$ bijectively to $U$.*

*(ii) Let $[\gamma], [\gamma'] \in \widetilde{X}_u$ be such that $\gamma(1), \gamma'(1) \in U$. Assume that $[\gamma'] \in \widetilde{U}_{[\gamma]}$. Then $\widetilde{U}_{[\gamma]} = \widetilde{U}_{[\gamma']}$.*

*(iii) Let $[\gamma], [\gamma'] \in \widetilde{X}_u$ be such that $\gamma(1), \gamma'(1) \in U$. Assume that $[\gamma'] \notin \widetilde{U}_{[\gamma]}$. Then $\widetilde{U}_{[\gamma]}$ and $\widetilde{U}_{[\gamma']}$ are disjoint.*

*(iv) Let $U' \subset X$ be an sl1c subset with $U' \subset U$. Let $[\gamma], [\gamma'] \in \widetilde{X}_u$ be such that $\gamma(1) \in U$ and $\gamma'(1) \in U'$. Assume that $[\gamma'] \in \widetilde{U}_{[\gamma]}$. Then $\widetilde{U}'_{[\gamma']} \subset \widetilde{U}_{[\gamma]}$.*

PROOF. We prove (i), then (iv), then (ii), and then (iii):

CLAIM (i). *Let $[\gamma] \in \widetilde{X}_u$ be such that $\gamma(1) \in U$. Then the map $p_u$ takes $\widetilde{U}_{[\gamma]}$ bijectively to $U$.*

Since $U$ is an sl1c subset of $X$, for each $y \in U$ there exists a path $\delta$ in $U$ from $\gamma(1)$ to $y$ and $\delta$ is unique up to homotopy in $X$. The claim follows.

CLAIM (iv). *Let $U' \subset X$ be an sl1c subset with $U' \subset U$. Let $[\gamma], [\gamma'] \in \widetilde{X}_u$ be such that $\gamma(1) \in U$ and $\gamma'(1) \in U'$. Assume that $[\gamma'] \in \widetilde{U}_{[\gamma]}$. Then $\widetilde{U}'_{[\gamma']} \subset \widetilde{U}_{[\gamma]}$.*

Since $[\gamma'] \in \widetilde{U}_{[\gamma]}$, we can write $[\gamma'] = [\gamma \cdot \delta]$ with $\delta$ a path in $U$ such that $\delta(0) = \gamma(1)$. Each element of $\widetilde{U}'_{[\gamma']}$ is of the form $[\gamma' \cdot \delta']$ with $\delta'$ a path in $U'$ such that $\delta'(0) = \gamma'(1)$, so

$$[\gamma' \cdot \delta'] = [\gamma \cdot (\delta \cdot \delta')] \in \widetilde{U}_{[\gamma]}.$$

The claim follows.

CLAIM (ii). *Let $[\gamma], [\gamma'] \in \widetilde{X}_u$ be such that $\gamma(1), \gamma'(1) \in U$. Assume that $[\gamma'] \in \widetilde{U}_{[\gamma]}$. Then $\widetilde{U}_{[\gamma]} = \widetilde{U}_{[\gamma']}$.*

Claim (iv) implies that $\widetilde{U}_{[\gamma']} \subset \widetilde{U}_{[\gamma]}$. We also have $[\gamma] = [\gamma' \cdot \overline{\delta}]$, so $[\gamma] \in \widetilde{U}_{[\gamma']}$ and we can apply Claim (iv) again to see that $\widetilde{U}_{[\gamma]} \subset \widetilde{U}_{[\gamma']}$. The claim follows.

CLAIM (iii). *Let $[\gamma], [\gamma'] \in \widetilde{X}_u$ be such that $\gamma(1), \gamma'(1) \in U$. Assume that $[\gamma'] \notin \widetilde{U}_{[\gamma]}$. Then $\widetilde{U}_{[\gamma]}$ and $\widetilde{U}_{[\gamma']}$ are disjoint.*

Assume for the sake of contradiction that there is some $[\gamma''] \in \widetilde{X}_u$ that lies in both $\widetilde{U}_{[\gamma]}$ and $\widetilde{U}_{[\gamma']}$. By Claim (ii), we have $\widetilde{U}_{[\gamma]} = \widetilde{U}_{[\gamma'']}$ and $\widetilde{U}_{[\gamma']} = \widetilde{U}_{[\gamma'']}$, so $\widetilde{U}_{[\gamma]} = \widetilde{U}_{[\gamma']}$. This contradicts our assumption that $[\gamma'] \notin \widetilde{U}_{[\gamma]}$. □

We now prove that $\mathfrak{B}$ is the basis for a topology on $\widetilde{X}_u$:

LEMMA 10.6*.2. *Let the notation be as above. Then $\mathfrak{B}$ is the basis for a topology on $\widetilde{X}_u$.*

PROOF. Consider $\widetilde{U}_{[\gamma]}, \widetilde{U}'_{[\gamma']} \in \mathfrak{B}$. We must prove that $\widetilde{U}_{[\gamma]} \cap \widetilde{U}'_{[\gamma']}$ is the union of elements of $\mathfrak{B}$. Consider some $[\gamma''] \in \widetilde{U}_{[\gamma]} \cap \widetilde{U}'_{[\gamma']}$. We must find an element of $\mathfrak{B}$ that contains $[\gamma'']$ and lies in $\widetilde{U}_{[\gamma]} \cap \widetilde{U}'_{[\gamma']}$. By Lemma 10.4*.1, the sl1c subsets of $X$ form a basis for the topology on $X$. We can therefore find an sl1c subset $U''$ of $X$ such that $\gamma''(1) \in U''$ and $U'' \subset U \cap U'$. Applying Lemma 10.6*.1, we see that $\widetilde{U}''_{[\gamma'']} \subset \widetilde{U}_{[\gamma]}$ and $\widetilde{U}''_{[\gamma'']} \subset \widetilde{U}'_{[\gamma']}$, so $\widetilde{U}''_{[\gamma'']} \subset \widetilde{U}_{[\gamma]} \cap \widetilde{U}'_{[\gamma']}$, as desired. □

In light of Lemma 10.6*.2, we can endow $\widetilde{X}_u$ with the topology with basis $\mathfrak{B}$.

### 10.7*. The universal cover is a cover

We now prove that the space we constructed in the previous section is a cover:

LEMMA 10.7*.1. *Let $(X, x_0)$ be a pointed space. Assume that $X$ is path connected, locally path connected, and semilocally 1-connected. Let $p_u\colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ be the space and map constructed in §10.6*. Then $p_u$ is a covering space.*

PROOF. The key to the proof is the following:

CLAIM. *Let $U \subset X$ be an sl1c subset of $X$ and let $[\gamma] \in \widetilde{X}_u$ satisfy $\gamma(1) \in U$. Then the restriction of $p_u$ to $\widetilde{U}_{[\gamma]}$ is a homeomorphism from $\widetilde{U}_{[\gamma]}$ to $U$.*

PROOF OF CLAIM. Let $q\colon \widetilde{U}_{[\gamma]} \to U$ be the restriction of $p_u$ to $\widetilde{U}_{[\gamma]}$. Lemma 10.6*.1 says that $q$ is a bijection of sets. Lemma 10.4*.1 implies that the topology on $U$ has for a basis the set of all sl1c subset $U' \subset U$. Lemma 10.6*.1 implies that each of these lifts to a unique element of the basis for the topology on $\widetilde{U}_{[\gamma]}$. The claim follows. □

Now fix an sl1c subset $U$ of $X$. Since the sl1c subset of $X$ form a basis for the topology on $X$ (Lemma 10.4*.1), it is enough to prove that $U$ is a trivialized open set in $X$ for $p_u$. For $x \in X$, set

$$\mathcal{P}(x_0, x) = p_u^{-1}(x) = \{[\gamma] \mid \gamma \text{ a path in } X \text{ from } x_0 \text{ to } x\}.$$

We therefore have

$$p_u^{-1}(U) = \bigcup_{u \in U} \mathcal{P}(x_0, u).$$

Fix some $u_0 \in U$. Since $U$ is an sl1c subset, for each $u \in U$ and each $[\gamma'] \in \mathcal{P}(x_0, u)$, there exists $[\gamma] \in \mathcal{P}(x_0, u_0)$ and a path $\delta$ in $U$ from $u_0$ to $u$ such that $[\gamma'] = [\gamma \cdot \delta] \in \widetilde{U}_{[\gamma]}$. We conclude that

$$p_u^{-1}(U) = \bigsqcup_{[\gamma] \in \mathcal{P}(x_0, u_0)} \widetilde{U}_{[\gamma]}.$$

The fact that this is a disjoint union follows from Lemma 10.6*.1. For $[\gamma] \in \mathcal{P}(x_0, u_0)$, the above claim implies that $p_u$ takes $\widetilde{U}_{[\gamma]}$ homeomorphically to $U$. We conclude that $U$ is a trivialized open set for $p_u$ and the $\widetilde{U}_{[\gamma]}$ with $[\gamma] \in \mathcal{P}(x_0, u_0)$ are the sheets lying above $U$, as desired. □

### 10.8*. 1-connectedness of universal cover

We now complete the proof of Theorem 10.3.1, whose statement we recall:

THEOREM 10.3.1 (Existence of universal covers). *Let $(X, x_0)$ be a pointed space. Assume that $X$ is path connected, locally path connected, and semilocally 1-connected. Then there exists a universal cover of $(X, x_0)$, i.e., a pointed cover $p_u\colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ such that $\widetilde{X}_u$ is 1-connected. This cover is a regular cover with deck group $\pi_1(X, x_0)$.*

PROOF. Let $p_u\colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ be the space and map constructed in §10.6*. Lemma 10.7*.1 says that $p_u$ is a covering space map. Below we will prove that $\widetilde{X}_u$ is 1-connected. Assuming this, Theorem 9.5.1 will imply that $\widetilde{X}_u$ is regular with deck group $\pi_1(X, x_0)$.

It remains to prove that $\widetilde{X}_u$ is 1-connected. Theorem 9.1.1 implies that $(p_u)_*\colon \pi_1(\widetilde{X}_u, \widetilde{x}_u) \to \pi_1(X, x_0)$ is injective and its image consists of all $[\gamma] \in \pi_1(X, x_0)$ such that the following holds:

(†) Let $\widetilde{\gamma}$ be the lift of $\gamma$ to $\widetilde{X}_u$ with $\widetilde{\gamma}(0) = \widetilde{x}_u$. Then $\widetilde{\gamma}$ is a loop, i.e., $\widetilde{\gamma}(1) = \widetilde{x}_u = [\mathfrak{c}_{x_0}] \in \widetilde{X}_u$. Here we recall that $\mathfrak{c}_{x_0}$ is the constant path in $X$ at the point $x_0$.

To prove that $\pi_1(\widetilde{X}_u, \widetilde{x}_u) = 1$, it is therefore enough to prove the following. Consider some $[\gamma] \in \pi_1(X, x_0)$. Let $\widetilde{\gamma}$ be the lift of $\gamma$ to $\widetilde{X}_u$ with $\widetilde{\gamma}(0) = \widetilde{x}_u = [\mathfrak{c}_{x_0}]$. Then $\widetilde{\gamma}(1) = [\gamma] \in \widetilde{X}_u$.

In fact, this is true for all paths, not just loops. Let $\gamma\colon I \to X$ be any path in $X$ with $\gamma(0) = x_0$. Let $\widetilde{\gamma}\colon I \to \widetilde{X}_u$ be the lift of $\gamma$ to $\widetilde{X}_u$ with $\widetilde{\gamma}(0) = [\mathfrak{c}_{x_0}]$. For $t \in I$, let $\gamma_t\colon I \to X$ be the path

$$\gamma_t(s) = \gamma(ts) \quad \text{for } s \in I.$$

We therefore have $\gamma_0 = \mathfrak{c}_{x_0}$ and $\gamma_1 = \gamma$. We claim that

$$\widetilde{\gamma}(t) = [\gamma_t] \in \widetilde{X}_u \quad \text{for all } t \in I. \tag{10.8*.1}$$

This will imply in particular that $\widetilde{\gamma}(1) = [\gamma_1] = [\gamma]$, establishing the theorem.

It remains to verify (10.8*.1). Using the Lebesgue number lemma just like in our proof of path lifting for covers (cf. Lemma 3.4.1), we can divide the domain $I$ of $\gamma\colon I \to X$ into subintervals

$$0 = \epsilon_1 < \epsilon_2 < \cdots < \epsilon_n = 1$$

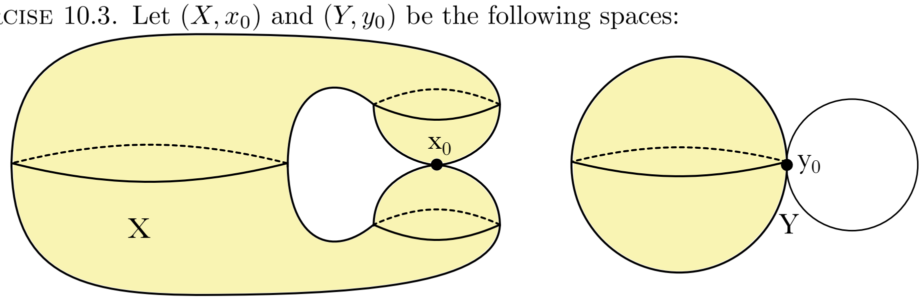
such that for all $1 \leq k < n$ the image $\gamma([\epsilon_k, \epsilon_{k+1}])$ is contained in an sl1c subset $U(k) \subset X$. For $t \in [\epsilon_k, \epsilon_{k+1}]$, we have $\gamma_t(1) = \gamma(t) \in U(k)$, so $[\gamma_t] \in \widetilde{U(k)}_{[\gamma_t]}$. The set $U(k)$ is a trivialized open set for $p_u$ and $\widetilde{U(k)}_{[\gamma_t]}$ is a sheet of $p_u$ lying above $U(k)$ (see the proof of Lemma 10.7*.1). We can therefore establish (10.8*.1) just like our proof of path lifting for covers (Lemma 3.4.1): we first first lift $\gamma$ on the interval $[\epsilon_1, \epsilon_2]$, then on the interval $[\epsilon_2, \epsilon_3]$, etc. □

## 10.9. Exercises

Exercise 10.1. Let $p\colon \widetilde{X} \to X$ be a cover. Prove that $X$ is semilocally 1-connected if and only if $\widetilde{X}$ is semilocally 1-connected. □

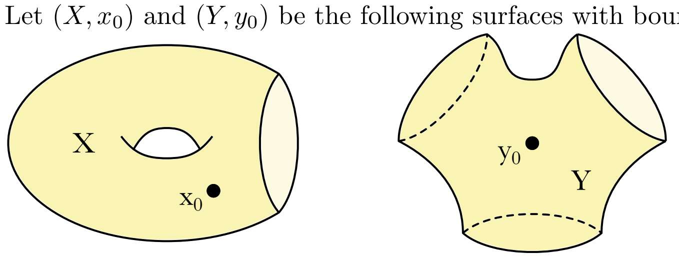
Exercise 10.2. Let $X$ be a semilocally 1-connected space and let $Y$ be a space that is homotopy equivalent to $X$. Prove that $Y$ is semilocally 1-connected. □

Exercise 10.3. Let $(X, x_0)$ and $(Y, y_0)$ be the following spaces:



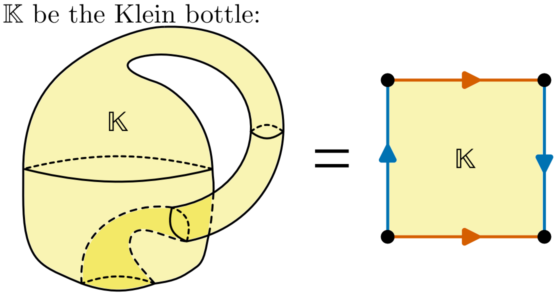

We proved in Example 8.6.1 that these spaces are homotopy equivalent, and also that $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$ are isomorphic to $\mathbb{Z}$. Explictly construct the universal covers of $X$ and $Y$. □

Exercise 10.4. Let $(X, x_0)$ and $(Y, y_0)$ be the following surfaces with boundary:




We proved in Example 8.2.1 that these spaces are homotopy equivalent, and also that $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$ are free groups on two generators. Explictly construct the universal covers of $X$ and $Y$. □

Exercise 10.5. Let $\mathbb{K}$ be the Klein bottle:

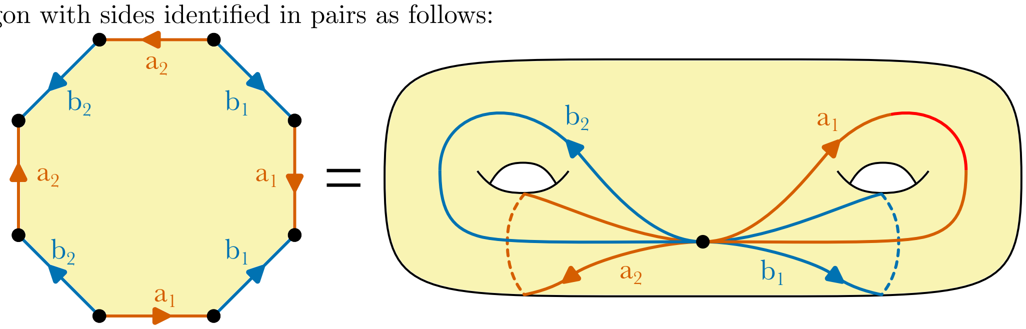



In Exercise 9.11, you proved that the fundamental group of $\mathbb{K}$ is the following group $\Gamma$:

- Let $\mathbb{Z}$ act on $\mathbb{Z}$ via the homomorphism $\phi\colon \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z})$ defined by
$$\phi(n)(m) = (-1)^n m \quad \text{for all } n \in \mathbb{Z} \text{ and } m \in \mathbb{Z}.$$
Then $\Gamma = \mathbb{Z} \rtimes \mathbb{Z}$, i.e., the semidirect product[1] of $\mathbb{Z}$ and $\mathbb{Z}$.

Construct a covering space action of $\Gamma$ on $\mathbb{R}^2$ such that $\mathbb{R}^2/\Gamma \cong \mathbb{K}$. This shows that the universal cover of $\mathbb{K}$ is $\mathbb{R}^2$, and also gives a new proof that the fundamental group of $\mathbb{K}$ is $\Gamma$. $\square$

Exercise 10.6. Let $\Sigma_2$ be a closed oriented genus 2 surface, which can be identified with an octagon with sides identified in pairs as follows:




Prove that the universal cover of $\Sigma_2$ is homeomorphic to $\mathbb{R}^2$. Hint: Let $P \cong \mathbb{D}^2$ be an octagon, so the above picture shows a surjection $f\colon P \to \Sigma_2$. The map $f$ is an open embedding on the interior of $P$, but is not injective on the boundary. Construct a space $\widetilde{S} \cong \mathbb{R}^2$ and a covering map $p\colon \widetilde{S} \to \Sigma_2$ by carefully gluing together infinitely many copies of $P$ and letting $p$ equal $f$ on each copy of $P$. We remark that this same argument will show that for all $g \geq 1$ the universal cover of a closed genus $g$ surface is homeomorphic to $\mathbb{R}^2$. $\square$

Exercise 10.7. Let $X$ be a space. Prove the following:

(a) Let $p\colon \widetilde{X} \to X$ be a cover with $X$ locally path connected and let $U \subset X$ be an sl1c-set. Prove that $U$ is a trivialized open set for $p\colon \widetilde{X} \to X$. Hint: imitate the proof of Theorem 4.6.1.

(b) Let $p\colon Y \to X$ and $q\colon Z \to Y$ be covers. Assume that $X$ is locally path connected and semilocally 1-connected. Prove that $p \circ q\colon Z \to X$ is a cover. We remark that the composition of two covering space maps is not always a covering space map (see Exercise 2.14**). $\square$

Exercise 10.8. Let $(X, x_0)$ be a pointed space that is path connected, locally path connected, and semilocally 1-connected. Let $p_u\colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ be the universal cover given by the construction from §10.6*, so the points of $\widetilde{X}_u$ are $[\gamma]$ with $\gamma$ a path in $X_0$ starting at $x_0$. Do the following:

(a) We know that the deck group of $p_u\colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ is isomorphic to $\pi_1(X, x_0)$. For $[\delta] \in \pi_1(X, x_0)$ and $[\gamma] \in \widetilde{X}_u$, give an explicit formula for the action of $[\delta]$ on $[\gamma]$. Make sure to prove that this indeed gives a deck transformation!

(b) Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be another cover. We know that there is a cover $q\colon (\widetilde{X}_u, \widetilde{x}_u) \to (\widetilde{X}, \widetilde{x}_0)$ such that $p \circ q = p_u$. For $[\gamma] \in \widetilde{X}_u$, given an explicit description of $q([\gamma]) \in \widetilde{X}$. Make sure to prove that this does indeed give a cover $q$. $\square$

Exercise 10.9. Let $(X, x_0)$ be a connected pointed space. Call a pointed cover $p_u\colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ a *universal cover* of $(X, x_0)$ if it satisfies the following condition:

- If $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a cover, then there exists a unique pointed cover $q\colon (\widetilde{X}_u, \widetilde{x}_u) \to (\widetilde{X}, \widetilde{x}_0)$ such that $p_u = p \circ q$, i.e., such that the following diagram commutes:

---

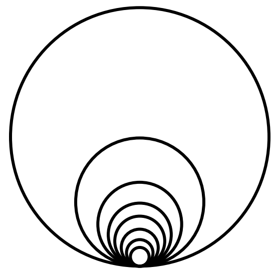
[1] Let $G$ and $H$ be groups such that $G$ acts on $H$ on the left. For $g \in G$ and $h \in H$, we will write ${}^g h$ for the image of $h$ under the action of $g$. The corresponding semidirect product $H \rtimes G$ is the following group:
- The elements of $H \rtimes G$ are pairs $(h, g)$ with $h \in H$ and $g \in G$.
- For $(h_1, g_1), (h_2, g_2) \in H \rtimes G$, their product is $(h_1, g_1)(h_2, g_2) = (h_1 \, {}^{g_1}h_2, g_1 g_2)$.

It is enlightening to prove that this is a group. If the $G$-action on $H$ is trivial, this is just the usual direct product $H \times G$.

$$\begin{tikzcd}(\widetilde{X}_u, \widetilde{x}_u) \arrow[r, "q"] \arrow[rr, bend right, "p_u"'] & (\widetilde{X}, \widetilde{x}_0) \arrow[r, "p"] & (X, x_0).\end{tikzcd}$$

We remark that these do not always exist (see Exercises 10.10–10.11). Assume that $p_u \colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ and $p'_u \colon (\widetilde{X}'_u, \widetilde{x}'_u) \to (X, x_0)$ are two universal covers of $(X, x_0)$. Prove that $p_u \colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ and $p'_u \colon (\widetilde{X}'_u, \widetilde{x}'_u) \to (X, x_0)$ are isomorphic covers. □

EXERCISE 10.10. For $n \geq 1$, let $C_n \subset \mathbb{R}^2$ be the circle of radius $1/n$ with center $(0, 1/n)$. Let $X = \cup_{n=1}^{\infty} C_n$, topologized as a subspace of $\mathbb{R}^2$. This is sometimes called the "earring space" or the "shrinking wedge of circles":



Prove the following:

(a) The space $X$ is path connected.
(b) The space $X$ is locally path connected.
(c) The space $X$ has no 1-connected cover.
(d) Defining a universal cover as in Exercise 10.9, for all $x_0 \in X$ prove that $(X, x_0)$ has no universal cover. □

EXERCISE 10.11. Let $X = \prod_{n=1}^{\infty} \mathbb{S}^1$, endowed with the product topology. Prove the following:

(a) The space $X$ is path connected.
(b) The space $X$ is locally path connected.
(c) The space $X$ has no 1-connected cover.
(d) Defining a universal cover as in Exercise 10.9, for all $x_0 \in X$ prove that $(X, x_0)$ has no universal cover. □

EXERCISE 10.12. This exercise uses the notion of the cone of a space from Exercise 6.2. Let $X$ be the earring space from Exercise 10.10 and let $x_0 \in X$ be the point $(0, 0) \in X$. By Exercise 6.2, the space $\mathrm{Cone}(X)$ is contractible. The contractible space $\mathrm{Cone}(X)$ contains the point $x_0$, which we emphasize is *not* the cone point. Let $(Y_1, y_1)$ and $(Y_2, y_2)$ be two copies of the pointed space $(\mathrm{Cone}(X), x_0)$. Let $(Z, z_0)$ be the wedge product of $(Y_1, y_1)$ and $(Y_2, y_2)$. Later in Exercise 16.8 you will prove that $\pi_1(Z, z_0)$ is nontrivial.[2] Prove the following:

(a) The space $Z$ is path connected.
(b) The space $Z$ is locally path connected.
(c) Every cover of $Z$ is trivial.

It follows that $Z$ is not semilocally 1-connected. However, by (c) the space $Z$ is its own universal cover in the generalized sense of Exercise 10.9 even though $Z$ is not 1-connected. □

EXERCISE 10.13. Let $(X, x_0)$ be a space that is path connected, locally path connected, and semilocally 1-connected. Let

$$C_0(I, (X, x_0)) = \{\gamma \colon I \to X \mid \gamma \text{ continuous and } \gamma(0) = x_0\}.$$

Give $C_0(I, (X, x_0))$ the compact open topology. Let $p_u \colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ be the universal cover given by the construction from §10.6* and let $f \colon C_0(I, (X, x_0)) \to \widetilde{X}_u$ be the map defined by $f(\gamma) = [\gamma]$. Prove that $f$ is a quotient map, so a set $V \subset \widetilde{X}_u$ is open if and only if $f^{-1}(V)$ is open. Warning: this is a fairly difficult problem. □

[2] This could be proved now, but will make more sense in the context of that exercise. We remark that this depends on the fact that the basepoint of $\mathrm{Cone}(X)$ is $x_0$, and would be false if we used the cone point as the basepoint.

CHAPTER 11

# The Galois correspondence for covers

We now describe one version of the classification of covering spaces.

## 11.1. Subgroup corresponding to cover

Let $(X, x_0)$ be a pointed space and let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a pointed cover. We proved in Theorem 9.1.1 that the induced map

$$p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$$

is injective. We call its image the subgroup of $\pi_1(X, x_0)$ corresponding to the cover.

Assume now that $X$ is path connected and locally path connected. Call $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ a connected pointed cover if $\widetilde{X}$ is path connected. We proved in Theorem 9.3.2 that a connected pointed cover of $(X, x_0)$ is determined up to isomorphism by its corresponding subgroup of $\pi_1(X, x_0)$.

Example 11.1.1. If $(X, x_0)$ is any pointed space, then the subgroup of $\pi_1(X, x_0)$ corresponding to the cover $\mathbb{1}\colon (X, x_0) \to (X, x_0)$ is the whole group $\pi_1(X, x_0)$. □

Example 11.1.2. If $(X, x_0)$ is path connected, locally path connected, and semilocally 1-connected then the subgroup of $\pi_1(X, x_0)$ corresponding to its universal cover $p_u\colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ is the trivial subgroup $1 < \pi_1(X, x_0)$. □

## 11.2. Galois correspondence

The first version of the classification of covering spaces is as follows. For reasons we will describe soon, it should be thought of as analogous to the classical Galois correspondence.

Theorem 11.2.1 (Galois correspondence). *Let $(X, x_0)$ be a pointed space that is path connected, locally path connected, and semilocally 1-connected. There is then a bijection between the following two sets:*

- *The set of isomorphism classes of connected pointed covers $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$.*
- *The set of subgroups of $\pi_1(X, x_0)$.*

*This bijection takes a connected pointed cover $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ to its corresponding subgroup, i.e., to the image $K$ of the map $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$. The degree of the cover $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is the index of $K$ in $\pi_1(X, x_0)$.*

Proof. Let $\Gamma = \pi_1(X, x_0)$ and let $K < \Gamma$ be a subgroup. Theorem 9.3.2 says that there is at most one isomorphism class of connected pointed covers corresponding to $K$, and Theorem 9.1.1 says that its degree is the index of $K$ in $\Gamma$. What we must prove is that there exists a connected pointed cover whose corresponding subgroup is $K$.

Let $p_u\colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ be the universal cover provided by Theorem 10.3.1. This is a regular cover with deck group $\Gamma$. We can therefore identify $X$ with $\widetilde{X}_u/\Gamma$. Set $\widetilde{X} = \widetilde{X}_u/K$, and let $\widetilde{x}_0$ be the image of $\widetilde{x}_u$ under the projection $q\colon \widetilde{X}_u \to \widetilde{X}_u/K = \widetilde{X}$. Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be the projection

$$\widetilde{X} = \widetilde{X}_u/K \twoheadrightarrow \widetilde{X}_u/\Gamma = X.$$

We claim that $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering space and that the image of $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is $K$.

To see that $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering space, consider $x \in X$. Let $U \subset X$ be a trivialized neighborhood for $p_u\colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ and let $\widetilde{U}_u \subset \widetilde{X}_u$ be the sheet lying above $U$ with $\widetilde{x}_u \in \widetilde{U}_u$. The deck group $\Gamma$ acts freely and transitively on the sheets of the universal cover lying above $U$, so

$$p_u^{-1}(U) = \bigsqcup_{g \in \Gamma} g\widetilde{U}_u.$$

Let $\{g_c \mid c \in \Gamma/K\}$ be a set of left coset representatives for $K$ in $\Gamma$. Recalling that $q\colon \widetilde{X}_u \to \widetilde{X}_u/K = \widetilde{X}$ is the projection, it follows that

$$p^{-1}(U) = \bigsqcup_{c \in \Gamma/K} q(g_c\widetilde{U})$$

and that that the restriction of $p$ to each $q(g_c\widetilde{U})$ is a homeomorphism. We conclude that $U$ is a trivialized neighborhood for $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ and that the $q(g_c\widetilde{U})$ are the sheets lying above $U$. In particular, $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering space.

It remains to prove that the image of $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is $K$. For this, recall that Theorem 9.1.1 says that the image of $p_*$ consists of all $[\gamma] \in \pi_1(X, x_0)$ such that the following holds:

(†) Let $\widetilde{\gamma}$ be the lift of $\gamma$ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$. Then $\widetilde{\gamma}$ is a loop, i.e., $\widetilde{\gamma}(1) = \widetilde{x}_0$.

Consider some $g = [\gamma] \in \pi_1(X, x_0)$. Let $\widetilde{\gamma}$ be the lift of $\gamma$ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$ and let $\widetilde{\gamma}_u$ be the lift of $\gamma$ to the universal cover $\widetilde{X}_u$ with $\widetilde{\gamma}_u(0) = \widetilde{x}_u$. The path $\widetilde{\gamma}_u$ projects to $\widetilde{\gamma}$, and $\widetilde{\gamma}_u(1) = g\widetilde{x}_u$. It follows that $\widetilde{\gamma}$ is a loop if and only if $g\widetilde{x}_u$ maps to $\widetilde{x}_0 \in \widetilde{X}$, i.e., if and only if $g \in K$, as desired. □

Before giving some examples of Theorem 11.2.1, we extract the following from its proof:

Corollary 11.2.2. *Let $(X, x_0)$ be a pointed space that is path connected, locally path connected, and semilocally 1-connected. Set $\Gamma = \pi_1(X, x_0)$, and let $K < \Gamma$ be a subgroup. Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be the connected pointed cover whose corresponding subgroup is $K$ and let $p_u\colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ be the universal cover. Then $\widetilde{X} = \widetilde{X}_u/K$.*

Proof. Immediate from the proof of Theorem 11.2.1. □

We now give some examples of Theorem 11.2.1:

Example 11.2.3 (Circle). We have $\pi_1(\mathbb{S}^1, 1) = \mathbb{Z}$. The subgroups of $\mathbb{Z}$ are all of the form $n\mathbb{Z}$ for some $n \geq 0$. These correspond to the following covers:

- For $n \geq 1$, the index $n$ subgroup $n\mathbb{Z} < \mathbb{Z}$ corresponds to the degree $n$ cover $p_n\colon (\mathbb{S}^1, 1) \to (\mathbb{S}^1, 1)$ defined by
$$p_n(z) = z^n \quad \text{for } z \in \mathbb{S}^1 \subset \mathbb{C}.$$
In particular, the whole group $1\mathbb{Z} = \mathbb{Z}$ corresponds to the trivial cover $\mathbb{1}_{\mathbb{S}^1}\colon (\mathbb{S}^1, 1) \to (\mathbb{S}^1, 1)$.
- The infinite index trivial subgroup $0 < \mathbb{Z}$ corresponds to the universal cover $p\colon \mathbb{R} \to \mathbb{S}^1$. □

Example 11.2.4 (Real projective space). Let $n \geq 2$ and let $x_0 \in \mathbb{RP}^n$ be a basepoint. We have $\pi_1(\mathbb{RP}^n, x_0) = C_2$. There are two subgroups of $C_2$:

- The whole group $C_2$ has index 1 and corresponds to the trivial cover $\mathbb{1}_{\mathbb{RP}^n}\colon (\mathbb{RP}^n, x_0) \to (\mathbb{RP}^n, x_0)$.
- The index 2 trivial group $0 < C_2$ corresponds to the degree 2 cover $p\colon (\mathbb{S}^n, \widetilde{x}_0) \to (\mathbb{RP}^n, x_0)$, where $\widetilde{x}_0 \in \mathbb{S}^n$ is a point projecting to $x_0$. This is the universal cover of $\mathbb{RP}^n$. □

### 11.3. Comparison with classical Galois correspondence

Theorem 9.3.2 should be viewed as an analogue for spaces of the classical Galois correspondence.[1] Letting $L/K$ be a finite Galois extension of fields, the classical Galois correspondence is a bijection between:

- fields $E$ with $K \subset E \subset L$; and

[1] Understanding this analogy is not essential for understanding covering spaces, so a reader should not worry if they are unfamiliar with the classical Galois correspondence.

- subgroups of the Galois group $\mathrm{Gal}(L/K)$, which we recall is the set of automorphisms of $L$ that fix the subfield $K$.

This bijection tales a subgroup $G$ of $\mathrm{Gal}(L/K)$ to the subfield $L^G = \{\ell \in L \mid g\ell = \ell \text{ for all } g \in G\}$. The degree of the field extension $L^G/K$ is the index of the subgroup $G$ of $\mathrm{Gal}(L/K)$.

We hope the analogy is clear: the field $K$ corresponds to the base of the cover, the Galois group $\mathrm{Gal}(L/K)$ corresponds to the fundamental group, and the field $L = L^1$ corresponds to the universal cover. Another feature of the classical Galois correspondence is that it is order-reversing:

- If $G_1, G_2 < \mathrm{Gal}(L/K)$ are subgroups with $G_1 < G_2$, then the corresponding fields satisfy $L^{G_2} < L^{G_1}$.

Something similar holds for covers, where the relation "is contained in" is replaced with the relation "covers". The following generalizes the fact that a 1-connected cover is universal, that is, covers all other covers (Lemma 10.1.1), which corresponds to the special case $K_1 = 1$.

THEOREM 11.3.1. *Let $(X, x_0)$ be a pointed space that is path connected, locally path connected, and semilocally 1-connected. Let $p_1\colon (\widetilde{X}_1, \widetilde{x}_1) \to (X, x_0)$ and $p_2\colon (\widetilde{X}_2, \widetilde{x}_2) \to (X, x_0)$ be connected pointed covers corresponding to subgroups $K_1, K_2 < \pi_1(X, x_0)$ satisfying $K_1 < K_2$. Then there is a pointed covering map $q\colon (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ such that the following diagram commutes:*

$$\begin{tikzcd} (\widetilde{X}_1, \widetilde{x}_1) \arrow[r, "q"] \arrow[rr, bend right, "p_1"'] & (\widetilde{X}_2, \widetilde{x}_2) \arrow[r, "p_2"] & (X, x_0). \end{tikzcd}$$

PROOF. Since $K_1 < K_2$, we can apply the lifting criterion (Theorem 9.2.1) to lift the map $p_1\colon (\widetilde{X}_1, \widetilde{x}_1) \to (X, x_0)$ to a map $q\colon (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ such that $p_1 = p_2 \circ q$. That $q$ is a covering space map follows from Lemma 10.1.2. □

Here is an example:

EXAMPLE 11.3.2 (Circle). For $n \geq 1$, the cover of $(\mathbb{S}^1, 1)$ corresponding to the subgroup $n\mathbb{Z} < \mathbb{Z} = \pi_1(\mathbb{S}^1, 1)$ is the cover $p_n\colon (\mathbb{S}^1, 1) \to (\mathbb{S}^1, 1)$ defined by $p_n(z) = z^n$. For $n, m \geq 1$, we have $n\mathbb{Z} < m\mathbb{Z}$ if and only if $m$ divides $n$. In this case, we have the covers

$$\begin{tikzcd} (\mathbb{S}^1, 1) \arrow[r, "p_{m/n}"] \arrow[rr, bend right, "p_m"'] & (\mathbb{S}^1, 1) \arrow[r, "p_n"] & (\mathbb{S}^1, 1) \end{tikzcd}$$

as in Theorem 11.3.1. □

## 11.4. Subgroups of free groups are free

We now explain an application of Theorem 11.2.1 to group theory. The following theorem was originally proved algebraically by Nielsen (who proved it for finitely generated subgroups) and Schreier (who proved it in general).

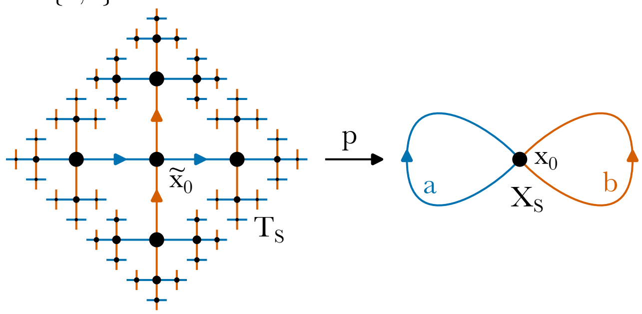
THEOREM 11.4.1. *Let $F(S)$ be the free group on a set $S$ and let $G$ be a subgroup of $F(S)$. Then $G$ is a free group.*

PROOF. Let $X_S$ be the graph with one vertex $x_0$ and with $|S|$ oriented edges, each labeled with an element of $S$. Identify each $s \in S$ with the corresponding loop in $X_S$ based at $x_0$. Theorem 7.8.1 says that $\pi_1(X_S, x_0)$ is a free group on $\{[s] \mid s \in S\}$. By Theorem 11.2.1, there is a pointed cover $q\colon (\widetilde{X}(G), \widetilde{x}_G) \to (X_S, x_0)$ whose corresponding subgroup is $G$. Since $X_S$ is a graph, one way to proceed would be to appeal to Exercise 4.8, which says that all covers of $X_S$ are also graphs. This would imply that $\widetilde{X}(G)$ is a graph, and thus that $G \cong \pi_1(\widetilde{X}(G), \widetilde{x}_G)$ is a free group (Theorem 8.8.1).

Instead of doing this, we give a more explicit approach that will later allow us to compute free generators for $G$. In the proof of Theorem 7.8.1, we identified the universal cover of $(X_S, x_0)$, though of course that term had not yet been defined. Namely, let $T_S$ to be an infinite tree each of whose vertices has valence $2|S|$. Label the oriented edges of $T_S$ by elements of $S$ such that for each vertex $v$ of $T_S$ there are:

- $|S|$ edges coming out of $v$ labeled by elements of $S$; and
- $|S|$ edges going into $v$ labeled by elements of $S$.

Fix a vertex $\widetilde{x}_0$ of $T_S$. There is a pointed covering space $p\colon (T_S, \widetilde{x}_0) \to (X_S, x_0)$ taking each vertex of $T_S$ to $x_0$ and each oriented edge of $T_S$ labeled by $s \in S$ to the corresponding loop in $X_S$ labeled by $s$. For instance, if $S = \{a, b\}$ this is the cover shown here:




The tree $T_S$ is contractible (Lemma 6.4.1), so $T_S$ is the universal cover of $X_S$. In particular, it is a regular cover with deck group $F(S)$. Let $\widetilde{X}(G) = T_S/G$ and let $\widetilde{x}_G \in \widetilde{X}(G)$ be the image of of $\widetilde{x}_0 \in T_S$. The map $p$ factors through a map $p_G\colon (\widetilde{X}(G), \widetilde{x}_G) \to (X_S, x_0)$. Corollary 11.2.2 says that $p_G\colon (\widetilde{X}(G), \widetilde{x}_G) \to (X_S, x_0)$ is the pointed connected cover corresponding to $G$. In particular,

$$\pi_1(\widetilde{X}(G), \widetilde{x}_G) \cong G.$$

Since $\widetilde{X}(G)$ is a graph, Theorem 8.8.1 says that $\pi_1(\widetilde{X}(G), \widetilde{x}_G)$ is a free group. The theorem follows. □

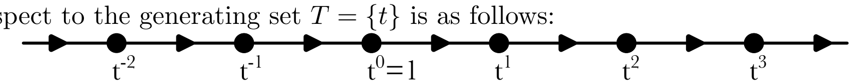
## 11.5. Computing free generators for subgroups of free groups

As in Theorem 11.4.1, let $F(S)$ be a free group on a set $S$ and let $G$ be a subgroup of $F(S)$. The proof of Theorem 11.4.1 actually gives an algorithm to compute free generators for $G$. We start with the following definition:

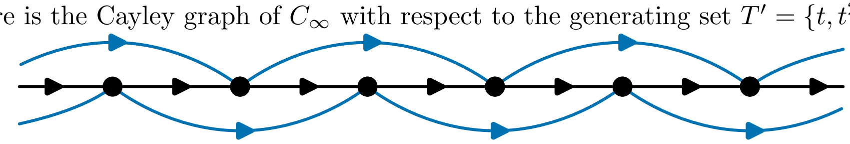
Definition 11.5.1. Let $H$ be a group with generating set $T$. The *Cayley graph* of $H$ with respect to $T$, denoted $\operatorname{Cay}(H, T)$, is the following graph:

- The vertices of $\operatorname{Cay}(H, T)$ are the elements of $H$.
- For each $h \in H$ and $t \in T$, there is an oriented edge of $\operatorname{Cay}(H, T)$ connecting the vertices $h$ and $ht$. □

Example 11.5.2. Consider the infinite cyclic group $C_\infty$ generated by $t$. The Cayley graph of $C_\infty$ with respect to the generating set $T = \{t\}$ is as follows:




The group $C_\infty$ has many other generating sets, and each of these has a different Cayley graph. For instance, here is the Cayley graph of $C_\infty$ with respect to the generating set $T' = \{t, t^2\}$:



Here the edges corresponding to $t^2$ are in blue. □

For a group $H$ with a generating set $T$, the graph $\operatorname{Cay}(H, T)$ is connected. Indeed, for $h \in H$ we can write $h = t_1^{e_1} \cdots t_n^{e_n}$ with $t_1, \ldots, t_n \in T$ and $e_1, \ldots, e_n \in \{\pm 1\}$. The following is then an edge path from the vertex $1 \in H$ to the vertex $h \in H$:

$$1, t_1^{e_1}, t_1^{e_1} t_2^{e_2}, t_1^{e_1} t_2^{e_2} t_3^{e_3}, \ldots, t_1^{e_1} \cdots t_n^{e_n} = h.$$

Here we are using the fact that for $h' \in H$ and $t \in T$ there is an oriented edge from $h' t^{-1}$ to $h'$. The group $H$ acts on $\operatorname{Cay}(H, T)$ on the left. This action is transitive on the vertices, and the orbits of the edges are in bijection with $T$.

Remark 11.5.3. In the above, we allow the generating set $T$ to have repeated elements. It is also allowed to contain the identity element $1 \in H$. The same is true in what we do below. □

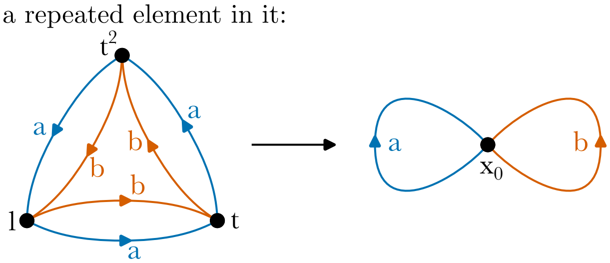
We now return to the setting of free groups. The tree $T_S$ constructed in the proof of Theorem 11.4.1 is exactly $\mathrm{Cay}(F(S), S)$. It follows that the graph $\widetilde{X}(G)$ from that proof with fundamental group $G < F(S)$ is $\mathrm{Cay}(F(S), S)/G$. If $G$ is a normal subgroup, then letting $\overline{S}$ be the image of $S$ in $F(S)/G$ we have $\mathrm{Cay}(F(S), S)/G \cong \mathrm{Cay}(F(S)/G, \overline{S})$. More generally, $\mathrm{Cay}(F(S), S)/G$ is the *Schreier graph* of $F(S)/G$ with respect to $S$, whose definition is as follows:

Definition 11.5.4. Let $H$ be a group with generating set $T$ and let $K < H$ be a subgroup. The *Schreier graph* of $H/K$ with respect to $T$, denoted $\mathrm{Sch}(H, K, T)$, is the following graph:

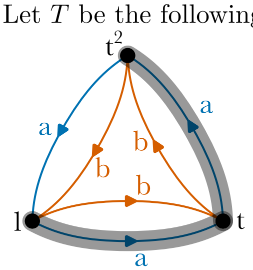
- The vertices of $\mathrm{Sch}(H, K, T)$ are the right cosets $Kh$ with $h \in H$.
- For each right coset $Kh$ and generator $t \in T$, there is an oriented edge of $\mathrm{Sch}(H, K, T)$ connecting the vertices $Kh$ and $Kht$. □

Here are three examples of how to use all this to compute free generators for subgroup $G$ of free group $F(S)$:

Example 11.5.5. Consider the free group $F(a, b)$ on $a$ and $b$. Let $C_3$ be the cyclic group of order 3 generated by $t$. Let $G$ be the kernel of the homomorphism $F(a, b) \to C_3$ taking $a$ and $b$ to $t$. Following the above recipe, the group $G$ is the fundamental group of the Cayley graph of $F(a, b)/G \cong C_3$ with respect to the generating set $\{a, b\}$. These map to the same element of $C_3$, so this generating set has a repeated element in it:

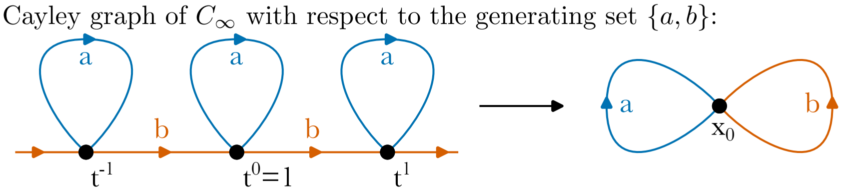



The basepoint is the vertex labeled 1. Let $T$ be the following maximal tree in this Cayley graph:

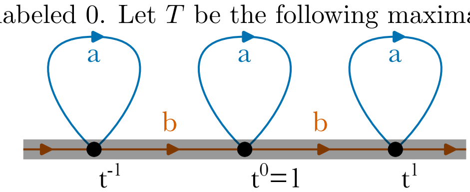



Using the recipe for computing fundamental groups of graphs from §8.9, we see that $G < F(a, b)$ is a free group on the following four generators: $\{ba^{-1}, aba^{-2}, a^3, a^2b\}$. □

Example 11.5.6. Consider the free group $F(a, b)$ on $a$ and $b$. Let $G$ be the normal subgroup of $F(a, b)$ generated by $a$. We thus have $F(a, b)/G \cong C_\infty$. Under this isomorphism, $a$ maps the generator $t \in C_\infty$ and $b$ maps to $t^0 = 1$. Following the above recipe, the group $G$ is the fundamental group of the Cayley graph of $C_\infty$ with respect to the generating set $\{a, b\}$:

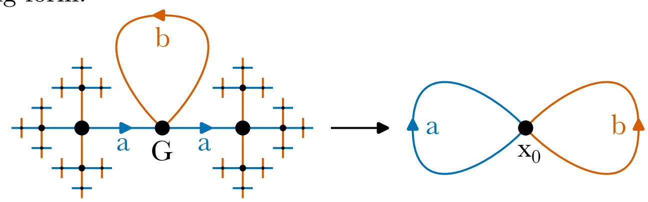



The basepoint is the vertex labeled 0. Let $T$ be the following maximal tree in this Cayley graph:




Using the recipe for computing fundamental groups of graphs from §8.9, we see that $G < F(a, b)$ is the free group on the following set of generators: $\{b^n a b^{-n} \mid n \in \mathbb{Z}\}$. □

Example 11.5.7. To illustrate this construction for a non-normal subgroup, let $F(a,b)$ be the free group on $a$ and $b$ and let $G$ be the cyclic subgroup generated by $b$. We already know that $G$ is free on the single generator $b$. The cosets of $G$ in $F(a,b)$ are of the form $Gw$ where $w \in F(a,b)$ is a reduced word that does not start with $b$ or $b^{-1}$. The Schreier graph of $G$ in $F(a,b)$ with respect to $S$ is thus of the following form:




For reasons of space, we only label the vertex corresponding to the trivial coset $G$, which is the basepoint. Note that this Schreier graph deformation retracts to the single loop labeled by $b$, so $G$ is indeed the free group on the single generator $b$. □

Remark 11.5.8. It is enlightening to re-interpret the examples in §9.6 from this point of view. □

## 11.6. Exercises

Exercise 11.1. Let $X$ be a space that is path connected, locally path connected, and semilocally 1-connected. Fix a basepoint $x_0 \in X$. Construct a bijection between the following two sets:

- The set of isomorphism classes of connected covers $p\colon \widetilde{X} \to X$.
- The set of conjugacy classes of subgroups of $\pi_1(X, x_0)$. □

Exercise 11.2. Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a pointed cover and let $f\colon (Z, z_0) \to (X, x_0)$ be a pointed map. As in Exercise 1.14, let $q\colon \widetilde{Z} \to Z$ be the pullback of $p\colon \widetilde{X} \to X$ along $f\colon Z \to X$, so

$$\widetilde{Z} = \left\{ (\widetilde{x}, z) \in \widetilde{X} \times Z \mid p(\widetilde{x}) = f(z) \right\}.$$

Set $\widetilde{z}_0 = (\widetilde{x}_0, z_0) \in \widetilde{Z}$, so $q\colon (\widetilde{Z}, \widetilde{z}_0) \to (Z, z_0)$ is a pointed cover fitting into a commutative diagram

$$\begin{tikzcd} (\widetilde{Z}, \widetilde{z}_0) \arrow[r] \arrow[d, "q"] & (\widetilde{X}, \widetilde{x}_0) \arrow[d, "p"] \\ (Z, z_0) \arrow[r, "f"] & (X, x_0). \end{tikzcd}$$

We call $q\colon (\widetilde{Z}, \widetilde{z}_0) \to (Z, z_0)$ the *pullback* of $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ along $f\colon (Z, z_0) \to (X, x_0)$. Prove the following:

- Let $G < \pi_1(X, x_0)$ be the subgroup corresponding to $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$. Then the subgroup of $\pi_1(Z, z_0)$ corresponding to $q\colon (\widetilde{Z}, \widetilde{z}_0) \to (Z, z_0)$ is the preimage of $G$ under the map $f_*\colon \pi_1(Z, z_0) \to \pi_1(X, x_0)$. □

Exercise 11.3. Let $(X, x_0)$ be a space. Let $p_1\colon (\widetilde{X}_1, \widetilde{x}_1) \to (X, x_0)$ and $p_2\colon (\widetilde{X}_2, \widetilde{x}_2) \to (X, x_0)$ be two covers. As in the previous exercise, let

$$\widetilde{Z} = \left\{ (\widetilde{x}, \widetilde{y}) \in \widetilde{X}_1 \times \widetilde{X}_2 \mid p_1(\widetilde{x}) = p_2(\widetilde{y}) \right\}$$

be the pullback of $p_2$ along $p_1$. Set $\widetilde{z}_0 = (\widetilde{x}_1, \widetilde{x}_2) \in \widetilde{Z}$. Let $q_1\colon (\widetilde{Z}, \widetilde{z}_0) \to (\widetilde{X}_1, \widetilde{x}_1)$ and $q_2\colon (\widetilde{Z}, \widetilde{z}_0) \to (\widetilde{X}_2, \widetilde{x}_2)$ be the projections. These are now both covers, and fit into a commutative diagram

$$\begin{tikzcd} (\widetilde{Z}, \widetilde{z}_0) \arrow[r, "q_1"] \arrow[d, "q_2"] & (\widetilde{X}_1, \widetilde{x}_1) \arrow[d, "p_1"] \\ (\widetilde{X}_2, \widetilde{x}_2) \arrow[r, "p_2"] & (X, x_0). \end{tikzcd}$$

Set $r = p_1 \circ q_1 = p_2 \circ q_2$, so $r$ is a map $r\colon (\widetilde{Z}, \widetilde{z}_0) \to (X, x_0)$. Prove the following:

(a) The map $r\colon (\widetilde{Z}, \widetilde{z}_0) \to (X, x_0)$ is a covering space. Hint: it is not necessarily the case that the composition of two covering maps is a covering map (see Exercise 2.14**), so you will have to prove this directly.

(b) For $i = 1, 2$, let $G_i < \pi_1(X, x_0)$ be the subgroup corresponding to $p_i\colon (\widetilde{X}_i, \widetilde{x}_i) \to (X, x_0)$. Prove that the subgroup of $\pi_1(X, x_0)$ corresponding to $r\colon (\widetilde{Z}, \widetilde{z}_0) \to (X, x_0)$ is $G_1 \cap G_2$. □

EXERCISE 11.4. Let $\mathbb{T}^2 = (\mathbb{S}^1)^{\times 2}$ be the 2-torus. Fix a basepoint $x_0 \in \mathbb{T}^2$, so $\pi_1(\mathbb{T}^2, x_0) \cong \mathbb{Z}^2$.

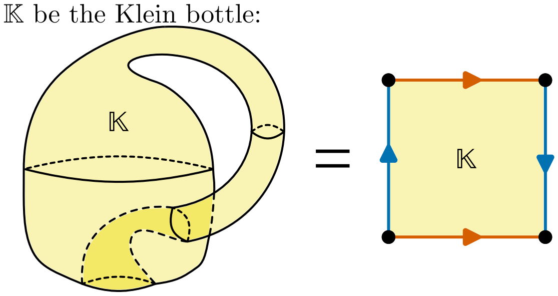
(a) Construct the universal cover of $\mathbb{T}^2$.

(b) Let $K < \mathbb{Z}^2$ be a finite-index subgroup and let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (\mathbb{T}^2, x_0)$ be the cover whose corresponding subgroup is $K$. Prove that $\widetilde{X} \cong \mathbb{T}^2$.

(c) Let $K < \mathbb{Z}^2$ be a nontrivial subgroup of infinite index and let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (\mathbb{T}^2, x_0)$ be the cover whose corresponding subgroup is $K$. Prove that $\widetilde{X} \cong \mathbb{S}^1 \times \mathbb{R}$. □

EXERCISE 11.5. Let $(X, x_0)$ be a path connected pointed space such that $\pi_1(X, x_0)$ is finite and let $p\colon (X, x_0) \to (X, x_0)$ be a covering space. Prove that $p$ is a homeomorphism, i.e., has degree 1. Hint: since we are not assuming anything about the local topology of $X$, you cannot use the results in this chapter. However, you can still think about the subgroup of $\pi_1(X, x_0)$ corresponding to $p$. □

EXERCISE 11.6. Let $\mathbb{K}$ be the Klein bottle:




In Exercise 9.11, you proved that the fundamental group of $\mathbb{K}$ is the following group $\Gamma$:

- Let $\mathbb{Z}$ act on $\mathbb{Z}$ via the homomorphism $\phi\colon \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z})$ defined by
$$\phi(n)(m) = (-1)^n m \quad \text{for all } n \in \mathbb{Z} \text{ and } m \in \mathbb{Z}.$$
Then $\Gamma = \mathbb{Z} \rtimes \mathbb{Z}$, i.e., the semidirect product[2] of $\mathbb{Z}$ and $\mathbb{Z}$.

Also, in Exercise 10.5 you proved that the universal cover of $\mathbb{K}$ is $\mathbb{R}^2$ and determined the action of $\Gamma$ on $\mathbb{R}^2$ such that $\mathbb{R}^2/\Gamma \cong \mathbb{K}$. Do the following:

(a) Fixing a basepoint $x_0 \in \mathbb{K}$, construct a irregular cover $p\colon (\mathbb{K}, x_0) \to (\mathbb{K}, x_0)$ and identify the corresponding subgroup of $\pi_1(\mathbb{K}, x_0)$.

(b) Letting $\mathbb{T}^2$ be the 2-torus and $y_0 \in \mathbb{T}^2$ be a basepoint, construct an irregular cover $q\colon (\mathbb{T}^2, y_0) \to (\mathbb{K}, x_0)$ and identify the corresponding subgroup of $\pi_1(\mathbb{K}, x_0)$. □

EXERCISE 11.7. As in Exercise 11.6, let $\mathbb{K}$ be the Klein bottle. Identify all the possible homeomorphism types of connected spaces $Z$ such that there is a cover $p\colon Z \to \mathbb{K}$. □

EXERCISE 11.8. Let $X$ be a graph with one vertex $x_0$ and two edges labeled $a$ and $b$. We can therefore identify $\pi_1(X, x_0)$ with the free group $F(a, b)$. Classify all the connected degree 2 and degree 3 covers of $X$, and for each cover determine if it is regular and give generators for its corresponding subgroup of $\pi_1(X, x_0) = F(a, b)$. □

EXERCISE 11.9. Let $F = F(a, b)$ be the free group on $a$ and $b$. Recall from Exercise 7.12 that the abelianization of $F$ is $\mathbb{Z}^2$. The kernel of the abelianization map $F \to \mathbb{Z}^2$ is the commutator subgroup $[F, F]$. Compute a free basis for the free group $[F, F]$. □

[2] Let $G$ and $H$ be groups such that $G$ acts on $H$ on the left. For $g \in G$ and $h \in H$, we will write ${}^g h$ for the image of $h$ under the action of $g$. The corresponding semidirect product $H \rtimes G$ is the following group:

– The elements of $H \rtimes G$ are pairs $(h, g)$ with $h \in H$ and $g \in G$.

– For $(h_1, g_1), (h_2, g_2) \in H \rtimes G$, their product is $(h_1, g_1)(h_2, g_2) = (h_1\, {}^{g_1}h_2, g_1 g_2)$.

It is enlightening to prove that this is a group. If the $G$-action on $H$ is trivial, this is just the usual direct product $H \times G$.

EXERCISE 11.10. Let $F(a,b)$ be the free group on $a$ and $b$ and let $w \in F$ satisfy $w \neq 1$. Prove that there is a finite-index subgroup $G < F(a,b)$ such that $w \notin G$ (such groups are called *residually finite*[3]). Hint: Let $X$ be a graph with one vertex $x_0$ and two edges labeled $a$ and $b$. We can therefore identify $\pi_1(X, x_0)$ with the free group $F(a,b)$. It is enough to find a finite pointed cover $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that $w$ is not in the subgroup of $\pi_1(X, x_0) = F(a,b)$ corresponding to $p$. Write $w$ as a reduced word:

$$w = s_1^{\epsilon_1} \cdots s_n^{\epsilon_n} \quad \text{with } s_1, \ldots, s_n \in \{a, b\} \text{ and } \epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}.$$

Let $Z$ be the graph with vertices $\{v_0, \ldots, v_n\}$ and $n$ oriented edges $\{e_1, \ldots, e_n\}$, with $e_i$ labeled by $s_i \in \{a, b\}$ and connecting:

- $v_{i-1}$ to $v_i$ if $\epsilon = 1$; and
- $v_i$ to $v_{i-1}$ if $\epsilon = -1$.

Letting $\widetilde{x}_0 = v_0$, show that you can add oriented labeled edges to $(Z, \widetilde{x}_0)$ to make it a finite graph $(\widetilde{X}, \widetilde{x}_0)$ whose edge-labels define a cover of $(X, x_0)$. We remark that this same argument works for free groups of arbitrary finite rank, not just $F(a,b)$. $\square$

[3]Another way of defining residually finite groups is as follows. A group $\Gamma$ is *residually finite* if the intersection of all finite-index subgroups of $\Gamma$ is the trivial subgroup 1.

CHAPTER 12*

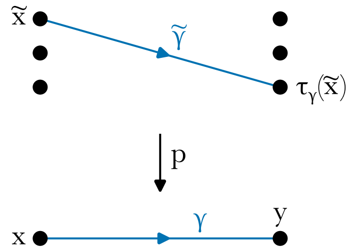
# Monodromy and the classification of covers

In Chapter 11, we gave a classification of covers in terms of subgroups of the fundamental group. This chapter gives an alternate classification terms of actions of the fundamental group on sets. We have marked this chapter with an $*$ to indicate that a first-time reader might want to skip it.

## 12*.1. Monodromy action of the fundamental group

Let $p\colon \widetilde{X} \to X$ be a cover. For $x \in X$, consider the fiber $p^{-1}(x)$ over $x$. If $\gamma$ is a path in $X$ from $x \in X$ to $y \in X$, then in §4.3 we defined a map $\tau_\gamma\colon p^{-1}(x) \to p^{-1}(y)$ as follows:

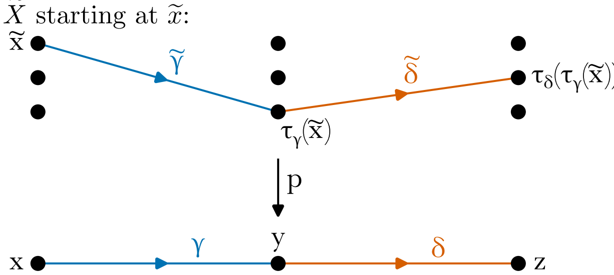
- For $\widetilde{x} \in p^{-1}(x)$, let $\widetilde{\gamma}$ be the lift of $\gamma$ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}$. We then set $\tau_\gamma(\widetilde{x}) = \widetilde{\gamma}(1) \in p^{-1}(y)$.




Lemma 4.2.1 says that $\tau_\gamma$ only depends on the homotopy class of $\gamma$, and Lemma 4.3.1 says that $\tau_\gamma$ is a bijection. These compose as follows:

Lemma 12*.1.1. *Let $p\colon \widetilde{X} \to X$ be a cover. $\gamma$ be a path in $X$ from $x \in X$ to $y \in X$ and let $\delta$ be a path in $X$ from $y \in X$ to $z \in X$. Then $\tau_{\gamma\cdot\delta} = \tau_\delta \circ \tau_\gamma$.*

Proof. Consider some $\widetilde{x} \in p^{-1}(x)$. Let $\widetilde{\gamma}$ be the lift of $\gamma$ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}$. We thus have $\widetilde{\gamma}(1) = \tau_\gamma(\widetilde{x})$. Let $\widetilde{\delta}$ be the lift of $\delta$ to $\widetilde{X}$ with $\widetilde{\delta}(0) = \widetilde{\gamma}(1)$. As the following figure shows, the path $\widetilde{\gamma}\cdot\widetilde{\delta}$ is the lift of $\gamma\cdot\delta$ to $\widetilde{X}$ starting at $\widetilde{x}$:




It follows that $\tau_{\gamma\cdot\delta} = \widetilde{\delta}(1) = \tau_\delta(\widetilde{\delta}(0)) = \tau_\delta(\tau_\gamma(\widetilde{x}))$, as desired. □

Fix a basepoint $x_0 \in X$. Consider the set $p^{-1}(x_0)$. Define a right action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$ as follows:

$$\widetilde{x}[\gamma] = \tau_\gamma(\widetilde{x}) \quad \text{for all } \widetilde{x} \in p^{-1}(x_0) \text{ and } [\gamma] \in \pi_1(X, x_0).$$

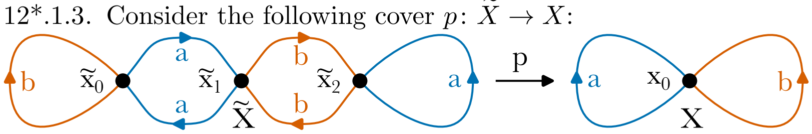
The fact that this is an action uses Lemma 12*.1.1: for $\widetilde{x} \in p^{-1}(x_0)$ and $[\gamma], [\delta] \in \pi_1(X, x_0)$, we have

$$\widetilde{x}[\gamma\cdot\delta] = \tau_{\gamma\cdot\delta}(\widetilde{x}) = \tau_\delta(\tau_\gamma(\widetilde{x})) = \tau_\gamma(\widetilde{x})[\delta] = \widetilde{x}[\gamma][\delta].$$

This is called the *monodromy action* of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$.

Remark 12*.1.2. In Lemma 12*.1.1, the order of $\gamma$ and $\delta$ are reversed in $\tau_{\gamma\cdot\delta} = \tau_\delta \circ \tau_\gamma$. This order reversal is why we get a right action rather than a left action. It happens because we compose paths from left to right, but functions from right to left. □

Example 12*.1.3. Consider the following cover $p\colon \widetilde{X} \to X$:

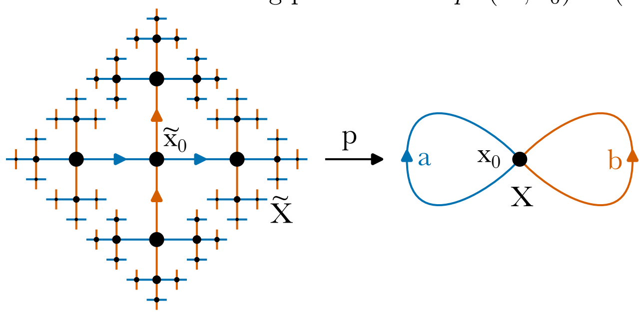



This is an irregular cover with trivial deck group (see Example 2.7.3). Let $x_0 \in X$ be the indicated basepoint, so $p^{-1}(x_0) = \{\widetilde{x}_0, \widetilde{x}_1, \widetilde{x}_2\}$. We have $\pi_1(X, x_0) \cong F(a, b)$, and the right monodromy action of $F(a, b)$ on $p^{-1}(x_0)$ is as follows:

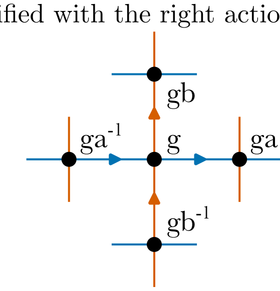
$$\begin{aligned} \widetilde{x}_0 a &= \widetilde{x}_1 & \widetilde{x}_1 a &= \widetilde{x}_0 & \widetilde{x}_2 a &= \widetilde{x}_2 \\ \widetilde{x}_0 b &= \widetilde{x}_0 & \widetilde{x}_1 b &= \widetilde{x}_2 & \widetilde{x}_2 b &= \widetilde{x}_1 \end{aligned}$$

Since the deck group of this cover is trivial, this action has nothing to do with the action of the deck group. □

Example 12*.1.4. Consider the following pointed cover $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$:




This is the universal cover of $(X, x_0)$, so its deck group can be identified with $\pi_1(X, x_0) \cong F(a, b)$. Using the left action of the deck group $F(a, b)$, we can identify $p^{-1}(x_0)$ with $F(a, b)$ by letting $g \in F(a, b)$ correspond to $g\widetilde{x}_0$. As the following shows, under this identification the right monodromy action of $F(a, b)$ on $p^{-1}(x_0)$ is identified with the right action of $F(a, b)$ on itself:




On the other hand, the left action of the deck group on $p^{-1}(x_0)$ corresponds to the left action of $F(a, b)$ on itself. □

## 12*.2. Monodromy and $G$-sets

Let $G$ be a group. A *right $G$-set* is a set $S$ equipped with a right $G$-action. If $S$ and $T$ are right $G$-sets, then a *$G$-equivariant map* is a set map $\phi\colon S \to T$ with $\phi(sg) = \phi(s)g$ for all $s \in S$ and $g \in G$. A $G$-equivariant map $\phi\colon S \to T$ is a *$G$-equivariant isomorphism* if it is a bijection, in which case its inverse is $G$-equivariant. If a $G$-equivariant isomorphism from $S$ to $T$ exists, then we say that $S$ and $T$ are *isomorphic* as right $G$-sets.

Now let $(X, x_0)$ be a pointed space and let $G = \pi_1(X, x_0)$. If $p\colon \widetilde{X} \to X$ is a cover, then the fiber $S = p^{-1}(x_0)$ is a right $G$-set via the monodromy action. For $X$ path connected, locally path connected, and semilocally 1-connected, our main goal in this chapter is to prove that in a suitable sense covers of $X$ are the "same" as right $G$-sets. Proving this will require some preliminary results.

Before moving on to those preliminary results, we make a remark about the nature of the monodromy action. Continue to let $(X, x_0)$ be a pointed space and $G = \pi_1(X, x_0)$. Let $S$ be an abstract set and let $\mathfrak{S}(S)$ be the symmetric group of $S$, that is, the group of all bijections $S \to S$. If $S$ is endowed with the structure of a right $G$-set, then we can define a homomorphism $\sigma\colon G \to \mathfrak{S}(S)$

via the following formula:

$$\sigma(g)(s) = sg^{-1} \quad \text{for all } g \in G \text{ and } s \in S. \tag{12*.2.1}$$

The inverse is there to convert the right action into a left action. Conversely, if $\sigma\colon G \to \mathfrak{S}(S)$ is a homomorphism then we can make $S$ into a right $G$-set via (12*.2.1). The upshot is that right $G$-set structures on $S$ are the same as homomorphisms $\sigma\colon G \to \mathfrak{S}(S)$.

Once we have proved that for appropriate $X$ covers of $X$ can be identified with right $G$-sets, it will follow that covers of $X$ whose fiber over $x_0$ is the set $S$ can be identified with homomorphisms $\sigma\colon G \to \mathfrak{S}(S)$. These homomorphisms are called the *monodromy representations* of the cover. We remark that these covers need not be regular, and the monodromy representations are very different from the lifting homomorphisms that characterize regular covers.

Remark 12*.2.1. Let $(X, x_0)$ be a pointed space. Consider a degree $n$ cover $p\colon \widetilde{X} \to X$. Let $S = p^{-1}(x_0)$, and enumerate $S$ as $S = \{\widetilde{x}_1, \ldots, \widetilde{x}_n\}$. This enumeration allows us to identify $\mathfrak{S}(S)$ with the symmetric group $\mathfrak{S}_n$ on $n$ letters, so the monodromy representation of $p$ takes the form $\sigma\colon \pi_1(X, x_0) \to \mathfrak{S}_n$. In the 19th century before the concept of a cover was formalized, degree $n$ covers were constructed from data like this. The homomorphism $\sigma$ characterized how you transitioned between different “sheets” of the cover as you went around loops. $\square$

## 12*.3. Morphisms of covers

Let $(X, x_0)$ be a pointed space. Consider covers $p_1\colon \widetilde{X}_1 \to X$ and $p_2\colon \widetilde{X}_2 \to X$. Set $S_1 = p_1^{-1}(x_0)$ and $S_2 = p_2^{-1}(x_0)$. A *morphism* of covers from $p_1$ to $p_2$ is a cover $q\colon \widetilde{X}_1 \to \widetilde{X}_2$ such that the following diagram commutes:

$$\begin{tikzcd} \widetilde{X}_1 \arrow[rr, "q"] \arrow[dr, "p_1"'] & & \widetilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

commutes. Given a morphism $q\colon \widetilde{X}_1 \to \widetilde{X}_2$ of covers, let $q_*\colon S_1 \to S_2$ denote the restriction of $q$ to $S_1$. This map $q_*$ is $G$-equivariant:

Lemma 12*.3.1. *Let $(X, x_0)$ be a pointed space and let $G = \pi_1(X, x_0)$. Let $p_1\colon \widetilde{X}_1 \to X$ and $p_2\colon \widetilde{X}_2 \to X$ be covers. Set $S_1 = p_1^{-1}(x_0)$ and $S_2 = p_2^{-1}(x_0)$. Let $q\colon \widetilde{X}_1 \to \widetilde{X}_2$ be a morphism of covers. Then $q_*\colon S_1 \to S_2$ is $G$-equivariant.*

Proof. Consider $s \in S_1$ and $[\gamma] \in G$. Our goal is to prove that $q(s[\gamma]) = q(s)[\gamma]$. Let $\widetilde{\gamma}_2$ be the lift of $\gamma$ through $p_2\colon \widetilde{X}_2 \to X$ with $\widetilde{\gamma}_2(0) = q(s)$. We thus have $q(s)[\gamma] = \widetilde{\gamma}_2(1)$. Let $\widetilde{\gamma}_1$ be the lift of $\widetilde{\gamma}_2$ through $q\colon \widetilde{X}_1 \to \widetilde{X}_2$ with $\widetilde{\gamma}_1(0) = s$. We thus have $q(\widetilde{\gamma}_1(1)) = \widetilde{\gamma}_2(1)$. Since $p_1 = p_2 \circ q$, the path $\widetilde{\gamma}_1$ is also the lift of $\gamma$ through $p_1\colon \widetilde{X}_1 \to X$ with $\widetilde{\gamma}_1(0) = s$. We thus have $s[\gamma] = \widetilde{\gamma}_1(1)$. Putting all our equalities together, we have $q(s[\gamma]) = q(\widetilde{\gamma}_1(1)) = \widetilde{\gamma}_2(1) = q(s)[\gamma]$. $\square$

## 12*.4. Deck groups and monodromy

Let $H$ and $G$ be groups. An $(H, G)$*-set* is a set $S$ equipped with a left action of $H$ and a right action of $G$ such that

$$(hs)g = h(sg) \quad \text{for all } h \in H \text{ and } s \in S \text{ and } g \in G.$$

We define $(H, G)$-equivariant isomorphisms between $(H, G)$-sets in the obvious way. These arise from covering spaces as follows. Let $(X, x_0)$ be a pointed space and let $p\colon \widetilde{X} \to X$ be a cover. Set $G = \pi_1(X, x_0)$ and $H = \mathrm{Deck}(\widetilde{X})$ and $S = p^{-1}(x_0)$. The set $S$ is a right $G$-set via the monodromy action. It is also a left $H$-set via the action of the deck group $H$ on $S$. Lemma 12*.3.1 implies that $H$ acts on $S$ via $G$-equivariant isomorphisms. It follows that these two actions make $S$ into an $(H, G)$-set. The following generalizes the pattern from Example 12*.1.4:

Lemma 12*.4.1. *Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a regular pointed cover with $\widetilde{X}$ connected. Set $G = \pi_1(X, x_0)$ and $H = \mathrm{Deck}(\widetilde{X})$, and let $f\colon G \to H$ be the lifting map from §7.2. Finally, let $S = p^{-1}(x_0)$. Then the $(H, G)$-set $S$ is isomorphic to $H$ with the following $(H, G)$-set structure:*

- *The group $H$ acts on $H$ on the left via its usual group multiplication.*
- *The group $G$ acts on $H$ on the right as $hg = hf(g)$ for all $h \in H$ and $g \in G$.*

PROOF. Since $\widetilde{X}$ is connected, $H$ acts freely and transitively on $S$. Each element of $S$ can thus be written as $h\widetilde{x}_0$ for a unique $h \in H$. Define a set map $\phi\colon S \to H$ as follows: for $s \in S$, let

$$\phi(s) = h \text{ for the } h \in H \text{ with } s = h\widetilde{x}_0.$$

The map $\phi$ is a bijection and satisfies $\phi(hs) = h\phi(s)$ for all $h \in H$ and $s \in S$. Fixing some $s \in S$ and $[\gamma] \in G = \pi_1(X, x_0)$, we must prove that $\phi(s[\gamma]) = \phi(s)f([\gamma])$.

Write $s = h\widetilde{x}_0$ with $h \in H$, so $\phi(s) = h$. Let $\widetilde{\gamma}$ be the lift of $\gamma$ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = h\widetilde{x}_0$, so $s[\gamma] = \widetilde{\gamma}(1)$. To prove that $\phi(s[\gamma]) = \phi(s)f([\gamma])$, we must prove that $\widetilde{\gamma}(1) = hf([\gamma])\widetilde{x}_0$. Letting the deck group $H$ act on paths in $\widetilde{X}$, the path $h^{-1}\widetilde{\gamma}$ goes from $h^{-1}s = \widetilde{x}_0$ to $h^{-1}\widetilde{\gamma}(1)$. By the definition of the lifting map from §7.2, we have $h^{-1}\widetilde{\gamma}(1) = f([\gamma])\widetilde{x}_0$. Applying $h$ to this, we conclude that $\widetilde{\gamma}(1) = hf([\gamma])\widetilde{x}_0$, as desired. □

## 12*.5. Covers are the same thing as right $G$-sets, I

Recall that our goal is to prove that if $(X, x_0)$ is a pointed space with suitable local properties and $G = \pi_1(X, x_0)$, then covers of $X$ are in some sense the "same" as right $G$-sets. The following is a first version of this. Using category theoretic language, we will give a more precise statement in the next chapter.

THEOREM 12*.5.1. *Let $(X, x_0)$ be a pointed space that is path connected, locally path connected, and semilocally 1-connected. Set $G = \pi_1(X, x_0)$. The following hold:*

(i) *For all right $G$-sets $S$, there exists a cover $p\colon \widetilde{X} \to X$ such that the right $G$-set $p^{-1}(x_0)$ is isomorphic to $S$.*
(ii) *Let $p_1\colon \widetilde{X}_1 \to X$ and $p_2\colon \widetilde{X}_2 \to X$ be covers. Set $S_1 = p_1^{-1}(x_0)$ and $S_2 = p_2^{-1}(x_0)$, and assume that the right $G$-sets $S_1$ and $S_2$ are isomorphic. Then $p_1\colon \widetilde{X}_1 \to X$ and $p_2\colon \widetilde{X}_2 \to X$ are isomorphic covers.*

PROOF. We divide the proof into two steps:

(a) We first prove (i) by giving a construction that takes a right $G$-set $S$ and produces a cover $p\colon \widetilde{X} \to X$ such that $p^{-1}(x_0)$ is isomorphic to $S$.
(b) We then prove that if $p\colon \widetilde{X} \to X$ is a cover and $S = p^{-1}(x_0)$, then $p\colon \widetilde{X} \to X$ is isomorphic to the result of performing the construction from the first step to $S$. This will imply (ii).

STEP 1. *Let $S$ be a right $G$-set. There then exists a cover $p\colon \widetilde{X} \to X$ such that the fiber $p^{-1}(x_0)$ is isomorphic to $S$ as a right $G$-set.*

By Theorem 10.3.1, there exists a universal cover $p_u\colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$. This is a regular cover with deck group $G$. Regarding $G$ as a $(G, G)$-set via the left- and right-actions of $G$ on itself, we can therefore apply Lemma 12*.4.1 to see that $p_u^{-1}(x_0)$ is isomorphic to $G$ as a $(G, G)$-set. In particular, $p_u^{-1}(x_0)$ is isomorphic to $G$ as a right $G$-set.

Endow $S$ with the discrete topology, and let $q\colon S \times \widetilde{X}_u \to X$ be the projection defined by $q(s, \widetilde{x}) = p_u(\widetilde{x})$ for $s \in S$ and $\widetilde{x} \in \widetilde{X}_u$. The map $q$ is a covering space. We have $q^{-1}(x_0) = S \times p_u^{-1}(x_0)$. However, we cannot combine with this fact that $p_u^{-1}(x_0)$ is isomorphic to $G$ as a right $G$-set to deduce that $q^{-1}(x_0)$ is isomorphic to $S \times G$ as a right $G$-set. The issue is that the right action of $G$ on $S$ plays no role in $q$. Instead, letting $S_{\text{triv}}$ be the right $G$-set with the same underlying set as $S$ but with the trivial $G$-action, we have an isomorphism $q^{-1}(x_0) \cong S_{\text{triv}} \times G$ of right $G$-sets.

Define a left action of $G$ on $S \times \widetilde{X}_u$ as follows:

$$g(s, \widetilde{x}) = (sg^{-1}, g\widetilde{x}) \quad \text{for all } s \in S \text{ and } g \in G \text{ and } \widetilde{x} \in \widetilde{X}_u.$$

The inverse is there to convert the right action of $G$ on $S$ into a left action. This action is a covering space action; indeed, $G$ acts on $S \times \widetilde{X}_u$ via deck transformations of the cover $q\colon S \times \widetilde{X}_u \to X$. Since we have both left- and right-actions, we will write $\widetilde{X} = G \backslash (S \times \widetilde{X}_u)$ for the quotient of $S \times \widetilde{X}_u$ by the left action of $G$. The cover $q\colon S \times \widetilde{X}_u \to X$ factors through a cover $p\colon \widetilde{X} \to X$.

We claim that this is the cover we are looking for, i.e., that the right $G$-set $p^{-1}(x_0)$ is isomorphic to $S$. Since $q^{-1}(x_0) \cong S_{\text{triv}} \times G$, we have $p^{-1}(x_0) \cong G\backslash(S_{\text{triv}} \times G)$. To prove that this is isomorphic to $S$, define a set map $\Phi\colon S_{\text{triv}} \times G \to S$ via the formula

$$\Phi(s,g) = sg \quad \text{for } s \in S_{\text{triv}} \text{ and } g \in G.$$

This map is $G$-equivariant for the right action of $G$ on $S_{\text{triv}} \times G$. Moreover, for $s \in S_{\text{triv}}$ and $g_1, g_2 \in G$ we have

$$\Phi(g_1(s,g_2)) = \Phi(sg_1^{-1}, g_1 g_2) = sg_1^{-1} g_1 g_2 = sg_2 = \Phi(s,g_2).$$

It follows that $\Phi$ is $G$-invariant for the left action of $G$ on $S_{\text{triv}} \times G$. We conclude that $\Phi$ factors through a map $\phi\colon G\backslash(S_{\text{triv}} \times G) \to S$ that is equivariant with respect to the right action of $G$. The map $\phi$ is a bijection with inverse the map taking $s \in S$ to the image of $(s,1)$ in $G\backslash(S_{\text{triv}} \times G)$. It follows that $\phi$ is an isomorphism of right $G$-sets, as desired.

Step 2. *Let $p\colon \widetilde{X} \to X$ be a cover and $S = p^{-1}(x_0)$. Then $p\colon \widetilde{X} \to X$ is isomorphic to the result of performing the construction from Step 1 to $S$.*

As in Step 1, let $p_u\colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ be the universal cover, endow $S$ with the discrete topology, and let $q\colon S \times \widetilde{X}_u \to X$ be the covering space defined by $q(s,\widetilde{x}) = p_u(\widetilde{x})$ for $s \in S$ and $\widetilde{x} \in \widetilde{X}_u$. For $s \in S = p^{-1}(x_0)$, the universality of the universal cover (Lemma 10.1.1) implies that there exists a unique covering map $r_s\colon (\widetilde{X}_u, \widetilde{x}_u) \to (\widetilde{X}, s)$ such that $p_u = p \circ r_s$. These assemble into a covering space $r\colon S \times \widetilde{X}_u \to \widetilde{X}$ defined by

$$r(s,\widetilde{x}) = r_s(\widetilde{x}) \quad \text{for all } (s,\widetilde{x}) \in S \times \widetilde{X}_u.$$

The fact that this is a covering space uses the fact that $X$ (and hence $\widetilde{X}$) is locally connected and semilocally 1-connected. Indeed, this implies that $\widetilde{X}$ can be covered by sl1c subsets $\widetilde{U}$. These are trivialized open sets for all the $r_s$ (see Lemma 10.4*.1), and hence are trivialized open sets for $r$.

As we observed in Step 1, the group $G$ acts on $S \times \widetilde{X}_u$ on the left by deck transformations for $q\colon S \times \widetilde{X}_u \to X$:

$$g(s,\widetilde{x}) = (sg^{-1}, g\widetilde{x}) \quad \text{for all } s \in S \text{ and } g \in G \text{ and } \widetilde{x} \in \widetilde{X}_u.$$

What we must prove is that these are also deck transformations for $r\colon S \times \widetilde{X}_u \to \widetilde{X}$ and that they act transitively on the fibers of $r$. Indeed, this will imply that we can identify $\widetilde{X}$ with $G\backslash(S \times \widetilde{X}_u)$, showing that $\widetilde{X}$ is isomorphic to the result of performing the construction from Step 1 to $S$.

In fact, this is quite formal. By Lemma 12*.4.1, the fiber $q^{-1}(x_0)$ is isomorphic as a right $G$-set to $S_{\text{triv}} \times G$. We also have $S = p^{-1}(x_0)$. When we restrict $r$ to the fiber over $x_0$, it is the map $\Phi\colon S_{\text{triv}} \times G \to S$ defined via the formula

$$\Phi(s,g) = sg \quad \text{for } s \in S_{\text{triv}} \text{ and } g \in G.$$

This map also appeared in Step 1. This map $\Phi$ is invariant under the restriction of the action of $G$ on $q^{-1}(x_0) \cong S_{\text{triv}} \times G$. What we must prove now follows formally. Indeed, in Exercise 12*.7 you will prove the following more general fact:

- Let $(X, x_0)$ be a path connected pointed space. Set $G = \pi_1(X, x_0)$. Consider covers $p\colon Y \to X$ and $q\colon Z \to X$ and $r\colon Z \to Y$ such that $p = q \circ r$. Set $S = p^{-1}(x_0)$ and $T = q^{-1}(x_0)$, so $S$ and $T$ are right $G$-sets. Then the following hold:
  (a) The map $r$ restricts to a $G$-equivariant map $R\colon T \to S$ of right $G$-sets.
  (b) Let $H$ be a group acting on $Z$ on the left by deck transformations of $q\colon Z \to X$. The set $T$ is thus an $(H,G)$-set. Assume that the map $R\colon T \to S$ is $H$-invariant. Then $H$ acts by deck transformations of $r\colon Z \to Y$.

Just apply this with $Z = \widetilde{X}_u$ and $Y = \widetilde{X}$ and $H = G$. □

### 12*.6. Properties of covers from monodromy

We now discuss how properties of covers are reflected in their monodromy representations. We start by characterizing when a cover is connected. If $G$ is a group, a right $G$-set $S$ is *transitive* if for all $s, s' \in S$ there exists some $g \in G$ with $sg = s'$.

LEMMA 12*.6.1. *Let $p\colon \widetilde{X} \to X$ be a cover with $X$ path connected and let $x_0 \in X$. The path components of $\widetilde{X}$ are in bijection with the $\pi_1(X, x_0)$-orbits of the right $\pi_1(X, x_0)$-set $p^{-1}(x_0)$. In particular, $\widetilde{X}$ is path connected if and only if the right $\pi_1(X, x_0)$-set $p^{-1}(x_0)$ is transitive.*

PROOF. For $\widetilde{x} \in \widetilde{X}$, a path $\delta$ in $X$ from $p(\widetilde{x})$ to $x_0$ lifts to a path from $\widetilde{x}$ to a point of $p^{-1}(x_0)$. It follows that each path component of $\widetilde{X}$ contains at least one point of $p^{-1}(x_0)$. To prove the lemma, it is therefore enough to prove that two points of $p^{-1}(x_0)$ can be connected by a path if and only if they are in the same $\pi_1(X, x_0)$-orbit.

Consider $\widetilde{x}, \widetilde{x}' \in p^{-1}(x_0)$. A path $\widetilde{\gamma}$ from $\widetilde{x}$ to $\widetilde{x}'$ projects to a loop $\gamma$ with $[\gamma] \in \pi_1(X, x_0)$ such that $\widetilde{x}[\gamma] = \widetilde{x}'$. Conversely, if $[\gamma] \in \pi_1(X, x_0)$ satisfies $\widetilde{x}[\gamma] = \widetilde{x}'$ then the lift of $\gamma$ to $\widetilde{X}$ starting at $\widetilde{x}$ is a path from $\widetilde{x}$ to $\widetilde{x}'$. It follows that $\widetilde{x}$ and $\widetilde{x}'$ can be connected by a path if and only if they are in the same $\pi_1(X, x_0)$-orbit, as desired. □

We next discuss the subgroup of the fundamental group corresponding to a cover. If $G$ is a group and $S$ is a right $G$-set, for $s \in S$ we will write $G_s$ for the stabilizer subgroup of $s$, i.e., $G_s = \{g \in G \mid sg = s\}$. We then have:

LEMMA 12*.6.2. *Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a pointed cover and let $G = \pi_1(X, x_0)$. Then the subgroup of $G$ corresponding to $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is $G_{\widetilde{x}_0}$.*

PROOF. Recall that the subgroup of $G = \pi_1(X, x_0)$ corresponding to the cover is the image of the induced map $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$. By Theorem 9.1.1, this subgroup consists of all $[\gamma] \in \pi_1(X, x_0)$ such that the following holds:

(†) Let $\widetilde{\gamma}$ be the lift of $\gamma$ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$. Then $\widetilde{\gamma}$ is a loop, i.e., $\widetilde{\gamma}(1) = \widetilde{x}_0$.

By definition, these are exactly the $[\gamma]$ such that $\widetilde{x}_0[\gamma] = \widetilde{x}_0$. □

We now turn to the deck group. For a right $G$-set $S$, let $\mathrm{Aut}_G(S)$ be the group of $G$-equivariant isomorphisms of $S$. If $X$ has suitable local properties, the deck group of a cover of $X$ is isomorphic to the automorphism group of its fibers:

LEMMA 12*.6.3. *Let $p\colon \widetilde{X} \to X$ be a cover and let $x_0 \in X$. Assume that $X$ is path connected, locally path connected, and semilocally 1-connected. Set $S = p^{-1}(x_0)$. Then $\mathrm{Deck}(\widetilde{X}) \cong \mathrm{Aut}_G(S)$.*

PROOF. Consider the map $\rho\colon \mathrm{Deck}(\widetilde{X}) \to \mathrm{Aut}_G(S)$ obtained by restricting deck transformations to $S = p^{-1}(x_0)$. We must prove that $\rho$ is injective and surjective:

CLAIM 1. *The map $\rho\colon \mathrm{Deck}(\widetilde{X}) \to \mathrm{Aut}_G(S)$ is injective.*

If $\widetilde{X}$ were connected, this would follow from Lemma 2.2.1, which says that for connected covers the action of the deck group is determined by what it does to a single point. However, we are only assuming that $X$ is path connected. As we observed in Exercise 2.2, in this case the proof of Lemma 2.2.1 goes through to show that the action of the deck group is determined by what it does to a single fiber. The lemma follows.

CLAIM 2. *The map $\rho\colon \mathrm{Deck}(\widetilde{X}) \to \mathrm{Aut}_G(S)$ is surjective.*

Let $p_u\colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$ be the universal cover. As in Step 1 of the proof of Theorem 12*.5.1, give $S$ the discrete topology and let $\widetilde{X}' = G\backslash(S \times \widetilde{X}_u)$, where $G$ acts on $S \times \widetilde{X}_u$ as follows:

$$g(s, \widetilde{x}) = (sg^{-1}, g\widetilde{x}) \quad \text{for all } s \in S \text{ and } g \in G \text{ and } \widetilde{x} \in \widetilde{X}_u.$$

The composition $S \times \widetilde{X}_u \to \widetilde{X}_u \to X$ factors through a cover $p'\colon \widetilde{X}' \to X$. Since $G$ acts freely and transitively on $p_u^{-1}(x_0)$, we have $(p')^{-1}(x_0) = G\backslash(S \times G\widetilde{x}_u)$. For $s \in S$ and $g \in G$, let $[s, g]$ denote the image of $(s, g\widetilde{x}_u)$ in $(p')^{-1}(x_0)$.

The argument in Step 2 of the proof of Theorem 12*.5.1 gives an isomorphism of covers $\lambda\colon \widetilde{X}' \to \widetilde{X}$. The restriction of $\lambda$ to $(p')^{-1}(x_0)$ is the following bijection $(p')^{-1}(x_0) \to p^{-1}(x_0) = S$:

$$\lambda([s,g]) = sg \in S \quad \text{for } s \in S \text{ and } g \in G. \tag{12*.6.1}$$

Consider some $\phi \in \mathrm{Aut}_G(S)$. We will construct a deck transformation $q'$ of $\widetilde{X}'$ and then use $\lambda$ to build a deck transformation $q$ of $\widetilde{X}$ with $\rho(q) = \phi$.

Define a homeomorphism $\mu\colon S \times \widetilde{X}_u \to S \times \widetilde{X}_u$ via the formula

$$\mu(s, \widetilde{x}) = (\phi(s), \widetilde{x}) \quad \text{for all } s \in S_1 \text{ and } \widetilde{x} \in \widetilde{X}_u.$$

The map $\mu$ is $G$-equivariant; indeed, for $g \in G$ and $s \in S$ and $\widetilde{x} \in \widetilde{X}_u$ we have

$$\mu(g(s,\widetilde{x})) = \mu(sg^{-1}, g\widetilde{x}) = (\phi(sg^{-1}), g\widetilde{x}) = (\phi(s)g^{-1}, g\widetilde{x}) = g(\phi(s), \widetilde{x}) = g\mu(s, \widetilde{x}).$$

It follows that $\mu$ descends to a homeomorphism $q'\colon \widetilde{X}' \to \widetilde{X}'$. On $(p')^{-1}(x_0)$, the map $\mu$ takes the form $\mu([s,g]) = [\phi(s), g]$ for $s \in S$ and $g \in G$. Set $q = \lambda \circ q' \circ \lambda^{-1}\colon \widetilde{X} \to \widetilde{X}$, so $q \in \mathrm{Deck}(\widetilde{X})$. Using (12*.6.1), for $s \in S = p^{-1}(x_0)$ we have

$$q(s) = \lambda \circ q'([s,1]) = \lambda([\phi(s), 1]) = \phi(s).$$

It follows that $\rho(q) = \phi$. $\square$

REMARK 12*.6.4. It was a little annoying in the proof of Claim 2 above that we had to use the explicit constructions from the proof of Theorem 12*.5.1. In §13*.4 we will show how to use category theory to package these constructions together in way that will make Lemma 12*.6.3 obvious. $\square$

## 12*.7. From monodromy to the Galois correspondence

We close this chapter by explaining how to recover the Galois correspondence (Theorem 11.2.1) from our classification of covers in terms of monodromy. We recall the statement of what we must prove:

THEOREM 11.2.1 (Galois correspondence). *Let $(X, x_0)$ be a pointed space that is path connected, locally path connected, and semilocally 1-connected. There is then a bijection between the following two sets:*

- *The set of isomorphism classes of connected pointed covers $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$.*
- *The set of subgroups of $\pi_1(X, x_0)$.*

*This bijection takes a connected pointed cover $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ to its corresponding subgroup, i.e., to the image $K$ of the map $p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$. The degree of the cover $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is the index of $K$ in $\pi_1(X, x_0)$.*

PROOF. Set $G = \pi_1(X, x_0)$. A *pointed right $G$-set* is a pair $(S, s_0)$ with $S$ a right $G$-set and $s_0 \in S$. Two pointed right $G$-sets $(S_1, s_1)$ and $(S_2, s_2)$ are isomorphic if there exists an isomorphism $\phi\colon S_1 \to S_2$ of right $G$-sets such that $\phi(s_1) = s_2$. If $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a pointed cover, then the pair $(S, s_0) = (p^{-1}(x_0), \widetilde{x}_0)$ is a pointed right $G$-set. Lemma 12*.6.1 says that $\widetilde{X}$ is connected if and only if $S$ is a transitive right $G$-set. Theorem 12*.5.1 implies that there is a bijection between the following two sets:

- The set of isomorphism classes of connected pointed covers $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$.
- The set of isomorphism classes of pointed right $G$-sets $(S, s_0)$ with $S$ transitive.

Consider a pointed right $G$-set $(S, s_0)$ with $S$ transitive. Set $K = G_{s_0}$. Associated to the subgroup $K < G$ is the transitive right $G$-set $K\backslash G$ consisting of the right cosets $Kg$ of $K$. Letting $k_0 = K$ be the trivial right coset, we have an isomorphism of pointed right $G$-sets $(S, s_0) \cong (K\backslash G, k_0)$ (see Exercise 12*.4). Combining all of this, we see that there is a bijection between the following two sets:

- The set of isomorphism classes of pointed right $G$-sets $(S, s_0)$ with $S$ transitive.
- The set of subgroups of $G$.

This bijection takes $(S, s_0)$ to $K = G_{s_0}$. Putting our two bijections together gives a bijection between the following two sets:

- The set of isomorphism classes of connected pointed covers $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$.

- The set of subgroups of $G$.

This bijection takes a connected pointed cover $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ to the subgroup $K = G_{\widetilde{x}_0}$. Lemma 12*.6.2 implies that this $K$ is the subgroup corresponding to $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$. The theorem follows. $\square$

## 12*.8. Exercises

EXERCISE 12*.1. Let $X$ be a graph with one vertex $x_0$ and $n$ oriented edges labeled $x_1, \ldots, x_n$, so $G = \pi_1(X, x_0)$ is a free group on generators $x_1, \ldots, x_n$. Let $p\colon \widetilde{X} \to X$ be a cover. Set $S = p^{-1}(x_0)$, so $S$ is right $G$-set. The set $S$ is the set of vertices of the graph $\widetilde{X}$. Describe the set of edges of $\widetilde{X}$ in terms of the right $G$-set structure on $S$. $\square$

EXERCISE 12*.2. Let $(X, x_0)$ be a pointed space that is path connected, locally path connected, and semilocally 1-connected. Let $K < \pi_1(X, x_0)$ be a subgroup and let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be the connected cover whose corresponding subgroup of $\pi_1(X, x_0)$ is $K$ (see Theorem 11.2.1). Let $S = p^{-1}(x_0)$ and let $\sigma\colon \pi_1(X, x_0) \to \mathfrak{S}(S)$ be the monodromy representation of $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$. Do the following:

(a) Describe $\ker(\sigma) < \pi_1(X, x_0)$ in terms of $K < \pi_1(X, x_0)$.

(b) Assume that $S$ is a finite set. Prove that the image of $\sigma$ has cardinality $|S|$ if and only if $K$ is a normal subgroup. $\square$

EXERCISE 12*.3. As in Example 1.6*.4, let $\mathrm{Poly}_n$ be the space of degree-$n$ monic polynomials over $\mathbb{C}$. Such an $f \in \mathrm{Poly}_n$ can be written as

$$f(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n \quad \text{with } a_1, \ldots, a_n \in \mathbb{C},$$

so $\mathrm{Poly}_n \cong \mathbb{C}^n$. Let

$$\mathrm{Poly}_n^{\mathrm{sf}} = \{f \in \mathrm{Poly}_n \mid f \text{ has } n \text{ distinct roots}\},$$
$$\mathrm{RPoly}_n^{\mathrm{sf}} = \left\{(f, x) \in \mathrm{Poly}_n^{\mathrm{sf}} \times \mathbb{C} \mid f(x) = 0\right\}.$$

We showed in Example 1.6*.4 that the projection $p\colon \mathrm{RPoly}_n^{\mathrm{sf}} \to \mathrm{Poly}_n^{\mathrm{sf}}$ is a degree $n$ cover. Do the following:

(a) Let
$$f_0 = (x-1)(x-2)\cdots(x-n) \in \mathrm{Poly}_n^{\mathrm{sf}}.$$
Set $S = p^{-1}(f_0)$, so
$$S = \{(f_0, 1), (f_0, 2), \ldots, (f_0, n)\}.$$
Let $\sigma\colon \pi_1(\mathrm{Poly}_n^{\mathrm{sf}}, f_0) \to \mathfrak{S}(S)$ be the monodromy representation of the cover $p\colon \mathrm{RPoly}_n^{\mathrm{sf}} \to \mathrm{Poly}_n^{\mathrm{sf}}$. Prove that $\sigma$ is surjective. Hint: the ideas from the proof of Lemma 3.3.1 might be useful.

(b) Prove that $p\colon \mathrm{RPoly}_n^{\mathrm{sf}} \to \mathrm{Poly}_n^{\mathrm{sf}}$ is an irregular cover for $n \geq 3$. Hint: Exercise 12*.2 might be useful. We remark that another proof of this can be found in Exercise 9.9. $\square$

EXERCISE 12*.4. Let $G$ be a group and let $(S, s_0)$ be a pointed right $G$-set. Set $K = G_{s_0}$, and let $k_0 = K$. Then $(S, s_0)$ is isomorphic to $(K\backslash G, k_0)$. $\square$

EXERCISE 12*.5. Let $(X, x_0)$ be a pointed space, let $p\colon \widetilde{X} \to X$ be a cover, and let $f\colon (Z, z_0) \to (X, x_0)$ be a pointed map. As in Exercise 1.14, let $q\colon \widetilde{Z} \to Z$ be the pullback of $p\colon \widetilde{X} \to X$ along $f\colon Z \to X$, so

$$\widetilde{Z} = \left\{(\widetilde{x}, z) \in \widetilde{X} \times Z \mid p(\widetilde{x}) = f(z)\right\}.$$

Let $S = p^{-1}(x_0)$ and $T = q^{-1}(x_0)$. Do the following:

- Letting $\widetilde{f}\colon \widetilde{Z} \to \widetilde{X}$ be the projection onto the first factor. Prove that $\widetilde{f}$ restricts to a bijection $\widetilde{f}_*\colon T \to S$ such that
$$\widetilde{f}_*(t[\gamma]) = \widetilde{f}_*(t) f_*([\gamma]) \quad \text{for all } t \in T \text{ and } [\gamma] \in \pi_1(Z, z_0). \qquad \square$$

Exercise 12*.6. Let $(X, x_0)$ be a space. Let $p_1 \colon \widetilde{X}_1 \to X$ and $p_2 \colon \widetilde{X}_2 \to X$ be two covers. As in Exercise 1.14, let

$$\widetilde{Z} = \left\{ (\widetilde{x}, \widetilde{y}) \in \widetilde{X}_1 \times \widetilde{X}_2 \mid p_1(\widetilde{x}) = p_2(\widetilde{y}) \right\}$$

be the pullback of $p_2$ along $p_1$. Let $q_1 \colon \widetilde{Z} \to \widetilde{X}_1$ and $q_2 \colon \widetilde{Z} \to \widetilde{X}_2$ be the projections. These are now both covers, and fit into a commutative diagram

$$\begin{tikzcd}
(\widetilde{Z}, \widetilde{z}_0) \arrow[r, "q_1"] \arrow[d, "q_2"] & (\widetilde{X}_1, \widetilde{x}_1) \arrow[d, "p_1"] \\
(\widetilde{X}_2, \widetilde{x}_2) \arrow[r, "p_2"] & (X, x_0).
\end{tikzcd}$$

Set $r = p_1 \circ q_1 = p_2 \circ q_2$, so $r$ is a map $r \colon \widetilde{Z} \to X$. You proved in Exercise 11.3 that $r$ is a cover. Set $G = \pi_1(X, x_0)$ and $S_1 = p_1^{-1}(x_0)$ and $S_2 = p_2^{-1}(x_0)$, so $S_1$ and $S_2$ are right $G$-sets. Prove the following:

(a) As a set, $r^{-1}(x_0) = S_1 \times S_2$.
(b) The right $G$-set structure on $r^{-1}(x_0) = S_1 \times S_2$ is the product of the right $G$-set structures on $S_1$ and $S_2$, i.e., $(s_1, s_2)[\gamma] = (s_1[\gamma], s_2[\gamma])$ for all $(s_1, s_2) \in S_1 \times S_2$ and $[\gamma] \in G$. $\square$

Exercise 12*.7. Let $(X, x_0)$ be a path connected pointed space. Set $G = \pi_1(X, x_0)$. Consider covers $p \colon Y \to X$ and $q \colon Z \to X$ and $r \colon Z \to Y$ such that $p = q \circ r$. Set $S = p^{-1}(x_0)$ and $T = q^{-1}(x_0)$, so $S$ and $T$ are right $G$-sets. Prove the following:

(a) The map $r$ restricts to a $G$-equivariant map $R \colon T \to S$ of right $G$-sets.
(b) Let $H$ be a group acting on $Z$ on the left by deck transformations of $q \colon Z \to X$. The set $T$ is thus an $(H, G)$-set. Assume that the map $R \colon T \to S$ is $H$-invariant. Prove that $H$ acts by deck transformations of $r \colon Z \to Y$. Hint: We must prove that $r$ restricts to an $H$-invariant map on all the fibers of $q \colon Z \to X$ from the fact that it does so on $q^{-1}(x_0)$. Use the maps $\tau_\gamma$ from §12*.1 to connect the different fibers of $q \colon Z \to X$. $\square$

CHAPTER 13*

# Categorical frameworks for covers and Grothendieck's $\pi_1$

We now recast our theory of monodromy in more categorical terms. This chapter has three families of results:

- a strengthening of our identification of covers and right $G$-sets; and
- a description of Grothendieck's approach to the fundamental group, which makes no reference to paths and can be applied in contexts where they make no sense; and
- a slightly more general version of the classification of covering spaces in terms of the fundamental groupoid.

The arguments in this chapter are more abstract and categorical than those in the other chapters of this book. We have therefore marked it with an $*$ to indicate that a reader might want to skip it on a first pass through the subject.

## 13*.1. Reminder about categories and functors

Since we have not used much categorical language for a while, we remind the reader of some basic terminology. Recall that a category $\mathbf{C}$ consists of the following data:

- A collection of objects. We will write $A \in \mathbf{C}$ to indicate that $A$ is an object of $\mathbf{C}$.
- For all objects $A, B \in \mathbf{C}$, a set $\mathbf{C}(A, B)$ of morphisms. We will often write $f\colon A \to B$ to indicate that $f$ is a morphism from $A$ to $B$.
- For all objects $A \in \mathbf{C}$, an identity morphism $\mathbb{1}_A\colon A \to A$.

These morphisms can be composed: if $f\colon A \to B$ and $g\colon B \to C$ are morphisms, then we have a morphism $g \circ f\colon A \to C$. This composition should be associative in the sense that if $f\colon A \to B$ and $g\colon B \to C$ and $h\colon C \to D$ are morphisms, then $(f \circ g) \circ h = f \circ (g \circ h)$. Under this composition, the identity morphisms should be units: if $f\colon A \to B$ is a morphism, then $f \circ \mathbb{1}_A = f$ and $\mathbb{1}_B \circ f = f$.

Example 13*.1.1. The collection of all groups and group homomomorphisms forms a category $\mathtt{Group}$. □

Example 13*.1.2. The collection of all pointed topological spaces and pointed maps forms a category $\mathtt{Top}_*$. □

Example 13*.1.3. For a group $G$, there is a category $\mathbf{G}$ with one object $\bullet$ and with $\mathbf{G}(\bullet, \bullet) = G$. □

Next, recall that a functor is defined as follows. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. A *functor* $F$ from $\mathbf{C}$ to $\mathbf{D}$ consists of the following data:

- For each object $c \in \mathbf{C}$, an object $F(c) \in \mathbf{D}$.
- For each morphism $f\colon c \to c'$ in $\mathbf{C}$, a morphism $F(f)\colon F(c) \to F(c')$ in $\mathbf{D}$.

This data should respect composition in the sense that if $f\colon c \to c'$ and $g\colon c' \to c''$ are morphisms in $\mathbf{C}$, then $F(g \circ f) = F(g) \circ F(f)$. We write $F\colon \mathbf{C} \to \mathbf{D}$ to indicate that $F$ is a functor from $\mathbf{C}$ to $\mathbf{D}$.

Example 13*.1.4. There is a functor $A\colon \mathtt{Group} \to \mathtt{Group}$ taking $G \in \mathtt{Group}$ to its abelianization $G^{\mathrm{ab}}$. For a group homomorphism $f\colon G \to H$, the the functor $A$ takes the value $A(f) = f_{\mathrm{ab}}$, where $f_{\mathrm{ab}}\colon G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ is the induced map on the abelianization. □

Example 13*.1.5. There is a functor $\pi_1\colon \mathtt{Top}_* \to \mathtt{Group}$ taking $(X, x_0) \in \mathtt{Top}_*$ to $\pi_1(X, x_0)$. □

Remark 13*.1.6. Like we did for $\pi_1$, we will often describe functors by just saying what they do to objects if their behavior on morphisms is obvious. □

### 13*.2. Natural transformations

We now introduce a sort of morphism between two functors. Let $\mathbf{C}$ and $\mathbf{D}$ be categories and let $F, G\colon \mathbf{C} \to \mathbf{D}$ be functors. A *natural transformation* $\mu$ from $F$ to $G$ consists of the following data:

- For each $c \in \mathbf{C}$, a morphism $\mu(c)\colon F(c) \to G(c)$ in $\mathbf{D}$.

These morphisms should be natural in the following sense: for all morphisms $f\colon c \to c'$ in $\mathbf{C}$, the following diagram should commute:

$$\begin{tikzcd} F(c) \arrow[r, "F(f)"] \arrow[d, "\lambda(c)"] & F(c') \arrow[d, "\lambda(c')"] \\ G(c) \arrow[r, "G(f)"] & G(c') \end{tikzcd}$$

In other words, $\lambda(c') \circ F(f) = G(f) \circ \lambda(c)$. We write $\lambda\colon F \Rightarrow G$ to indicate that $\lambda$ is a natural transformation from $F$ to $G$.

Example 13*.2.1. Let $A\colon \texttt{Group} \to \texttt{Group}$ be the functor taking a group $G$ to its abelianization and let $\mathrm{Id}\colon \texttt{Group} \to \texttt{Group}$ be the identity functor. There is a natural transformation $\mu\colon \mathrm{Id} \Rightarrow A$ defined as follows:

- For a group $G$, we must define a group homomorphism $\mu(G)$ from $\mathrm{Id}(G) = G$ to $A(G) = G^{\mathrm{ab}}$. Let $\mu(G)\colon G \to G^{\mathrm{ab}}$ be the abelianization homomorphism.

The naturality of $\mu$ is the assertion that if $f\colon G \to H$ is a group homomorphism, then the diagram

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\mu(G)"] & H \arrow[d, "\mu(H)"] \\ G^{\mathrm{ab}} \arrow[r, "f_{\mathrm{ab}}"] & H^{\mathrm{ab}} \end{tikzcd}$$

commutes, which is immediate. □

Example 13*.2.2. Let $\texttt{Ring}$ be the category of commutative rings and let $n \geq 1$. Define two functors $\mathrm{GL}_n\colon \texttt{Ring} \to \texttt{Group}$ and $U\colon \texttt{Ring} \to \texttt{Group}$ as follows:

- For a commutative ring $R$, let $\mathrm{GL}_n(R)$ be the rank-$n$ general linear group over $R$.
- For a commutative ring $R$, let $U(R)$ be the group of units of $R$, i.e., the group of all $x \in R$ such that there exists some $y \in R$ with $xy = 1$.

We can define a natural transformation $\mu\colon \mathrm{GL}_n \Rightarrow U$ as follows:

- For a commutative ring $R$, let $\mu(R)\colon \mathrm{GL}_n(R) \to U(R)$ be the homomorphism taking a matrix to its determinant.

The naturality of $\mu$ is the assertion that if $f\colon R \to R'$ is a ring homomorphism, then the diagram

$$\begin{tikzcd} \mathrm{GL}_n(R) \arrow[r, "f"] \arrow[d, "\det"] & \mathrm{GL}_n(R') \arrow[d, "\det"] \\ U(R) \arrow[r, "f"] & U(R') \end{tikzcd}$$

commutes, which is immediate. In our future examples of natural transformations, we will generally not expand out the meaning of the assertion that they are natural, but leave it as an exercise to the reader. □

For functors $F, G\colon \mathbf{C} \to \mathbf{D}$, a natural transformation $\mu\colon F \Rightarrow G$ is a *natural isomorphism* if the morphisms $\mu(c)\colon F(c) \to G(c)$ are isomorphisms for all $c \in \mathbf{C}$. In this case, we say that $F$ and $G$ are *naturally isomorphic*.

## 13*.3. Equivalences of categories

The final piece of abstract nonsense we need is the notion of an equivalence of categories. One way that categories $\mathbf{C}$ and $\mathbf{D}$ can be the "same" is if they are isomorphic, i.e., there exist functors $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$ such that $G \circ F\colon \mathbf{C} \to \mathbf{C}$ and $F \circ G\colon \mathbf{D} \to \mathbf{D}$ are both the identity functor. However, it turns out that this is too restrictive.

Weakening this, functors $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$ form an *equivalence of categories* if $G \circ F\colon \mathbf{C} \to \mathbf{C}$ and $F \circ G\colon \mathbf{D} \to \mathbf{D}$ are both naturally isomorphic to the identity functor. This is the right notion of equivalence in category theory. We will also say that a functor $F\colon \mathbf{C} \to \mathbf{D}$ is an equivalence of categories if a functor $G\colon \mathbf{D} \to \mathbf{C}$ as above exists, though be warned that $G$ is typically not unique.

For our first example of this, we need two pieces of terminology. Let $\mathbf{A}$ be a category. A *subcategory* of $\mathbf{A}$ is a category $\mathbf{B}$ such that:

- all objects $b \in \mathbf{B}$ are also objects of $\mathbf{A}$; and
- for all $b, b' \in \mathbf{B}$ the morphisms in $\mathbf{B}$ from $b$ to $b'$ are also morphisms of $\mathbf{A}$, i.e., we have $\mathbf{B}(b, b') \subset \mathbf{A}(b, b')$; and
- the composition rule for morphisms of $\mathbf{B}$ is the same as that of the corresponding morphisms of $\mathbf{A}$.

These conditions imply that there is a natural inclusion functor $\mathbf{B} \to \mathbf{A}$. If for all $b, b' \in \mathbf{B}$ we have $\mathbf{B}(b, b') = \mathbf{A}(b, b')$, then we say that $\mathbf{B}$ is a *full* subcategory of $\mathbf{A}$.

EXAMPLE 13*.3.1. Regard $\spadesuit$ as a formal symbol. Let $\mathtt{Set}$ be the category of sets and let $\mathtt{Set}_{\times\spadesuit}$ be the full subcategory of $\mathtt{Set}$ consisting of sets that can be written as $X \times \{\spadesuit\}$ for some $X \in \mathtt{Set}$. Let $F\colon \mathtt{Set}_{\times\spadesuit} \to \mathtt{Set}$ be the inclusion and let $G\colon \mathtt{Set} \to \mathtt{Set}_{\times\spadesuit}$ be the functor $G(X) = X \times \{\spadesuit\}$. The functors $F$ and $G$ form an equivalence of categories. Indeed:

- The functor $F \circ G\colon \mathtt{Set} \to \mathtt{Set}$ is the functor taking $X \in \mathtt{Set}$ to $X \times \{\spadesuit\}$. This is naturally isomorphic to the identity functor $\mathrm{Id}\colon \mathtt{Set} \to \mathtt{Set}$ via the natural isomorphism $\mu\colon F \circ G \Rightarrow \mathrm{Id}$ defined as follows:
  - For $X \in \mathtt{Set}$, we must define a set map $\mu(X)$ from $F \circ G(X) = X \times \{\spadesuit\}$ to $X$. This map takes $(x, \spadesuit) \in X \times \{\spadesuit\}$ to $x \in X$.
- The functor $G \circ F\colon \mathtt{Set}_{\times\spadesuit} \to \mathtt{Set}_{\times\spadesuit}$ is the functor taking $X \times \{\spadesuit\} \in \mathtt{Set}_{\times\spadesuit}$ to $(X \times \{\spadesuit\}) \times \{\spadesuit\}$. This is naturally isomorphic to the identity functor $\mathrm{Id}\colon \mathtt{Set}_{\times\spadesuit} \to \mathtt{Set}_{\times\spadesuit}$ via the natural isomorphism $\mu\colon G \circ F \Rightarrow \mathrm{Id}$ defined as follows:
  - For $X \times \{\spadesuit\} \in \mathtt{Set}_{\times\spadesuit}$, we must define a set map $\mu(X)$ from $G \circ F(X) = (X \times \{\spadesuit\}) \times \{\spadesuit\}$ to $X \times \{\spadesuit\}$. This map takes $((x, \spadesuit), \spadesuit) \in (X \times \{\spadesuit\}) \times \{\spadesuit\}$ to $(x, \spadesuit) \in X \times \{\spadesuit\}$. $\square$

Though the inclusion $\mathtt{Set}_{\times\spadesuit} \to \mathtt{Set}$ is not an isomorphism of categories, the categories $\mathtt{Set}_{\times\spadesuit}$ and $\mathtt{Set}$ are actually isomorphic. This might make Example 13*.3.1 a bit unsatisfying. The following is a more typical example of an equivalence of categories where the two categories are not isomorphic.

EXAMPLE 13*.3.2. Let $\mathtt{FinSet}$ be the category of finite sets. For $n \geq 0$, set $\underline{n} = \{1, \ldots, n\} \in \mathtt{FinSet}$. Let $\mathbf{C}$ be the full subcategory of $\mathtt{FinSet}$ whose objects are the sets of the form $\underline{n}$ for $n \geq 0$. Let $F\colon \mathbf{C} \to \mathtt{FinSet}$ be the inclusion functor. We claim that $F$ is an equivalence of categories. The corresponding morphism $G\colon \mathtt{FinSet} \to \mathbf{C}$ is constructed as follows:

- For a finite set $X \in \mathtt{FinSet}$, set $n = |X|$ and let $G(X) = \underline{n}$. For later use, we also choose a bijection $\sigma_X : X \to \underline{n}$. To simplify things, we choose this such that if $X = \underline{n}$ then $\sigma_X$ is the identity $\mathbb{1}\colon \underline{n} \to \underline{n}$. We remark that this depends on the axiom of choice.[1]
- For finite sets $X$ and $Y$ and a set map $f\colon X \to Y$, define $G(f)\colon G(X) \to G(Y)$ as follows. Set $n = |X|$ and $m = |Y|$, and let $G(f)\colon \underline{n} \to \underline{m}$ be the following composition:

$$\underline{n} \xrightarrow{\sigma_X^{-1}} X \xrightarrow{f} Y \xrightarrow{\sigma_Y} \underline{m}.$$

[1] There is a small foundational issue here. The collection of all finite sets is too large to be a set, so the usual axiom of choice is not strong enough to make all these choices. Instead, the collection of finite sets is what is called a proper class. The version of the axiom of choice we need is what is called the axiom of *global* choice. Roughly speaking, it says that it is possible to consistently choose an element from each nonempty set. Since in this book we do not focus on foundational issues, we refer the reader to books on set theory for a precise statement of this.

We check that $F \circ G$ and $G \circ F$ are both naturally isomorphic to the identity as follows:

- Since $\sigma_{\underline{n}} = \mathbb{1}$ for all $n \geq 0$, the functor $F \circ G \colon \mathbf{C} \to \mathbf{C}$ equals the identity functor.
- Consider a finite set $X$. Letting $n = |X|$, the functor $G \circ F \colon \mathtt{FinSet} \to \mathtt{FinSet}$ satisfies $G \circ F(X) = \underline{n}$. We can define a natural isomorphism $\mu \colon G \circ F \Rightarrow \mathrm{Id}$ by letting $\mu(X) \colon \underline{n} \to X$ be the bijection $\sigma_X^{-1}$. □

The subcategory $\mathbf{C}$ of $\mathtt{FinSet}$ from Example 13*.3.2 is an example of a *skeleton* of a category, that is, a full subcategory containing exactly one representative of each isomorphism class of objects. The argument from Example 13*.3.2 shows that the inclusion of a skeleton into a category is an equivalence of categories.

In fact, even more is true. This requires some terminology. Let $\mathbf{C}$ and $\mathbf{D}$ be categories and let $F \colon \mathbf{C} \to \mathbf{D}$ be a functor. For $c, c' \in \mathbf{C}$ the functor $F$ induces a map $F \colon \mathbf{C}(c, c') \to \mathbf{D}(F(c), F(c'))$. We make the following definitions.

- The functor $F$ is *faithful* if for all $c, c' \in \mathbf{C}$ the induced map $F \colon \mathbf{C}(c, c') \to \mathbf{D}(F(c), F(c'))$ is injective.
- The functor $F$ is *full* if for all $c, c' \in \mathbf{C}$ the induced map $F \colon \mathbf{C}(c, c') \to \mathbf{D}(F(c), F(c'))$ is surjective.
- The functor $F$ is *essentially surjective* if for all $d \in \mathbf{D}$ there exists some $c \in \mathbf{C}$ such that $F(c)$ is isomorphic to $d$.

If $F \colon \mathbf{C} \to \mathbf{D}$ is the inclusion of a subcategory then $F$ is faithful, and if $F$ is the inclusion of a full subcategory then $F$ is full and faithful. The inclusion of a skeleton of a category is full, faithful, and essentially surjective. The following lemma helps explain the meaning of an equivalence of categories:

LEMMA 13*.3.3. *Let $\mathbf{C}$ and $\mathbf{D}$ be categories and let $F \colon \mathbf{C} \to \mathbf{D}$ be a functor. Then $F$ is an equivalence of categories if and only if $F$ is full, faithful, and essentially surjective.*

PROOF. The main idea of the proof is similar to that of Example 13*.3.2, so we leave it as an exercise (see Exercise 13*.5). □

## 13*.4. Covers are the same thing as right $G$-sets, II

We now return to covering spaces. Let $(X, x_0)$ be a pointed space. Set $G = \pi_1(X, x_0)$. Define the following categories:

- Let $\mathtt{RSet}_G$ be the category whose objects are right $G$-sets and whose morphisms are $G$-equivariant maps.
- Let $\mathtt{Cover}_X$ be the category whose objects are covers $p \colon \widetilde{X} \to X$ and where a morphism from a cover $p_1 \colon \widetilde{X}_1 \to X$ to a cover $p_2 \colon \widetilde{X}_2 \to X$ is a cover $q \colon \widetilde{X}_1 \to \widetilde{X}_2$ such that the following diagram commutes:

$$\begin{tikzcd} \widetilde{X}_1 \arrow[rr, "q"] \arrow[dr, "p_1"'] & & \widetilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

The *fiber functor* is the functor $\mathcal{F} \colon \mathtt{Cover}_X \to \mathtt{RSet}_G$ defined by

$$\mathcal{F}(p \colon \widetilde{X} \to X) = p^{-1}(x_0) \in \mathtt{RSet}_G \quad \text{for a cover } p \colon \widetilde{X} \to X.$$

The fact that this is a functor follows from Lemma 12*.3.1. Our first main result in this chapter is the following, which strenthens Theorem 12*.5.1:

THEOREM 13*.4.1. *Let $(X, x_0)$ be a pointed space that is path connected, locally path connected, and semilocally 1-connected. Set $G = \pi_1(X, x_0)$. Then the fiber functor $\mathcal{F} \colon \mathtt{Cover}_X \to \mathtt{RSet}_G$ is an equivalence of categories.*

For a category $\mathbf{C}$ and $c \in \mathbf{C}$, let $\mathrm{Aut}_{\mathbf{C}}(c)$ denote the group of invertible morphisms $f \colon c \to c$ in $\mathbf{C}(c, c)$. For $(X, x_0)$ as in Theorem 13*.4.1, that theorem together with Lemma 13*.3.3 implies the following. Consider a cover $p \colon \widetilde{X} \to X$, and set $S = p^{-1}(x_0)$. Then

$$\mathrm{Deck}(\widetilde{X}) = \mathrm{Aut}_{\mathtt{Cover}_X}(p \colon \widetilde{X} \to X) \cong \mathrm{Aut}_{\mathtt{RSet}_G}(p^{-1}(x_0)) = \mathrm{Aut}_G(S).$$

This fact was proved earlier in Lemma 12*.6.3. It is enlightening to go through the proof of Lemma 12*.6.3 and see that secretly it was using the same ingredients as the proof of Theorem 13*.4.1 below.

Proof of Theorem 13*.4.1. The proof uses the same ideas as the proof of Theorem 12*.5.1. We divide it into three steps: we first construct a functor $\mathcal{R}\colon \mathtt{RSet}_G \to \mathtt{Cover}_X$ that we call the *realization functor*, then we prove that $\mathcal{F}\circ\mathcal{R}\colon \mathtt{RSet}_G \to \mathtt{RSet}_G$ is naturally isomorphic to the identity, and then finally we prove that $\mathcal{R}\circ\mathcal{F}\colon \mathtt{Cover}_X \to \mathtt{Cover}_X$ is naturally isomorphic to the identity. Throughout the proof, we fix a universal cover $p_u\colon (\widetilde{X}_u, \widetilde{x}_u) \to (X, x_0)$, which we recall is a regular cover with deck group $G$ (see Theorem 10.3.1).[2]

Step 1. *We construct the realization functor* $\mathcal{R}\colon \mathtt{RSet}_G \to \mathtt{Cover}_X$.

Let $S$ be a right $G$-set. Endow $S$ with the discrete topology, and define a left action of $G$ on $S \times \widetilde{X}_u$ as follows:

$$g(s, \widetilde{x}) = (sg^{-1}, g\widetilde{x}) \quad \text{for all } s \in S \text{ and } g \in G \text{ and } \widetilde{x} \in \widetilde{X}_u.$$

Consider the space $G\backslash(S \times \widetilde{X}_u)$. The group $G$ is acting on $S \times \widetilde{X}_u$ by deck transformations of the cover $q\colon S \times \widetilde{X}_u \to X$ given by

$$S \times \widetilde{X}_u \xrightarrow{\text{proj}} \widetilde{X}_u \xrightarrow{p_u} X,$$

so we get an induced cover $p\colon G\backslash(S \times \widetilde{X}_u) \to X$. Define $\mathcal{R}(S) = p$. This is clearly functorial.

Step 2. *We prove that* $\mathcal{F}\circ\mathcal{R}\colon \mathtt{RSet}_G \to \mathtt{RSet}_G$ *is naturally isomorphic to the identity.*

Let $S$ be a right $G$-set. To construct a natural isomorphism $\mu\colon \mathcal{F}\circ\mathcal{R} \Rightarrow \mathrm{Id}$, we must define a natural isomorphism $\mu(S)\colon \mathcal{F}\circ\mathcal{R}(S) \to S$. Examining the construction from Step 1, we see that

$$\mathcal{F}\circ\mathcal{R}(S) = G\backslash(S \times p_u^{-1}(x_0)) = G\backslash(S \times G\widetilde{x}_u).$$

Let $\widetilde{\mu(S)}\colon S \times G\widetilde{x}_u \to S$ be the map

$$\widetilde{\mu(S)}(s, g\widetilde{x}_u) = sg \quad \text{for all } s \in S \text{ and } g \in G.$$

The map $\widetilde{\mu(S)}$ is $G$-invariant, so it factors through a map $\mu(S)\colon G\backslash(S \times G\widetilde{x}_u) \to S$. The map $\mu(S)$ is a bijection whose inverse takes $s \in S$ to the image of $(s, \widetilde{x}_u)$ in $G\backslash(S \times G\widetilde{x}_u)$.

Step 3. *We prove that* $\mathcal{R}\circ\mathcal{F}\colon \mathtt{Cover}_X \to \mathtt{Cover}_X$ *is naturally isomorphic to the identity.*

Let $p\colon \widetilde{X} \to X$ be a cover. To construct a natural isomorphism $\mu\colon \mathcal{R}\circ\mathcal{F} \Rightarrow \mathrm{Id}$, we must define an isomorphism of covers $\mu(p)\colon \mathcal{R}\circ\mathcal{F}(p) \to p$. Let $S = \mathcal{F}(S) = p^{-1}(x_0)$. The cover $\mathcal{R}\circ\mathcal{F}(p) = \mathcal{R}(S)$ is of the form $p'\colon G\backslash(S \times \widetilde{X}_u) \to X$. Our goal is to construct a homeomorphism $\lambda\colon G\backslash(S \times \widetilde{X}_u) \to \widetilde{X}$ such that the following diagram commutes:

$$\begin{tikzcd} G\backslash(S \times \widetilde{X}_u) \arrow[rr, "\lambda"] \arrow[dr, "p'"'] & & \widetilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd} \tag{13*.4.1}$$

We can then define $\mu(p) = \lambda$. It will be clear from our construction that this $\mu$ is natural.

For $s \in S = p^{-1}(x_0)$, the universality of the universal cover (Lemma 10.1.1) implies that there exists a unique covering map $r_s\colon (\widetilde{X}_u, \widetilde{x}_u) \to (\widetilde{X}, s)$ such that $p_u = p \circ r_s$. These assemble into a covering space $r\colon S \times \widetilde{X}_u \to \widetilde{X}$ defined by

$$r(s, \widetilde{x}) = r_s(\widetilde{x}) \quad \text{for all } (s, \widetilde{x}) \in S \times \widetilde{X}_u.$$

The fact that this is a covering space uses the fact that $X$ (and hence $\widetilde{X}$) is locally connected and semilocally 1-connected. Indeed, this implies that $\widetilde{X}$ can be covered by sl1c subsets $\widetilde{U}$. These are trivialized open sets for all the $r_s$ (see Lemma 10.4*.1), and hence are trivialized open sets for $r$.

[2]The universal cover is unique up to isomorphism. We are fixing a representative of its isomorphism class.

Let $q\colon S \times \widetilde{X}_u \to X$ be the cover from Step 1. The group $G$ acts on $S \times \widetilde{X}_u$ on the left by deck transformations for $q\colon S \times \widetilde{X}_u \to X$:

$$g(s, \widetilde{x}) = (sg^{-1}, g\widetilde{x}) \quad \text{for all } s \in S \text{ and } g \in G \text{ and } \widetilde{x} \in \widetilde{X}_u.$$

Just like in Step 2 of the proof of Theorem 12*.5.1, it follows from Exercise 12*.7 that the action of $G$ on $S \times \widetilde{X}_u$ is by deck transformations of $r\colon S \times \widetilde{X}_u \to \widetilde{X}$ and that $G$ acts transitively on the fibers of $r$. We conclude that $r$ factors through a homeomorphism $\lambda\colon G\backslash(S \times \widetilde{X}_u) \to \widetilde{X}$ such that (13*.4.1) commutes, as desired. □

## 13*.5. Grothendieck's approach to the fundamental group

Let $\mathbf{C}$ and $\mathbf{D}$ be categories. Define $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$ be the following category:

- The objects of $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$ consist of functors $F\colon \mathbf{C} \to \mathbf{D}$.
- For functors $F, G\colon \mathbf{C} \to \mathbf{D}$, the morphisms from $F$ to $G$ consist[3] of natural transformations $\mu\colon F \Rightarrow G$.

For $F \in \mathrm{Fun}(\mathbf{C}, \mathbf{D})$, the $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$-automorphism group consists of all natural isomorphisms $\mu\colon F \Rightarrow F$. We call this the *automorphism group* of the functor $F$.

Grothendieck showed how to describe the fundamental group in this language. Let $(X, x_0)$ be a pointed space. Let $\mathcal{F}_{\mathrm{set}}\colon \mathtt{Cover}_X \to \mathtt{Set}$ be the functor defined by

$$\mathcal{F}_{\mathrm{set}}(p\colon \widetilde{X} \to X) = p^{-1}(x_0) \in \mathtt{Set} \quad \text{for a cover } p\colon \widetilde{X} \to X.$$

The difference beetween this and the fiber functor from the last section is that $\mathcal{F}_{\mathrm{set}}$ takes values in sets, not in right $G$-sets for $G = \pi_1(X, x_0)$. The functor $\mathcal{F}_{\mathrm{set}}$ makes no reference to $\pi_1(X, x_0)$; however, in favorable situations we can recover $\pi_1(X, x_0)$ from it:

THEOREM 13*.5.1 (Grothendieck). *Let $(X, x_0)$ be a pointed space that is path connected, locally path connected, and semilocally 1-connected. Then $\pi_1(X, x_0)$ is isomorphic to the automorphism group of the functor $\mathcal{F}_{set}$.*

PROOF. Concretely, an element $\sigma$ of the automorphism group of $\mathcal{F}_{\mathrm{set}}$ consists of the following data:

- For each cover $p\colon \widetilde{X} \to X$, a bijection $\sigma_{\widetilde{X}}$ of the fiber $p^{-1}(x_0)$.

These bijections must be compatible in the sense that if $p_1\colon \widetilde{X}_1 \to X$ and $p_2\colon \widetilde{X}_2 \to X$ are covers and $q\colon \widetilde{X}_1 \to \widetilde{X}_2$ is a morphism of covers, then

$$q(\sigma_{\widetilde{X}_1}(\widetilde{x})) = \sigma_{\widetilde{X}_2}(q(\widetilde{x})) \quad \text{for all } \widetilde{x} \in p_1^{-1}(x_0).$$

We now turn to the proof of the theorem. Set $G = \pi_1(X, x_0)$. Let $\mathcal{F}\colon \mathtt{Cover}_X \to \mathtt{RSet}_G$ be the fiber functor. The functor $\mathcal{F}_{\mathrm{set}}\colon \mathtt{Cover}_X \to \mathtt{Set}$ factors as

$$\mathtt{Cover}_X \xrightarrow{\mathcal{F}} \mathtt{RSet}_G \xrightarrow{\mathcal{S}} \mathtt{Set},$$

where $\mathcal{S}\colon \mathtt{RSet}_G \to \mathtt{Set}$ takes a right $G$-set and forgets the right action of $G$. Since $\mathcal{F}$ is an equivalence of categories (Theorem 13*.4.1), the automorphism group of $\mathcal{F}_{\mathrm{set}}$ is isomorphic to the automorphism group of $\mathcal{S}$ (see Exercise 13*.6). We can define a map $\phi$ from $G$ to the automorphism group of $\mathcal{S}\colon \mathtt{RSet}_G \to \mathtt{Set}$: for $g \in G$, let $\phi(g)\colon \mathcal{S} \Rightarrow \mathcal{S}$ be the following natural transformation:

- For $S \in \mathtt{RSet}_G$, let $\phi(g)(S)\colon S \to S$ be the bijection defined by $\phi(g)(S)(s) = sg^{-1}$ for $g \in G$.

We must prove that this is an isomorphism. This is a purely algebraic result about right $G$-sets, so we leave it as an exercise (see Exercise 13*.4). □

Theorem 13*.5.1 is an interesting result since it describes the fundamental group entirely in terms of covers. There are contexts where "covers" make sense but loops do not. Grothendieck was interested in defining the fundamental group of an algebraic variety $Z$ over an arbitrary field. In algebraic geometry, infinite covers do not make sense. However, finite covers $p\colon \widetilde{Z} \to Z$ do make

[3]There is a small set theoretic issue here. In our definition of a category, the collection of objects was not required to be a set but the collection of morphisms between two objects was required to be a set. This need not hold here in general, but in our examples it will always hold.

sense, and are what the algebraic geometers call *finite étale morphisms*. Grothendieck used these to define his *étale fundamental group*. Roughly speaking, what he did was as follows.

Fix a basepoint $z_0 \in Z$. Grothendieck defined a category $\text{Étale}_Z$ of finite étale morphisms $p\colon \widetilde{Z} \to Z$. There is a fiber functor $\mathcal{F}\colon \text{Étale}_Z \to \mathtt{Set}$ that takes a finite étale morphism $p\colon \widetilde{Z} \to Z$ and returns its fiber $p^{-1}(z_0)$. Grothendieck defined $\pi_1^{\text{ét}}(Z, z_0)$ to be the automorphism group of the functor $\mathcal{F}\colon \text{Étale}_Z \to \mathtt{Set}$. Since $\text{Étale}_Z$ only includes the analogues of finite covers, $\pi_1^{\text{ét}}(Z, z_0)$ has a natural topology making it into what is called a profinite group. The group $\pi_1^{\text{ét}}(Z, z_0)$ plays the same role in the topology of varieties that the classical fundamental group plays in the topology of spaces.

## 13*.6. Reminder about fundamental groupoid

Let $X$ be a space. Our final result in this chapter uses the fundamental groupoid of $X$, which is a category that encodes the collection of homotopy classes of paths between points of $X$. Since we have not discussed the fundamental groupoid since introducing it in Chapter 5, we recall some basic facts about it.

As we discussed in §5.8*, in a category morphisms are composed right-to-left like functions. However, our conventions for composing paths goes from left to right: if $\gamma$ is a path from $x$ to $y$ and $\gamma'$ is a path from $y$ to $z$, then $\gamma \cdot \gamma'$ is the path from $x$ to $z$ that first traverses $\gamma$ and then traverses $\gamma'$. To fix this mismatch of conventions, we introduced the following notation:

Notation 13*.6.1. Let $X$ be a space. For points $x, y, z \in X$, let $\gamma$ be a path in $X$ from $x$ to $y$ and let $\delta$ be a path in $X$ from $y$ to $z$. We then define $\gamma' * \gamma = \gamma \cdot \gamma'$. This descends to homotopy classes of paths, and we also write $[\gamma'] * [\gamma] = [\gamma' * \gamma]$. □

With this notation, the fundamental groupoid of $X$, denoted $\Pi(X)$, is the following category:

- The objects of $\Pi(X)$ are the points of $X$.
- For points $x$ and $y$, the $\Pi(X)$-morphisms from $x$ to $y$ are the set of all homotopy classes of paths from $x$ to $y$. For a path $\gamma$ from $x$ to $y$, we will write $[\gamma]\colon x \to y$ for the corresponding morphism from $x$ to $y$.
- If $\gamma$ is a path from $x$ to $y$ and $\gamma'$ is a path from $y$ to $z$, then the composition of the morphisms $[\gamma]\colon x \to y$ and $[\gamma']\colon y \to z$ is the morphism $[\gamma'] * [\gamma]\colon x \to z$.
- For a point $x \in X$, the identity morphism of $x$ is the constant path $[\mathfrak{c}_x]\colon x \to x$.

## 13*.7. Global fiber functors

Let $X$ be a space and let $p\colon \widetilde{X} \to X$ be a covering space. The *global fiber functor* of $p\colon \widetilde{X} \to X$ is the following functor $\mathcal{G}_p\colon \Pi(X) \to \mathtt{Set}$. For an object $x \in \widetilde{X}$, we define $\mathcal{G}_p(x) = p^{-1}(x)$. For a morphism $[\gamma]\colon x \to y$ in $\Pi(X)$, we define $\mathcal{G}_p([\gamma])\colon \mathcal{G}_p(x) \to \mathcal{G}_p(y)$ to be the map $\tau_\gamma$ we discussed in §12*.1. To see that this is a functor, consider morphisms $[\gamma]\colon x \to y$ and $[\delta]\colon y \to z$ in $\Pi(X)$. We must prove that $\mathcal{G}_p([\delta] * [\gamma]) = \mathcal{G}_p([\delta]) \circ \mathcal{G}_p([\gamma])$, i.e., that

$$\tau_{\gamma \cdot \delta} = \tau_\delta \circ \tau_\gamma.$$

This is exactly Lemma 12*.1.1.

Remark 13*.7.1. Using $*$ instead of the concatanation product $\cdot$ in the fundamental groupoid accomplished the same thing for the fiber functor that using right actions did for the monodromy action of the fundamental group. □

## 13*.8. Realizing fiber functors

Let $X$ be a space that is locally path connected and semilocally 1-connected. Note that we are not assuming that $X$ is path connected. Our goal is to prove that covers of $X$ are the "same" as functors from $\Pi(X)$ to $\mathtt{Set}$. The easiest version of this says the following:

- For all functors $F\colon \Pi(X) \to \mathtt{Set}$, there is a cover $p\colon \widetilde{X} \to X$ whose global fiber functor is naturally isomorphic to $F$.
- If $p_1\colon \widetilde{X}_1 \to X$ and $p_2\colon \widetilde{X}_2 \to X$ are covers whose global fiber functors are naturally isomorphic, then $p_1\colon \widetilde{X}_1 \to X$ and $p_2\colon \widetilde{X}_2 \to X$ are isomorphic.

The more precise form is an equivalence of categories.

Recall from §13*.5 that $\mathrm{Fun}(\Pi(X), \mathtt{Set})$ is the category of functors $F\colon \Pi(X) \to \mathtt{Set}$. The morphisms in $\mathrm{Fun}(\Pi(X), \mathtt{Set})$ are natural transformations. It is immediate from the definitions that the map $\mathcal{G}\colon \mathtt{Cover}_X \to \mathrm{Fun}(\Pi(X), \mathtt{Set})$ taking a cover $p\colon \widetilde{X} \to X$ to its global fiber functor $\mathcal{G}_p$ is a functor that we will also call the global fiber functor. We will prove:

THEOREM 13*.8.1. *Let $X$ be a space that is locally path connected and semilocally 1-connected. The global fiber functor $\mathcal{G}\colon \mathtt{Cover}_X \to \mathrm{Fun}(\Pi(X), \mathtt{Set})$ is an equivalence of categories.*

PROOF. It is enough to prove this on each path component of $X$, so we can assume without loss of generality that $X$ is path connected. Fix a basepoint $x_0 \in X$. Let $\mathbf{C}$ be the full subcategory of $\Pi(X)$ with the single object $x_0$. This is a skeleton of $\Pi(X)$. Indeed, consider $x \in X$. We must prove that there is an isomorphism in $\Pi(X)$ from $x$ to $x_0$. Since all morphisms in $\Pi(X)$ are invertible, this follows from the fact that $X$ is path connected, so in particular there exists a path in $X$ from $x_0$ to $x$.

Since $\mathbf{C}$ is a skeleton of $\Pi(X)$, the inclusion functor $\mathbf{C} \to \Pi(X)$ is an equivalence of categories (see Lemma 13*.3.3). This implies that restricting functors to $\mathbf{C}$ gives an equivalence of categories $\mathrm{Fun}(\Pi(X), \mathtt{Set}) \to \mathrm{Fun}(\mathbf{C}, \mathtt{Set})$; see Exercise 13*.6. Composing $\mathcal{G}$ with this gives a functor $\mathcal{F}\colon \mathtt{Cover}_X \to \mathrm{Fun}(\mathbf{C}, \mathtt{Set})$, and it is enough to prove that this is an equivalence of categories.

As we discussed in Example 13*.1.3, a group $G$ can be identified with a category $\mathbf{G}$ with one object $\bullet$ and with $\mathbf{G}(\bullet, \bullet) = G$. Setting $G = \pi_1(X, x_0)$, under this identification $\mathbf{G}$ equals the category $\mathbf{C}$. A functor $F \in \mathrm{Fun}(\mathbf{G}, \mathtt{Set})$ is determined by the following data:

- The set $S$ obtained by evaluating $F$ on the single object $\bullet$ of $\mathbf{G}$.
- For each $g \in G$, a map $F(g)\colon S \to S$. Since $G$ is a group, the element $g$ is invertible and $F(g)$ is a bijection.

Using this, it is easy to show that $\mathrm{Fun}(\mathbf{G}, \mathtt{Set})$ is isomorphic to the category $\mathtt{RSet}_G$ of right $G$-sets (see Exercise 13*.3). The functor

$$\mathcal{F}\colon \mathtt{Cover}_X \to \mathrm{Fun}(\mathbf{C}, \mathtt{Set}) = \mathrm{Fun}(\mathbf{G}, \mathtt{Set}) = \mathtt{RSet}_G$$

is exactly the fiber functor that we proved is an equivalence of categories in Theorem 13*.4.1. The theorem follows. ☐

## 13*.9. Exercises

EXERCISE 13*.1. Expand out what it means for the natural transformations in Examples 13*.3.1 and 13*.3.2 to be natural. ☐

EXERCISE 13*.2. Let $G$ be a group, and let $\mathtt{LSet}_G$ and $\mathtt{RSet}_G$ be the categories of left $G$-sets and right $G$-sets. Prove that $\mathtt{LSet}_G$ and $\mathtt{RSet}_G$ are isomorphic categories. ☐

EXERCISE 13*.3. Let $G$ and $H$ be groups and let $\mathbf{G}$ and $\mathbf{H}$ be the corresponding categories with one object $\bullet$. Do the following:

(a) Prove that the category $\mathrm{Fun}(\mathbf{G}, \mathtt{Set})$ is isomorphic both to the category $\mathtt{LSet}_G$ of left $G$-sets and to the category $\mathtt{RSet}_G$ of right $G$-sets.
(b) Prove that a functor $F\colon \mathbf{G} \to \mathbf{H}$ can be identified with a group homomorphism $f\colon G \to H$.
(c) Let $F_1, F_2\colon \mathbf{G} \to \mathbf{H}$ be two functors, and let $f_1, f_2\colon G \to H$ be the corresponding group homomorphisms as in (b). Prove that there exists a natural transformation $\mu\colon F_1 \Rightarrow F_2$ if and only there exists some $h \in H$ such that $f_1(g) = h f_2(g) h^{-1}$ for all $g \in G$. ☐

EXERCISE 13*.4. Let $G$ be a group and let $\mathcal{S}\colon \mathtt{RSet}_G \to \mathtt{Set}$ be the functor that takes a right $G$-set and forgets the right action of $G$. Prove that the automorphism group of $\mathcal{S}$ is isomorphic to $G$. ☐

EXERCISE 13*.5. Let $\mathbf{C}$ and $\mathbf{D}$ be categories and let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor. Prove that $F$ is an equivalence of categories if and only if $F$ is full, faithful, and essentially surjective. To avoid set-theoretic issues, you may assume that the collection of objects of $\mathbf{D}$ forms a set. ☐

EXERCISE 13*.6. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. Consider the category $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$ of functors from $\mathbf{C}$ to $\mathbf{D}$. Do the following:

(a) Let $\mathbf{C}'$ be a category and $E\colon \mathbf{C}' \to \mathbf{C}$ be an equivalence of categories. Prove that composing with $E$ gives an equivalence of categories $\mathrm{Fun}(\mathbf{C}, \mathbf{D}) \to \mathrm{Fun}(\mathbf{C}', \mathbf{D})$. In particular, for $F \in \mathrm{Fun}(\mathbf{C}, \mathbf{D})$ the automorphism groups of the functors $F\colon \mathbf{C} \to \mathbf{D}$ and $F \circ E\colon \mathbf{C}' \to \mathbf{D}$ are isomorphic.

(b) Let $\mathbf{D}'$ be a category and $E\colon \mathbf{D} \to \mathbf{D}'$ be an equivalence of categories. Prove that composing with $E$ gives an equivalence of categories $\mathrm{Fun}(\mathbf{C}, \mathbf{D}) \to \mathrm{Fun}(\mathbf{C}, \mathbf{D}')$. In particular, for $F \in \mathrm{Fun}(\mathbf{C}, \mathbf{D})$ the automorphism group of the functors $F\colon \mathbf{C} \to \mathbf{D}$ and $E \circ F\colon \mathbf{C} \to \mathbf{D}'$ are isomorphic. □

Exercise 13*.7. Let $(X, x_0)$ be a pointed topological space that is path connected, locally path connected, and semilocally 1-connected. Describe the functor $F\colon \Pi(X) \to \mathtt{Set}$ that is the global fiber functor of the universal cover of $X$. □

Exercise 13*.8. Let $X$ be a topological space that is locally path connected and semilocally 1-connected. Let $F\colon \Pi(X) \to \mathtt{Set}$ be a functor. By Theorem 13*.8.1, the functor $F$ is the global fiber functor of a cover $p\colon \widetilde{X} \to X$. As a set, we know that the points of $\widetilde{X}$ are in bijection with the elements of the sets $F(x)$ as $x$ ranges over points of $X$. Define

$$\widetilde{Y} = \{(x, \widetilde{x}) \mid x \in X \text{ and } \widetilde{x} \in F(x)\}\,.$$

This is a set, and by the above the points of $\widetilde{Y}$ are in bijection with the points of $\widetilde{X}$. Let $q\colon \widetilde{Y} \to X$ be the projection on to the first factor. Without using Theorem 13*.8.1, explitely construct a topology on $\widetilde{Y}$ such that $q\colon \widetilde{Y} \to X$ is a cover and prove that the cover $q\colon \widetilde{Y} \to X$ is isomorphic to $p\colon \widetilde{X} \to X$. Hint: try to use the ideas from the construction of the topology on the universal cover in Chapter 10. □

# Calculating fundamental groups

CHAPTER 14

# Preliminaries on group presentations

Our final goal is to show how to compute the fundamental groups of more complicated spaces than the ones we have thus far discussed. Before we can do this, we need a language to describe an arbitrary group. The key notion is that of a group presentation, which we describe in this chapter.

## 14.1. Some basic groups

There are a vast range of groups that arise throughout mathematics. For instance:

Example 14.1.1 (Cyclic groups). The cyclic groups include the infinite cyclic group $C_\infty \cong \mathbb{Z}$ and the finite cyclic group $C_n \cong \mathbb{Z}/n$ of order $n$. We will typically write the generator of these cyclic group by $t$, so $C_\infty = \{t^n \mid n \in \mathbb{Z}\}$ and $C_n = \{1, t, \ldots, t^{n-1}\}$. □

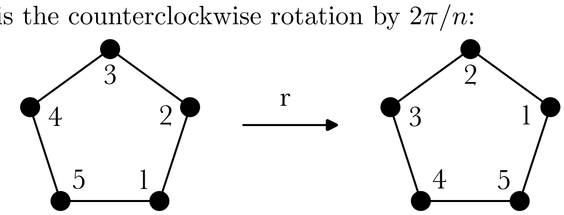
Example 14.1.2 (Abelian groups). Let $A$ be an abelian group. One possibility is that $A$ might be cyclic. When we are thinking of the cyclic groups as abelian groups we will often write them as $\mathbb{Z}$ and $\mathbb{Z}/n$. If $A$ is a finitely generated abelian group, then we can write

$$A \cong \mathbb{Z}^n \oplus \bigoplus_{i=1}^{k} \mathbb{Z}/p_i^{d_i}$$

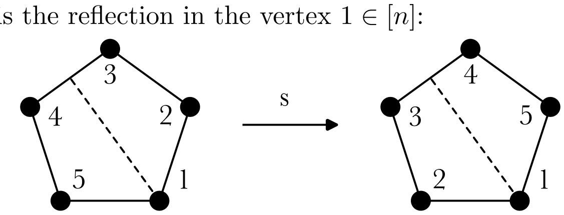
with $n \geq 0$ and $p_1, \ldots, p_k$ prime and $d_1, \ldots, d_k \geq 1$. However, if $A$ abelian but not finitely generated then is no hope for any kind of simple classification. □

Example 14.1.3 (Finite dihedral groups). For $n \geq 3$, the dihedral group $D_{2n}$ of order $2n$ is the isometry group of a regular $n$-gon $P_n$. Enumerate the vertices of $P_n$ counterclockwise as $[n] = \{1, \ldots, n\}$. The group $D_{2n}$ acts on $[n]$, and an element $\sigma \in D_{2n}$ is determined by $\sigma(1) \in [n]$ and whether or not $\sigma$ preserves the orientation of $P_n$. From this, we see that $D_{2n} = \{1, r, \ldots, r^{n-1}, s, rs, \ldots, r^{n-1}s\}$ where $r$ and $s$ are as follows:

- The element $r$ is the counterclockwise rotation by $2\pi/n$:




- The element $s$ is the reflection in the vertex $1 \in [n]$:




These elements satisfy the relations $r^n = 1$ and $s^2 = 1$ and $srs^{-1} = r^{-1}$. There is a homomorphism $\sigma\colon D_{2n} \to C_2$ that records whether or not an element preserves orientation, so $\sigma(r) = t^0 = 1 \in C_2$ and $\sigma(s) = t \in C_2$. We have $\ker(\sigma) = \{1, r, \ldots, r^{n-1}\} \cong C_n$, and $\sigma$ splits via the map $C_2 \to D_{2n}$ taking $t \in C_2$ to $s$. We therefore have a semidirect product decomposition[1] $D_{2n} \cong C_n \rtimes C_2$ where the generator of $C_2$ acts on $C_n$ by the automorphism taking $t^k \in C_n$ to $t^{-k}$. □

[1]Let $G$ and $H$ be groups such that $G$ acts on $H$ on the left. For $g \in G$ and $h \in H$, we will write ${}^g h$ for the image of $h$ under the action of $g$. The corresponding semidirect product $H \rtimes G$ is the following group:

EXAMPLE 14.1.4 (Infinite dihedral group). The infinite dihedral group $D_\infty$ is the group of isometries of $\mathbb{R}$ that preserve the subspace $\mathbb{Z}$. We have $D_\infty = \{r^k, sr^k \mid k \in \mathbb{Z}\}$, where:

- The element $r\colon \mathbb{R} \to \mathbb{R}$ is the translation map $r(x) = x + 1$.
- The element $s\colon \mathbb{R} \to \mathbb{R}$ is the reflection map $s(x) = -x$.

These elements satisfy the relations $s^2 = 1$ and $srs^{-1} = r^{-1}$. Just like for the finite dihedral groups, we have $D_\infty \cong C_\infty \rtimes C_2$ where $C_2$ acts on $C_\infty$ by the automorphism taking $t^k$ to $t^{-k}$. There are also surjections $f_n\colon D_\infty \to D_{2n}$ taking $r \in D_\infty$ to $r \in D_{2n}$ and $s \in D_\infty$ to $s \in D_{2n}$. □

EXAMPLE 14.1.5 (Symmetric groups). Fix $n \geq 2$. Let $[n] = \{1, \ldots, n\}$ and let $\mathfrak{S}_n$ be the symmetric group on $[n]$, i.e., the group of bijections $\sigma\colon [n] \to [n]$. The group $\mathfrak{S}_n$ contains every finite group $G$ of order $n$ as a subgroup. Indeed, enumerate $G$ as

$$G = \{g_1, \ldots, g_n\}.$$

For $g \in G$, let $\sigma_g \in \mathfrak{S}_n$ be the permutation such that $gg_i = g_{\sigma_g(i)}$ for $1 \leq i \leq n$. The map $f\colon G \to \mathfrak{S}_n$ defined by $f(g) = \sigma_g$ for $g \in G$ is then an injective homomorphism whose image is isomorphic to $G$. More generally, if $G$ is an arbitrary group then an action of $G$ on the set $[n]$ is the same as a homomorphism $G \to \mathfrak{S}_n$. For instance, there is an injective homomorphism $D_{2n} \to \mathfrak{S}_n$ arising from the action of $D_{2n}$ on the $n$ vertices of the regular $n$-gon $P_n$. □

EXAMPLE 14.1.6 (Linear groups). For a field $\mathbf{k}$, the group $\mathrm{GL}_n(\mathbf{k}) = \mathrm{Aut}(\mathbf{k}^n)$ plays a basic role in linear algebra. It contains many interesting subgroups; for instance, the orthogonal groups of quadratic forms on $\mathbf{k}^n$. It also contains a copy of $\mathfrak{S}_n$ consisting of the permutation matrices. A subgroup of $\mathrm{GL}_n(\mathbf{k})$ for some $n$ and $\mathbf{k}$ is called a *linear group*. For instance, since $\mathfrak{S}_n$ can be realized as a linear group and all finite groups are subgroups of $\mathfrak{S}_n$ for some $n$, it follows that all finite groups can be realized as linear groups. □

EXAMPLE 14.1.7 (Free groups). Let $S$ be a set and let $F(S)$ be the free group on $S$ (see §7.6). The group $F(S)$ is generated by $S$, and each element $w \in F(S)$ can be uniquely expressed as a reduced word $w = s_1^{\epsilon_1} \cdots s_n^{\epsilon_n}$ in $S$. Recall that this means that:

- $s_i \in S$ and $\epsilon_i \in \{\pm 1\}$ for all $1 \leq i \leq n$; and
- for all $1 \leq i < n$ we do not have $s_i^{\epsilon_i} s_{i+1}^{\epsilon_{i+1}} \in \{ss^{-1},\, s^{-1}s \mid s \in S\}$. □

## 14.2. Presentations for groups

The examples in the previous section barely scratch the surface of the world of groups. To write down an arbitrary group, we introduce the notation of a presentation. We start with some notation:

NOTATION 14.2.1. Let $G$ be a group. If $C \subset G$ is a subset, then $\langle C \rangle$ denotes the subgroup generated by $C$ and $\langle\!\langle C \rangle\!\rangle$ denotes the normal subgroup generated by $C$. We therefore have $\langle\!\langle C \rangle\!\rangle = \langle gcg^{-1} \mid c \in C \text{ and } g \in G \rangle$. □

This allows us to make the following key definition:

DEFINITION 14.2.2. Let $S$ be a set and let $R$ be a subset of the free group $F(S)$ on $S$. Define $G = \langle S \mid R \rangle$ to be the quotient $F(S)/\langle\!\langle R \rangle\!\rangle$. We call $S$ the *generators* and $R$ the *relations* for the presentation, and $\langle S \mid R \rangle$ is called a *group presentation* for $G$. If both $S$ and $R$ are finite, then $\langle S \mid R \rangle$ is a *finite presentation* for $G$. A *finitely presented group* is a group with a finite presentation. □

EXAMPLE 14.2.3. For a set $S$, we have $F(S) = \langle S \mid \emptyset \rangle$. This is often written $\langle S \mid \rangle$. If $S$ is finite, then this is a finite presentation and $F(S)$ is a finitely presented group. □

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- The elements of $H \rtimes G$ are pairs $(h, g)$ with $h \in H$ and $g \in G$.
- For $(h_1, g_1), (h_2, g_2) \in H \rtimes G$, their product is $(h_1, g_1)(h_2, g_2) = (h_1\, {}^{g_1}h_2, g_1 g_2)$.

It is enlightening to prove that this is a group. If the $G$-action on $H$ is trivial, this is just the usual direct product $H \times G$. Identify $H$ and $G$ with the subgroups of $H \rtimes G$ consisting of elements of the form $(h, 1)$ and $(1, g)$, respectively. The subgroup $H$ is normal, and every $x \in H \rtimes G$ can be uniquely written as $x = hg$ with $h \in H$ and $g \in G$.

Conversely, let $\Gamma$ be a group and $H, G < \Gamma$ be subgroups such that $H$ is normal and every $x \in \Gamma$ can be uniquely written as $x = hg$ with $h \in H$ and $g \in G$. Since $H$ is a normal subgroup of $\Gamma$, the group $\Gamma$ acts on $H$ by conjugation. This restricts to an action of $G$ on $H$, and $\Gamma \cong H \rtimes G$. This isomorphism takes $(h, g) \in H \rtimes G$ to $hg \in \Gamma$. One way this can arise is if $\Gamma$ is a group and $p\colon \Gamma \to G$ is a surjection that splits via a map $\sigma\colon G \to \Gamma$. Identify $G$ with its image in $\Gamma$ under the injective map $\sigma$, and set $H = \ker(p)$. We are then in the above situation, so $\Gamma \cong H \rtimes G$.

EXAMPLE 14.2.4. As a special case of the previous example, $C_\infty \cong \langle t \mid \rangle$. Similarly, for $n \geq 2$ we have $C_n \cong \langle t \mid t^n \rangle$. Both $C_\infty$ and $C_n$ are therefore finitely presented. □

REMARK 14.2.5. Not all presentations of finitely presented groups are finite. For instance, $C_\infty = \langle t \mid \rangle$ can also be written $C_\infty = \langle t, x_1, x_2, \ldots \mid x_1, x_2, \ldots \rangle$. □

Every group has some presentation:

LEMMA 14.2.6. *Every group $G$ can be written as $G \cong \langle S \mid R \rangle$.*

PROOF. Let $S$ be a generating set for $G$; e.g., $S = G$. The map $S \to G$ extends to a surjection $\phi\colon F(S) \to G$. Set $R = \ker(\phi)$, so $\phi$ induces an isomorphism from $F(S)/R \cong \langle S \mid R \rangle$ to $G$. □

Before we give more examples, we introduce some notation. Let $G = \langle S \mid R \rangle$ be a group equipped with a presentation. For $w \in F(S)$, we write $\overline{w}$ for its image in $G$. A *relation* in $G$ is an element $r \in \langle\!\langle R \rangle\!\rangle \subset F(S)$. The relations of the presentation are thus relations in $G$, but except in degenerate cases there are many other relations in $G$. For instance, here are some relations in $G = \left\langle a, b \mid a^2, b^3 \right\rangle$:

$$a^{-2}, a^{10}, a^2b^3a^2, ab^3a^{-1}a^2 = ab^3a.$$

If $w, v \in F(S)$ are such that $\overline{w} = \overline{v}$, then $wv^{-1}$ is a relation. We will sometimes write this relation as $w = v$. This convention will also be used when giving presentations. For instance,

$$\left\langle a, b, c \mid ab = c, b^2 = 1, cab = a \right\rangle = \left\langle a, b, c \mid abc^{-1}, b^2, caba^{-1} \right\rangle.$$

For $w, v \in F(S)$, we will also write $w \equiv v$ to mean that $\overline{w} = \overline{v}$, i.e., that $w = v$ is a relation.

## 14.3. Mapping from groups with presentations

Let $G = \langle S \mid R \rangle$ be a group given by a presentation. For any group $H$ that is well-understood, it is easy to construct homomorphisms $\Phi\colon G \to H$. Indeed, choose a set map $\phi\colon S \to H$. The map $\phi$ extends to a homomorphism $\phi\colon F(S) \to H$. To check if $\phi$ descends to a homomorphism on $G = F(S)/\langle\!\langle R \rangle\!\rangle$, we must only verify that $\phi(r) = 1$ for all $r \in R$. In summary, to construct a homomorphism $\Phi\colon G \to H$ we must choose where the generators go and then verify that relations go to relations. Here is an example of this:

EXAMPLE 14.3.1. Let $G$ be a group. Let $S$ be the set of formal symbols $\{s_g \mid g \in G\}$ and let $R = \{s_g s_h = s_{gh} \mid g, h \in G\}$. Set $\Gamma = \langle S \mid R \rangle$. We claim that $\Gamma \cong \langle S \mid R \rangle$. Indeed, the map $s_g \mapsto g$ gives a map $S \to G$ taking each relation $s_g s_h = s_{gh}$ to a relation in $G$. We thus get a surjective map $\Phi\colon \Gamma \to G$.

To see that $\Phi$ is injective, consider $w \in F(S)$. For $g \in G$ the relation $s_g^{-1} \equiv s_{g^{-1}}$ holds in $\Gamma$; indeed, $s_1 \equiv 1$ since $s_1 s_1 \equiv s_1$, so $s_g s_{g^{-1}} \equiv s_1 \equiv 1 \equiv s_g s_g^{-1}$. It follows that $w$ is equivalent to a word in $S$ in which only positive powers of generators appear. Using the relations in $\Gamma$, we then see that $w$ is equivalent to a single generator $s_g$. We conclude that $\Gamma = \{\overline{s}_g \mid g \in G\}$. Since $\Phi$ takes $\{\overline{s}_g \mid g \in G\}$ bijectively to $G$, we conclude that $\Phi$ is injective. □

REMARK 14.3.2. If $G$ is a finite group, then the presentation from Example 14.3.1 is finite and thus $G$ is finitely presented. □

## 14.4. Normal forms

We abstract the argument from Example 14.3.1. Let $G$ be a group with a generating set $S$. For each $g \in G$, pick $w_g \in F(S)$ such that $w_g$ maps to $g$ under the natural map $F(S) \to G$. The collection of elements $\{w_g \in F(S) \mid g \in G\}$ is called a *normal form* for $G$. Assume now that $R \subset F(S)$ is a collection of relations that hold in $G$. Letting $\Gamma = \langle S \mid R \rangle$, we want to prove that $\Gamma \cong G$.

Since $S$ generates $G$ and each relation in $R$ holds in $G$, we get a surjective homomorphism $\Phi\colon \Gamma \to G$. We want to prove that $\Phi$ is an isomorphism. Observe that $\Phi$ restricts to a bijection from $\{\overline{w}_g \in F(S) \mid g \in G\}$ to $G$. It is therefore enough to prove that $\{\overline{w}_g \in F(S) \mid g \in G\} = \Gamma$. In other words, we must prove that every $w \in F(S)$ can be reduced to some $w_g$ using the relations in $R$. Here are some examples:

Example 14.4.1 (Free abelian group). Fix $n \geq 1$. For $1 \leq i \leq n$, let $x_i \in \mathbb{Z}^n$ be the element with a 1 in position $i$ and zeros elsewhere. The elements $S = \{x_1, \ldots, x_n\}$ generate $\mathbb{Z}^n$. They satisfy the relations

$$R = \{x_i x_j = x_j x_i \mid 1 \leq i, j \leq n\}.$$

We claim that $\mathbb{Z}^n \cong \langle S \mid R\rangle$. Indeed, $\mathbb{Z}^n$ has the normal form $\left\{x_1^{d_1} \cdots x_n^{d_n} \mid d_1, \ldots, d_n \in \mathbb{Z}\right\}$. We must prove that an arbitrary $w \in F(S)$ can be reduced to an element in this normal form. For this, use the relations in $R$ to move the $x_1$ terms to the left, then the $x_2$ terms to the left, etc. Here is an example:[2]

$$x_3 x_2^{-1} x_1 x_2^2 x_1^{-3} \equiv x_1 x_1^{-3} x_3 x_2^{-1} x_2^2 \equiv x_1 x_1^{-3} x_2^{-1} x_2^2 x_3 \equiv x_1^{-2} x_2 x_3. \qquad \square$$

Example 14.4.2 (Finite dihedral group). Fix $n \geq 3$. As in Example 14.1.3, let $D_{2n}$ be the dihedral group of order $2n$. This group is generated by the rotation $r \in D_{2n}$ and the reflection $s \in D_{2n}$. Set $S = \{r, s\}$. The elements $r$ and $s$ satisfy the relations

$$R = \{r^n = 1, s^2 = 1, srs^{-1} = r^{-1}\}.$$

We claim that $D_{2n} \cong \langle S \mid R\rangle$. Indeed, as we observed in Example 14.1.3 the group $D_{2n}$ has the normal form $\left\{r^d, sr^d \mid 0 \leq d \leq n-1\right\}$. We must prove that an arbitrary $w \in F(S)$ can be reduced to an element in this normal form. Using the relation $s^2 \equiv 1$, we can replace all $s^{-1}$ terms in $w$ with $s$. Also using $s^2 \equiv 1$, the relation $srs^{-1} \equiv r^{-1}$ can be rearranged to $rs \equiv sr^{-1}$. This implies that we also have $r^{-1}s \equiv sr$. Applying all of these, we can pull all the $s$-terms in $w$ to the left and reduce $w$ to $s^e r^d$ as in the following example:

$$sr^{-1}sr^3sr^2 \equiv ssr^{-1}r^3sr^2 \equiv sssr^{-1}r^3r^2 = s^3r^4.$$

Using the fact that $s^2 \equiv 1$ and $r^n \equiv 1$, we see that $s^e r^d$ is equivalent to either $r^d$ or $sr^d$ with $0 \leq d \leq n-1$, as desired. $\square$

Example 14.4.3 (Infinite dihedral group). An argument identical to the one in the previous example shows that $D_\infty \cong \left\langle r, s \mid s^2 = 1, srs^{-1} = r^{-1}\right\rangle$. $\square$

Remark 14.4.4. See Exercise 14.2 for a presentation of the symmetric group $\mathbb{S}_n$. $\square$

### 14.5. Abelianization

As we will elaborate on below, it is hard to determine much about a group $G$ from a presentation. The only easy thing to calculate is its abelianization $G^{\text{ab}}$, which we recall is the largest abelian group on which $G$ surjects. Letting $[G,G]$ be the commutator subgroup[3] of $G$, we have $G^{\text{ab}} = G/[G,G]$. For instance, if $S$ is an $n$-element set, then $F(S)^{\text{ab}} \cong \mathbb{Z}^n$ (see Exercise 7.12).

Assume that $G$ has a finite presentation $G = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m\rangle$. Let $V \cong \mathbb{Z}^n$ be the free abelian group with basis $\{X_1, \ldots, X_n\}$. For $1 \leq i \leq m$, let $R_i \in V$ be the following element:

- Write $r_i = x_{j_1}^{e_1} \cdots x_{j_k}^{e_k}$ with $1 \leq j_1, \ldots, j_k \leq n$ and $e_1, \ldots, e_k \in \{\pm 1\}$. Then $R_i = e_1 X_{j_1} + \cdots + e_k X_{j_k} \in V$.

It is immediate from the definitions that $G^{\text{ab}}$ is the quotient of $V$ by $\langle R_1, \ldots, R_m\rangle$. As is discussed in most treatements of the classification of finitely generated abelian groups, this quotient can be calculated using tools like Smith normal form. However, it is often easier to work with it directly. Here are some examples.

Example 14.5.1 (Infinite dihedral group). As we saw in Example 14.4.3, we have

$$D_\infty \cong \left\langle r, s \mid s^2 = 1, srs^{-1} = r^{-1}\right\rangle.$$

Following the above recipe, the abelianization of $D_\infty$ is the quotient of the free abelian group with basis $R$ and $S$ by the following two relations:[4]

- $2S = 0$ and $S + R - S = -R$. This second relation can be rewritten $2R = 0$.

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[2]Here we are also using relations like $x_i^{-1} x_j \equiv x_j x_i^{-1}$, which can be obtained from $x_j x_i \equiv x_i x_j$ by multiplying both sides on the left by $x_i^{-1}$ and on the right by $x_i^{-1}$.

[3]The commutator subgroup $[G,G]$ is the subgroup generated by commutators $[g,h] = ghg^{-1}h^{-1}$.

[4]Here quotienting an abelian group by a relation $A = B$ should be interpreted as quotienting by $A - B$.

We thus see that the abelianization of $D_\infty$ is $(\mathbb{Z}/2) \oplus (\mathbb{Z}/2)$. □

EXAMPLE 14.5.2 (Finite dihedral group). Fix $n \geq 3$. As we saw in Example 14.4.2, we have

$$D_{2n} = \left\langle r, s \mid r^n = 1, s^2 = 1, srs^{-1} = r^{-1} \right\rangle.$$

Following the above recipe, the abelianization of $D_{2n}$ is the quotient of the free abelian group with basis $R$ and $S$ by the relations $nR = 0$ and $2S = 0$ and $S + R - S = -R$. This last relation can be rearranged to $2R = 0$. From this, we see that there are two cases:

(a) If $n$ is even, then the relations reduce to $2S = 0$ and $2R = 0$, so the abelianization is $(\mathbb{Z}/2) \oplus (\mathbb{Z}/2)$.

(b) If $n$ is odd, then $nR = 0$ and $2R = 0$ combine to give that $R = 0$, so the abelianization is $\mathbb{Z}/2$. □

## 14.6. Some cautionary examples

As we said in the last section, it is not easy to extract information from a group presentation. This section contains a number of cautionary examples. The first is as follows:

EXAMPLE 14.6.1. Consider the group $G = \langle a, b \mid babab = 1 \rangle$. What kind of group is this? We first determine its abelianization. Following the recipe in the previous section, $G^{\text{ab}}$ is the quotient of the free abelian group with basis $\{A, B\}$ subject to the relation $B + A + B + A + B = 0$, or $2A + 3B = 0$. After making the change of basis $A = X - Y$ and $B = Y$, this relation becomes $2(X - Y) + 3Y = 0$, or $-2X = Y$. Eliminating the variable $Y$, we conclude that $G^{\text{ab}} \cong \mathbb{Z}$.

In fact, it turns out that $G$ is an abelian group, so $G \cong \mathbb{Z}$. To see this, note that we can conjugate $babab \equiv 1$ by $(bab)^{-1}$ and get

$$1 \equiv (bab)^{-1}(babab)(bab) = abbab.$$

It follows that

$$1 \equiv (abbab)(babab)^{-1} \equiv (abbab)(b^{-1}a^{-1}b^{-1}a^{-1}b^{-1}) = aba^{-1}b^{-1},$$

so $a$ and $b$ commute and $G$ is indeed abelian. □

REMARK 14.6.2. Another way to see that $G = \langle a, b \mid babab = 1 \rangle$ is isomorphic to $\mathbb{Z}$ is as follows. Let $H = \left\langle x, y \mid x^2 y = 1 \right\rangle$. It is clear that $H \cong \mathbb{Z}$ since $y \equiv x^{-2}$. Define $f \colon H \to G$ via the formulas

$$f(x) = ba \quad \text{and} \quad f(y) = b.$$

This works since $f$ takes the relation $x^2 y = 1$ to $(ba)^2 b = 1$, which is the relation in $G$. The map $f$ is an isomorphism with inverse the map $g \colon G \to H$ defined by

$$g(a) = y^{-1}x \quad \text{and} \quad g(b) = y.$$

Again, this works since $g$ takes the relation $babab = 1$ to $y(y^{-1}x)y(y^{-1}x)y = 1$, which reduces to the relation $x^2 y = 1$ in $H$. □

For our next examples, for $n \geq 1$ define

$$G_n = \left\langle x_1, \ldots, x_n \mid x_{i+1} x_i x_{i+1}^{-1} = x_i^2 \text{ for } 1 \leq i \leq n \right\rangle.$$

Here the subscripts should be taken modulo $n$. These are called the *Higman groups*. All of these groups have trivial abelianizations:

LEMMA 14.6.3. *For all $n \geq 1$, we have $G_n^{\text{ab}} = 0$.*

PROOF. The abelianization $G_n^{\text{ab}}$ is the free abelian group with basis $X_1, \ldots, X_n$ modulo the following relations:

- For $1 \leq i \leq n$, we have $X_{i+1} + X_i - X_{i+1} = 2X_i$. This reduces to $X_i = 0$.

The lemma follows. □

The first three $G_n$ are trivial:

LEMMA 14.6.4. *We have $G_1 = 1$.*

PROOF. We have $G_1 \cong \left\langle x_1 \mid x_1 x_1 x_1^{-1} = x_1^2 \right\rangle = \langle x_1 \mid x_1 = 1 \rangle = 1$. □

Lemma 14.6.5. *We have $G_2 = 1$.*

Proof. Rewrite the relations in $G_2$ as $x_2 x_1 \equiv x_1^2 x_2$ and $x_1 x_2 \equiv x_2^2 x_1$. These imply that

$$x_1^2 x_2^2 \equiv x_2 x_1 x_2 \equiv x_2^3 x_1 \equiv x_1^8 x_2^3,$$

so $x_2 \equiv x_1^{-6}$. We conclude that $x_1$ and $x_2$ commute, so $G_2$ is abelian and $G_2 \cong G_2^{\mathrm{ab}} = 1$. □

Lemma 14.6.6. *We have $G_3 = 1$.*

Proof. The proof is a more elaborate version of the calculation we used to prove Lemma 14.6.6, so we leave it as Exercise 14.5. □

However, it turns out that $G_n$ is infinite for $n \geq 4$. We will discuss the proof in Essay YYY. The reason this proof is difficult is that it is hard to map $G_n$ to other well-understood groups. In particular, we have the following theorem of Higman:

Theorem 14.6.7. *For $n \geq 4$, any homomorphism from $G_n$ to a finite group is trivial.*

Proof. Let $H_n$ be the image of $G_n$ in some finite group. Each element of $H_n$ has finite order. We can therefore write $H_n = \langle y_1, \ldots, y_n \mid R \rangle$ where $R$ contains the following relations:

- For $1 \leq i \leq n$, the relation $y_{i+1} y_i y_{i+1}^{-1} = y_i^2$. Here the subscripts should be interpreted modulo $n$.
- For $1 \leq i \leq n$, a relation of the form $y_i^{d_i} = 1$ for some $d_i \geq 1$. Choose the $d_i$ to be as small as possible, so $d_i$ is the order of $y_i$.

There might also be some other relations. We will prove that these relations force $H_n$ to be trivial.

Assume that $H_n$ is nontrivial. We must have $d_i \geq 2$ for all $i$. Indeed, if $d_{i_0} = 1$ for some $i_0$, then

$$y_{i_0-1} = y_{i_0} y_{i_0-1} y_{i_0}^{-1} = y_{i_0-1}^2,$$

so $y_{i_0-1} = 1$ and thus $d_{i_0-1} = 1$. Iterating this, we deduce that $d_i = 1$ for all $i$ and hence that $H_n = 1$, contrary to our assumptions.

Let $p \geq 2$ be the smallest prime dividing some $d_i$. Since everything is invariant under cyclic permutations of the generators, we can assume without loss of generality that $p$ divides $d_1$. Since $d_2$ is the order of $y_2$, we have

$$y_1 = y_2^{d_2} y_1 y_2^{-d_2} = y_1^{2^{d_2}}$$

and thus $y_1^{2^{d_2}-1} = 1$. Since $d_1$ is the order of $y_1$, it follows that $d_1$ and hence $p$ divides $2^{d_2} - 1$. This implies that $p$ is odd. Since $2^{d_2} \equiv 1 \pmod{p}$, we see that $d_2$ is a multiple of the order $p-1$ of 2 in $(\mathbb{Z}/p)^{\times}$. This implies that a prime smaller than $p$ divides $d_2$, contradicting the fact that $p$ is the smallest prime dividing some $d_i$. □

Remark 14.6.8. Malcev proved that if $\mathbf{k}$ is a field and $H$ is a nontrivial finitely generated subgroup of $\mathrm{GL}_m(\mathbf{k})$, then $H$ has many nontrivial finite quotients. In fact, $H$ is *residually finite*: for any $h \in H$ with $h \neq 1$, there exists a finite group $F$ and a homomorphism $f \colon H \to F$ with $f(h) \neq 1$. It therefore follows from Theorem 14.6.7 that all homomorphisms $f \colon G_n \to \mathrm{GL}_m(\mathbf{k})$ are trivial for $n \geq 4$. We outline a simple proof of this for $\mathbf{k} = \mathbb{C}$ in Exercise 14.6. □

## 14.7. Decision problems for groups

In 1911, Dehn posed the following three problems about group presentations.

Problem 14.7.1 (Word problem). *Let $G = \langle S \mid R \rangle$ be a finitely presented group. Give an algorithm that for $w, v \in F(S)$ determines whether or not $\overline{w} = \overline{v}$ in $G$.*

Problem 14.7.2 (Conjugacy problem). *Let $G = \langle S \mid R \rangle$ be a finitely presented group. Give an algorithm that for $w, v \in F(S)$ determines whether or not $\overline{w} \in G$ and $\overline{v} \in G$ are conjugate.*[5]

Problem 14.7.3 (Isomorphism problem). *Give an algorithm that for finitely presented group $G = \langle S \mid R \rangle$ and $G' = \langle S' \mid R' \rangle$ determines whether or not $G$ and $G'$ are isomorphic.*

Here are some simple observations about the relationship between these problems.

[5] Recall that $g, h \in G$ are conjugate if there exists some $k \in G$ such that $g = khk^{-1}$.

(a) Let $G = \langle S \mid R \rangle$ be a finitely presented group. To solve the word problem for $G$, it is enough to give an algorithm that for $w \in F(S)$ determines whether or not $\overline{w} = 1$. Indeed, for $w, v \in F(S)$ such an algorithm allows us to determine whether or not $\overline{wv^{-1}} = 1$, and thus whether or not $\overline{w} = \overline{v}$.

(b) Let $G = \langle S \mid R \rangle$ be a finitely presented group. If we can solve the conjugacy problem for $G$, then we can also solve the word problem for $G$. Indeed, an element $w \in F(S)$ satisfies $\overline{w} = 1$ if and only if $\overline{w}$ is conjugate to 1.

(c) A special case of the isomorphism problem is the triviality problem: give an algorithm that for a finitely presented group $G = \langle S \mid R \rangle$ determines whether or not $G = 1$. The group satisfies $G = 1$ if and only if for all $s \in S$ we have $\overline{s} = 1$, so a solution to the word problem gives a solution to the triviality problem.

While these problems can be solved in many special cases, they are quite difficult and our knowledge about them is limited. For instance, consider a one-relator group $G$, that is, a group of the form $G = \langle S \mid r \rangle$ where $S$ is finite and $r \in F(S)$. In 1932, Magnus showed how to solve the word problem for one-relator groups. However, we still do not know how to solve the conjugacy problem for one-relator groups. We also do not know an algorithm to determine if two one-relator groups are isomorphic. For groups with two relations, even the word problem is open.

One reason that these problems are difficult is that they are unsolvable in general. Indeed, in the 1950's Novikov and Boone independently proved the following theorem:

THEOREM 14.7.4 (Novikov–Boone). *There exists*[6] *a finitely presented group $G = \langle S \mid R \rangle$ for which there does not exist an algorithm solving the word problem.*

REMARK 14.7.5. To make sense of Theorem 14.7.4, the notion of an "algorithm" must be formally defined. Roughly speaking, an "algorithm" here means a program in any standard computer language (C, Python, LISP, etc.) run on a computer with unlimited memory. This algorithm must terminate in finite time for any input. The formal definition involves the notion of a Turing machine, and is discussed in any book on computability theory. □

One observation about Theorem 14.7.4 is that the hard part in it is checking that some $w \in F(S)$ does not represent the identity. More precisely, we have the following:

LEMMA 14.7.6. *Let $G = \langle S \mid R \rangle$ be a finitely presentable group. There exists an algorithm that takes as input $w \in F(S)$ and does the following:*

- *If $\overline{w} = 1$, then the algorithm terminates and certifies that $\overline{w} = 1$.*
- *If $\overline{w} \neq 1$, then the algorithm does not terminate.*

PROOF. Each $w \in F(S)$ can be uniquely be written as a reduced word $w = s_1^{\epsilon_1} \cdots s_n^{\epsilon_n}$ with $s_i \in S$ and $\epsilon_i \in \{\pm 1\}$ for all $1 \leq i \leq n$, and we define the *length* of $w$ to be $\ell(w) = n$. Set $B_n(S) = \{w \in F(S) \mid \ell(w) \leq n\}$. Define $B_n(R)$ to be the set of all elements of $F(S)$ that for some $m \leq n$ can be expressed as

$$(w_1 r_1 w_1^{-1})^{\epsilon_1} \cdots (w_m r_m w_m^{-1})^{\epsilon_m}$$

with $r_i \in R$ and $w_i \in B_n(S)$ and $\epsilon_i \in \{\pm 1\}$ for $1 \leq i \leq m$. Note that the above is not necessarily a reduced word. The set $B_n(R)$ is a finite set, and can be effectively enumerated on a computer. We have

$$B_1(R) \subset B_2(R) \subset B_3(R) \subset \cdots,$$

and the normal closure $\langle\!\langle R \rangle\!\rangle$ is $\cup_{n=1}^{\infty} B_n(R)$. Our algorithm is as follows:

- Start at Step $n = 1$.
- At Step $n$, enumerate $B_n(R)$ and check whether or not $w \in B_n(R)$. If it does, terminate. Otherwise, go on to Step $n + 1$.

This terminates if and only if $w \in \langle\!\langle R \rangle\!\rangle$, i.e., if and only if $\overline{w} = 1$. □

We close with the following observation whose proof illustrates the fact that Theorem 14.7.4 touches on deep logical issues that seemingly have nothing to do with computers:

[6] The proof of this theorem is not merely an existence proof. It gives an explicit such group. There are now several such groups in the literature; for instance, in 1969 Borisov constructed such a group with 12 relations.

THEOREM 14.7.7. *Let $G = \langle S \mid R \rangle$ be the group with an unsolvable word problem from Theorem 14.7.4. There exists some $w \in F(S)$ with the following two properties:*

- *We have $\overline{w} \neq 1$ in $G$.*
- *There is no proof in ZFC*[7] *that $\overline{w} \neq 1$ in $G$.*

PROOF. Consider $w \in F(S)$. Using two computers (or two threads on one computer), run the following two algorithms on $w$ at the same time, terminating if either of the two terminate:

- Computer one runs the algorithm from Lemma 14.7.6, and thus terminates if and only if $\overline{w} = 1$ in $G$.
- Computer two systematically enumerates all possible proofs in ZFC. For each proof, it first determines whether or not it is a valid proof. If it is, it checks whether it proves that $\overline{w} \neq 1$ in $G$. If it does, then this algorithm terminates. Otherwise, it keeps going.

This algorithm will terminate if $\overline{w} = 1$ in $G$ or if there is a proof in ZFC that $\overline{w} \neq 1$ in $G$. Since there is no algorithm to solve the word problem in $G$, there must be some $w \in F(S)$ for which this algorithm does not terminate. The theorem follows. □

REMARK 14.7.8. Theorem 14.7.7 is not effective, and there is no way to determine which $w \in F(S)$ satisfies its conclusions since doing so would in particular provide a proof that $\overline{w} \neq 1$. □

## 14.8. Exercises

EXERCISE 14.1. Let

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}.$$

The group $H$ is often called the integer Heisenberg group.

(a) Set

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

As notation, set $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$. Prove that $H = \langle x, y \mid [[x,y],x] = 1, [[x,y],y] = 1 \rangle$. Hint: work out a normal form.

(b) Calculate the abelianization of $H$. □

EXERCISE 14.2. Fix $n \geq 2$. Let $\mathfrak{S}_n$ be the symmetric group on $[n] = \{1, \ldots, n\}$. For $1 \leq i \leq n-1$, let $\tau_i \in \mathfrak{S}_n$ be the transposition $(i, i+1)$. Let

$$\begin{aligned} S &= \{\tau_i \mid 1 \leq i \leq n-1\}, \\ R &= \left\{\tau_i^2 = 1 \mid 1 \leq i \leq n-1\right\} \cup \{\tau_i \tau_j = \tau_j \tau_i \mid 1 \leq i, j \leq n, \, |i-j| \geq 2\} \\ &\quad \cup \{\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \mid 1 \leq i \leq n-2\}. \end{aligned}$$

Prove the following:

(a) The group $\mathfrak{S}_n$ is generated by $S$.

(b) Each element of $R$ is a relation in $\mathfrak{S}_n$.

(c) We have $\mathfrak{S}_n = \langle S \mid R \rangle$. Hint: For $1 \leq i \leq j \leq n$, let $\sigma_{i,j} = \tau_{j-1} \tau_{j-1} \cdots \tau_i$. For $i = j$, our convention is that $\sigma_{i,j} = 1$. The key feature of $\sigma_{i,j}$ is that as an element of $\mathfrak{S}_n$, it takes $i \in [n]$ to $j \in [n]$. Prove that $\mathfrak{S}_n$ has the normal form

$$\left\{\sigma_{n-1,i_{n-1}} \sigma_{n-2,i_{n-2}} \cdots \sigma_{1,i_1} \mid i_j \geq j \text{ for } 1 \leq j \leq n-1\right\}.$$

Use this normal form to prove the result.

(d) Calculate the abelianization of $\mathfrak{S}_n$. □

[7]There is nothing special about ZFC, and the same argument works in other foundational systems that are rich enough to express facts about group theory and unsolvability. We focus on ZFC since that is the set of foundations we are assuming in this book.

EXERCISE 14.3. Let $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$. Let $R_C$ be the following set of relations in $F(S_1 \sqcup S_2)$:

$$R_C = \{s_1 s_2 = s_2 s_1 \mid s_1 \in S_1 \text{ and } s_2 \in S_2\}.$$

Set $\Gamma = \langle S_1 \sqcup S_2 \mid R_1 \sqcup R_2 \sqcup R_C \rangle$. Prove that $\Gamma \cong G_1 \times G_2$. Hint: use normal forms. □

EXERCISE 14.4. Let $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$. Assume that $G_2$ acts on $G_1$ on the left. Determine a presentation for the resulting semidirect product $G_1 \rtimes G_2$. Hint: For each $s \in S_1$ and $t \in S_2$, start by writing ${}^t s$ as a word $w_{s,t}$ in $S_1$. □

EXERCISE 14.5. Recall that the third Higman group is

$$G_3 = \left\langle x_1, x_2, x_3 \mid x_2 x_1 x_2^{-1} = x_1^2, x_3 x_2 x_3^{-1} = x_2^2, x_1 x_3 x_1^{-1} = x_3^2 \right\rangle.$$

Prove that $G_3 = 1$. Hint: Start by applying the relations to $x_1^2 x_2^2 x_3$ to find an identity that lets you write $x_3$ in terms of $x_1$ and $x_2$. Plug this into $x_3 x_2 x_3^{-1} \equiv x_2^2$ and manipulate the result to show that $x_2$ equals a power of $x_1$, and hence that $x_2$ and $x_1$ commute. □

EXERCISE 14.6. Recall from Remark 14.6.8 that if $\mathbf{k}$ is a field and $G_n$ is the Higman group, then all homomorphisms $f\colon G_n \to \mathrm{GL}_m(\mathbf{k})$ are trivial. This exercise explains how to prove this for $\mathbf{k} = \mathbb{C}$. Let $H < \mathrm{GL}_m(\mathbb{C})$ be the image of such a homomorphism. Let $y_1, \ldots, y_n \in H$ be the images of the generators $x_1, \ldots, x_n$ of $G_n$. By Theorem 14.6.7, it is enough to prove that $y_i$ has finite order for all $1 \le i \le n$. Do the following:

(a) Prove that all the eigenvalues of each $y_i$ are roots of unity. Hint: use the relation $y_{i+1} y_i y_{i+1} = y_i^2$.
(b) Prove that there is some polynomial $p \in \mathbb{C}[z]$ such that for all $1 \le i \le n$ and all $k \ge 1$, if $c \in \mathbb{C}$ is one of the entries of the matrix $y_i^k$ then $|c| \le p(k)$. Hint: use Jordan normal form along with (a).
(c) Prove that the matrix $y_i$ is diagonalizable for $1 \le i \le n$. Hint: think about the Jordan normal form of $y_i$, and consider the identity $p_{i+1}^k p_i p_{i+1}^{-k} = p_i^{2^k}$. You will use part (b).
(d) Prove that the matrix $y_i$ has finite order for $1 \le i \le n$. □

EXERCISE 14.7. Recall that a group $G$ is residually finite if for all $g \in G$ with $g \neq 1$, there exists a finite group $F$ and a homomorphism $f\colon G \to F$ with $f(g) \neq 1$. Prove the following:

(a) The group $G = \mathrm{GL}_n(\mathbb{Z})$ is residually finite. Hint: think about reducing matrices modulo $p$ for various primes $p$.
(b) Let $G$ be a residually finite group and let $G' < G$ be a subgroup. Prove that $G'$ is residually finite.
(c) Let $G$ be a finitely generated group. For all $g \in G$ with $g \neq 1$, assume that there exists a finite-index subgroup $H < G$ with $g \notin H$. Prove that $G$ is residually finite. Hint: the problem is that $H$ might not be normal, so $G/H$ might not be a group. Try to intersect conjugates of $H$ to get a smaller finite-index normal subgroup.
(d) Let $F(S)$ be a free group on a finite set $S$. Prove that $F(S)$ is residually finite. Hint: it might be easier to first use part (b) to reduce to the case of the free group $F(a,b)$ on $a$ and $b$. For $w \in F(a,b)$ with $w \neq 1$, try to use covering spaces to produce a finite-index subgroup $H$ of $F(a,b)$ as in part (c).
(e) Let $G = \langle S \mid R \rangle$ be a finitely generated residually finite group. Prove that there is an algorithm to solve the word problem in $G$. Hint: just like in the proof of Theorem 14.7.7, it is enough to produce an algorithm that terminates for $w \in F(S)$ with $w \neq 1$. Try to systematically list all finite groups $F$ and all homomorphisms $f\colon G \to F$. One key observation is that a group structure on a finite set $X$ is determined by its multiplication table. □

CHAPTER 15

# The effect on the fundamental group of attaching cells

In this chapter, we calculate presentations for the fundamental groups of surfaces and show that all groups can be realized as the fundamental groups of spaces. The key to this is understanding how the fundamental group changes under a geometric operation called attaching a cell.

## 15.1. Restricting covers to subspaces

Before we can make these computations, we need to understand how covers behave when we pass to subspaces. Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a pointed cover and let $X' \subset X$ be a subspace. Define $\widetilde{X}' = p^{-1}(X)$ and $p' = p|_{\widetilde{X}'}$, so $p'\colon \widetilde{X}' \to X'$ is a cover (see Exercise 1.12). We call $p'\colon \widetilde{X}' \to X$ the *restriction* of $p\colon \widetilde{X} \to X$ to $X'$. If $x_0 \in X'$ then $\widetilde{x}_0 \in \widetilde{X}'$. We therefore have a pointed cover $p'\colon (\widetilde{X}', \widetilde{x}_0) \to (X', x_0)$, again called the *restriction* of $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ to $X'$.

We need two properties of this. The first explains how this interacts with the Galois correspondence:

Lemma 15.1.1. *Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a pointed cover with corresponding subgroup $K < \pi_1(X, x_0)$. Let $X' \subset X$ be a subspace with $x_0 \in X'$ and let $p'\colon (\widetilde{X}', \widetilde{x}_0) \to (X', x_0)$ be the restriction of $p$ to $X'$. Let $\iota\colon (X', x_0) \to (X, x_0)$ be the inclusion and let $\iota_*\colon \pi_1(X', x_0) \to \pi_1(X, x_0)$ be the induced map. Then the subgroup of $\pi_1(X', x_0)$ corresponding to $p'$ is $\iota_*^{-1}(K)$.*

Proof. As we observed in Theorem 9.1.1, the subgroup $K < \pi_1(X, x_0)$ corresponding to $p$ consists of all $[\gamma] \in \pi_1(X, x_0)$ such that the following holds:

- Let $\widetilde{\gamma}$ be the lift of $\gamma$ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$. Then $\widetilde{\gamma}$ is a loop, i.e., $\widetilde{\gamma}(1) = \widetilde{x}_0$.

A similar result holds for $p'\colon (\widetilde{X}', \widetilde{x}_0) \to (X', x_0)$. For $[\gamma] \in \pi_1(X', x_0)$, we can view $\gamma$ as a loop in $X'$ or in the larger space $X$, so the lift $\widetilde{\gamma}$ starting at $\widetilde{x}_0$ is the same. In particular, whether or not that lift is a loop is the same in $\widetilde{X}'$ or in $\widetilde{X}$. It follows that $[\gamma]$ lies in the subgroup corresponding to $p'$ if and only if its image $\iota_*([\gamma]) = [\gamma] \in \pi_1(X, x_0)$ lies in the subgroup corresponding to $p$. $\square$

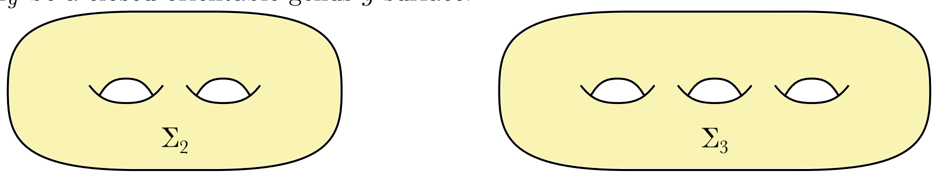
If $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a connected cover, it is not necessarily the case that the restriction of $p$ to a subspace of $X$ is connected. For instance, the restriction to the one-point subspace $x_0$ is the projection $p^{-1}(x_0) \to x_0$. For the universal cover, we have the following:

Lemma 15.1.2. *Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a pointed cover with corresponding subgroup $K < \pi_1(X, x_0)$. Let $X' \subset X$ be a subspace with $x_0 \in X'$ and let $p'\colon (\widetilde{X}', \widetilde{x}_0) \to (X', x_0)$ be the restriction of $p$ to $X'$. Let $\iota\colon (X', x_0) \to (X, x_0)$ be the inclusion and let $\iota_*\colon \pi_1(X', x_0) \to \pi_1(X, x_0)$ be the induced map. Then $\iota_*$ is surjective if and only if $\widetilde{X}'$ is path connected.*

Proof. Let $[\gamma] \in \pi_1(X, x_0)$ and let $\widetilde{\gamma}$ be the lift of $\gamma$ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$. Set $\widetilde{x}_1 = \widetilde{\gamma}(1)$. Let $[\gamma'] \in \pi_1(X', x_0)$ and let $\widetilde{\gamma}'$ be the lift of $\gamma'$ to $\widetilde{X}'$ with $\widetilde{\gamma}'(0) = \widetilde{x}_0$. If $\iota_*([\gamma']) = [\gamma]$, then $\widetilde{\gamma}'(1) = \widetilde{x}_1$. Also, since $\widetilde{X}$ is 1-connected any path in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}_1$ will be homotopic to $\gamma$. It follows that $\iota_*([\gamma']) = [\gamma]$ if and only if $\widetilde{\gamma}(1) = \widetilde{x}_1$. We can find such a $[\gamma'] \in \pi_1(X', x_0)$ if and only if $\widetilde{x}_0$ and $\widetilde{x}_1$ are in the same path component of $\widetilde{X}'$. Therefore, $\iota_*$ is surjective if and only if all points of $(p')^{-1}(x_0)$ are in the same path component of $\widetilde{X}'$, which holds if and only if $\widetilde{X}'$ is path connected. $\square$

## 15.2. Orientable surfaces

Let $\Sigma_g$ be a closed orientable genus $g$ surface:




We calculate the fundamental group of $\Sigma_g$ as follows. For a group $G$ and $x, y \in G$, recall that $[x, y] = xyx^{-1}y^{-1}$.

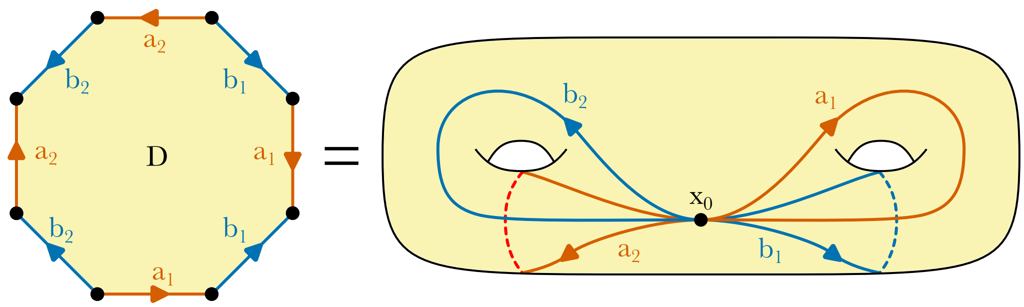
Theorem 15.2.1. *For some $g \geq 0$, let $x_0 \in \Sigma_g$ be a basepoint. Then*

$$\pi_1(\Sigma_g, x_0) \cong \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle .$$

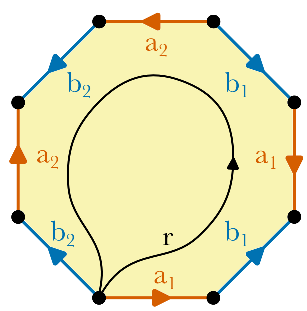
Proof. Let $D \cong \mathbb{D}^2$ be a regular $4g$-gon. Label the edges of $D$ going clockwise with the following labels:

$$a_1, b_1, a_1, b_1, \ldots, a_g, b_g, a_g, b_g.$$

Orient the edges such that the first $a_i$ and $b_i$ go clockwise and the second $a_i$ and $b_i$ go counterclockwise. Gluing the edges with the same label together according to these orientations, we get $\Sigma_g$:




As is shown here, all the vertices of $D$ get identified to a single point, which we can assume is the basepoint $x_0$. The $a_i$ and $b_i$ become loops based at $x_0$, and will form the generators for our group presentation. The element $r = [a_1, b_1] \cdots [a_g, b_g]$ is a relation between the $a_i$ and $b_i$; indeed, as an element of the fundamental group it traces out the boundary of $D$ and can be homotoped to the following loop in $\Sigma_g$:




Let $X \subset \Sigma_g$ be the graph with vertex $x_0$ and edges the loops $a_i$ and $b_i$. Let $z_0 \in \Sigma_g$ be the image of the center $c_0 \in D$. Set $Y = \Sigma_g \setminus z_0$. Since $D \setminus c_0 \cong \mathbb{D}^2 \setminus 0$ deformation retracts to its boundary $\partial D \cong \mathbb{S}^1$, the space $Y$ deformation retracts to $X$. It follows that

$$\pi_1(Y, x_0) = \pi_1(X, x_0) = F(a_1, b_1, \ldots, a_g, b_g).$$

To prove the theorem, it is enough to prove two things:

(i) The inclusion $\iota\colon (Y, x_0) \hookrightarrow (\Sigma_g, x_0)$ induces a surjective map $\iota_*\colon \pi_1(Y, x_0) \to \pi_1(\Sigma_g, x_0)$.
(ii) The kernel of $\iota_*$ is the normal subgroup generated by $r = [a_1, b_1] \cdots [a_g, b_g]$.

Let $p\colon (\widetilde{\Sigma}_g, \widetilde{x}_0) \to (\Sigma_g, x_0)$ be the universal cover and let $q\colon (\widetilde{Y}, \widetilde{x}_0) \to (Y, x_0)$ be the restriction of $p$ to $Y$. By Lemma 15.1.2, to prove (i) we must prove that $\widetilde{Y}$ is path connected. We have

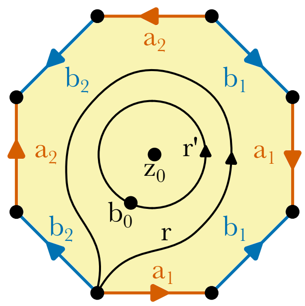
$\widetilde{Y} = \widetilde{\Sigma}_g \setminus p^{-1}(z_0)$. Let $B \cong \mathrm{Int}(\mathbb{D}^2)$ be the image in $\Sigma_g$ of the interior of $D$, so $z_0 \in B$. Since $B$ is contractible, the restriction of $p$ to $B$ must be trivial. We can therefore write

$$p^{-1}(B) = \bigsqcup_{i \in I} \widetilde{B}_i,$$

where $p$ takes each $\widetilde{B}_i$ homeomorphically to $B$. The fiber $p^{-1}(z_0)$ contains exactly one point $\widetilde{z}_i$ in each $\widetilde{B}_i$. Since $\widetilde{B}_i \setminus \widetilde{z}_i \cong \mathrm{Int}(\mathbb{D}^2) \setminus 0$ is path connected, it follows that removing these points $\widetilde{z}_i$ to form $\widetilde{Y}$ does not disconnect $\widetilde{\Sigma}_g$. We conclude that $\widetilde{Y}$ is path connected, and thus that (i) holds, i.e., that the inclusion $\iota\colon (Y, x_0) \hookrightarrow (\Sigma_g, x_0)$ induces a surjective map $\iota_*\colon \pi_1(Y, x_0) \to \pi_1(\Sigma_g, x_0)$.

It remains to prove that $K = \ker(\iota_*)$ is the normal subgroup generated by $r$. By Lemma 15.1.1, the cover $q\colon (\widetilde{Y}, \widetilde{x}_0) \to (Y, x_0)$ is the connected cover corresponding to $K$. Let $K' < \pi_1(Y, x_0)$ be the normal subgroup generated by $r$, so $K' < K$. Let $q'\colon (\widetilde{Y}', \widetilde{y}_0) \to (Y, x_0)$ be the connected cover corresponding to $K'$. We will prove below that $q'$ is the restriction to $Y$ of a cover $p'\colon (\widetilde{\Sigma}'_g, \widetilde{x}'_0) \to (\Sigma_g, x_0)$. Letting $L < \pi_1(\Sigma_g, x_0)$ be the subgroup corresponding to $p'$, Lemma 15.1.1 will imply that $K' = \iota_*^{-1}(L)$. Since $K' < K = \ker(\iota_*)$, this will allow us to conclude that $L = 1$ and $K' = K$, as desired.

To see that $q'$ is the restriction to $Y = \Sigma_g \setminus z_0$ of a cover of $\Sigma_g$, consider the subspace $B \setminus z_0 \cong \mathrm{Int}(\mathbb{D}^2) \setminus 0$ of $Y$. For a basepoint $b_0 \in B \setminus z_0$, we have $\pi_1(B \setminus z_0, b_0) \cong \mathbb{Z}$. This $\mathbb{Z}$ is generated by the loop $r'$ shown here:




Since the subgroup $K'$ corresponding to $q'$ contains all conjugates of $r$, it follows that all lifts to $\widetilde{Y}'$ of a loop representing $r$ lift are loops. From the above figure, this implies that all lifts of $r'$ to the cover are also loops. We conclude that the restriction of $q'$ to $B \setminus z_0$ is trivial, so

$$(q')^{-1}(B \setminus z_0) = \bigcup_{j \in J} \widetilde{B}'_j$$

with $q'$ taking $\widetilde{B}'_j$ homeomorphically to $B \setminus z_0 \cong \mathrm{Int}(\mathbb{D}^2) \setminus 0$. By adding the missing central point to each $\widetilde{B}'_j$, we can extend $q'$ to a cover of $\Sigma_g$, as desired. □

Corollary 15.2.2. *For $g, g' \geq 0$ with $g \neq g'$, the surfaces $\Sigma_g$ and $\Sigma_{g'}$ are not homotopy equivalent. In particular, they are not homeomorphic.*

Proof. For $g \geq 0$, let $\Gamma_g$ be the fundamental group of $\Sigma_g$. Theorem 15.2.1 says that

$$\Gamma_g \cong \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle .$$

The abelianization $\Gamma_g^{\mathrm{ab}}$ is thus the abelian group with generators $A_1, B_1, \ldots, A_g, B_g$ subject to the single relation

$$(A_1 + B_1 - A_1 - B_1) + \cdots + (A_g + B_g - A_g - B_g) = 0.$$

The left hand side of this is already 0, so this relation is trivial. We conclude that $\Gamma_g^{\mathrm{ab}} \cong \mathbb{Z}^{2g}$. Consequently, for $g, g' \geq 0$ with $g \neq g'$ the groups $\Gamma_g$ and $\Gamma_{g'}$ are not isomorphic, and thus $\Sigma_g$ and $\Sigma_{g'}$ are not homotopy equivalent. □

Remark 15.2.3. We could have proved earlier that these surfaces are not homeomorphic by observing that for $z_0 \in \Sigma_g$ the surface $\Sigma_g \setminus z_0$ deformation retracts onto a graph with one vertex and $2g$ edges (cf. the proof of Theorem 15.2.1 above). Consequently, the fundamental group of $\Sigma_g \setminus z_0$ is a free group on $2g$ generators. □

## 15.3. Attaching 2-cells

The key to our calculation of the fundamental groups of surfaces was the observation that $\Sigma_g$ can be obtained by gluing a disk $\mathbb{D}^2$ to a graph. We generalize this as follows. Let $X$ be a space and let $\{D_i^2\}_{i\in I}$ be a collection of 2-cells, i.e., spaces $D_i^2$ with $D_i^2 = \mathbb{D}^2$. We thus have $\partial D_i^2 = \partial \mathbb{D}^2 = \mathbb{S}^1$. For $i \in I$, let $\phi_i \colon \partial D_i^2 \to X$ be a map that we will call the *attaching map* for $D_i^2$. The result of of *attaching* the 2-cells $\{D_i^2\}_{i\in I}$ via the attaching maps $\{\phi_i \colon \partial D_i^2 \to X\}$ is the space

$$Z = \left( X \sqcup \bigsqcup_{i\in I} D_i^2 \right) / \sim,$$

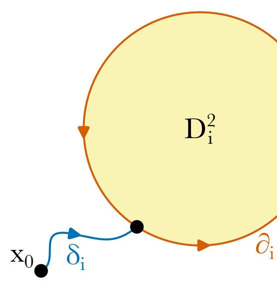
where $\sim$ is the equivalence relation that for $i \in I$ and $z \in \partial D_i^2$ identifies $z \sim \phi_i(z) \in X$.

Example 15.3.1. Let $X$ be a graph with one vertex and $2g$ edges. As we observed in the proof of Theorem 15.2.1 above, $\Sigma_g$ can be obtained from $X$ by attaching a single 2-cell. □

Let $x_0 \in X$ be a basepoint. For $i \in I$, a *loop around* $D_i^2$ in $(X, x_0)$ is a loop $r_i$ of the form $r_i = \delta_i \cdot \partial_i \cdot \overline{\delta}_i$, where $\delta_i$ and $\partial_i$ are as follows:

- The path $\delta_i$ is a path in $X$ from $x_0$ to $\phi_i(1)$. Here are are identifying $\partial D_i^2$ with $\mathbb{S}^1 \subset \mathbb{C}$.
- The path $\gamma_i \colon I \to X$ is the loop in $X$ based at $\phi_i(1)$ defined by the formula $\gamma_i(s) = \phi_i(e^{2\pi i s})$ for $s \in I$.

We have $[r_i] \in \pi_1(X, x_0)$. When we attach the cells $\{D_i^2\}_{i\in I}$ to $X$ to form $Z$, the loops $r_i = \delta_i \cdot \partial_i \cdot \overline{\delta}_i$ become null-homotopic:

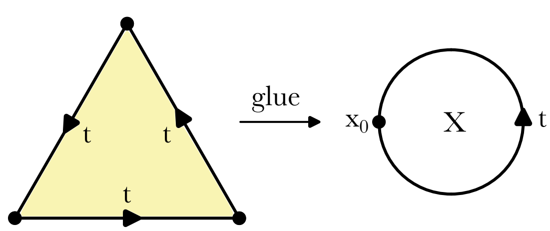



It follows that $[r_i]$ is in the kernel of the map $\pi_1(X, x_0) \to \pi_1(Z, x_0)$. It turns out that they normally generate the kernel of this map. In fact:

Theorem 15.3.2. *Let $(X, x_0)$ be a pointed space. Assume that $X$ is path connected, locally path connected, and semilocally 1-connected.*[1] *Let $Z$ be the result of attaching a collection $\{D_i^2\}_{i\in I}$ of 2-cells to $X$ via attaching maps $\{\phi_i \colon \partial D_i^2 \to X\}$. For each $i \in I$, let $r_i = \delta_i \cdot \partial_i \cdot \overline{\delta}_i$ be a loop around $D_i^2$ in $(X, x_0)$. Then $\pi_1(Z, x_0)$ is the quotient of $\pi_1(X, x_0)$ by the normal subgroup generated by $\{[r_i] \mid i \in I\}$.*

Before proving Theorem 15.3.2, we give an example.

Example 15.3.3. Let $X \cong \mathbb{S}^1$ be a graph with one vertex $x_0$ and one loop labeled $t$. Let $D^2 \cong \mathbb{D}^2$, identified with a triangle. Let $Z$ be the space obtained by attaching $D^2$ to $X$ as indicated in the following figure:




Then it follows from Theorem 15.3.2 that $\pi_1(Z, x_0)$ is the quotient of $\pi_1(X, x_0) = \mathbb{Z} = \{t^n \mid n \in \mathbb{Z}\}$ by the normal subgroup generated by $r = t^3$. In other words, $\pi_1(X, x_0) = C_3$. □

[1] As we will prove in Chapter 19* using the Seifert–van Kampen theorem, it is not necessary to assume that $X$ is locally path connected and semilocally 1-connected.

Proof of Theorem 15.3.2. The proof is almost exactly the same as our calculation of the fundamental group of $\Sigma_g$ in Theorem 15.2.1, so we just sketch it. For $i \in I$, let $z_i \in Z$ be the image of the central point $0 \in D_i^2$. Set $Y = Z \setminus \{z_i \mid i \in I\}$. Since $D_i^2 \setminus 0$ deformation retracts to $\partial D_i^2$, the space $Y$ deformation retracts to $X$. We thus have $\pi_1(Y, x_0) \cong \pi_1(X, x_0)$. We must prove the following:

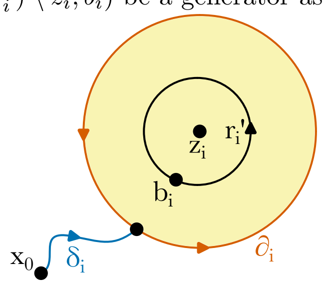
(i) The inclusion $\iota\colon (Y, x_0) \hookrightarrow (Z, x_0)$ induces a surjective map $\iota_*\colon \pi_1(Y, x_0) \to \pi_1(Z, x_0)$.
(ii) The kernel of $\iota_*$ is the normal subgroup generated by $\{[r_i] \mid i \in I\}$.

Let $p\colon (\widetilde{Z}, \widetilde{x}_0) \to (Z, x_0)$ be the universal cover. Let

$$\widetilde{Y} = \widetilde{Z} \setminus \bigcup_{i \in I} p^{-1}(z_i)$$

and let $q = p|_{\widetilde{Y}}$, so $q\colon (\widetilde{Y}, \widetilde{x}_0) \to (Y, x_0)$ is the restriction of $p$ to $Y$. Just like in the proof of Theorem 15.2.1, removing the points in $p^{-1}(z_i)$ does not disconnect $\widetilde{Z}$, so $\widetilde{Y}$ is path connected. By Lemma 15.1.2, it follows that $\iota_*\colon \pi_1(Y, x_0) \to \pi_1(Z, x_0)$, as was claimed by (i).

For (ii), let $K = \ker(\iota_*)$ and let $K'$ be the normal subgroup of $\pi_1(Y, x_0)$ generated by $\{[r_i] \mid i \in I\}$, so $K' < K$. Let $q'\colon (\widetilde{Y}', \widetilde{y}_0) \to (Y, x_0)$ be the connected cover corresponding to $K'$. Just like in the proof of Theorem 15.2.1, it is enough to prove that $q'$ is the restriction of a cover of $(Z, x_0)$. For $i \in I$, identify $\mathrm{Int}(D_i^2)$ with its image in $Z$, so $z_i \in \mathrm{Int}(D_i^2)$. Following the proof of Theorem 15.2.1, we must prove that the restriction of $q'$ to each $\mathrm{Int}(D_i^2) \setminus z_i$ is trivial. Fixing a basepoint $b_i \in \mathrm{Int}(D_i^2) \setminus z_i$, let $[r_i'] \in \pi_1(\mathrm{Int}(D_i^2) \setminus z_i, b_i)$ be a generator as in the following figure:




Just like in the proof of Theorem 15.2.1, the fact that each lift of $r_i = \delta_i \cdot \partial_i \cdot \overline{\delta}_i$ to $\widetilde{Y}'$ is a loop implies that each lift of $r_i'$ to $\widetilde{Y}'$ is a loop. This implies that the restriction of $q'$ to each $\mathrm{Int}(D_i^2) \setminus z_i$ is trivial, as desired. The theorem follows. $\square$

## 15.4. Attaching 2-cells: local properties

We will soon give an interesting example of the the above result, but first we prove that attaching 2-cells does not turn a nice space into a terrible one:

Lemma 15.4.1. *Let $X$ be a space that is locally path connected and semilocally 1-connected. Let $Z$ be the result of attaching a collection $\{D_i^2\}_{i \in I}$ of 2-cells to $X$ via attaching maps $\{\phi_i\colon \partial D_i^2 \to X\}$. Then $Z$ is locally path connected and semilocally 1-connected.*

Proof. We will prove that $Z$ is semilocally 1-connected and leave the proof that it is locally path connected as an exercise (see Exercise 15.4). Let $z \in Z$. It is enough to find an open neighborhood $U$ of $z$ that is semilocally 1-connected. There are two cases:

- The first is $z \notin X$. There is an $i \in I$ such that $z$ is in the image $U$ of $\mathrm{Int}(D_i^2)$. The open set $U \cong \mathrm{Int}(\mathbb{D}^2)$ is contractible, so in particular $U$ is semilocally 1-connected.
- If second is $z \in X$. For each $i \in I$, let $z_i \in Z$ be the image of the central point $0 \in D_i^2$. Let $U = Z \setminus \{z_i \mid i \in I\}$. The subspace $U$ is an open neighborhood of $z$ that deformation retracts to $X$ (cf. the the proof of Theorem 15.3.2). Since $X$ is semilocally 1-connected, it follows that $U$ is as well (see Exercise 10.2). $\square$

## 15.5. All groups are fundamental groups

As an application of the above ideas, we prove that all groups can be realized as fundamental groups of spaces. We remark that an alternate proof of this was sketched in Exercise 7.11.

THEOREM 15.5.1. *Let $G$ be a group. Then there exists a pointed space $(Z, x_0)$ with $\pi_1(Z, x_0) \cong G$. Moreover, $Z$ can be chosen to be path connected, locally path connected, and semilocally 1-connected.*

PROOF. Choose a presentation $G = \langle S \mid R \rangle$. Let $X$ be a graph with 1 vertex $x_0$ and $|S|$ oriented edges, each labeled by an element of $S$. We therefore have $\pi_1(X, x_0) = F(S)$. For each $r \in R \subset F(S)$, we can therefore identify $r$ with an element of $\pi_1(X, x_0)$. For each $r \in R$, let $D_r^2 = \mathbb{D}^2$ be a 2-cell and $\phi_r \colon \mathbb{S}^1 \to X$ be an attaching map such that the following holds:

- Let $Z$ be the result of attaching the collection $\{D_r^2\}_{r \in I}$ of 2-cells to $X$ via the attaching maps $\{\phi_r \colon \partial D_r^2 \to X\}$. Then each $r \in R$ is the homotopy class of a loop around $D_r^2$ in $(X, x_0)$.

It then follows from Theorem 15.3.2 that $\pi_1(Z, x_0)$ is the quotient of $\pi_1(X, x_0) = F(S)$ by the normal subgroup generated by $R$. In other words, $\pi_1(Z, x_0) \cong G$. Since $X$ is path connected, locally path connected, and semilocally 1-connected it follows from Lemma 15.4.1 that $Z$ is as well. □

## 15.6. Attaching higher-dimensional cells

We close this chapter by explaining how attaching higher dimensional cells affects the fundamental group. Let $X$ be a space and let $\{D_i^{n_i}\}_{i \in I}$ be a collection of cells, i.e., spaces $D_i^{n_i}$ with $D_i^{n_i} = \mathbb{D}^{n_i}$ for some $n_i \geq 0$. We thus have $\partial D_i^{n_i} = \partial \mathbb{D}^{n_i} = \mathbb{S}^{n_i - 1}$. For $i \in I$, let $\phi_i \colon \partial D_i^{n_i} \to X$ be a map that we will call the *attaching map* for $D_i^{n_i}$. The result of of *attaching* the cells $\{D_i^{n_i}\}_{i \in I}$ via the attaching maps $\{\phi_i \colon \partial D_i^{n_i} \to X\}$ is the space

$$Z = \left( X \sqcup \bigsqcup_{i \in I} D_i^{n_i} \right) / \sim,$$

where $\sim$ is the equivalence relation that for $i \in I$ and $z \in \partial D_i^{n_i}$ identifies $z \sim \phi_i(z) \in X$.

EXAMPLE 15.6.1. Let $X = \mathbb{D}^n$ and let $Z$ be the space obtained by attaching a single $n$-cell $D^n = \mathbb{D}^n$ to $X$ via the attaching map $\phi \colon \partial D^n \to X$ given by the composition

$$\partial D^n = \mathbb{S}^{n-1} = \partial \mathbb{D}^n \hookrightarrow \mathbb{D}^n = X.$$

We then have $Z \cong \mathbb{S}^n$ with $X$ the lower hemisphere and the $n$-cell $D^n$ the upper hemisphere. □

It turns out that attaching cells of dimension greater than 2 to a space does not change its fundamental group. In light of Example 15.6.1, this gives a new proof that $\mathbb{S}^n$ is 1-connected for $n \geq 2$.

THEOREM 15.6.2. *Let $(X, x_0)$ be a pointed space. Assume that $X$ is path connected, locally path connected, and semilocally 1-connected.*[2] *Let $Z$ be the result of attaching a collection $\{D_i^{n_i}\}_{i \in I}$ of cells to $X$ via attaching maps $\{\phi_i \colon \partial D_i^{n_i} \to X\}$. For each $i \in I$, assume that $n_i \geq 3$. Then $\pi_1(X, x_0) \cong \pi_1(Z, x_0)$.*

PROOF. The proof is very similar to that of Theorem 15.3.2, so we leave it as an exercise (see Exercise 15.6). □

REMARK 15.6.3. As we will see in later volumes, the most important spaces in algebraic topology are the CW complexes. These are spaces $X$ such that $X = \cup_{n=0}^{\infty} X^{(n)}$, where $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching a collection of $n$-cells. The starting point is $X^{(0)}$, which is obtained from $X^{(-1)} = \emptyset$ by attaching a collection of 0-cells, i.e., adjoining a discrete collection of points. The subspace $X^{(n)}$ is called the *n-skeleton*. We discussed the point set topology of these spaces in Volume 1, and will say more about their algebraic topology in Essay YYY. □

[2] Just like for 2-cells, we will prove in Chapter 19* using the Seifert–van Kampen theorem that it is not necessary to assume that $X$ is locally path connected and semilocally 1-connected.

## 15.7. Exercises

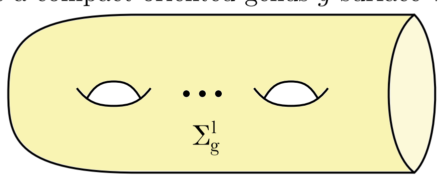
EXERCISE 15.1. Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a connected pointed cover and let $f\colon (Y, y_0) \to (X, x_0)$ be a map. Assume that $X$ and $Y$ are path connected and locally path connected. Define

$$\widetilde{Y} = \left\{ (y, \widetilde{x}) \in Y \times \widetilde{X} \mid f(y) = p(\widetilde{x}) \right\},$$
$$\widetilde{y}_0 = (y_0, \widetilde{x}_0) \in \widetilde{Y},$$

and let $q\colon (\widetilde{Y}, \widetilde{y}_0) \to (y, y_0)$ be the projection onto the first factor. You proved in Exercise 1.14 that $q\colon (\widetilde{Y}, \widetilde{y}_0) \to (Y, y_0)$ is a covering space. Let $\widetilde{Y}'$ be the path component of $\widetilde{Y}$ containing $\widetilde{y}_0$ and let $q' = q|_{\widetilde{Y}'}$. Prove the following:

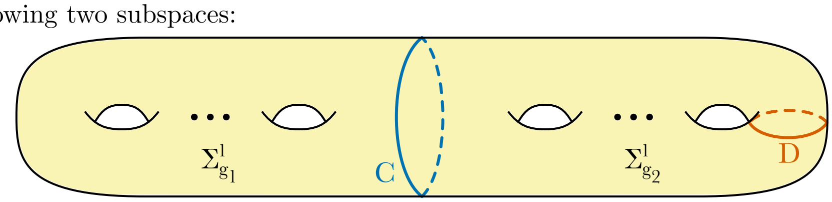
(a) The map $q'\colon (\widetilde{Y}', \widetilde{y}_0) \to (Y, y_0)$ is a cover.
(b) Let
$$K = \operatorname{Im}(p_*\colon \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0))$$
be the subgroup corresponding to $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ and let
$$K' = f_*^{-1}(K) \subset \pi_1(Y, y_0).$$
Prove that $K'$ is the subgroup corresponding to $q'\colon (\widetilde{Y}', \widetilde{y}_0) \to (y, y_0)$. □

EXERCISE 15.2. Let $\Sigma_g^1$ be a compact oriented genus $g$ surface with 1 boundary component:



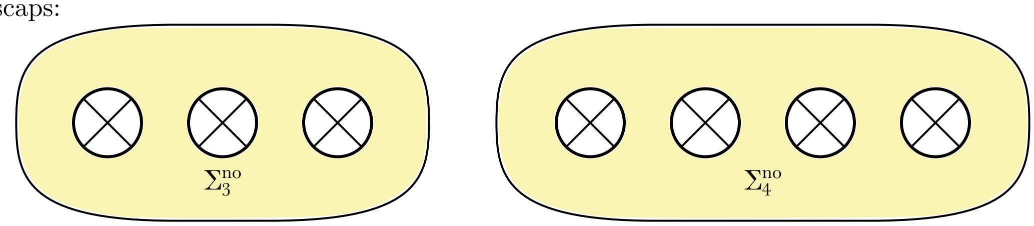

Its boundary $\partial\Sigma_g^1$ is thus homeomorphic to $\mathbb{S}^1$. Do the following:

(a) Prove that $\pi_1(\Sigma_g^1, x_0)$ is isomorphic to a free group $F_{2g}$ on $2g$ generators.
(b) Let $x_0 \in \mathbb{S}^1$ be a basepoint and let $\iota\colon (\partial\Sigma_g^1, x_0) \to (\Sigma_g^1, x_0)$ be the inclusion. The induced map $\iota_*$ thus goes from $\pi_1(\partial\Sigma_g^1, x_0) \cong \mathbb{Z}$ to $\pi_1(\Sigma_g^1, x_0) \cong F_{2g}$. Passing to the abelianizations, the map $\iota_*$ thus induces a map from $\mathbb{Z}$ to $\mathbb{Z}^{2g}$. Prove that this induced map on abelianizations is the trivial map.
(c) Prove that there does not exist a retraction $r\colon \Sigma_g^1 \to \partial\Sigma_g^1$.
(d) Assume that $g \geq 2$. For some $g_1, g_1 \geq 1$ with $g_1 + g_2 = g$, let $C \subset \Sigma_g$ and $D \subset \Sigma_g$ be the following two subspaces:




Prove that $\Sigma_g$ does not retract onto $C$, but that $\Sigma_g$ does retract onto $D$. □

EXERCISE 15.3. Let $\Sigma_n^{\mathrm{no}}$ be a closed non-orientable surface of genus $n \geq 1$, i.e., a sphere with $n$ crosscaps:



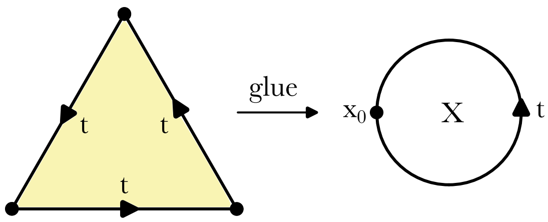

Fix $x_0 \in \Sigma_n^{\mathrm{no}}$. Do the following:

(a) Prove that $\pi_1(\Sigma_n^{\mathrm{no}}, x_0) \cong \langle a_1, \ldots, a_n \mid a_1^2 \cdots a_n^2 = 1 \rangle$.
(b) Calculate the abelianization of $\pi_1(\Sigma_n^{\mathrm{no}}, x_0)$.
(c) Prove that $\Sigma_n^{\mathrm{no}}$ is not homotopy equivalent to $\Sigma_m^{\mathrm{no}}$ for any $m \geq 1$ with $m \neq n$.

(d) Prove that $\Sigma_n^{\mathrm{no}}$ is not homotopy equivalent to any closed orientable surface. □

EXERCISE 15.4. Let $X$ be a space that is locally path connected. Let $Z$ be the result of attaching a collection $\{D_i^2\}_{i\in I}$ of 2-cells to $X$ via attaching maps $\{\phi_i \colon \partial D_i^2 \to X\}$. Prove that $Z$ is locally path connected. □

EXERCISE 15.5. Let $X \cong \mathbb{S}^1$ be a graph with one vertex $x_0$ and one loop labeled $t$. Let $D^2 \cong \mathbb{D}^2$, identified with a triangle. Let $Z$ be the space obtained by attaching $D^2$ to $X$ as indicated in the following figure:




As we observed in Example 15.3.3, we have $\pi_1(Z, x_0) = C_3$. Describe the universal cover $p \colon \widetilde{Z} \to Z$ as well as the $C_3$-action on $\widetilde{Z}$. □

EXERCISE 15.6. Let $(X, x_0)$ be a pointed space. Assume that $X$ is path connected, locally path connected, and semilocally 1-connected. Let $Z$ be the result of attaching a collection $\{D_i^{n_i}\}_{i\in I}$ of cells to $X$ via attaching maps $\{\phi_i \colon \partial D_i^{n_i} \to X\}$. For each $i \in I$, assume that $n_i \geq 3$. Prove that $\pi_1(X, x_0) \cong \pi_1(Z, x_0)$. □

CHAPTER 16

# Free products and regular $G$-covers

This chapter introduces free products and explains how to compute the fundamental groups of wedges of spaces. Along the way, we introduce the notion of a regular $G$-cover, which will play a key role our proof of the Seifert–van Kampen theorem in the next chapter.

## 16.1. Free products

We start by defining free products. The definitions parallel those of free groups from §7.6. Let $\{A_i\}_{i \in I}$ be a collection of groups. Roughly speaking, a free product of the $A_i$ is a group $\Gamma$ such that maps from $\Gamma$ to a group $G$ are the same as collections of maps from the $A_i$ to $G$. The formal definition is as follows:

DEFINITION 16.1.1. Let $\{A_i\}_{i \in I}$ be a collection of groups. A *free product* of the $A_i$ is a group $\Gamma$ equipped with homomorphisms $\iota_i \colon A_i \to \Gamma$ for each $i \in I$ such that the following holds:

- (†) Let $G$ be a group and for each $i \in I$ let $h_i \colon A_i \to G$ be a homomorphism. Then there is a unique homomorphism $H \colon \Gamma \to G$ such that $h_i = H \circ \iota_i$ for all $i \in I$. □

The maps $\iota_i$ in the definition of a free product are necessarily split injective:

LEMMA 16.1.2. *Let $\{A_i\}_{i \in I}$ be a collection of groups and let $\Gamma$ be a free product of the $A_i$. Then for each $i \in I$ the associated map $\iota_i \colon A_i \to \Gamma$ is split injective.*

PROOF. Fix some $i_0 \in I$. For each $i \in I$, let $h_i \colon A_i \to A_{i_0}$ be the identity if $i = i_0$ and the trivial homomorphism if $i \neq i_0$. By the universal property (†), there is a homomorphism $H \colon \Gamma \to G$ such that $h_i = H \circ \iota_i$. In particular, $\mathbb{1}_{A_{i_0}} = h_{i_0} = H \circ \iota_{i_0}$, so $\iota_{i_0}$ is split injective. □

Let $\Gamma$ be a free product of the groups $\{A_i\}_{i \in I}$. By Lemma 16.1.2, we can identify each $A_i$ with a subgroup of $\Gamma$ via the corresponding map $\iota_i$. Having done this, we can now rephrase (†) as follows:

- (†′) Let $G$ be a group and for each $i \in I$ let $h_i \colon A_i \to G$ be a homomorphism. Then the $h_i$ extend uniquely to a homomorphism $H \colon \Gamma \to G$.

Whenever we work with free products in this book, we will identify the $A_i$ with subgroups of $\Gamma$ and use (†′) as the defining property of the free product. The subgroups $A_i$ generate the free product:

LEMMA 16.1.3. *Let $\{A_i\}_{i \in I}$ be a collection of groups and let $\Gamma$ be a free product of the $A_i$. Then $\Gamma$ is generated by its subgroups $\{A_i\}_{i \in I}$.*

PROOF. Let $G < \Gamma$ be the subgroup generated by the $A_i$. By (†′), the inclusion maps $h_i \colon A_i \hookrightarrow G$ extend uniquely to a homomorphism $H \colon \Gamma \to G$. The composition

$$\Gamma \xrightarrow{H} G \hookrightarrow \Gamma$$

and the identity map $\mathbb{1}_{F(S)} \colon \Gamma \to \Gamma$ both extend the identity maps $\mathbb{1}_{A_i} \colon A_i \to A_i$, and thus must be equal. We conclude that $G = \Gamma$, as desired. □

We now prove that free products exist and are unique in an appropriate sense:

LEMMA 16.1.4. *Let $\{A_i\}_{i \in I}$ be a collection of groups. The following hold:*

- *(i) There exists a free product of the $A_i$.*
- *(ii) Let $\Gamma$ and $\Gamma'$ be two free products of the $A_i$. There is then a unique isomorphism $\phi \colon \Gamma \to \Gamma'$ with $\phi|_{A_i} = \mathbb{1}_{A_i}$ for all $i \in I$.*

PROOF. We divide the proof into two parts:

STEP 1. *There exists a free product of the $A_i$.*

For $i \in I$, choose a presentation $A_i = \langle S_i \mid R_i \rangle$. Set

$$\Gamma = \langle \sqcup_{i\in I} S_i \mid \sqcup_{i \in I} R_i \rangle . \tag{16.1.1}$$

For $i \in I$, the evident map $A_i \to \Gamma$ splits via the map taking $S_i$ to itself by the identity and $S_{i'}$ for $i' \in I$ with $i' \neq i$ to 1. We can therefore identify $A_i$ with a subgroup of $\Gamma$. We verify ($\dagger'$) for $\Gamma$ as follows. Let $G$ be a group and for each $i \in I$ let $h_i \colon A_i \to G$ be a homomorphism. For $i \in I$, let $f_i \colon S_i \to G$ be the composition

$$S_i \hookrightarrow F(S_i) \twoheadrightarrow A_i \xrightarrow{h_i} G.$$

The $f_i$ assemble into a set map $f \colon \sqcup_{i\in I} S_i \to G$. By the universal property of a free group, the map $f$ extends to homomorphism $F \colon F(\sqcup_{i\in I} S_i) \to G$. Since each $F(S_i) \to G$ factors through $A_i$, the map $F$ takes each $r \in R_i$ to $1 \in G$. We conclude that $F$ factors through a map $H \colon \Gamma \to G$, as desired.

STEP 2. *Let $\Gamma$ and $\Gamma'$ be two free products of the $A_i$. There is then a unique isomorphism $\phi \colon \Gamma \to \Gamma'$ with $\phi|_{A_i} = \mathbb{1}_{A_i}$ for all $i \in I$.*

Applying ($\dagger'$) to the identity maps between the subgroups $A_i \subset \Gamma$ and $A_i \subset \Gamma'$, we get:

- a homomorphism $\phi \colon \Gamma \to \Gamma'$ such that $\phi|_{A_i} = \mathbb{1}_{A_i}$ for all $i \in I$; and
- a homomorphism $\phi' \colon \Gamma' \to \Gamma$ such that $\phi'|_{A_i} = \mathbb{1}_{A_i}$ for all $i \in I$.

The composition $\phi' \circ \phi \colon \Gamma \to \Gamma$ fixes each element of subgroups $A_i$. Since these generate $\Gamma$, we have $\phi' \circ \phi = \mathbb{1}_{\Gamma}$. Similarly, $\phi \circ \phi' = \mathbb{1}_{\Gamma'}$. We conclude that $\phi$ and $\phi'$ are inverse isomorphisms. □

In light of this lemma, it makes sense to talk about *the* free product. If each $A_i$ is given by a presentation $A_i = \langle S_i \mid R_i \rangle$, then this free product has the presentation (16.1.1). We introduce the following notation:

NOTATION 16.1.5. Let $\{A_i\}_{i\in I}$ be a collection of groups. We denote their free product by $*_{i\in I} A_i$. For the free product of finitely many groups $A_1, \ldots, A_n$, we will also use the notation $A_1 * \cdots * A_n$ for their free product. □

## 16.2. Free products: reduced words

Let $\{A_i\}_{i\in I}$ be a collection of groups. A *word* in the $A_i$ is a formal expression $w = a_1 \cdots a_n$ such that for all $1 \le j \le n$ we have $a_j \in A_{i_j}$ for some $i_j \in I$. Since the $A_i$ are subgroups of $*_{i\in I} A_i$, a word corresponds to an element of this free product. The word is *reduced* if the follow two conditions hold:

- For all $1 \le j \le n$, we have $a_j \neq 1$.
- For all $1 \le j \le n-1$, we have $i_j \neq i_{j+1}$. In other words, $a_j$ and $a_{j+1}$ lie in different $A_i$'s.

By repeatedly deleting $a_j$ with $a_j = 1$ and combining adjacent terms that lie in the same $A_i$, each word can be reduced to a reduced word. It follows that each element of $*_{i\in I} A_i$ can be written as a reduced word. The main theorem about free products is that this reduced word is unique:

THEOREM 16.2.1. *Let $\{A_i\}_{i\in I}$ be a collection of groups. Then every element of $*_{i\in I} A_i$ can be represented by a unique reduced word.*

See Exercise 16.7 for an outline of the standard algebraic proof of this fact. At the end of this chapter, we will explain how to prove it geometrically. This requires first finding a space whose fundamental group is a given free product. To verify the universal property of the free product, for a pointed space $(X, x_0)$ we will need a way to understand geometrically a homomorphism $f \colon \pi_1(X, x_0) \to G$. We will do this using the notion of a regular $G$-cover, which is our next topic.

## 16.3. Regular $G$-covers

Let $X$ be a space and $G$ be a group. A *regular $G$-cover*[1] of $X$ is a covering space $p \colon \widetilde{X} \to X$ together with a left action of $G$ on $\widetilde{X}$ by deck transformation such that the following holds:

[1] These are also sometimes called principal $G$-bundles, but for principal $G$-bundles the group action is typically on the right, so we do not use that terminology.

- For all $x \in X$, the action of $G$ on the fiber $p^{-1}(x)$ is free and transitive.

For $x \in X$ and $\widetilde{x} \in p^{-1}(x)$, this implies that the orbit map $G \to p^{-1}(x)$ taking $g \in G$ to $g\widetilde{x}$ is a bijection. Here are two examples:

EXAMPLE 16.3.1. Let $p\colon \widetilde{X} \to X$ be a regular cover with $\widetilde{X}$ connected. For each $x \in X$, the deck group $\mathrm{Deck}(\widetilde{X})$ acts freely and transitively on the fiber $p^{-1}(x)$ (cf. Remark 2.3.2). Endowing $p\colon \widetilde{X} \to X$ with the structure of a regular $G$-cover is then the same as fixing an isomorphism $G \cong \mathrm{Deck}(\widetilde{X})$. □

EXAMPLE 16.3.2. Regard $G$ as a discrete set. The *trivial* regular $G$-cover of $X$ is the trivial cover $p\colon X \times G \to X$ together with the following action of $G$:

$$g(x, h) = (x, gh) \quad \text{for all } g \in G \text{ and } (x, h) \in X \times G. \qquad \square$$

REMARK 16.3.3. If $p\colon \widetilde{X} \to X$ is a regular $G$-cover and $\widetilde{X}$ is not connected, then the inclusion $G \hookrightarrow \mathrm{Deck}(\widetilde{X})$ coming from the action of $G$ by deck transformations will in general not be surjective. For instance, if $|G| \geq 3$ then in Example 16.3.2 there will be deck transformations not coming from the action of $G$. □

A *pointed* regular $G$-cover is a pointed cover $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that $p\colon \widetilde{X} \to X$ is a regular cover. If $p_1\colon (\widetilde{X}_1, \widetilde{x}_1) \to (X, x_0)$ and $p_2\colon (\widetilde{X}_2, \widetilde{x}_2) \to (X, x_0)$ are two pointed regular $G$-covers, then an *isomorphism* between them is a pointed covering space isomorphism $\widetilde{f}\colon (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ that is $G$-equivariant, i.e., such that

$$\widetilde{f}(g\widetilde{x}) = g\widetilde{f}(\widetilde{x}) \quad \text{for all } g \in G \text{ and } \widetilde{x} \in \widetilde{X}_1.$$

Here recall that a pointed covering space isomorphism is a homeomorphism $\widetilde{f}$ such that following diagram commutes:

$$\begin{tikzcd}
(\widetilde{X}_1, \widetilde{x}_1) \arrow[rr, "\widetilde{f}"] \arrow[dr, "p_1"] & & (\widetilde{X}_2, \widetilde{x}_2) \arrow[dl, "p_2"] \\
& (X, x_0) &
\end{tikzcd}$$

If an isomorphism between pointed regular $G$-covers exists, we say that they are *isomorphic*. Our next goal in this chapter will be to classify pointed regular $G$-covers up to isomorphism.

## 16.4. The $G$-lifting map

This requires some preliminaries. Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a pointed regular $G$-cover. Define a set map $f\colon \pi_1(X, x_0) \to G$ as follows:

- Consider $[\gamma] \in \pi_1(X, x_0)$. Let $\widetilde{\gamma}$ be the lift of $\gamma$ to $\widetilde{X}$ with $\widetilde{\gamma}(0) = \widetilde{x}_0$. Since $G$ acts freely and transitively on $p^{-1}(x_0)$, there exists a unique $g \in G$ with $\widetilde{\gamma}(1) = g\widetilde{x}_0$. We then set $f([\gamma]) = g$.

When $\widetilde{X}$ is path connected, we proved in Lemma 7.2.1 that this is a surjective homomorphism and we called $f$ the lifting map. In that lemma, $G$ was the whole deck group and we used the fact that $\widetilde{X}$ is path connected to ensure that $G$ acted freely and transitively on $p^{-1}(x_0)$. In our context, we have fixed an action of $G$ by deck transformations that is free and transitive on each fiber, and the proof of Lemma 7.2.1 goes through to show that $f\colon \pi_1(X, x_0) \to G$ is a homomorphism. However, if $\widetilde{X}$ is not path connected then $f$ need not be surjective.[2] We call $f\colon \pi_1(X, x_0) \to G$ the *$G$-lifting map*.

[2] If $X$ is path connected, then $f$ will be surjective if and only if $\widetilde{X}$ is path connected.

### 16.5*. Classification theorem via monodromy

We will give two approaches to our classification theorem. The shortest approach uses results about the monodromy action from Chapter 12*. Since that chapter was marked with an $*$ to indicate that a first-time reader might want to skip it, this section is also marked with an $*$. We will give another approach using the Galois correspondence next.

For a pointed space $(X, x_0)$ satisfying appropriate hypotheses, the following theorem classifies pointed regular $G$-covers of $(X, x_0)$:

THEOREM 16.5*.1. *Let $(X, x_0)$ be a pointed space and $G$ be a group. Assume that $X$ is path connected, locally path connected, and semilocally 1-connected. There is then a bijection between the following two sets:*

*(i) The set of isomorphism classes of pointed regular $G$-covers $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$.*
*(ii) The set of homomorphisms $f\colon \pi_1(X, x_0) \to G$.*

*This bijection takes a pointed regular $G$-cover to its $G$-lifting map.*

PROOF. Set $H = \pi_1(X, x_0)$. As we observed in §12*.7, Theorem 12*.5.1 implies that there is a bijection between the following two sets:

(a) The set of isomorphism classes of pointed covers $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$.
(b) The set of isomorphism classes of pointed right $H$-sets $(S, s_0)$.

We must incorporate the regular $G$-cover structure into this bijection. Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a pointed cover and let $(S, s_0)$ be the corresponding pointed right $H$-set. By Lemma 12*.6.3, an action of $G$ on $\widetilde{X}$ by deck transformations is the same as a left action of $G$ on $S$ by $H$-equivariant bijections. This will correspond to the structure of a regular $G$-cover if the action of $G$ on $S$ is free and transitive. We conclude that there is a bijection between the following two sets:

(i) The set of isomorphism classes of pointed regular $G$-covers $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$.
(ii') The set of isomorphism classes of pointed right $H$-sets $(S, s_0)$ such that $S$ is equipped with a free and transitive left $G$-action by $H$-equivariant bijections.

The isomorphisms in (ii') must respect the action of $G$. Now consider $(S, s_0)$ as in (ii'). Since $G$ acts freely and transitively on $S$, each element of $S$ can be uniquely written as $gs_0$ for some $g \in G$. Define a set map $f\colon H \to G$ as follows: for $h \in H$, let $f(h) \in G$ be the unique element such that $s_0 h = f(h)s_0$. This map $f$ is a homomorphism; indeed, since $G$ acts on $S$ by $H$-equivariant bijections it follows that for $h, h' \in H$ we have

$$f(hh')s_0 = s_0(hh') = f(h)s_0 h' = f(h)f(h')s_0.$$

We have therefore constructed a homomorphism $f\colon H \to G$ from each element of (ii'). Conversely, given a homomorphism $f\colon H \to G$ we can construct an element of (ii') as follows:

- Set $(S, s_0) = (G, 1)$, where $1 \in G$ is the identity. Equip $S$ with the left $G$-action coming from the left action of $G$ on itself and the right $H$-action induced by $f$ as follows:

$$sh = sf(h) \quad \text{for all } h \in H.$$

We conclude that the set (ii') is in bijection with the set of homomorphisms $f\colon H \to G$. This establishes a bijection between the sets (i) and (ii) in the theorem, and tracing through the constructions we see that this bijection takes a pointed regular $G$-cover to its $G$-lifting map. □

### 16.6. Inducing actions and covers

Before describing the approach to the classification using the Galois correspondence, we need some general results about group actions. We will call a space $X$ equipped with an action of a group $G$ a *$G$-space*. Two $G$-spaces $X$ and $Y$ are isomorphic if there exists a homeomorphism $f\colon X \to Y$ that is $G$-equivariant, i.e., such that

$$f(gx) = gf(x) \quad \text{for all } g \in G \text{ and } x \in X.$$

If $X$ is a $G$-space and $H < G$ is a subgroup, then $\operatorname{Res}^G_H X$ will denote the $H$-space obtained by forgetting the whole $G$-action on $X$ and just remembering the $H$-action.

Now let $G$ be a group and let $Z$ be an $H$-space for a subgroup $H < G$. Regard $G$ as a discrete space. Define

$$\operatorname{Ind}_H^G Z = (G \times Z)/\sim$$

where $\sim$ is the following equivalence relation:

$$(gh, z) \sim (g, hz) \quad \text{for all } g \in G \text{ and } h \in H \text{ and } z \in Z.$$

We endow $\operatorname{Ind}_H^G Z$ with the quotient topology. The group $G$ acts on $G \times Z$ via

$$g(g', z) = (gg', z) \quad \text{for } g, g' \in G \text{ and } z \in Z.$$

This descends to an action of $G$ on $\operatorname{Ind}_H^G Z$. We call $\operatorname{Ind}_H^G Z$ the *induced $G$-space* of the $H$-space $Z$. The subspace $1 \times Z$ of $G \times Z$ maps homeomorphically to a subspace of $\operatorname{Ind}_H^G Z$. We will identify this subspace with $Z$. The space $\operatorname{Ind}_H^G Z$ satisfies the following universal property:[3]

LEMMA 16.6.1. *Let $G$ be a group and $H < G$ be a subgroup. Let $Z$ be an $H$-space and let $X$ be a $G$-space. There is then a bijection between:*

- *$H$-equivariant maps $f\colon Z \to \operatorname{Res}_H^G X$; and*
- *$G$-equivariant maps $F\colon \operatorname{Ind}_H^G Z \to X$.*

*This bijection takes a $G$-equivariant map $F\colon \operatorname{Ind}_H^G Z \to X$ to $f = F|_Z$.*

PROOF. Since $\operatorname{Ind}_H^G Z = \cup_{g \in G}(gZ)$, a $G$-equivariant map $F\colon \operatorname{Ind}_H^G Z \to X$ is determined by its restriction to $Z$. To prove the lemma, we must prove that an $H$-equivariant map $f\colon Z \to \operatorname{Res}_H^G X$ can be extended to a $G$-equivariant map $F\colon \operatorname{Ind}_H^G Z \to X$. For this, define $f'\colon G \times Z \to X$ via the formula

$$f'(g, z) = gf(z) \quad \text{for all } (g, z) \in G \times Z.$$

Since $f'(gh, z) = f'(g, hz)$ for all $g \in G$ and $h \in H$ and $z \in Z$, the map $f'$ factors through a map $F\colon \operatorname{Ind}_H^G Z \to X$ with $F|_Z = f$. $\square$

For $g \in G$, we have a subspace $gZ$ of $\operatorname{Ind}_H^G Z$ that is homeomorphic to $Z$. The space $\operatorname{Ind}_H^G Z$ is the disjoint union of a collection of such subspaces indexed by the left cosets $G/H$. Indeed, let $\{c_i\}_{i \in I}$ be a complete set of left $H$-coset representatives in $G$, so $G = \sqcup_{i \in I} c_i H$. We have

$$\operatorname{Ind}_H^G Z = \bigsqcup_{i \in I} c_i Z.$$

The action of $G$ permutes these subspaces transitively, and the stabilizer of $Z$ is $H$. Conversely, we have the following:

LEMMA 16.6.2. *Let $G$ be a group and let $X$ be a $G$-space. Let $Z \subset X$ be a subspace and let $H$ be the subgroup stabilizing $Z$. Letting $\{c_i\}_{i \in I}$ be a complete set of left $H$-coset representatives in $G$, assume that $X = \bigsqcup_{i \in I} c_i Z$. Then the $G$-space $X$ is isomorphic to $\operatorname{Ind}_H^G Z$.*

PROOF. The map $G \times Z \to X$ taking $(g, z) \in G \times Z$ to $gz \in X$ factors through a $G$-equivariant map $f\colon \operatorname{Ind}_H^G Z \to X$. Our assumptions imply that $f$ is a homeomorphism. $\square$

For a pointed space $(X, x_0)$, let $q\colon (\widetilde{Y}, \widetilde{y}_0) \to (X, x_0)$ be a pointed regular $H$-cover. Regard $X$ as a $G$-space with the trivial $G$ action, so $q$ is an $H$-equivariant map $q\colon \widetilde{Y} \to \operatorname{Res}_H^G X$. Let $\widetilde{X} = \operatorname{Ind}_H^G \widetilde{Y}$ and let $p\colon \widetilde{X} \to X$ be the map given by the universal property from Lemma 16.6.1. Finally, let $\widetilde{x}_0 \in \widetilde{X}$ be the point $\widetilde{y}_0$ in the subspace $\widetilde{Y} \subset \widetilde{X}$, so we have a pointed map $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$. We call $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ the *induced regular $G$-cover* of the regular $H$-cover $q\colon (\widetilde{Y}, \widetilde{y}_0) \to (X, x_0)$. As is suggested by its name, it is a regular $G$-cover:

[3]In an obvious way, $\operatorname{Res}_H^G$ is a functor from $G$-spaces to $H$-spaces and $\operatorname{Ind}_H^G$ is a functor from $H$-spaces to $G$-spaces. A categorically inclined reader will recognize that this lemma says that $\operatorname{Ind}_H^G$ is a left adjoint to $\operatorname{Res}_H^G$, i.e., that for $H$-spaces $Z$ and $G$-spaces $X$ we have a natural identification $\operatorname{Hom}_H(Z, \operatorname{Res}_H^G X) = \operatorname{Hom}_G(\operatorname{Ind}_H^G Z, X)$. See Exercise 16.2 for the right adjoint to $\operatorname{Res}_H^G$.

LEMMA 16.6.3. *Let $G$ be a group and let $H < G$ be a subgroup. For a space $X$, let $q\colon (\widetilde{Y}, \widetilde{y}_0) \to (X, x_0)$ be a pointed regular $H$-cover. Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be the induced regular $G$-cover of $q\colon (\widetilde{Y}, \widetilde{y}_0) \to (X, x_0)$. Then $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a regular $G$-cover.*

PROOF. The only nontrivial thing to prove is that $p\colon \widetilde{X} \to X$ is a cover. Consider $x \in X$. Let $U \subset X$ be a trivialized neighborhood of $x$ for the cover $q\colon \widetilde{Y} \to X$ and let $\widetilde{U} \subset \widetilde{Y}$ be a sheet lying above $U$. The sheets of $q\colon \widetilde{Y} \to X$ lying above $U$ are then $q^{-1}(U) = \sqcup_{h \in H}(h\widetilde{U})$. We have $\widetilde{U} \subset \widetilde{Y} \subset \widetilde{X}$, and by construction we have $p^{-1}(U) = \sqcup_{g \in G}(g\widetilde{U})$. Each $g\widetilde{U}$ maps homeomorphically to $U$, so $U$ is a trivialized neighborhood of $x$ for $p\colon \widetilde{X} \to X$. □

The following is a sort of converse to Lemma 16.6.3:

LEMMA 16.6.4. *Let $G$ be a group and let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a pointed regular $G$-cover. Assume that $X$ is locally path connected. Let $\widetilde{Y}$ be the path component of $\widetilde{X}$ with $\widetilde{x}_0 \in \widetilde{Y}$, let $\widetilde{y}_0 = \widetilde{x}_0$, and let $H < G$ be the subgroup stabilizing $\widetilde{y}$. Set $q = p|_{\widetilde{Y}}$. Then:*

*(i) The map $q\colon (\widetilde{Y}, \widetilde{y}_0) \to (X, x_0)$ is a regular $H$-cover.*
*(ii) The regular $G$-cover $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is isomorphic to the regular $G$-cover induced from $q\colon (\widetilde{Y}, \widetilde{y}_0) \to (X, x_0)$.*

PROOF. That $q\colon (\widetilde{Y}, \widetilde{y}_0) \to (X, x_0)$ is a pointed cover follows from the fact that $X$ is locally path connected (see Exercise 1.10), and that it is a regular $H$-cover is immediate. This proves conclusion (i). Conclusion (ii) is immediate from Lemma 16.6.2. □

## 16.7. Classification theorem via Galois correspondence

We now prove our classification theorem using the Galois correspondence:

THEOREM 16.5*.1. *Let $(X, x_0)$ be a pointed space and $G$ be a group. Assume that $X$ is path connected, locally path connected, and semilocally 1-connected. There is then a bijection between the following two sets:*

*(i) The set of isomorphism classes of pointed regular $G$-covers $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$.*
*(ii) The set of homomorphisms $f\colon \pi_1(X, x_0) \to G$.*

*This bijection takes a pointed regular $G$-cover to its $G$-lifting map.*

PROOF. We divide the proof into two steps.

STEP 1. *Let $f\colon \pi_1(X, x_0) \to G$ be a homomorphism. Then $f$ is the $G$-lifting map of a pointed regular $G$-cover $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$.*

Let $H = \operatorname{Im}(f)$ and $K = \ker(f)$. By the Galois correspondence (Theorem 11.2.1), there is a pointed connected cover $q\colon (\widetilde{Y}, \widetilde{y}_0) \to (X, x_0)$ whose corrsponding subgroup is $K$. Let $f'\colon \pi_1(X, x_0) \to H$ be the surjection obtained by restricting the codomain of $f$ to $H = \operatorname{Im}(f)$. We will prove below that $q\colon (\widetilde{Y}, \widetilde{y}_0) \to (X, x_0)$ is a regular $H$-cover with lifting map $f'$. The desired regular $G$-cover can then be obtained by inducing $q\colon (\widetilde{Y}, \widetilde{y}_0) \to (X, x_0)$ to $G$ (see Lemma 16.6.3).

It remains to prove that $q\colon (\widetilde{Y}, \widetilde{y}_0) \to (X, x_0)$ is a regular $H$-cover with lifting map $f'$. Since $K$ is a normal subgroup of $\pi_1(X, x_0)$, the cover $q\colon (\widetilde{Y}, \widetilde{y}_0) \to (X, x_0)$ is regular and the lifting map $g\colon \pi_1(X, x_0) \to \operatorname{Deck}(\widetilde{Y})$ is a surjection with $\ker(g) = K$ (Theorem 9.5.1). Since both $f'\colon \pi_1(X, x_0) \to H$ and $g\colon \pi_1(X, x_0) \to \operatorname{Deck}(\widetilde{Y})$ are surjections with kernel $K$, there is an isomorphism $\phi\colon H \to \operatorname{Deck}(\widetilde{Y})$ making the following diagram commute:

$$\begin{tikzcd} & H \arrow[dd, "\phi", "\cong"'] \\ \pi_1(X, x_0) \arrow[ur, "f'"] \arrow[dr, "g"'] & \\ & \operatorname{Deck}(\widetilde{Y}) \end{tikzcd}$$

Using the isomorphism $\phi\colon H \to \mathrm{Deck}(\widetilde{Y})$, we can endow $q\colon (\widetilde{Y}, \widetilde{y}_0) \to (X, x_0)$ with the structure of a regular $H$-cover (cf. Example 16.3.1). By construction, $f'$ is the $H$-lifting map of $q\colon (\widetilde{Y}, \widetilde{y}_0) \to (X, x_0)$.

STEP 2. *Let $f\colon \pi_1(X, x_0) \to G$ be a homomorphism. For $i = 1, 2$, let $p_i\colon (\widetilde{X}_i, \widetilde{x}_i) \to (X, x_0)$ be a regular $G$-cover with lifting map $f$. Then $p_1$ and $p_2$ are isomorphic.*

Using Lemma 16.6.4, we can reduce to the case where the $\widetilde{X}_i$ are path connected and $f$ is surjective. Since the $p_i\colon (\widetilde{X}_i, \widetilde{x}_i) \to (X, x_0)$ are pointed connected covers with corresponding subgroup $K = \ker(f)$, the Galois correspondence (Theorem 11.2.1) implies that they are isomorphic as pointed covers. Let $\widetilde{f}\colon (\widetilde{X}_1, \widetilde{x}_1) \to (\widetilde{X}_2, \widetilde{x}_2)$ be the[4] isomorphism. We will prove that $\widetilde{f}$ is an isomorphism of regular $G$-covers, i.e., that $\widetilde{f}$ is $G$-equivariant. By Theorem 9.5.1, the lifting maps induce isomorphisms

$$\pi_1(X, x_0)/K \xrightarrow{\cong} \mathrm{Deck}(\widetilde{X}_1) \quad \text{and} \quad \pi_1(X, x_0)/K \xrightarrow{\cong} \mathrm{Deck}(\widetilde{X}_2).$$

Using these two isomorphisms, we get an isomorphism $\lambda\colon \mathrm{Deck}(\widetilde{X}_1) \to \mathrm{Deck}(\widetilde{X}_2)$. By its construction, the isomorphism $\widetilde{f}$ is natural with respect to the isomorphism $\lambda$ in the sense that

$$\widetilde{f}(d\widetilde{x}) = \phi(d)\widetilde{f}(\widetilde{x}) \quad \text{for } d \in \mathrm{Deck}(\widetilde{X}_1) \text{ and } \widetilde{x} \in \widetilde{X}_1. \tag{16.7.1}$$

We now use the fact that the $G$-lifting map for both covers is $f\colon \pi_1(X, x_0) \to G$. This gives an identification between $G$ and $\mathrm{Deck}(\widetilde{X}_i)$ for $i = 1, 2$. By construction, $\lambda\colon \mathrm{Deck}(\widetilde{X}_1) \to \mathrm{Deck}(\widetilde{X}_2)$ is simply the composition of the identifications $\mathrm{Deck}(\widetilde{X}_1) = G$ and $G = \mathrm{Deck}(\widetilde{X}_2)$. We therefore see that (16.7.1) implies that

$$\widetilde{f}(g\widetilde{x}) = g\widetilde{f}(\widetilde{x}) \quad \text{for all } g \in G \text{ and } \widetilde{x} \in \widetilde{X}_1,$$

as desired. □

## 16.8. Wedges of spaces and free products

We now recall some terminology from §8.6. For pointed spaces $\{(X_i, x_i)\}_{i \in I}$, the *wedge product* of the $X_i$ is the space $Y = \vee_{i \in I}(X_i, x_i)$ obtained from the disjoint union of the $X_i$ by identifying all the basepoints $x_i$ together to a single point $y_0$. The point $y_0$ serves as a basepoint, so $(Y, y_0)$ is a pointed space. We will often omit explicit mention of the basepoints $x_i$ and just write $Y = \vee_{i \in I} X_i$, and if $I$ is finite we will use the $\vee$ like a sum and e.g. write $X_1 \vee X_2$.

For each $i \in I$, the pointed space $(Y, y_0)$ retracts onto its subspace $(X_i, x_i)$. This retraction takes each $X_j$ with $j \neq i$ to the basepoint $y_0$. It follows that $\pi_1(X_i, x_i)$ is a subgroup of $\pi_1(Y, y_0)$. The main theorem in this section says that if the $X_i$ have reasonable local properties, then $\pi_1(Y, y_0)$ is the free product of its subgroups $\pi_1(X_i, x_i)$:

THEOREM 16.8.1. *Let $\{(X_i, x_i)\}_{i \in I}$ be a collection of pointed spaces and let $y_0$ be the basepoint for $\vee_{i \in I} X_i$. Assume that each $X_i$ is path connected, locally path connected, and semilocally 1-connected. Then $\pi_1(\vee_{i \in I} X_i, y_0) \cong *_{i \in I} \pi_1(X_i, x_i)$.*

PROOF. We verify the universal property of the free product from §16.1. Let $G$ be a group, and for each $i \in I$ let $h_i\colon \pi_1(X_i, x_i) \to G$ be a homomorphism. We must prove that there is a unique homomorphism $H\colon \pi_1(\vee X_i, y_0) \to G$ that restricts to $h_i$ on each subgroup $\pi_1(X_i, x_i)$. For each $i \in I$, Theorem 16.5*.1 implies that there is a unique pointed regular $G$-cover $p_i\colon (\widetilde{X}_i, \widetilde{x}_i) \to (X_i, x_i)$ with $G$-lifting map $h_i$. Define

$$\widetilde{Y} = \bigsqcup_{i \in I} \widetilde{X}_i / \sim,$$

where $\sim$ is the equivalence relation that for $g \in G$ and $i, j \in I$ identifies the points $g\widetilde{x}_i \in \widetilde{X}_i$ and $g\widetilde{x}_j \in \widetilde{X}_j$. The actions of $G$ on each $\widetilde{X}_i$ induce an action of $G$ on $\widetilde{Y}$. Let $\widetilde{y}_0 \in \widetilde{Y}$ be common image of the points $\widetilde{x}_i \in \widetilde{X}_i$. The $p_i$ glue together to give a map $p\colon (\widetilde{Y}, \widetilde{y}_0) \to (\vee_{i \in I} X_i, y_0)$. This map is clearly a regular $G$-cover. The corresponding $G$-lifting map $H\colon \pi_1(\vee X_i, y_0) \to G$ is the desired extension of the $h_i$. Since no choices were made in this construction and the correspondence in Theorem 16.5*.1 between regular $G$-covers and homomorphisms to $G$ is a bijection, this map $H$ is unique. □

[4] The isomorphism is unique since the $\widetilde{X}_i$ are path connected and we require $\widetilde{f}(\widetilde{x}_1) = \widetilde{f}(\widetilde{x}_2)$.

### 16.9. Reduced words in fundamental groups of wedges of spaces

We now use covering spaces to prove Theorem 16.2.1, whose statement we recall:

THEOREM 16.2.1. *Let $\{A_i\}_{i \in I}$ be a collection of groups. Then every element of $*_{i \in I} A_i$ can be represented by a unique reduced word.*

PROOF. By Theorem 15.5.1, for each $i \in I$ we can find a pointed space $(X_i, x_i)$ with $\pi_1(X_i, x_i) = A_i$. Moreover, we can ensure that $X_i$ is path connected, locally path connected, and semilocally 1-connected. Let $Y = \vee_{i \in I} X_i$ and let $y_0$ be the basepoint of $Y$, so $\pi_1(Y, y_0) = *_{i \in i} A_i$. For $i \in I$, let $p_i\colon (\widetilde{X}_i, \widetilde{x}_i) \to (X_i, x_i)$ be the universal cover. Let $\mathcal{R}$ be the set of all reduced words in $*_{i \in I} A_i$. For $w \in \mathcal{R}$, let $\overline{w} \in \pi_1(Y, y_0) = *_{i \in I} A_i$ be the corresponding element. We will construct a pointed cover $p\colon (\widetilde{Y}, \widetilde{y}_0) \to (Y, y_0)$ such that:

(♠) For each $w \in \mathcal{R}$, let $\lambda(w) \in \widetilde{Y}$ be the endpoint of the lift to $\widetilde{Y}$ starting at $\widetilde{y}_0$ of a loop $\gamma$ such that $[\gamma] \in \pi_1(Y, y_0)$ represents $\overline{w}$. Then $\lambda(w_1) \neq \lambda(w_2)$ for all $w_1, w_2 \in \mathcal{R}$ with $w_1 \neq w_2$.

Since every element of $\pi_1(Y, y_0) = *_{i \in I} A_i$ can be represented by some reduced word in $\mathcal{R}$, this will immediately imply the theorem. We remark (♠) will also imply that no nontrivial element of $\pi_1(Y, y_0)$ lifts to a loop in $\widetilde{Y}$ based at $\widetilde{y}_0$, so the subgroup corresponding to $p\colon (\widetilde{Y}, \widetilde{y}_0) \to (Y, y_0)$ is trivial. Since $\widetilde{Y}$ will be path connected, this implies that it is the universal cover of $Y$.

For $i \in I$, recall that $\pi_1(X_i, x_i) = A_i$ and that $p_i\colon (\widetilde{X}_i, \widetilde{x}_i) \to (X_i, x_i)$ is the universal cover of $(X_i, x_i)$. The group $A_i$ acts freely on $\widetilde{X}_i$ by deck transformations, and $p_i^{-1}(x_i) = \{a\widetilde{x}_i \mid a \in A_i\}$. Let $\mathcal{R}_i$ consist of the empty reduced word 1 along with the set of reduced words $a_1 \cdots a_n$ whose final letter $a_n$ does *not* lie in $A_i$. We thus have $\mathcal{R} = \cup_{i \in I} \mathcal{R}_i$ and $\mathcal{R}_i \cap \mathcal{R}_j = \{1\}$ for distinct $i, j \in I$. Define

$$\widetilde{Y} = \left( \bigsqcup_{i \in I} \mathcal{R}_i \times \widetilde{X}_i \right) / \sim,$$

where $\sim$ is the equivalence relation that identifies the following points together:

- For all $i, j \in I$, set $(1, \widetilde{x}_i) \sim (1, \widetilde{x}_i)$. Let $\widetilde{y}[1] \in \widetilde{Y}$ denote this common point.
- For all $i \in I$ and $w \in \mathcal{R}_i$ and $a \in A_i \setminus 1$, we have $wa \in \mathcal{R}_j$ for all $j \in I$ with $j \neq i$. Set $(w, a\widetilde{x}_i) \sim (wa, \widetilde{x}_j)$ for all $j \in I$ with $j \neq i$. Let $\widetilde{y}[wa] \in \widetilde{Y}$ denote this common point.

Recall that $Y = \vee_{i \in I} X_i$ and that $y_0 \in Y$ is the basepoint obtained by identifying the $x_i \in X_1$ together. The maps

$$\mathcal{R}_i \times \widetilde{X}_i \twoheadrightarrow \widetilde{X}_i \xrightarrow{p_i} X_i$$

assemble into a map $p\colon \widetilde{Y} \to Y$. By construction,

$$p^{-1}(y_0) = \{\widetilde{y}[w] \mid w \in \mathcal{R}\}.$$

Set $\widetilde{y}_0 = \widetilde{y}[1]$, so we have a map $p\colon (\widetilde{Y}, \widetilde{y}_0) \to (Y, y_0)$. We will prove that $p\colon (\widetilde{Y}, \widetilde{y}_0) \to (Y, y_0)$ is a pointed covering space satisfying (♠):

CLAIM 1. *The map $p\colon \widetilde{Y} \to Y$ is a pointed covering space.*

The only point of $Y$ for which there might be any issue is the basepoint $y_0$. To find a trivialized open neighborhood of $y_0$, for $i \in I$ let $U_i \subset X_i$ be a trivialized open neighborhood of $x_i$ for $p_i\colon (\widetilde{X}_i, \widetilde{x}_i) \to (X_i, x_i)$. Set $V = \vee_{i \in I} U_i$, so $V$ is an open subset of $Y = \vee_{i \in I} X_i$. We claim that $V$ is a trivialized open set for $p\colon \widetilde{Y} \to Y$.

To see this, for $i \in I$ let $\widetilde{U}_i$ be the sheet of $p_i$ lying above $U$ such that $\widetilde{x}_i \in \widetilde{U}_i$. Since $p_i$ is a regular cover with deck group $A_i$, the sheets of $p_i$ lying above $U_i$ are of the form $a\widetilde{U}_i$ with $a \in A_i$. For $w \in \mathcal{R}_i$ and a subspace $Z \subset \widetilde{X}_i$, let $Z[w]$ be the image of $w \times Z \subset w \times \widetilde{X}_i$ in $\widetilde{Y}$. The sheets of $p$ lying above $V$ are then the following:

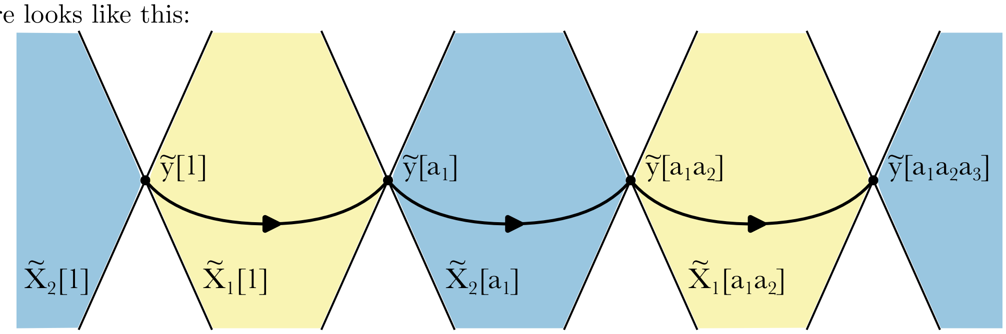
- For $i \in I$, the set $\widetilde{U}_i[1]$ contains the point $\widetilde{y}[1]$. Wedging these sets together at this wedge point, the open set $\vee_{i \in I} \widetilde{U}_i[1]$ is a sheet.

- Consider $w \in \mathcal{R}_i$ for some $i \in I$ and $a \in A_i \setminus 1$. The set $a\widetilde{U}_i[w]$ contains $\widetilde{y}[wa]$, and for $j \in J$ with $j \neq i$ the set $\widetilde{U}_j[wa]$ also contains $\widetilde{y}[wa]$. Wedging these sets together at this wedge point, the open set
$$a\widetilde{U}_i[w] \vee (\bigvee_{\substack{j\in I\\ j\neq i}} \widetilde{U}_j[wa])$$
is a sheet.

CLAIM 2. *The pointed covering space $p\colon (\widetilde{Y}, \widetilde{y}_0) \to (Y, y_0)$ satisfies (♠), which we recall says the following:*

(♠) *For each $w \in \mathcal{R}$, let $\lambda(w) \in \widetilde{Y}$ be the endpoint of the lift to $\widetilde{Y}$ starting at $\widetilde{y}_0$ of a loop $\gamma$ such that $[\gamma] \in \pi_1(Y, y_0)$ represents $\overline{w}$. Then $\lambda(w_1) \neq \lambda(w_2)$ for all $w_1, w_2 \in \mathcal{R}$ with $w_1 \neq w_2$.*

In fact, for $w \in \mathcal{R}$ we have $\lambda(w) = \widetilde{y}[w]$. To see this, write $w = a_1 \cdots a_n$ with $a_j \in A_{i_j} \setminus 1$ for $1 \leq j \leq n$. The lifted path starts at $\widetilde{y}_0 = \widetilde{y}[1]$, goes in $\widetilde{X}_{i_1}[1]$ to $\widetilde{y}[a_1]$, then goes in $\widetilde{X}_{i_2}[a_1]$ to $\widetilde{y}[a_1a_2]$, etc. For instance, if $I = \{1, 2\}$ and $w = a_1a_2a_3$ with $a_1 \in A_1 \setminus 1$ and $a_2 \in A_2 \setminus 1$ and $a_3 \in A_1 \setminus 1$ the picture looks like this:




At the final step, the lifted path goes in $\widetilde{X}_{i_n}[a_1 \cdots a_{n-1}]$ from $\widetilde{y}[a_1 \cdots a_{n-1}]$ to $\widetilde{y}[a_1 \cdots a_n] = \widetilde{y}[w]$, as desired. □

## 16.10. Exercises

EXERCISE 16.1. Do the following:

(a) Let $S$ and $T$ be sets. Prove that $F(S) * F(T) = F(S \sqcup T)$.
(b) If $S = \{a_1, \ldots, a_n\}$, then prove that $F(S)$ is isomorphic to the free product of $n$ copies of $\mathbb{Z}$. □

EXERCISE 16.2. Let $G$ be a group and let $H < G$ be a subgroup. For an $H$-space $Z$, regard $G$ as a discrete space and endow the set $\mathcal{C}(G, Z)$ of continuous functions $\phi\colon G \to Z$ with the compact open topology. We then define
$$\operatorname{Coind}_H^G Z = \{\phi\colon G \to Z \mid \phi(h\gamma) = h\phi(\gamma) \text{ for all } h \in H \text{ and } \gamma \in G\} \subset \mathcal{C}(G, Z).$$
As you will prove below, this is a $G$-space called the *coinduced $G$-space* of the $H$-space $Z$. Prove the following:

(a) Let $Z$ be a $G$-space. For $g \in G$ and a map $\phi\colon G \to Z$ in $\operatorname{Coind}_H^G Z$, define $g\phi\colon G \to Z$ to be the map
$$g\phi(\gamma) = \phi(\gamma g^{-1}) \quad \text{for all } \gamma \in G.$$
Prove that $g\phi \in \operatorname{Coind}_H^G Z$ and that this defines an action of $G$ on $\operatorname{Coind}_H^G Z$ by continuous maps. In other words, this makes $\operatorname{Coind}_H^G Z$ into a $G$-space.
(b) Let $Z$ be an $H$-space and let $X$ be a $G$-space. Prove that there is a bijection between:
   - $H$-equivariant maps $f\colon \operatorname{Res}_H^G X \to Z$; and
   - $G$-equivariant maps $F\colon X \to \operatorname{Coind}_H^G Z$.

   This bijection takes a $G$-equivariant map $F\colon X \to \operatorname{Coind}_H^G Z$ to the map $f\colon \operatorname{Res}_G^H X \to Z$ defined by $f(x) = F(x)(1)$.

(c) Prove that as a space we have $\mathrm{Coind}_H^G Z \cong \prod_{G/H} Z$. Hint: first prove that if $Z$ is any space and $D$ is a discrete space, then $\mathcal{C}(D, Z) \cong \prod_{d \in D} Z$. $\square$

EXERCISE 16.3. Recall that $C_n$ is the cyclic group of order $n$. In this exercise, you will determine all subgroups of $C_2 * C_2$.

(a) Let $x_0 \in \mathbb{RP}^2 \vee \mathbb{RP}^2$ be the basepoint, so $\pi_1(\mathbb{RP}^2 \vee \mathbb{RP}^2, x_0) \cong C_2 * C_2$. Enumerate all connected pointed covers of $(\mathbb{RP}^2 \vee \mathbb{RP}^2, x_0)$.
(b) As notation, let $a \in C_2 * C_2$ be the generator of the first factor and $b \in C_2 * C_2$ be the generator of the second factor, so $C_2 * C_2 = \langle a, b \mid a^2 = 1, b^2 = 1 \rangle$. For each cover $p\colon (\widetilde{X}, \widetilde{x}_0) \to (\mathbb{RP}^2 \vee \mathbb{RP}^2, x_0)$ from part (a), determine the following:
   - Generators for the corresponding subgroup of $\pi_1(\mathbb{RP}^2 \vee \mathbb{RP}^2, x_0) \cong C_2 * C_2$.
   - The isomorphism type of the group $\pi_1(\widetilde{X}, \widetilde{x}_0)$. $\square$

EXERCISE 16.4. Let $G$ be a finite group. In this exercise, you will give construct a space $X$ whose fundamental group is $G$ and whose universal cover is a product of finitely many copies of $\mathbb{S}^3$. Do the following:

(a) For $g \in G$, let $\langle g \rangle$ be the corresponding cyclic subgroup. Construct a free action of $\langle g \rangle$ on $\mathbb{S}^3$. Let $\mathbb{S}^3_g$ be the corresponding $\langle g \rangle$-space.
(b) For each $g \in G$, by Exercise 16.2 we have a $G$-space $\mathrm{Coind}_{\langle g \rangle}^G \mathbb{S}^3_g$ which is homeomorphic to a product of finitely many copies of $\mathbb{S}^3$. Define

$$\widetilde{X} = \prod_{g \in G} \mathrm{Coind}_{\langle g \rangle}^G \mathbb{S}^3_g.$$

This is a product of $G$-spaces, and thus is a $G$-space. Prove that $G$ acts freely on $\widetilde{X}$.

This shows that $G$ acts freely on a product $\widetilde{X}$ of finitely many copies of $\mathbb{S}^3$. Since $G$ is finite this is a covering space action. Moreover, since $\mathbb{S}^3$ is 1-connected the space $\widetilde{X}$ is 1-connected, so letting $X = \widetilde{X}/G$ and letting $x_0 \in X$ be a basepoint we have $\pi_1(X, x_0) \cong G$. $\square$

EXERCISE 16.5. Recall that the *center* of a group $G$ is the subgroup

$$Z(G) = \{c \in G \mid cg = gc \text{ for all } g \in G\}.$$

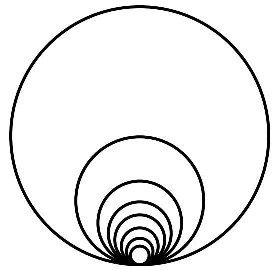
Let $A$ and $B$ be nontrivial groups. Prove that $Z(A * B) = 1$. Hint: use the fact that elements of $A * B$ can be represented by unique reduced words. $\square$

EXERCISE 16.6. Let $A$ and $B$ be groups. Let $v \in A * B$ be an element of finite order, so $v^d = 1$ for some $d \geq 1$. Prove that $v$ is conjugate to an element of either $A$ or $B$. Hint: use the fact that elements of $A * B$ can be represented by unique reduced words. $\square$

EXERCISE 16.7. Let $\{A_i\}_{i \in I}$ be a collection of groups. This exercise outlines the classical algebraic proof that every element of $\Gamma = *_{i \in I} A_i$ is represented by a unique reduced word.

(a) Let $\mathcal{W}$ be the set of reduced words. Use the universal property of the free product to construct a left action of $\Gamma$ on $\mathcal{W}$ that for $a \in A_i$ multiplies a word $w \in \mathcal{W}$ by $a$ to get $aw$ and then reduces appropriately to get a reduced word. Hint: an action is the same as a homomorphism $\Gamma \to \mathrm{Sym}(\mathcal{W})$ where $\mathrm{Sym}(\mathcal{W})$ is the symmetric group consisting of all bijections of $\mathcal{W}$. You can construct such homomorphisms using the universal property of $\Gamma = *_{i \in I} A_i$.
(b) Prove that every element of $\Gamma$ is represented by a unique reduced word. Hint: consider $w \in \Gamma$, and think about where $w$ takes the trivial word $1 \in \mathcal{W}$. $\square$

EXERCISE 16.8. This exercise uses the notion of the cone of a space from Exercise 6.2. For $n \geq 1$, let $C_n \subset \mathbb{R}^2$ be the circle of radius $1/n$ with center $(0, 1/n)$. Let $X = \cup_{n=1}^{\infty} C_n$, topologized as a subspace of $\mathbb{R}^2$. This is sometimes called the "earring space" or the "shrinking wedge of circles":



Let $x_0 \in X$ be the point $(0,0) \in X$. By Exercise 6.2, the space $\mathrm{Cone}(X)$ is contractible. The contractible space $\mathrm{Cone}(X)$ contains the point $x_0$, which we emphasize is *not* the cone point. Let $(Y_1, y_1)$ and $(Y_2, y_2)$ be two copies of the pointed space $(\mathrm{Cone}(X), x_0)$ and let $z_0$ be the basepoint of $Y_1 \vee Y_2$. Prove that $\pi_1(Y_1 \vee Y_2, z_0)$ is nontrivial even though both $Y_1$ and $Y_2$ are contractible. We remark that this depends on the fact that the basepoint of $\mathrm{Cone}(X)$ is $x_0$, and would be false if we used the cone point as the basepoint. $\square$

CHAPTER 17

# The Seifert–van Kampen theorem

We finally come to the Seifert–van Kampen theorem. For a space $X$ decomposed into open sets as $X = U \cup V$, this theorem shows how to write the fundamental group of $X$ in terms of the fundamental groups of $U$ and $V$ and $U \cap V$.

## 17.1. Naturality of regular $G$-covers

We first briefly describe how pointed regular $G$-covers restrict to subspaces. Let $(X, x_0)$ be a pointed space, let $G$ be a group, and let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a pointed regular $G$-cover. For a subspace $X' \subset X$ with $x_0 \in X'$, let $\widetilde{X}' = p^{-1}(X')$ and let $p' = p|_{\widetilde{X}'}$, so $\widetilde{x}_0 \in \widetilde{X}'$. The map $p'\colon (\widetilde{X}', \widetilde{x}_0) \to (X', x_0)$ is a cover called the restriction of $p$ to $X'$ (see §15.1). The $G$-action on $\widetilde{X}$ restricts to a $G$-action on $\widetilde{X}'$, so $p'\colon (\widetilde{X}', \widetilde{x}_0) \to (X', x_0)$ is a regular $G$-cover.

Associated to $p$ and $p'$ are $G$-lifting maps $f\colon \pi_1(X, x_0) \to G$ and $f'\colon \pi_1(X', x_0) \to G$. If $X$ and $X'$ are both path connected, locally path connected, and semilocally 1-connected the maps $f$ and $f'$ characterize $p$ and $p'$ up to isomorphism (see Theorem 16.5*.1). They are related as follows:

Lemma 17.1.1. *Let $(X, x_0)$ be a pointed space, let $G$ be a group, and let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a pointed regular $G$-cover. Let $X' \subset X$ be a subspace with $x_0 \in X'$ and let $p'\colon (\widetilde{X}', \widetilde{x}_0) \to (X, x_0)$ be the restriction of $p$ to $X'$. Let $f\colon \pi_1(X, x_0) \to G$ and $f'\colon \pi_1(X', x_0) \to G$ be the $G$-lifting maps of $p$ and $p'$. Finally, let $\iota\colon (X', x_0) \to (X, x_0)$ be the inclusion, so we have an induced map $\iota_*\colon \pi_1(X', x_0) \to (X, x_0)$. Then $f' = f \circ \iota_*$.*

Proof. Immediate from the definition of the $G$-lifting map. ☐

## 17.2. Pushouts of groups

The Seifert–van Kampen theorem expresses the fundamental group of a space as a pushout of groups, which we define here. This requires a certain amount of notation to set up. Let $X$ and $Y$ and $Z$ be groups and $f_1\colon Z \to X$ and $f_2\colon Z \to Y$ be homomorphisms. Denote this data by the pair $[f_1, f_2]$. For a group $Q$, a *map* $h\colon [f_1, f_2] \to Q$ is a pair of homomorphisms $h_1\colon X \to Q$ and $h_2\colon Y \to Q$ such that the diagram

$$\begin{tikzcd} Z \arrow[r, "f_1"] \arrow[d, "f_2"'] & X \arrow[d, "h_1"] \\ Y \arrow[r, "h_2"] & Q \end{tikzcd}$$

commutes, i.e., such that $h_1 \circ f_1 = h_2 \circ f_2$. A *pushout* of $[f_1, f_2]$ is a group $P$ and a map $g\colon [f_1, f_2] \to P$ such that if $h\colon [f_1, f_2] \to Q$ is another map, then there exists a unique homomorphism $\phi\colon P \to Q$ such that the following diagram commutes:

$$\begin{tikzcd} Z \arrow[r, "f_1"] \arrow[d, "f_2"'] & X \arrow[d, "g_1"'] \arrow[ddr, bend left, "h_1"] & \\ Y \arrow[r, "g_2"] \arrow[drr, bend right, "h_2"'] & P \arrow[dr, "\phi"] & \\ & & Q \end{tikzcd}$$

In this case, we will say that

$$\begin{tikzcd} Z \arrow[r, "f_1"] \arrow[d, "f_2"'] & X \arrow[d, "g_1"] \\ Y \arrow[r, "g_2"] & P \end{tikzcd}$$

is a *pushout*. The same argument we used for free group and free products shows that pushouts are unique up to isomorphism if they exist (see Exercise 17.1). In fact, they always exist and can be easily calculated from a presentation:

LEMMA 17.2.1. *Let $X$ and $Y$ and $Z$ be groups and $f_1\colon Z \to X$ and $f_2\colon Z \to Y$ be homomorphisms. Let $X = \langle S_1 \mid R_1\rangle$ and $Y = \langle S_1 \mid R_2\rangle$ be presentations and let $T$ be a generating set for $Z$. For $t \in T$, choose $u_t \in F(S_1)$ and $v_t \in F(S_2)$ such that $f_1(t) = \overline{u}_1 \in X$ and $f_2(t) = \overline{v}_t \in Y$. Define*

$$P = \langle S_1 \sqcup S_2 \mid R_1 \sqcup R_2 \sqcup \{u_t = v_t \mid t \in T\}\rangle .$$

*Then $P$ is isomorphic to a pushout of $[f_1, f_2]$.*

PROOF. Let $g_1\colon X \to P$ and $g_2\colon Y \to P$ be the evident homomorphisms. Let $Q$ be a group and let $h\colon [f_1, f_2] \to Q$ be a map. We must show that there is a unique homomorphism $\phi\colon P \to Q$ such that the following diagram commutes:

$$\begin{tikzcd} Z \arrow[r, "f_1"] \arrow[d, "f_2"'] & X \arrow[d, "g_1"] \arrow[ddr, bend left, "h_1"] & \\ Y \arrow[r, "g_2"] \arrow[drr, bend right, "h_2"'] & P \arrow[dr, "\phi"] & \\ & & Q \end{tikzcd}$$

For $s \in S_1 \sqcup S_2$, define

$$\phi(s) = \begin{cases} h_1(\overline{s}) & \text{if } s \in S_1, \\ h_2(\overline{s}) & \text{if } s \in S_1. \end{cases}$$

We must check that this takes the relations in $R_1 \sqcup R_2 \sqcup \{u_t = v_t \mid t \in T\}$ to relations in $Q$. The map $\phi$ takes the relations in $R_1$ to relations since $h_1$ does, and it take the relations in $R_2$ to relations since $h_2$ does. As for the others, consider $t \in T$. We must check that $\phi(u_t) = \phi(v_t)$. Since $h\colon [f_1, f_2] \to Q$ is a map, we have $\phi(u_t) = h_1(\overline{u}_t) = f_1 \circ h_1(t) = f_2 \circ h_2(t) = h_2(\overline{v}_t) = \phi(v_t)$. □

To help the reader understand what this means, here are two examples:

EXAMPLE 17.2.2. Let $A$ and $B$ be groups. We then have a pushout

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & A * B. \end{tikzcd}$$

This can be proved directly from the universal property. It is also immediate from Lemma 17.2.1. □

EXAMPLE 17.2.3. Let $f\colon A \to B$ be a homomorphism of groups. Letting $\langle\!\langle f(A)\rangle\!\rangle$ be the normal subgroup of $B$ geneated by $f(A)$, we have a pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ 1 \arrow[r] & B/\langle\!\langle f(A)\rangle\!\rangle. \end{tikzcd}$$

Again, this can be proved directly from the universal property or by using Lemma 17.2.1. □

## 17.3. The Seifert–van Kampen theorem

The following is our main theorem. In it, the maps $f_i$ and $g_i$ are induced by subspace inclusions.

THEOREM 17.3.1 (Seifert–van Kampen). *Let $(X, x_0)$ be a pointed space. Let $U, V \subset X$ be open sets such that $X = U \cup V$ and $x_0 \in U \cap V$. Assume that $U$ and $V$ and $X \cap V$ are path connected. We then have a pushout*

$$\begin{tikzcd} \pi_1(U \cap V, x_0) \arrow[r, "f_1"] \arrow[d, "f_2"'] & \pi_1(U, x_0) \arrow[d, "g_1"] \\ \pi_1(V, x_0) \arrow[r, "g_2"] & \pi_1(X, x_0) \end{tikzcd}$$

We will prove this under the assumption that the spaces $\{X, U, V, U \cap V\}$ are all path connected, locally path connected, and semilocally 1-connected, which we will call the "tame" case. This suffices for most applications and has a simple proof due to Grothendieck. See Chapter 19* for the general case.[1]

PROOF OF TAME CASE OF THEOREM 17.3.1. Let $Q$ be a group and let $h\colon [f_1, f_2] \to Q$ be a map. We must prove that there is a unique homomorphism $\phi\colon \pi_1(X, x_0) \to Q$ such that the following diagram commutes:

$$\begin{tikzcd} \pi_1(U \cap V, x_0) \arrow[r, "f_1"] \arrow[d, "f_2"'] & \pi_1(U, x_0) \arrow[d, "g_1"] \arrow[ddr, bend left, "h_1"] & \\ \pi_1(V, x_0) \arrow[r, "g_2"] \arrow[drr, bend right, "h_2"'] & \pi_1(X, x_0) \arrow[dr, "\phi"] & \\ & & Q \end{tikzcd} \tag{17.3.1}$$

By Theorem 16.5*.1, we can find the following:

- a pointed regular $G$-cover $p_1\colon (\widetilde{U}, \widetilde{x}_u) \to (U, x_0)$ with $G$-lifting map $h_1\colon \pi_1(U, x_0) \to G$; and
- a pointed regular $G$-cover $p_2\colon (\widetilde{V}, \widetilde{x}_v) \to (V, x_0)$ with $G$-lifting map $h_2\colon \pi_1(V, x_0) \to G$.

Let $q_1\colon (p_1^{-1}(U \cap V), \widetilde{x}_u) \to (U \cap V, x_0)$ and $q_2\colon (p_2^{-1}(U \cap V), \widetilde{x}_v) \to (U \cap V, x_0)$ be the restrictions of $p_1$ and $p_2$ to $U \cap V$, respectively. By Lemma 17.1.1, the $G$-lifting map of $q_1$ is $h_1 \circ f_1$ and the $G$-lifting map of $q_2$ is $h_2 \circ f_2$. Since $h_1 \circ f_1 = h_2 \circ f_2$, these $G$-lifting maps are the same. Theorem 16.5*.1 therefore implies that $q_1$ and $q_2$ are isomorphic pointed regular $G$-covers.

Let $\lambda\colon (p_1^{-1}(U \cap V), \widetilde{x}_u) \to (p_2^{-1}(U \cap V), \widetilde{x}_v)$ be an isomorphism between these covers, so $\lambda$ is a $G$-equivariant homeomorphism such that the following diagram commutes:

$$\begin{tikzcd} (p_1^{-1}(U \cap V), \widetilde{x}_u) \arrow[rr, "\lambda"] \arrow[dr, "q_1"] & & (p_2^{-1}(U \cap V), \widetilde{x}_v) \arrow[dl, "q_2"] \\ & (U \cap V, x_0) & \end{tikzcd}$$

We use $\lambda$ to glue $p_1$ and $p_2$ together into a pointed regular $G$-cover $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ as follows:

- Set $\widetilde{X} = \widetilde{U} \sqcup \widetilde{V} / \sim$, where $\sim$ is the equivalence relation that for $\widetilde{u} \in p_1^{-1}(U \cap V)$ identifies $\widetilde{u} \sim \lambda(\widetilde{u}) \in p_2^{-1}(U \cap V)$. Let $\widetilde{x}_0 \in \widetilde{X}$ be the image of $\widetilde{u}_0 \sim \widetilde{v}_0$.
- Since $X = U \cup V$, the maps $p_1$ and $p_2$ glue together to give a map $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$. The map $p$ is easily seen to be a pointed cover (see Exercise 17.3).
- Since $\lambda$ is $G$-equivariant, the $G$-actions on $\widetilde{U}$ and $\widetilde{V}$ glue together to give a $G$-action on $\widetilde{X}$ by deck transformations. This makes $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ into a regular $G$-bundle.

Let $\phi\colon \pi_1(X, x_0) \to G$ be the $G$-lifting map of $p$. By the naturality of the $G$-lifting map (cf. Lemma 17.1.1), the diagram (17.3.1) commutes. Since the correspondence between isomorphism classes of

[1] Note that we will quote the general case in our applications to get the most general possible statements.

pointed regular $G$-bundles and maps from the fundamental group to $G$ in Theorem 16.5*.1 is a bijection, we made no arbitrary choices and $\phi$ is unique. The theorem follows. $\square$

REMARK 17.3.2. There is also a version of the Seifert–van Kampen theorem for arbitrary open covers $X = \cup_{i\in I} U_i$. In this more general version, we need to assume each $U_i$ is path connected, each $U_i \cap U_j$ is path connected, and each $U_i \cap U_j \cap U_k$ is path connected. The version we gave above suffices for almost all applications. We describe the more general version in Chapter 19*. $\square$

## 17.4. Wedges, redux

In many simple examples of the Seifert–van Kampen theorem, at least one of the groups in the pushout is trivial. Here is an example of this. In Theorem 16.8.1, we proved the following:

- Let $\{(X_i, x_i)\}_{i\in I}$ be a collection of pointed spaces and let $y_0$ be the basepoint for $\vee_{i\in I} X_i$. Assume that each $X_i$ is path connected, locally path connected, and semilocally 1-connected. Then $\pi_1(\vee_{i\in I} X_i, y_0) \cong *_{i\in I} \pi_1(X_i, x_i)$.

Using the Seifert–van Kampen theorem, we can prove a result like this under slighly different assumptions. For each $i \in I$, instead of assuming that $X_i$ is path connected, locally path connected, and semilocally 1-connected, what we will need is that for all $i \in I$, there exists an open neighborhood $N_i$ of the basepoint $x_i \in X_i$ such that $N_i$ deformation retracts onto $x_i$. Neither hypotheses implies the other, so this is neither stronger nor weaker than Theorem 16.8.1.

Here we explain how to prove this for wedges of finitely many spaces. For wedges of infinitely many spaces, what is needed is the Seifert–van Kampen theorem for covers with more than two open sets, which we describe in Chapter 19*.

THEOREM 17.4.1. *Let $\{(X_i, x_i)\}_{i\in I}$ be a finite collection of path connected pointed spaces and let $y_0$ be the basepoint for $\vee_{i\in I} X_i$. For each $i \in I$, assume that there exists an open neighborhood $N_i \subset X_i$ of $x_i$ such that $N_i$ deformation retracts onto $x_i$. Then $\pi_1(\vee_{i\in I} X_i, y_0) \cong *_{i\in I} \pi_1(X_i, x_i)$.*

PROOF. Identifying all the $X_i$ with subspaces of $\vee_{i\in I} X_i$ identifies $x_i$ with $y_0$ for all $i \in I$. By induction, it is enough to handle the wedge of two spaces. Assume, therefore, that $I = \{1, 2\}$. We must prove that $\pi_1(X_1 \vee X_2, y_0) \cong \pi_1(X_1, y_0) * \pi_1(X_2, y_0)$.

Let $U = X_1 \vee N_2$ and $V = N_1 \vee X_2$, so $X_1 \vee X_2 = U \cup V$ and $U \cap V = N_1 \vee N_2$. Since $N_i$ deformation retracts to $x_i$ for $i = 1, 2$, it follows that $U$ deformation retracts to $X_1$, that $V$ deformation retracts to $X_2$, and that $U \cap V$ deformation retracts to the basepoint $y_0$. It follows that $U$ and $V$ and $U \cap V$ are all path connected, so Theorem 17.3.1 (Seifert–van Kampen) applies. Applying our homotopy equivalences, we get a pushout

$$\begin{array}{ccc} \pi_1(U\cap V, y_0) & \longrightarrow & \pi_1(U, y_0) \\ \downarrow & & \downarrow \\ \pi_1(V, y_0) & \longrightarrow & \pi_1(X_1 \vee X_2, y_0) \end{array} \quad = \quad \begin{array}{ccc} 1 & \longrightarrow & \pi_1(X_1, y_0) \\ \downarrow & & \downarrow \\ \pi_1(X_2, y_0) & \longrightarrow & \pi_1(X_1 \vee X_2, y_0). \end{array}$$

As we described in Example 17.2.2, this implies that $\pi_1(X_1 \vee X_2, y_0) \cong \pi_1(X_1, y_0) * \pi_1(X_2, y_0)$. $\square$

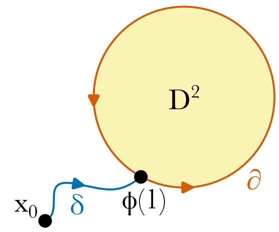
REMARK 17.4.2. It is tempting to try to prove Theorem 17.4.1 using the cover $X_1 \vee X_2 = X_1 \cup X_2$, where $X_1 \cap X_2$ is the basepoint. The problem is that this is not an open cover. As in the above proof, it is often useful to "thicken" a non-open cover to an open cover. However, care must be taken to ensure that the relevant fundamental groups are not changed. $\square$

## 17.5. Attaching 2-cells, redux

Let $(X, x_0)$ be a path connected pointed space. Let $D^2 \cong \mathbb{D}^2$ be a 2-cell, let $\phi\colon \partial D^2 \to X$ be an attaching map, and let $Z$ be the result of attaching $D^2$ to $X$ via $\phi$. Recall that a *loop around* $D^2$ in $(X, x_0)$ is a loop of the form $r = \delta \cdot \partial \cdot \overline{\delta}$, where:

- The path $\delta$ is a path in $X$ from $x_0$ to $\phi_i(1)$. Here are are identifying $\partial D_i^2$ with $\mathbb{S}^1 \subset \mathbb{C}$.
- The path $\gamma\colon I \to X$ is the loop in $X$ based at $\phi(1)$ defined by the formula $\gamma(s) = \phi_i(e^{2\pi i s})$.

See here:




In Theorem 15.3.2, we proved the following result:

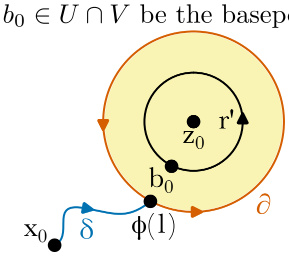
- Let $(X, x_0)$ be path connected, locally path connected, and semilocally 1-connected pointed space. Let $Z$ be the result of attaching a collection $\{D_i^2\}_{i \in I}$ of 2-cells to $X$ via attaching maps $\{\phi_i : \partial D_i^2 \to X\}$. For each $i \in I$, let $r_i = \delta_i \cdot \partial_i \cdot \overline{\delta}_i$ be a loop around $D_i^2$ in $(X, x_0)$. Then $\pi_1(Z, x_0)$ is the quotient of $\pi_1(X, x_0)$ by the normal subgroup generated by $\{[r_i] \mid i \in I\}$.

Using the Seifert–van Kampen theorem, we can prove this without the assumption that $X$ is locally path connected and semilocally 1-connected. Here we will do this in the case where $Z$ is obtained by attaching finitely many 2-cells. The general case not only requires the Seifert–van Kampen theorem for covers by more than one open set, but also a new idea to get around some basepoint issues. We describe it in Chapter 19*.

Theorem 17.5.1. *Let $(X, x_0)$ be a path connected pointed space. Let $Z$ be the result of attaching a finite collection $\{D_i^2\}_{i \in I}$ of 2-cells to $X$ via attaching maps $\{\phi_i : \partial D_i^2 \to X\}$. For each $i \in I$, let $r_i = \delta_i \cdot \partial_i \cdot \overline{\delta}_i$ be a loop around $D_i^2$ in $(X, x_0)$. Then $\pi_1(Z, x_0)$ is the quotient of $\pi_1(X, x_0)$ by the normal subgroup generated by $\{[r_i] \mid i \in I\}$.*

Proof. By induction, it is enough to handle the case where $Z$ is obtained by attaching a single 2-cell $D^2$ via an attaching map $\phi : \partial D^2 \to X$. Let $r = \delta \cdot \partial \cdot \overline{\delta}$ be a loop around $D_i^2$ in $(X, x_0)$. Our goal is to prove that $\pi_1(Z, x_0)$ is the quotient of $\pi_1(X, x_0)$ by the normal subgroup generated by $[r] \in \pi_1(X, x_0)$.

Let $z_0 \in Z$ be the image of the central point $0 \in D^2$. Let $U = Z \setminus z_0$ and let $V \subset Z$ be the image of $\mathrm{Int}(D^2)$, so $V \cong \mathrm{Int}(\mathbb{D}^2)$ and $U \cap V \cong \mathrm{Int}(\mathbb{D}^2) \setminus 0$. Since $D^2 \setminus 0$ deformation retracts to $\partial D^2$, the open set $U$ deformation retracts to $X$. It follows that $\pi_1(U, x_0) = \pi_1(X, x_0)$. We would like to apply the Seifert–van Kampen theorem to the open cover $X = U \cup V$. The only issue is that the basepoint $x_0$ does not lie in $V$. To fix this, let $b_0 \in U \cap V$ be the basepoint and $r'$ be the loop shown here:




If we change the basepoint from $x_0$ to $b_0$, the element $[r] \in \pi_1(X, x_0)$ goes to an element that is conjugate to $[r'] \in \pi_1(X, b_0)$. It follows that it is enough to prove that $\pi_1(Z, b_0)$ is isomorphic to the quotient of $\pi_1(X, x_0)$ by the normal closure of $[r'] \in \pi_1(X, x_0)$.

We can now apply the Seifert–van Kampen theorem. Just like above, we have $\pi_1(U, b_0) \cong \pi_1(X, b_0)$. Also, since $V \cong \mathrm{Int}(\mathbb{D}^2)$ and $U \cap V \cong \mathrm{Int}(\mathbb{D}^2) \setminus 0$, we have $\pi_1(V, b_0) = 1$ and $\pi_1(U \cap V, b_0) \cong \mathbb{Z}$. Theorem 17.3.1 (Seifert–van Kampen) therefore gives us a pushout

$$\begin{array}{ccccccc}
\pi_1(U \cap V, b_0) & \xrightarrow{\phi} & \pi_1(U, b_0) & & \mathbb{Z} & \xrightarrow{\phi} & \pi_1(X, b_0) \\
\downarrow & & \downarrow & = & \downarrow & & \downarrow \\
\pi_1(V, b_0) & \longrightarrow & \pi_1(X, b_0) & & 1 & \longrightarrow & \pi_1(X, b_0).
\end{array}$$

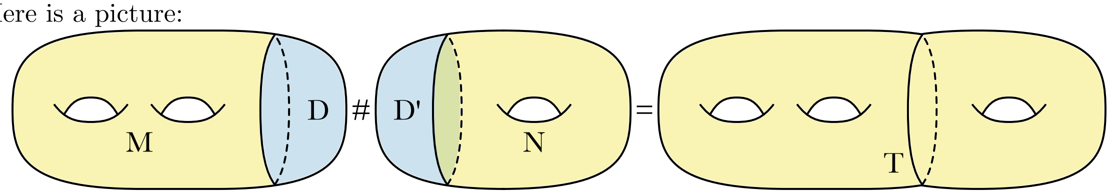
As in Example 17.2.3, we deduce that $\pi_1(X, b_0)$ is isomorphic to the quotient of $\pi_1(X, b_0)$ by the normal closure of the image of $\phi$. Since $[r'] \in \pi_1(U \cap V, b_0)$ is a generator, the image of $\phi$ is generated by $[r'] \in \pi_1(X, b_0)$. The theorem follows. $\square$

## 17.6. Connect sums

Our final two examples of the Seifert–van Kampen theorem require a definition from manifold topology. Let $M$ and $N$ be connected $n$-dimensional manifolds, possibly with boundary. A *connect sum* of $M$ and $N$ is an $n$-manifold $M\#N$ obtained as follows:

- Let $D \subset M$ and $D' \subset N$ be embedded $n$-dimensional disks, so $D \cong D' \cong \mathbb{D}^n$.
- Let $M_0 = M \setminus \mathrm{Int}(D)$ and $N_0 = N \setminus \mathrm{Int}(D')$, so $M_0$ and $N_0$ are $n$-manifolds with boundary. Let $S = \partial D$ and $S' = \partial D'$, so $S \subset M_0$ and $S' \subset N_0$ are boundary components with $S \cong S' \cong \mathbb{S}^{n-1}$.
- Choose a homeomorphism $\phi\colon S \to S'$, and define $M\#N = M_0 \sqcup N_0/\sim$, where $\sim$ is the equivalence relation that for $x \in S$ identifies $x \sim \phi(x) \in S'$.

Here is a picture:




As in the above figure, let $T \cong \mathbb{S}^{n-1}$ be the common image of $S$ and $S'$ in $M\#N$. If $M$ and $N$ are oriented, then orienting $S$ and $S'$ in the usual way one usually requires $\phi\colon S \to S'$ to be orientation-reversing. This makes $M\#N$ into an orientable manifold with $M_0$ on one side of $T$ and $N_0$ on the other.

Remark 17.6.1. With the above conventions, it is a difficult theorem that the homeomorphism type of $M\#N$ is independent of our choices if $M$ and $N$ are orientable. The non-orientable case is more subtle. However, for our purposes this will not be important. □

## 17.7. Fundamental groups of connect sums, higher dimensions

For manifolds of dimension $n \geq 3$, the fundamental group of a connect sum has a simple form:

Theorem 17.7.1. *Let $M$ and $N$ be connected manifolds of dimension $n \geq 3$, possibly with boundary, and let $M\#N$ be a connect sum of $M$ and $N$. Let $x_0 \in M$ and $y_0 \in N$ and $z_0 \in M\#N$ be basepoints. We then have $\pi_1(M\#N, z_0) \cong \pi_1(M, x_0) * \pi_1(N, y_0)$.*

Proof. As in the definition of a connect sum, let $D \subset M$ and $D' \subset N$ be embedded $n$-dimensional disks. Set $M_0 = M \setminus \mathrm{Int}(D)$ and $N_0 = N \setminus \mathrm{Ind}(D')$. Both $M_0$ and $N_0$ are subspaces of $M\#N$, and $M_0 \cap N_0 = T \cong \mathbb{S}^{n-1}$. Changing all of our basepoints, we can assume that $x_0 = y_0 = z_0 \in T$. Since boundary components of manifolds have collar neighborhoods, there is an open neighborhood $W$ of $T$ in $M\#N$ along with a homeomorphism $\lambda\colon T \times (-1,1) \to W$ such that:

$$T = \lambda(T \times 0) \quad \text{and} \quad \lambda(T \times (-1,0]) \subset M_0 \quad \text{and} \quad \lambda(T \times [0,1)) \subset N_0.$$

Set $U = M_0 \cup W$ and $V = N_0 \cup W$, so $M\#N = U \cup V$. Since $T \times (-1,1)$ deformation retracts to $T \times (-1,0]$, the open set $U$ deformation retracts to $M_0$. Similarly, $V$ deformation retracts to $N_0$ and $U \cap V = W$ deformation retracts to $T \cong \mathbb{S}^{n-1}$. All of these are path connected, so Theorem 17.3.1 (Seifert–van Kampen) gives a pushout

$$\begin{tikzcd}
\pi_1(U \cap V, z_0) \arrow[r] \arrow[d] & \pi_1(U, z_0) \arrow[d] \\
\pi_1(V, z_0) \arrow[r] & \pi_1(M\#N, z_0)
\end{tikzcd}
=
\begin{tikzcd}
1 \arrow[r] \arrow[d] & \pi_1(M_0, z_0) \arrow[d] \\
\pi_1(N_0, z_0) \arrow[r] & \pi_1(M\#N, z_0).
\end{tikzcd}$$

Here we are using the fact that $\pi_1(U \cap V, z_0) = \pi_1(T, z_0) = 1$, which holds since $T \cong \mathbb{S}^{n-1}$ and $n \geq 3$. As in Example 17.2.2, we deduce that

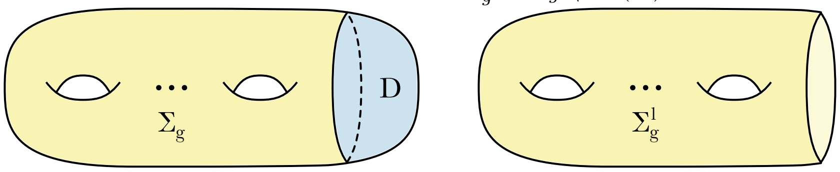
$$\pi_1(M\#N, z_0) \cong \pi_1(M_0, z_0) * \pi_1(N_0, z_0). \tag{17.7.1}$$

The manifold $M$ can be obtained from $M_0$ by attaching an $n$-cell. Since $n \geq 3$, attaching an $n$-cell does not change the fundamental group (see Theorem 15.6.2), so $\pi_1(M_0, z_0) \cong \pi_1(M, z_0)$. Similarly,

$\pi_1(N_0, z_0) \cong \pi_1(N, z_0)$. Recalling that $x_0 = y_0 = z_0$, we can plug these two facts into (17.7.1) and conclude that $\pi_1(M\#N, z_0) \cong \pi_1(M, x_0) * \pi_1(N, y_0)$, as desired. □

## 17.8. Fundamental groups of connect sums, dimension 2

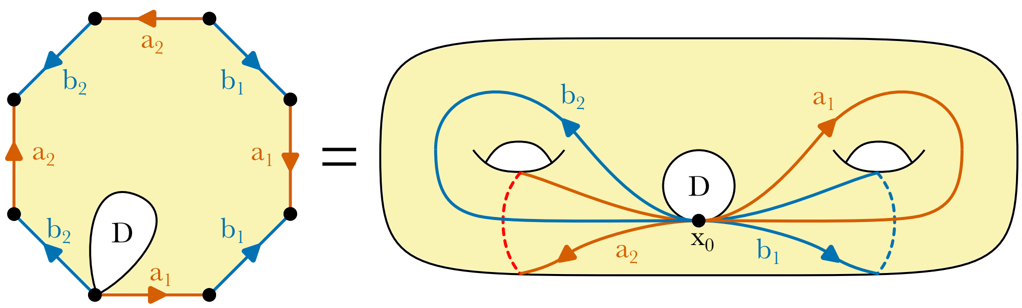
The proof of Theorem 17.7.1 fails for $n = 2$. As a final example, we discuss what the Seifert–van Kampen theorem says in this case. Fix some $g \geq 1$. Let $\Sigma_g$ be a closed oriented genus-$g$ surface, let $D \subset \Sigma_g$ be an embedded 2-dimensional disk, and let $\Sigma_g^1 = \Sigma_g \setminus \mathrm{Int}(D)$:




The surface $\Sigma_g$ is obtained from $\Sigma_g^1$ by attaching a 2-cell. One thing that goes wrong with the proof of Theorem 17.7.1 is that attaching a 2-cell does change the fundamental group. Indeed, it follows from the discussion in §15.2 that:

- $\Sigma_g^1$ deformation retracts onto a graph with one vertex $x_0$ and $2g$ loops $a_1, b_1, \ldots, a_g, b_g$; and
- $\pi_1(\Sigma_g, x_0) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$.

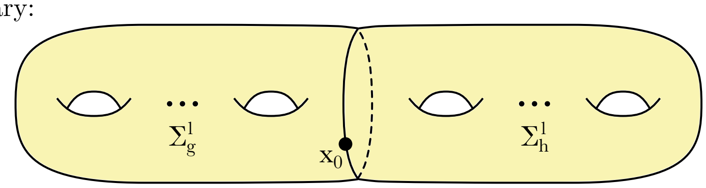
See here for the case $g = 2$:




As in this figure, we can choose the basepoint $x_0 \in \Sigma_g$ such that $x_0 \in \partial D$. It can therefore also serve as a basepoint for $\Sigma_g^1$. Let $r \in \pi_1(\Sigma_g^1, x_0)$ be the loop going around $\partial D$ with $\Sigma_g^1$ on its right. The group $\pi_1(\Sigma_g^1, x_0)$ is free on $\{a_1, b_1, \ldots, a_g, b_g\}$, and it follows from the above figure that

$$r = [a_1, b_1] \cdots [a_g, b_g].$$

Now let $h \geq 1$. Set $X = \Sigma_g \# \Sigma_h$. The surface $X \cong \Sigma_{g+h}$ is obtained by gluing $\Sigma_g^1$ and $\Sigma_h^1$ together along their boundary:




Identifying $\Sigma_g^1$ and $\Sigma_h^1$ with subsurfaces of $X$, the basepoint $x_0$ for $\Sigma_g^1$ we chose above also lies in $\Sigma_h^1$. Just like for $\Sigma_g^1$, the group $\pi_1(\Sigma_h^1, x_0)$ is free on $2h$ generators. To make our notation match up, we call these generators $\{a_{g+1}, b_{g+1}, \ldots, a_{g+h}, b_{g+h}\}$. Letting $r' \in \pi_1(\Sigma_h^1, x_0)$ be the loop around the boundary with $\Sigma_h^1$ on its right, we can choose these free generators such that

$$r' = [a_{g+1}, b_{g+1}] \cdots [a_{g+h}, b_{g+h}].$$

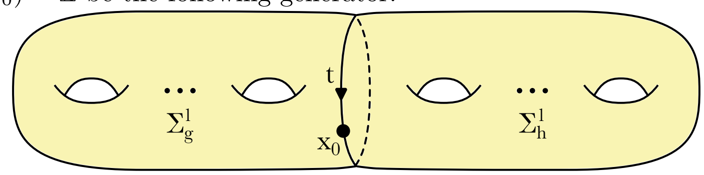
We can choose small open neighborhoods $U \subset X$ of $\Sigma_g^1$ and $V \subset X$ of $\Sigma_h^1$ such that:

- $U$ deformation retracts to $\Sigma_g^1$ and $V$ deformation retracts to $\Sigma_h^1$; and
- $U \cap V$ deformation retracts to the common boundary $T \cong \mathbb{S}^1$ of $\Sigma_g^1$ and $\Sigma_h^1$, so $\pi_1(U \cap V, x_0) = \mathbb{Z}$.

Applying Theorem 17.3.1 (Seifert–van Kampen), we get a pushout

$$\begin{tikzcd} \pi_1(U\cap V, x_0) \arrow[r, "f_1"] \arrow[d, "f_2"'] & \pi_1(U, x_0) \arrow[d] \\ \pi_1(V, x_0) \arrow[r] & \pi_1(X, x_0) \end{tikzcd} \quad = \quad \begin{tikzcd} \mathbb{Z} \arrow[r, "f_1"] \arrow[d, "f_2"'] & \pi_1(\Sigma_g^1, x_0) \arrow[d] \\ \pi_1(\Sigma_h^1, x_0) \arrow[r] & \pi_1(X, x_0). \end{tikzcd}$$

Let $t \in \pi_1(U \cap V, x_0) \cong \mathbb{Z}$ be the following generator:




Since $\Sigma_g^1$ lies to the right of $t$ and $\Sigma_h^1$ lies to the left of $t$, we have $f_1(t) = r$ and $f_2(t) = r'$. Since $\pi_1(\Sigma_g^1, x_0)$ and $\pi_1(\Sigma_h^1, x_0)$ are free groups, it follows from Lemma 17.2.1 that

$$\pi_1(X, x_0) = \langle a_1, b_1, \ldots, a_{g+h}, b_{g+h} \mid f_1(t) = f_2(t)\rangle = \left\langle a_1, b_1, \ldots, a_{g+h}, b_{g+h} \mid r = (r')^{-1} \right\rangle .$$

The single relation can be rewritten as $rr' = 1$. Expanding this out, we get the single surface relation

$$rr' = [a_1, b_1] \cdots [a_g, b_g][a_{g+1}, b_{g+1}] \cdots [a_{g+h}, b_{g+h}]$$

for the fundamental group of $X \cong \Sigma_{g+h}$, as in Theorem 15.2.1.

## 17.9. Exercises

Exercise 17.1. Let $X$ and $Y$ and $Z$ be group and $f_1 \colon Z \to X$ and $f_2 \colon Z \to Y$ be homomorphisms. Consider two different pushouts of $[f_1, f_2]$:

$$\begin{tikzcd} Z \arrow[r, "f_1"] \arrow[d, "f_2"'] & X \arrow[d, "g_1"'] \\ Y \arrow[r, "g_2"] & P \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Z \arrow[r, "f_1"] \arrow[d, "f_2"'] & X \arrow[d, "g_1'"'] \\ Y \arrow[r, "g_2'"] & P' \end{tikzcd}$$

Prove that there is a unique isomorphism $\psi \colon P \to P'$ such that the following diagram commutes:

$$\begin{tikzcd} Z \arrow[r, "f_1"] \arrow[d, "f_2"'] & X \arrow[d, "g_1"'] \arrow[ddr, bend left, "g_1'"] & \\ Y \arrow[r, "g_2"] \arrow[drr, bend right, "g_2'"'] & P \arrow[dr, "\psi", "\cong"'] & \\ & & P' \end{tikzcd}$$

Hint: see the proofs of Lemmas 7.6.5 and 16.1.4.(ii). The use of generating sets in those proofs should be replaced by the uniqueness statement in the definition of a pushout. □

Exercise 17.2. Let

$$\begin{tikzcd} Z \arrow[r, "f_1"] \arrow[d, "f_2"'] & X \arrow[d, "g_1"'] \\ Y \arrow[r, "g_2"] & P \end{tikzcd}$$

be a pushout of groups. Prove the following:

(a) If $f_1$ is an isomorphism, then $g_1$ is an isomorphism.
(b) If $f_1$ is a surjection, then $g_1$ is a surjection. □

Exercise 17.3. Let $Z$ be a space, let $p_1 \colon \widetilde{Z}_1 \to Z$ and $p_2 \colon \widetilde{Z}_2 \to Z$ be covers, and let $\lambda \colon \widetilde{Z}_1 \to \widetilde{Z}_2$ be a covering space isomorphism. Define $\widetilde{Z} = \widetilde{Z}_1 \sqcup \widetilde{Z}_2/\sim$, where $\sim$ is the equivalence relation that for $\widetilde{z} \in \widetilde{Z}_1$ identifies $\widetilde{z} \sim \lambda(\widetilde{z}) \in \widetilde{Z}_2$. The maps $p_1$ and $p_2$ glue together to a map $p \colon \widetilde{Z} \to Z$. Prove that $p$ is a covering space. □

EXERCISE 17.4. Let $M^n$ be a connected $n$-dimensional manifold with $n \geq 3$ and let $x_0, x_1 \in M^n$ be distinct points. Prove that $\pi_1(M^n, x_0) \cong \pi_1(M^n \setminus x_1, x_0)$. $\square$

EXERCISE 17.5. Let $X_1, X_2 = \mathbb{T}^2 = (\mathbb{S}^1)^{\times 2}$ and let $Y = X_1 \sqcup X_2/\sim$ where $\sim$ identifies $\mathbb{S}^1 \times 1 \subset X_1$ with $\mathbb{S}^1 \times 1 \subset X_2$ via the evident homeomorphism. Here $1 \in \mathbb{S}^1 \subset \mathbb{C}$ is the basepoint. Calculate the fundamental group of $Y$. $\square$

EXERCISE 17.6. Let $(X, x_0)$ be a path connected pointed space and let $f\colon (X, x_0) \to (X, x_0)$ be a map. The *mapping torus* of $f$ is the space $M_f = X \times I/\sim$ where $\sim$ is the equivalence relation that identifies $(x, 1)$ with $(f(x), 0)$ for all $x \in X$. Let $y_0 \in M_f$ be the image of $x_0 \times 0$. Assume that $x_0$ has an open neighborhood that deformation retracts to $x_0$.

(a) Use the Seifert–van Kampen theorem to prove that $\pi_1(M_f, y_0)$ is isomorphic to the semidirect product of $\pi_1(X, x_0)$ with $C_\infty$, where $C_\infty$ acts on $\pi_1(X, x_0)$ by letting the generator $t \in C_\infty$ act via $f_*\colon \pi_1(X, x_0) \to \pi_1(X, x_0)$.
(b) Assume that $X$ is locally path connected. Give an alternate proof of (a) by constructing a regular cover $p\colon X \times \mathbb{R} \to M_f$ with deck group $\mathbb{Z}$ and then examining the resulting homomorphism $p_*\colon \pi_1(M_f, y_0) \to \mathbb{Z}$ (see Theorem 9.5.1). Hint: Recall from the footnote in Example 14.1.3 that semidirect products arise from split surjections. $\square$

EXERCISE 17.7. Fix a basepoint $x_0 \in \mathbb{RP}^3 \# \mathbb{RP}^3$. It follows from Theorem 17.7.1 that $\pi_1(\mathbb{RP}^3 \# \mathbb{RP}^3, x_0) = C_2 * C_2$. Do the following:

(a) Fix a basepoint $y_0 \in \mathbb{S}^2 \times \mathbb{S}^1$. Construct a cover $p\colon (\mathbb{S}^2 \times \mathbb{S}^1, y_0) \to (\mathbb{RP}^3 \# \mathbb{RP}^3, x_0)$ of degree 2.
(b) Explicly identify the subgroup of $\pi_1(\mathbb{RP}^3 \# \mathbb{RP}^3, x_0) = C_2 * C_2$ corresponding to the cover from (a). Hint: It might be helpful to let $a$ and $b$ be the generators of the two copies of $C_2$, so $\pi_1(\mathbb{RP}^3 \# \mathbb{RP}^3, x_0) = \langle a, b \mid a^2 = 1, b^2 = 1 \rangle$. You can then identify the subgroup by giving a generating set for it. $\square$

CHAPTER 18*

# More examples of Seifert–van Kampen

To understand a manifold, it is often useful to decompose it into the union of two submanifolds meeting along a common boundary. The Seifert–van Kampen theorem can then be used to calculate the fundamental group. This chapter contains some examples of such calculations for 3-manifolds.

## 18*.1. Gluing spaces together

We begin with a general discussion of gluing spaces together. Let $X$ and $Y$ be topological spaces. Let $X' \subset X$ and $Y' \subset Y$ be closed subsets and let $f\colon X' \to Y'$ be a homeomorphism. Define $\mathrm{Glue}(X, Y; f) = (X \sqcup Y)/\sim$, where $\sim$ is the equivalence relation that for $x \in X'$ identifies $x \sim f(x) \in Y'$. Both $X$ and $Y$ are closed subsets of $\mathrm{Glue}(X, Y; f)$, and $X \cap Y$ is the common image of $X'$ and $Y'$ in $\mathrm{Glue}(X, Y; f)$. There is a natural homeomorphism $\mathrm{Glue}(Y, X; f^{-1}) \cong \mathrm{Glue}(X, Y; f)$.

The following gives a criterion for homeomorphisms $f_1, f_2\colon X' \to Y'$ to give homeomorphic spaces $\mathrm{Glue}(X, Y; f_1)$ and $\mathrm{Glue}(X, Y; f_2)$:

Lemma 18*.1.1. *Let $X$ and $Y$ be topological spaces. Let $X' \subset X$ and $Y' \subset Y$ be closed subsets and let $f\colon X' \to Y'$ be a homeomorphism. The following hold:*

- *(i) Let $g\colon X' \to X'$ be a homeomorphism that extends to a homeomorphism $G\colon X \to X$. Then* $\mathrm{Glue}(X, Y; f \circ g) \cong \mathrm{Glue}(X, Y; f)$.
- *(ii) Let $h\colon Y' \to Y'$ be a homeomorphism that extends to a homeomorphism $H\colon Y \to Y$. Then* $\mathrm{Glue}(X, Y, h \circ f) \cong \mathrm{Glue}(X, Y; f)$.

Proof. Using the symmetry $\mathrm{Glue}(Y, X; f^{-1}) \cong \mathrm{Glue}(X, Y; f)$, conclusions (i) and (ii) are equivalent. It is thus enough to prove (i). Let $\sim_1$ and $\sim_2$ be the equivalence relations on $X \sqcup Y$ that for $x \in X'$ identify

$$x \sim_1 f \circ g(x) \in Y' \quad \text{and} \quad x \sim_2 f(x) \in Y'.$$

We therefore have $\mathrm{Glue}(X, Y; f \circ g) = (X \sqcup Y)/\sim_1$ and $\mathrm{Glue}(X, Y; f) = (X \sqcup Y)/\sim_2$. Let $\phi\colon X \sqcup Y \to X \sqcup Y$ be the homeomorphism $\phi = G \sqcup \mathbb{1}_Y$. For $z, w \in X \sqcup Y$, we have $z \sim_1 w$ if and only if $\phi(z) \sim_2 \phi(w)$. It follows that $\phi$ descends to a homeomorphism $\Phi\colon \mathrm{Glue}(X, Y; f \circ g) \to \mathrm{Glue}(X, Y; f)$. □

We now specialize this to manifolds. Let $M$ and $N$ be compact $n$-manifolds with boundary. Let $f\colon \partial M \to \partial N$ be a homeomorphism. The space $\mathrm{Glue}(M, N; f)$ is then a closed $n$-manifold. The easiest case of this when $M = N = \mathbb{D}^n$, so $\partial M = \partial N = \mathbb{S}^{n-1}$. The following lemma shows that in this case gluing $M$ to $N$ always gives $\mathbb{S}^n$:

Lemma 18*.1.2. *Let $f\colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ be a homeomorphism. The following then hold:*

- *(i) The homeomorphism $f$ extends to a homeomorphism $F\colon \mathbb{D}^n \to \mathbb{D}^n$.*
- *(ii) We have* $\mathrm{Glue}(\mathbb{D}^n, \mathbb{D}^n; f) \cong \mathbb{S}^n$.

Proof. For (i), each point of $\mathbb{D}^n$ other than 0 can be uniquely written as $tx$ for some $t \in (0, 1]$ and $x \in \mathbb{S}^{n-1}$. We can therefore define an extension $F\colon \mathbb{D}^n \to \mathbb{D}^n$ of $f\colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ via the formula

$$\begin{cases} F(tx) = tf(x) & \text{for } t \in (0,1] \text{ and } x \in \mathbb{S}^{n-1}, \\ F(0) = 0. \end{cases}$$

Note that by Lemma 18*.1.1, conclusion (i) implies that the homeomorphism type of $\mathrm{Glue}(\mathbb{D}^n, \mathbb{D}^n; f)$ is independent of $f$. Conclusion (ii) follows since $\mathbb{S}^n$ can be decomposed into the union of two copies of $\mathbb{D}^n$ meeting along their common boundary $\mathbb{S}^{n-1}$, so there exists some homeomorphism $f'\colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ with $\mathrm{Glue}(\mathbb{D}^n, \mathbb{D}^n; f') \cong \mathbb{S}^{n-1}$. □

## 18*.2. Decomposing $\mathbb{S}^3$ into a union of solid tori

Since $\mathbb{S}^3$ is the only 3-manifold that can be obtained by gluing two copies of $\mathbb{D}^3$ together along their common boundary, to obtain interesting 3-manifolds we will have to use more complicated pieces. Before we can do this, we need some more facts about $\mathbb{S}^3$. We will describe subsets of $\mathbb{S}^3$ in two ways:

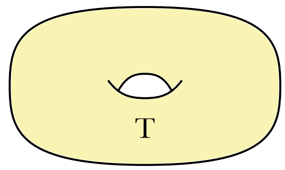
- Using coordinates in $\mathbb{S}^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$.
- By drawing subsets of $\mathbb{R}^3$. Letting $\infty = (0,i) \in \mathbb{S}^3$, subsets of $\mathbb{R}^3$ can be identified with points of $\mathbb{S}^3 \subset \mathbb{C}^2$ via the stereographic projection[1]

$$\mathbb{S}^3 \setminus \infty \xrightarrow{\cong} \mathbb{C} \times \mathbb{R} = \mathbb{R}^3.$$

The *standard torus* is the following subspace $T \cong \mathbb{T}^2$ of $\mathbb{S}^3$:

$$T = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 = |w|^2 = 1/2\}. \tag{18*.2.1}$$

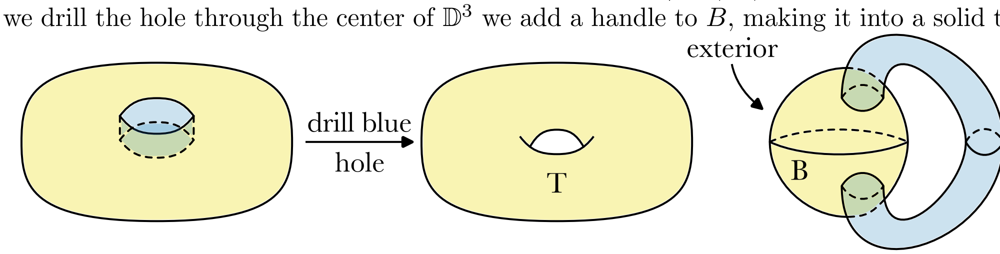
This is the boundary of a tubular neighborhood of the unit circle $\mathbb{S}^1 \times 0 \subset \mathbb{S}^3$. Under the stereographic projection, this unit circle goes to the unit circle $\mathbb{S}^1 \times 0 \subset \mathbb{R}^3$. Taking the boundary of an appropriate tubular neighborhood of this, we can draw $T$ as follows:




From this, it is clear that $T$ bounds a solid torus $X \cong \mathbb{S}^1 \times \mathbb{D}^2$, namely the closure of the bounded component of $\mathbb{R}^3 \setminus T$. We will call $X$ the *interior* solid torus. In fact, from (18*.2.1) we see that $T$ divides $\mathbb{S}^3$ into two solid tori $X \cong \mathbb{S}^1 \times \mathbb{D}^2$ and $Y \cong \mathbb{D}^2 \times \mathbb{S}^1$:

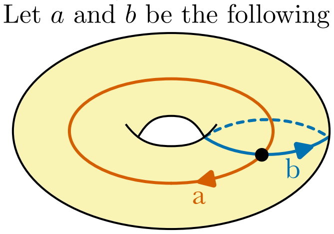
$$X = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 = 1 - |w|^2 \text{ and } |w|^2 \leq 1/2\},$$
$$Y = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 \leq 1/2 \text{ and } |w|^2 = 1 - |z|^2\}.$$

We will call $Y$ the *exterior* solid torus. Another way to see that $T \subset \mathbb{S}^3$ bounds solid tori on either side is as follows. Up to a homeomorphism of $\mathbb{R}^3$, the solid torus $X$ is obtained from the unit ball $\mathbb{D}^3 \subset \mathbb{R}^3$ by drilling a hole through its center. The exterior $\mathbb{S}^3 \setminus \mathrm{Int}(\mathbb{D}^3)$ is another solid ball $B$, and when we drill the hole through the center of $\mathbb{D}^3$ we add a handle to $B$, making it into a solid torus $Y$:




## 18*.3. Fundamental group and homeomorphisms of standard torus

Let $T = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 = |w|^2 = 1/2\}$ be the standard torus in $\mathbb{S}^3 \subset \mathbb{C}^2$. For its basepoint, we will always use $t_0 = (1/2, 1/2)$. Let $a$ and $b$ be the following loops on $T$:



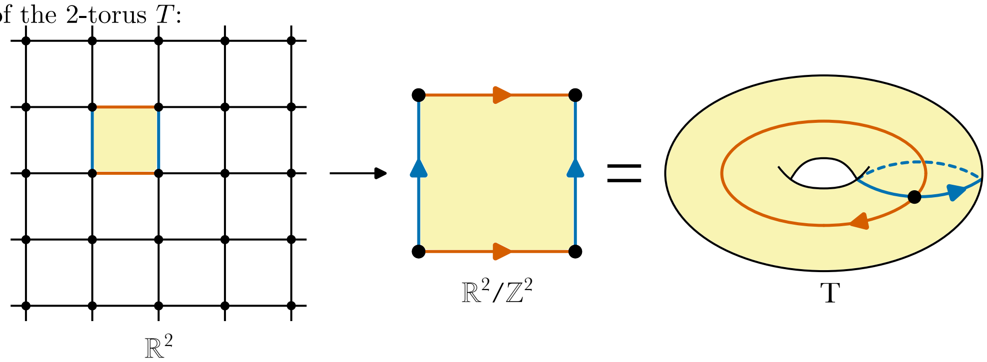

We will identify $\pi_1(T, t_0)$ with $\mathbb{Z}^2$ in such a way that $[a] \in \pi_1(T, t_0)$ corresponds to $(1,0) \in \mathbb{Z}^2$ and

---

[1] By definition, the stereographic projection takes a point $x \in \mathbb{S}^3 \setminus \infty$ to the following point of $\mathbb{C} \times \mathbb{R}$. Let $\ell_x = \{tx + (1-t)\infty \mid t \in \mathbb{R}\}$ be the line in $\mathbb{C}^2$ through $x$ and $\infty = (0,i)$. Then the stereographic projection of $x$ is the intersection point of $\ell_x$ with $\mathbb{C} \times \mathbb{R}$.

$[b] \in \pi_1(T, t_0)$ corresponds to $(0, 1) \in \mathbb{Z}^2$. Let $X \cong \mathbb{S}^1 \times \mathbb{D}^2$ be the interior solid torus and $Y \cong \mathbb{D}^2 \times \mathbb{S}^1$ be the exterior solid torus, so $X \cap Y = T$. We will identify $\pi_1(X, t_0)$ with $\mathbb{Z}$ such that $[a] \in \pi_1(X, t_0)$ corresponds to $1 \in \mathbb{Z}$, and we will also identify $\pi_1(Y, t_0)$ with $\mathbb{Z}$ such that $[b] \in \pi_1(Y, t_0)$ corresponds to $1 \in \mathbb{Z}$. With these conventions, the map $\pi_1(T, t_0) \to \pi_1(X, t_0)$ is the projection of $\mathbb{Z}^2$ onto its first coordinate and the map $\pi_1(T, t_0) \to \pi_1(Y, t_0)$ is the projection of $\mathbb{Z}^2$ onto its second coordinate.

Let $\pi\colon \mathbb{R}^2 \to T$ be the map $\pi(x, y) = (e^{2\pi i x}/\sqrt{2}, e^{2\pi i y}/\sqrt{2})$. This exhibits $\mathbb{R}^2$ as the universal cover of the 2-torus $T$:




For $M \in \mathrm{SL}_2(\mathbb{Z})$, let $\phi_M\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the associated linear map. The map $\phi_M$ commutes with the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$, and thus induces a homeomorphism $\lambda_M\colon (T, t_0) \to (T, t_0)$. With our identification $\pi_1(T, t_0) = \mathbb{Z}^2$, the induced map $(\lambda_M)_*\colon \pi_1(T, t_0) \to \pi_1(T, t_0)$ is multiplication by $M \in \mathrm{SL}_2(\mathbb{Z})$.

## 18*.4. Three-dimensional lens spaces

Let $T \cong \mathbb{T}^2$ be the standard torus in $\mathbb{S}^3$. Let $X \cong \mathbb{S}^1 \times \mathbb{D}^2$ be the interior solid torus and $Y \cong \mathbb{D}^2 \times \mathbb{S}^1$ be the exterior solid torus, so $X \cap Y = T$. We therefore have $\mathbb{S}^3 = \mathrm{Glue}(X, Y; \mathbb{1}_T)$. For $M \in \mathrm{SL}_2(\mathbb{Z})$, we wish to study the 3-manifold $\mathrm{Glue}(X, Y; \lambda_M)$. To understand how it depends on $M \in \mathrm{SL}_2(\mathbb{Z})$, we make the following observation:

LEMMA 18*.4.1. *Let $X$ and $Y$ and $T$ be as above. For $d \in \mathbb{Z}$, let*

$$A(d) = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix} \quad \text{and} \quad B(d) = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}.$$

*Then $\lambda_{A(d)}$ extends over $X$ and $\lambda_{B(d)}$ extends over $Y$.*

PROOF. The proofs for $\lambda_{A(d)}$ and $\lambda_{B(d)}$ are similar, so we will prove it for for $\lambda_{A(d)}$. Recall that

$$T = \left\{(z, w) \in \mathbb{C}^2 \mid |z|^2 = |w|^2 = 1/2\right\},$$
$$X = \left\{(z, w) \in \mathbb{C}^2 \mid |z|^2 = 1 - |w|^2 \text{ and } |w|^2 \le 1/2\right\}.$$

To make our formulas more attractive, rescale everything such that

$$T = \left\{(z, w) \in \mathbb{C}^2 \mid |z|^2 = |w|^2 = 1\right\},$$
$$X = \left\{(z, w) \in \mathbb{C}^2 \mid |z|^2 = 2 - |w|^2 \text{ and } |w| \le 1\right\}.$$

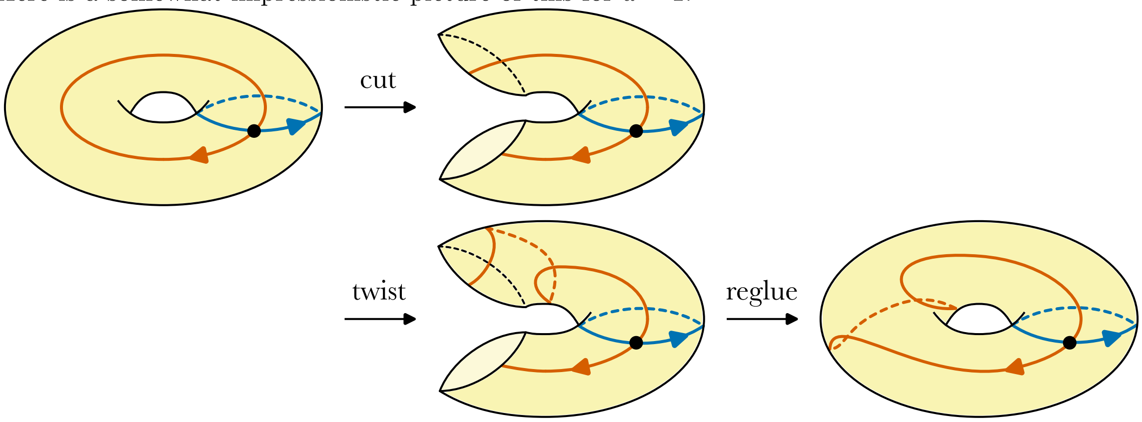
With these identifications, the map $\lambda_{A(d)}\colon T \to T$ is the map

$$\lambda_{A(d)}(z, w) = (z, zw^d) \quad \text{for } (z, w) \in T.$$

The desired extension to $X$ is then the map $F_{A(d)}\colon X \to X$ defined by

$$F_{A(d)}(z, w) = (z, w(z/|z|)^d) \quad \text{for } (z, w) \in X.$$

Here is a somewhat impressionistic picture of this for $d = 1$:




We remark that this is the $d^{\text{th}}$ power of what is called a *Dehn twist*. $\square$

We now make the following definition. Consider $p, q \in \mathbb{Z}$ with $\gcd(p, q) = 1$. We can therefore find $n, m \in \mathbb{Z}$ with $np + mq = 1$. Define $L(p; q)$ to be the 3-manifold $\mathrm{Glue}(X, Y; \lambda_M)$ with

$$M = \begin{pmatrix} n & -m \\ q & p \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

This matrix lies in $\mathrm{SL}_2(\mathbb{Z})$ since $np + mq = 1$. It might appear that this depends on the choice of $n, m \in \mathbb{Z}$ with $np + mq = 1$. To see that it does not, let $n', m' \in \mathbb{Z}$ also satisfy $n'p + m'q = 1$ and let

$$M' = \begin{pmatrix} n' & -m' \\ q & p \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

It is easy to see that for some $d \in \mathbb{Z}$ we have $n' = n + dq$ and $m' = m - dp$ (see Exercise 18*.1). Letting $B(d)$ be as in Lemma 18*.4.1, we have

$$M' = \begin{pmatrix} n' & -m' \\ q & p \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n & -m \\ q & p \end{pmatrix} = B(d)M.$$

Lemma 18*.4.1 says that $\lambda_{B(d)}$ extends over $Y$, so by Lemma 18*.1.1 we have

$$\mathrm{Glue}(X, Y; \lambda_{M'}) = \mathrm{Glue}(X, Y; \lambda_{B(d)} \circ \lambda_M) = \mathrm{Glue}(X, Y; \lambda_M),$$

as desired. The closed orientable 3-manifold $L(p; q)$ is what is called a 3-*dimensional lens space*.

### 18*.5. Fundamental groups of three-dimensional lens spaces

We finally come to our first fundamental group calculation:

Theorem 18*.5.1. *Let $p, q \in \mathbb{Z}$ satisfy $\gcd(p, q) = 1$. Let $x_0 \in L(p; q)$ be a basepoint. Then $\pi_1(L(p; q), x_0) \cong C_{|p|}$.*

Proof. Let $T \cong \mathbb{T}^2$ be the standard torus in $\mathbb{S}^3$. Let $X \cong \mathbb{S}^1 \times \mathbb{D}^2$ be the interior solid torus and $Y \cong \mathbb{D}^2 \times \mathbb{S}^1$ be the exterior solid torus, so $X \cap Y = T$. Pick $n, m \in \mathbb{Z}$ such that $np - mq = 1$, and let

$$M = \begin{pmatrix} n & -m \\ q & p \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

We therefore have $L(p; q) = \mathrm{Glue}(X, Y; \lambda_M)$.

Let $\overline{T}$ and $\overline{X}$ and $\overline{Y}$ be the images of $T$ and $X$ and $Y$ in $\mathrm{Glue}(X, Y; \lambda_M)$. Changing our basepoint, we can assume that $x_0 \in \mathrm{Glue}(X, Y; \lambda_M)$ is the image $\overline{t}_0 \in \overline{T}$ of the basepoint $t_0 \in T$ discussed in §18*.3. Thickening up $\overline{X}$ and $\overline{Y}$, we can find open neighborhoods $U$ of $\overline{X}$ and $V$ of $\overline{Y}$ such that:

- $U$ deformation retracts to $\overline{X} \cong \mathbb{D}^2 \times \mathbb{S}^1$; and
- $V$ deformation retracts to $\overline{Y} \cong \mathbb{S}^1 \times \mathbb{D}^2$; and
- $U \cap V$ deformation retracts to $\overline{T} \cong \mathbb{T}^2$, and satisfies $U \cap V \cong \overline{T} \times (-1, 1)$.

Applying Theorem 17.3.1 (Seifert–van Kampen), we get a pushout

$$\begin{tikzcd} \pi_1(U\cap V,\bar{t}_0) \arrow[r, "f_1"] \arrow[d, "f_2"'] & \pi_1(U,\bar{t}_0) \arrow[d] \\ \pi_1(V,\bar{t}_0) \arrow[r] & \pi_1(L(p;q),\bar{t}_0) \end{tikzcd} = \begin{tikzcd} \pi_1(\overline{T},\bar{t}_0) \arrow[r, "f_1"] \arrow[d, "f_2"'] & \pi_1(\overline{X},\bar{t}_0) \arrow[d] \\ \pi_1(\overline{Y},\bar{t}_0) \arrow[r] & \pi_1(L(p;q),\bar{t}_0) \end{tikzcd} = \begin{tikzcd} \mathbb{Z}^2 \arrow[r, "f_1"] \arrow[d, "f_2"'] & \mathbb{Z} \arrow[d] \\ \mathbb{Z} \arrow[r] & \pi_1(L(p;q),\bar{t}_0). \end{tikzcd}$$

Here the second equality identifies these fundamental groups as in §18*.3.

To use this to compute $\pi_1(L(p;q),\bar{t}_0)$, we must understand the maps $f_1$ and $f_2$. Using the formulas from §18*.3, the map $f_1\colon \mathbb{Z}^2\to\mathbb{Z}$ is the map $f_1(d_1,d_2)=d_1$. Since

$$\begin{pmatrix} n & -m \\ q & p \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} nd_1-md_2 \\ qd_1+pd_2 \end{pmatrix},$$

those same formulas show that $f_2\colon\mathbb{Z}^2\to\mathbb{Z}$ is the map $f_2(d_1,d_2)=qd_1+pd_2$. Let $x\in\pi_1(\overline{X},x_0)\cong\mathbb{Z}$ and $y\in\pi_1(\overline{Y},x_0)\cong\mathbb{Z}$ be the standard generators from §18*.3, and let $a=(1,0)\in\mathbb{Z}^2$ and $b=(0,1)\in\mathbb{Z}^2$. It follows from Lemma 17.2.1 that

$$\begin{aligned}\pi_1(L(p;q),x_0) &\cong \langle x,y \mid f_1(a)=f_2(a), f_1(b)=f_2(b)\rangle \\ &\cong \langle x,y \mid x=y^q, 1=y^p\rangle \cong \langle y \mid y^p=1\rangle \cong C_{|p|}. \end{aligned}$$ □

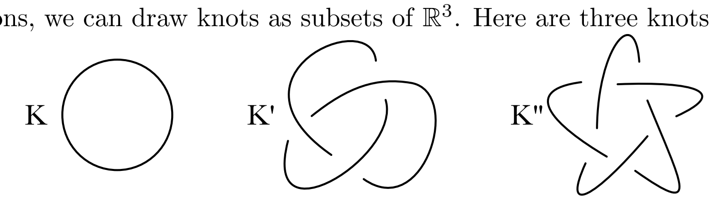
Remark 18*.5.2. The 3-dimensional lens spaces are interesting since they were the first examples found of closed manifolds where the classifications up to homeomorphism and up to homotopy equivalence differ. Indeed, consider $p,q\in\mathbb{Z}$ and $p',q'\in\mathbb{Z}$ with $\gcd(p,q)=\gcd(p',q')=1$. We then have:

- The spaces $L(p;q)$ and $L(p';q')$ are homeomorphic if and only if $|p|=|p'|$ and $q'\equiv\pm q \pmod p$. See Exercise 18*.2 for the easy direction of this.
- The spaces $L(p;q)$ and $L(p';q')$ are homotopy equivalent if and only if $|p|=|p'|$ and $q'q\equiv\pm d^2 \pmod p$ for some $d\in\mathbb{Z}$.

The proofs of these two results use more sophisticated tools from algebraic topology, and we will give them in subsequent volumes. □

## 18*.6. Torus knots

Our final example illustrates how to use the Seifert–van Kampen theorem to solve a classical problem in geometric topology. A *knot* in $\mathbb{S}^3$ is a subspace $K\subset\mathbb{S}^3$ with $K\cong\mathbb{S}^1$. Just like we did in the previous sections, we can draw knots as subsets of $\mathbb{R}^3$. Here are three knots $K$ and $K'$ and $K''$:




The knot $K$ is the *unknot*, the knot $K'$ is the *trefoil*, and the knot $K''$ is the *cinquefoil*.

Two knots $K_1,K_2\subset\mathbb{S}^3$ are *equivalent* if there exists a homeomorphism[2] $F\colon\mathbb{S}^3\to\mathbb{S}^3$ with $F(K_1)=K_2$. We will often confuse knots with their equivalence classes. One easy way to see that knots are equivalent is as follows. Assume that the $K_i$ are the images of smooth[3] embeddings $f_i\colon\mathbb{S}^1\to\mathbb{S}^3$. We say that $f_1$ is *smoothly isotopic* to $f_2$ if there is a homotopy $h_t\colon\mathbb{S}^1\to\mathbb{S}^3$ from $f_1$ to $f_2$ with the following properties:

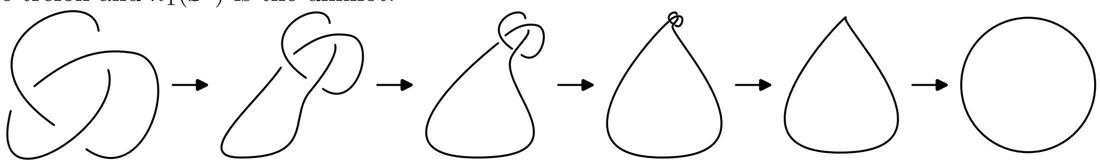
- the map $H\colon\mathbb{S}^1\times I\to\mathbb{S}^3$ defined by $H(x,t)=h_t(x)$ is a smooth map; and
- each $h_t\colon\mathbb{S}^1\to\mathbb{S}^3$ is a smooth embedding.

If $f_1$ is smoothly isotopic to $f_2$, then it follows from a standard result in manifold topology called the isotopy extension theorem that $K_1$ is equivalent to $K_2$.

[2]One often requires this homeomorphism to be orientation-preserving, but we will ignore this.

[3]Here we are regarding $\mathbb{S}^1$ and $\mathbb{S}^3$ as smooth manifolds.

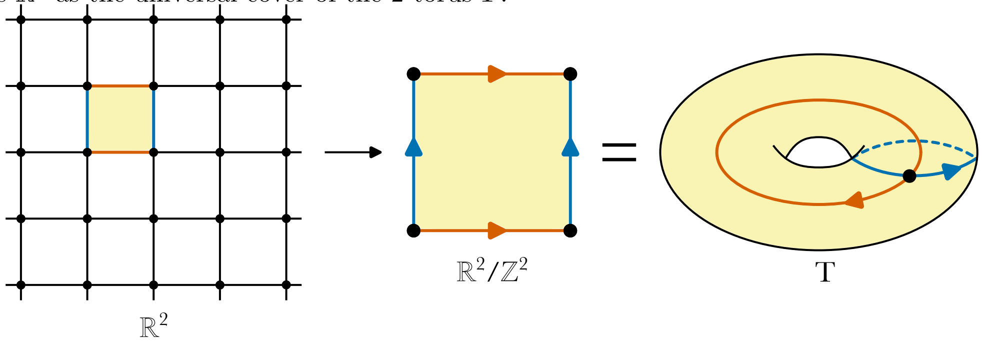
Remark 18*.6.1. The above is false without some assumption like smoothness. Indeed, the following depicts a homotopy $h_t \colon \mathbb{S}^1 \to \mathbb{S}^3$ with each $h_t$ a topological embedding such that $h_0(\mathbb{S}^1)$ is the trefoil and $h_1(\mathbb{S}^1)$ is the unknot:



However, we will soon see that the unknot and the trefoil are not equivalent. □

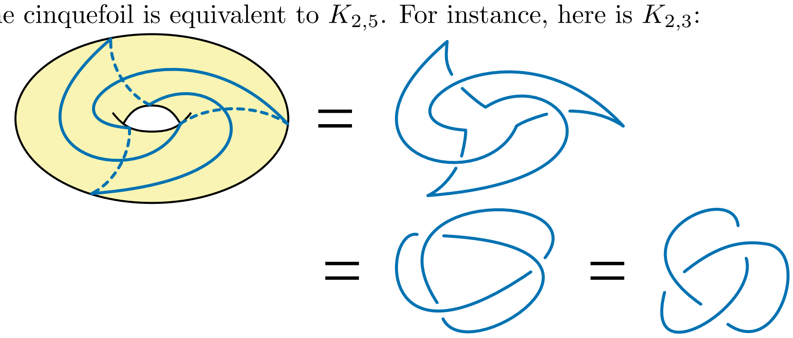
Our main goal is to prove that many knots are not equivalent. In particular, the unknot, the trefoil, and the cinquefoil are all pairwise non-equivalent. This requires constructing invariants of a knot $K$. The invariant we will use is the *knot group*, that is, the fundamental group of $\mathbb{S}^3 \setminus K$. In fact, we will be able to handle the more general class of torus knots, which are defined as follows.

Consider $p, q \in \mathbb{Z}$ with $\gcd(p,q) = 1$. Let $T = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 = |w|^2 = 1/2\}$ be the standard torus in $\mathbb{S}^3 \subset \mathbb{C}^2$. As in §18*.3, let $\pi \colon \mathbb{R}^2 \to T$ be the map $\pi(x,y) = (e^{2\pi i x}/\sqrt{2}, e^{2\pi i y}/\sqrt{2})$. This exhibits $\mathbb{R}^2$ as the universal cover of the 2-torus $T$:




Let $\ell_{p,q} \subset \mathbb{R}^2$ be any line of slope $q/p \in \mathbb{Q} \cup \{\infty\}$. The $(p,q)$-*torus knot*, denoted $K_{p,q}$, is $\pi(\ell_{p,q})$. We claim that $K_{p,q}$ is a circle embedded in $T \subset \mathbb{S}^3$. To see this, let $v = (p,q) \in \mathbb{R}^2$ and consider $x, y \in \ell_{p,q}$. We have $\pi(x) = \pi(y)$ if and only if $y - x \in \mathbb{Z}^2$. Since $\ell_{p,q}$ has slope $q/p$, we have $y - x = cv$ for some $c \in \mathbb{R}$. This will lie in $\mathbb{Z}^2$ if and only if $c \in \mathbb{Z}$. That $K_{p,q} = \pi(\ell_{p,q}) \cong \mathbb{S}^1$ follows. Any two choices of $\ell_{p,q}$ give smoothly isotopic $K_{p,q}$, so up to equivalence of knots $K_{p,q}$ does not depend on the choice of $\ell_{p,q}$. We will call the $K_{p,q}$ corresponding to choices of $\ell_{p,q}$ different *realization* of $K_{p,q}$.

The knot $K_{p,q}$ wraps $|p|$ times around the first factor of $T = \mathbb{S}^1 \times \mathbb{S}^1$ and $|q|$ times around the other factor. A bit of reflection shows that the unknot is equivalent to $K_{1,0}$, the trefoil is equivalent to $K_{2,3}$, and the cinquefoil is equivalent to $K_{2,5}$. For instance, here is $K_{2,3}$:



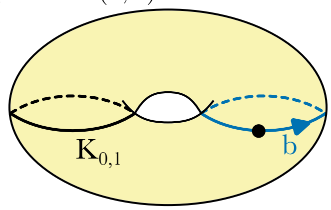
We will completely classify torus knots. To do this, we will first compute a presentation for the fundamental group of $\mathbb{S}^3 \setminus K_{p,q}$ and then carefully analyze its group-theoretic properties.

## 18*.7. Knot groups for torus knots

To calculate the fundamental group of $\mathbb{S}^3 \setminus K_{p,q}$, we need to understand the fundamental group of $T \setminus K_{p,q}$. We will use the basepoint $t_0 \in T$ and the identification $\pi_1(T, t_0) \cong \mathbb{Z}^2$ from §18*.3. Recall that $K_{p,q}$ is the image in $T$ of a line $\ell_{p,q}$ of slope $q/p \in \mathbb{Q} \cup \{\infty\}$ in $\mathbb{R}^2$. The integer points $\mathbb{Z}^2$ of $\mathbb{R}^2$ map to $t_0$, so by choosing $\ell_{p,q}$ to not pass through $\mathbb{Z}^2$ we can ensure that $t_0 \notin K_{p,q}$. We then have:

LEMMA 18*.7.1. *Let $p, q \in \mathbb{Z}$ satisfy* $\gcd(p, q) = 1$. *Let $T$ be the standard torus and let $K_{p,q} \subset T$ be a realization of the $(p, q)$-torus knot with $t_0 \notin K_{p,q}$. Then $\pi_1(T \setminus K_{p,q}, t_0) \cong \mathbb{Z}$ and the image of $\pi_1(T \setminus K_{p,q}, t_0)$ in $\pi_1(T, t_0) = \mathbb{Z}^2$ is generated by $(p, q) \in \mathbb{Z}^2$.*

PROOF. We first prove this for $(p, q) = (0, 1)$. Choose a realization of $K_{0,1}$ with $t_0 \notin K_{0,1}$. As the following figure shows, $T \setminus K_{0,1} \cong \mathbb{S}^1 \times (0, 1)$:




This implies that $\pi_1(T \setminus K_{0,1}, t_0) \cong \mathbb{Z}$. The above figure also shows that $T \setminus K_{0,1}$ contains the loop $b$ such that $[b] \in \pi_1(T, t_0)$ corresponds to $(0, 1) \in \mathbb{Z}^2$. The lemma follows in this case.

For the general case $K_{p,q}$, since $\gcd(p, q) = 1$ we can find $n, m \in \mathbb{Z}$ with $np + mq = 1$. Set

$$M = \begin{pmatrix} m & p \\ -n & q \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

As in §18*.3, let $\phi_M \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map associated to $M$ and let $\lambda_M \colon (T, t_0) \to (T, t_0)$ be the induced homeomorphism of $T = \mathbb{R}^2/\mathbb{Z}^2$. The linear map $\phi_M$ takes lines of slope $1/0 = \infty$ to lines of slope $q/p$. It follows that $\lambda_M$ takes realizations of $K_{0,1}$ to realizations of $K_{p,q}$. Applying $\lambda_M$, we reduce to the already proven case of $K_{0,1}$. $\square$

We now calculate the fundamental group of the complement of a torus knot:

THEOREM 18*.7.2. *Let $p, q \in \mathbb{Z}$ satisfy $\gcd(p, q) = 1$. Then for $x_0 \in \mathbb{S}^3 \setminus K_{p,q}$ we have $\pi_1(\mathbb{S}^3 \setminus K_{p,q}, x_0) \cong \langle x, y \mid x^p = y^q \rangle$.*

PROOF. Let $T \subset \mathbb{S}^3$ be the standard torus with its basepoint $t_0 \in T$. Choose a realizsation of $K_{p,q}$ such that $t_0 \notin K_{p,q}$. Changing the basepoint, we can assume that $x_0 = t_0$. As in §18*.2, let $X \cong \mathbb{S}^1 \times \mathbb{D}^2$ and $Y \cong \mathbb{D}^2 \times \mathbb{S}^1$ be the interior and exterior solid tori, so $X \cap Y = T$. Thickening up $X \setminus K_{p,q}$ and $Y \setminus K_{p,q}$, we can find open sets $U, V \subset \mathbb{S}^3 \setminus K_{p,q}$ such that:

- $U$ deformation retracts to $X \setminus K_{p,q}$; and
- $V$ deformation retracts to $Y \setminus K_{p,q}$; and
- $U \cap V$ deformation retracts to $T \setminus K_{p,q}$.

Applying Theorem 17.3.1 (Seifert–van Kampen), we get a pushout

$$\begin{tikzcd} \pi_1(U \cap V, t_0) \arrow[r, "f_1"] \arrow[d, "f_2"'] & \pi_1(U, t_0) \arrow[d] \\ \pi_1(V, t_0) \arrow[r] & \pi_1(S^3 \setminus K_{p,q}, t_0) \end{tikzcd} = \begin{tikzcd} \pi_1(T \setminus K_{p,q}, t_0) \arrow[r, "f_1"] \arrow[d, "f_2"'] & \pi_1(X \setminus K_{p,q}, t_0) \arrow[d] \\ \pi_1(Y \setminus K_{p,q}, t_0) \arrow[r] & \pi_1(S^3 \setminus K_{p,q}, t_0). \end{tikzcd}$$

The spaces $X \cong \mathbb{S}^1 \times \mathbb{D}^2$ and $X \setminus K_{p,q}$ both deformation retract to the same circle $\mathbb{S}^1 \times 0$, so we can identify their fundamental groups and see that

$$\pi_1(X \setminus K_{p,q}, t_0) = \pi_1(X, t_0) = \mathbb{Z}.$$

Similarly, $\pi_1(Y \setminus K_{p,q}, t_0) = \pi_1(Y, t_0) = \mathbb{Z}$. Lemma 18*.7.1 says that $\pi_1(T \setminus K_{p,q}, t_0) = \mathbb{Z}$, so the above becomes a pushout

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "f_1"] \arrow[d, "f_2"'] & \mathbb{Z} \arrow[d] \\ \mathbb{Z} \arrow[r] & \pi_1(S^3 \setminus K_{p,q}, t_0). \end{tikzcd}$$

Let $x \in \pi_1(X \setminus K_{p,q}, t_0) = \mathbb{Z}$ and $y = \pi_1(Y \setminus K_{p,q}, t_0) = \mathbb{Z}$ and $t \in \pi_1(T \setminus K_{p,q}, t_0) = \mathbb{Z}$ be generators. Lemma 17.2.1 says that

$$\pi_1(S^3 \setminus K_{p,q}, t_0) = \langle x, y \mid f_1(t) = f_2(t) \rangle .$$

Lemma 18*.7.1 implies that $f_1(t) = x^p$ and $f_2(t) = y^q$. The theorem follows. $\square$

## 18*.8. Classification of torus knots

We now classify torus knots. The easy part of the classification will be to show that certain torus knots are equivalent, and will be left as exercises:

LEMMA 18*.8.1. *For all $d \in \mathbb{Z}$, the knots $K_{1,d}$ and $K_{d,1}$ and $K_{-1,d}$ and $K_{d,-1}$ are all equivalent to the unknot.*

PROOF. See Exercise 18*.3. □

LEMMA 18*.8.2. *Let $p, q \in \mathbb{Z}$ satisfy $\gcd(p, q) = 1$. The following hold:*

*(i) The knots $K_{p,q}$ and $K_{q,p}$ are equivalent.*
*(ii) The knots $K_{p,q}$ and $K_{p,-q}$ and $K_{-p,q}$ and $K_{-p,-q}$ are all equivalent.*

PROOF. See Exercise 18*.4. □

By these two lemmas, it is enough to handle the knots $K_{p,q}$ for $p, q \geq 2$ with $\gcd(p, q) = 1$. Our main result is as follows:

THEOREM 18*.8.3. *For $i = 1, 2$, let $p_i, q_i \geq 2$ satisfy $\gcd(p_i, q_i) = 1$. Then $K_{p_1,q_1}$ is equivalent to $K_{p_2,q_2}$ if and only if either $(p_1, q_1) = (p_2, q_2)$ or $(p_1, q_1) = (q_2, p_2)$.*

PROOF. Let $p, q \geq 2$ satisfy $\gcd(p, q) = 1$ and let $x_0 \in \mathbb{S}^3 \setminus K_{p,q}$ be a basepoint. Set $G = \pi_1(\mathbb{S}^3 \setminus K_{p,q}, x_0)$. Theorem 18*.7.2 says that

$$G = \langle x, y \mid x^p = y^q \rangle .$$

The isomorphism type of the group $G$ only depends on the equivalence class of $K_{p,q}$. It is therefore enough to prove that we can recover the unordered pair of integers $\{p, q\}$ from group-theoretic properties of $G$.

Let $Z(G)$ denote the center of $G$:

$$Z(G) = \{c \in G \mid cg = gc \text{ for all } g \in G\} .$$

This is a normal subgroup of $G$, so we can quotient $G$ by it and get a group $G/Z(G)$. The key to the theorem is the following:

CLAIM. *We have $G/Z(G) \cong C_p * C_q$.*

PROOF OF CLAIM. For $w \in F(x, y)$, recall that $\overline{w}$ is its image in $G$. Since $\overline{x}^p$ commutes with $\overline{x}$ and $\overline{y}^q$ commutes with $\overline{y}$, the common value $c = \overline{x}^p = \overline{y}^q$ commutes with both $\overline{x}$ and $\overline{y}$, and thus lies in $Z(G)$. Letting $C = \langle c \rangle$, since $C$ is a subgroup of $Z(G)$ it follows that $C$ is a normal subgroup of $G$. We have

$$G/C \cong \langle x, y \mid x^p = 1, y^q = 1\rangle = \langle x \mid x^p = 1\rangle * \langle y \mid y^q = 1\rangle = C_p * C_q.$$

Since $p, q \geq 2$, the groups $C_p$ and $C_q$ are nontrivial and thus their free product $C_p * C_q$ has a trivial center (see Exercise 16.5). Under the surjection $G \to G/C$, elements of $Z(G)$ map to elements of $Z(G/C)$. We deduce that $Z(G)$ maps trivially to $G/C$, so $Z(G) < C$ and thus $Z(G) = C$. We conclude that $G/Z(G) = G/C = C_p * C_q$, as desired. □

It is thus enough to prove that we can recover $p$ and $q$ from group-theoretic properties of $C_p * C_q$. A first idea might be to look at the abelianization of $C_p * C_q$, but this does not work: since $\gcd(p, q) = 1$, the Chinese remainder theorem says that

$$(C_p * C_q)^{\text{ab}} \cong \mathbb{Z}/p \oplus \mathbb{Z}/q \cong \mathbb{Z}/pq,$$

so we can only recover $pq$ from the abelianization of $C_p * C_q$. Instead, we use the fact each $v \in C_p * C_q$ that has finite order is conjugate to either $C_p$ or $C_q$ (see Exercise 16.6). Letting

$$\mathcal{O} = \{d \geq 2 \mid \text{there exists some } v \in C_p * C_q \text{ of order } d\} ,$$

it follows that

$$\mathcal{O} = \{d \geq 2 \mid d \text{ divides } p\} \sqcup \{d \geq 1 \mid d \text{ divides } q\} .$$

This is a disjoint union since $\gcd(p, q) = 1$. We conclude that $p$ and $q$ are the maximal elements of $\mathcal{O}$ under the partial order of divisibility. The theorem follows. □

## 18*.9. Exercises

Exercise 18*.1. In this exercise, you will verify a number-theoretic fact we used to prove that 3-dimensional lens spaces are well-defined. Let $p, q \in \mathbb{Z}$ satisfy $\gcd(p, q) = 1$. Let $n, m \in \mathbb{Z}$ and $n', m' \in \mathbb{Z}$ satisfy $np - mq = 1$ and $n'p - m'q = 1$. Prove that there exists some $d \in \mathbb{Z}$ such that $n' = n + dq$ and $m' = m + dp$. $\square$

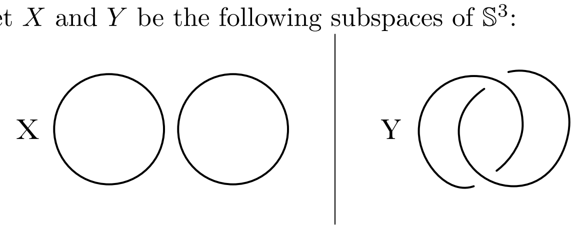
Exercise 18*.2. Consider $p, q \in \mathbb{Z}$ and $p', q' \in \mathbb{Z}$ with $\gcd(p, q) = \gcd(p', q') = 1$. Assume that $|p| = |p'|$ and that $q' \equiv \pm q \pmod p$. Prove that $L(p; q) \cong L(p'; q')$. $\square$

Exercise 18*.3. Let $d \in \mathbb{Z}$. Prove that the knots $K_{1,d}$ and $K_{d,1}$ and $K_{-1,d}$ and $K_{d,-1}$ are all equivalent to the unknot. $\square$

Exercise 18*.4. Let $p, q \in \mathbb{Z}$ satisfy $\gcd(p, q) = 1$. Do the following:

(a) Prove that the knots $K_{p,q}$ and $K_{q,p}$ are equivalent.
(b) Prove that the knots $K_{p,q}$ and $K_{p,-q}$ and $K_{-p,q}$ and $K_{-p,-q}$ are all equivalent. $\square$

Exercise 18*.5. Let $X$ and $Y$ be the following subspaces of $\mathbb{S}^3$:




Both $X$ and $Y$ are thus homeomorphic to the disjoint union of two copies of $\mathbb{S}^1$, with the two circles linked in $Y$ and not linked in $X$.

(a) Prove that the fundamental group of $\mathbb{S}^3 \setminus X$ is isomorphic to the free group on two generators.
(b) Prove that the fundamental group of $\mathbb{S}^3 \setminus Y$ is isomorphic to $\mathbb{Z}^2$.
(c) Conclude that there is no homeomorphism $f : \mathbb{S}^3 \to \mathbb{S}^3$ with $f(X) = Y$. $\square$

CHAPTER 19*

# The general Seifert–van Kampen theorem (to be written)

This chapter does two things. First, it explains how to generalize the Seifert–van Kampen theorem to covers by more than two open sets. Second, it describes the traditional proof of the Seifert–van Kampen theorem. Unlike our proof in Chapter 17, this proof does not require the space to be locally path connected or semilocally 1-connected. We have marked this chapter with an $*$ to indicate that a reader might want to skip it when first learning the subject.

# Looking ahead

CHAPTER 20*

# Higher homotopy groups (to be written)

This final chapter introduces higher homotopy groups. It proves that they are abelian, and using smooth methods proves that $\pi_d(\mathbb{S}^n, x_0) = 0$ for $d \leq n-1$. There aren't many other interesting calculations you can do at this point, though this chapter does discuss some of them without proofs (and constructs the Hopf fibration and proves it is non-trivial). To illustrate how powerful homotopy groups are, it proves that a CW complex is contractible if and only if its homotopy groups are all trivial and also proves Whitehead's theorem.

# Part 2

# Essays

# Essays on applications

ESSAY A

# Holomorphic functions, germs, and analytic continuation (to be written)

This essay will describe how covering spaces interact with complex analysis. It will include the following topics:

- The space of germs of holomorphic functions, and how this projects to $\mathbb{C}$ via a local homeomorphism.
- A bunch of examples of using this to study analytic continuation, including some where it gives a covering space and others where it does not.
- How the derivative map is a covering space from the space of germs to itself, and how path lifting and homotopy lifting can be applied to this covering space to clarify complex path integrals and their homotopy invariance.

ESSAY B

# Lie groups and covers (to be written)

This essay will describe how covering spaces interact with Lie groups. It will include the following topics:

- How to make covers of Lie groups into Lie groups, and how the kernel of the covering space map is a discrete central subgroup.
- Some examples of calculations of the fundamental group of Lie groups: $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SO}_n$. These can be done in an elementary way for $n \leq 3$, but for $n \geq 4$ we will need to use the long exact sequence in homotopy groups associated to a fiber bundle (discussed in a later essay).
- Lie algebras, and the proof that subalgebras of Lie algebras correspond to subgroups. This is a nice illustration of covering spaces. Examples will be included showing that covering spaces are really needed here.
- Related to the previous bullet point, the fact that Lie algebra homomorphism correspond to homomorphisms between Lie groups. Again, this is a nice illustration of covering spaces. Combined with our fundamental group computations, this also gives nice examples of non-linear Lie groups.

ESSAY C

# Orientations (to be written)

This essay will describe how covering spaces clarify orientations of manifolds. It will include the following topics:

- A quick review of what a local orientation is on a smooth manifold.
- To generalize this to topological manifolds, we need degrees of continuous maps $\mathbb{S}^n \to \mathbb{S}^n$. Explain how to deduce this from the corresponding smooth theory. Refer to Milnor and Gullemin–Pollack for smooth degree, but sketch the main idea. Mention that in the next volume the degree will be constructed directly using homology.
- Use this to define a local orientation on a topological manifold. Really, what you do is describe what it means for a homeomorphism $f\colon U \to V$ with $U, V \subset \mathbb{R}^n$ to be orientation preserving/reversing at a point $p \in U$, which can be done by looking at degrees of maps of small spheres around $p$.
- Construct the orientation double cover, and talk about the (non)-existence of sections of it.

ESSAY D

# The unsolvability of the quintic (to be written)

This essay will give Arnold's beautiful proof of the unsolvability of the quintic using covering spaces.

ESSAY E

# The Borsuk–Ulam theorem and its applications (to be written)

This essay will motivate and state the general Borsuk–Ulam theorem and then use the fundamental group to prove it in in dimensions $\leq 2$. We will then promise to prove the general case once we have cohomology available to us. We will then give several classic applications of the Borsuk–Ulam theorem in general,[1] including the ham sandwich theorem and the proof of Kneser's conjecture.

[1]These applications work the same no matter what dimension we are in, so there is no point in restricting the dimension here.

# Essays on examples

ESSAY F

# Free groups (to be written)

In the main part of the book we already explained the connection between free groups and graphs, and used it to prove that all subgroups of free groups are free. This essay will expand on this by discussing immersions and Stallings folding. This will allow us to actually calculate a free basis for a subgroup generated by a given set of elements, prove Marshall Hall's theorem, and prove that intersections of finitely generated subgroups of free groups are finitely generated.

ESSAY G

# Surfaces (to be written)

This essay will prove a bunch of basic results about the topology of surfaces:

- Using the fundamental group, it will prove that a closed oriented genus $g$ surface does not contain a topologically embedded Möbius band (which was assumed in the first volume when we classified surfaces). In the smooth category this is clear from orientability assumptions, but is not so trivial in the topological category. There is another proof of this in the essay on orientations (which does orientations on topological manifolds), but the proof here is more direct and self-contained.
- It will explain Dehn's solution to the word problem in surface groups.
- It will prove that non-compact connected surfaces have free fundamental groups. The proof I will give actually proves something much more general: for a non-compact connected $n$-manifold $M^n$ that can be triangulated, there is an $(n-1)$-dimensional spine $X \subset M^n$ to which $M^n$ deformation retracts (a theorem of Whitehead).
- As a consequence of the last fact, it will prove that a subgroup of a surface group is either a surface group or a free group.

ESSAY H

# Knot theory (to be written)

This essay will show how to use the fundamental group and covering spaces to prove a bunch of basic facts about knot theory. The focus mostly will be on examples. For instance, it will construct winding numbers to detect linking, prove the Wirtinger presentation, and explain the connection between knot colorings and maps from the fundamental group of the knot complement to dihedral groups.

There are two things that it will describe but not prove at this point:

- The loop theorem. The proof here is a nice illustration of covering spaces, but I can't figure out how to do it cleanly without Poincaré duality. I will include an essay with that proof in a later volume.
- The Alexander polynomial: while it will explain why there is a canonical infinite cyclic cover and show how to construct it using a Seifert surface, since we don't have Poincaré duality available it would be unnatural to construct the Alexander polynomial here. Again, this will be in a later volume.

ESSAY I

# Braid groups and configuration spaces (to be written)

This essay will introduce braid groups as fundamental groups of configuration spaces. After a bunch of examples, it will restrict to the classical case of the braid group on a 2-dimensional disk and prove two things: the usual presentation for the braid groups, and the fact that the configuration space of $n$ points on a disk is an Eilenberg–MacLane space. These will use fiber bundles, which are discussed in a later essay.

ESSAY J

# Some wild spaces (to be written)

This essay will discuss the fundamental groups and covers of some "wild" spaces: the Alexander horned ball, the earring space, and the harmonic archipelago.

# Essays on bundles and related topics

ESSAY K

# Fiber bundles and fibrations (to be written)

This chapter will introduce fiber bundles as a generalization of covering spaces. It will contain a large number of examples and prove the usual lifting results. It will then introduce fibrations as a weakening of a fiber bundle and give the path-loop fibration as an example. It will conclude by proving the long exact sequence in homotopy groups.

ESSAY L

# Diffeomorphisms, embeddings, and isotopies (to be written)

This chapter will give Lima's beautiful short proof of the Palais's theorem saying that in the smooth category, the restriction map on embedding spaces is a fiber bundle. Along the way, we will establish the isotopy extension theorem (used elsewhere in the book to show that smoothly isotopic knots are ambient isotopic).

ESSAY M

# Structure groups and monodromy (to be written)

This essay will introduce structure groups to our theory of fiber bundles (via a formalism using groupoids that seems to be more "canonical" than the traditional methods). We can then talk about things like vector bundles. Of course, there will be many examples. We will also explain how to construct appropriate monodromy representations on the fundamental group of the base of such a fiber bundle.

ESSAY N

# Flat bundles (to be written)

This will explain how flat bundles (i.e., fiber bundles with discrete structure groups) are the same thing as functors from the fundamental groupoid to spaces, or for connected base spaces the same as actions of the fundamental group on spaces. This will generalize our discussion of similar results for covering spaces.

ESSAY O

# Classifying spaces for bundles (to be written)

Let $G$ be a topological group. This essay construct a classifying space for bundles with structure group $G$.

To do this, we first have to show how to pull such bundles back and prove that the result is homotopy invariant. We then define classifying spaces. We state the existence theorem, and then explore some properties of them.

Using the clutching construction, we prove that if $BG$ is a classifying space for $G$, then $\pi_n(BG) = \pi_{n-1}(G)$. This lets us show that if $G$ is discrete, then $BG$ is an Eilenberg–MacLane space. From this, we get a new proof of the classification of pointed regular $G$-covers, at least for CW complexes.

We then have to prove the recognition principle: a classifying space $BG$ can be obtained by starting with a weakly contractible space $EG$ on which $G$ acts freely and properly, and taking $BG = EG/G$. Finally, we give Milnor's construction of such an $EG$.

# Essays on group theory

ESSAY P

# Eilenberg–Mac Lane spaces (to be written)

Let $G$ be a discrete group. This essay introduces $K(G,1)$ spaces. It will give many examples, prove existence, prove uniqueness up to homotopy equivalence, and prove that for a pointed CW complex $(X, x_0)$ a homotopy class of pointed map $X \to K(G,1)$ is the same thing as a homomorphism $\pi_1(X, x_0) \to G$.

ESSAY Q

# Combinatorial group theory and topology (to be written)

This essay contains more combinatorial group theory, showing how a mixture of topological and combinatorial arguments can help analyze many natural groups (for instance, Baumslag–Solitar groups). It also introduces HNN extensions, topologically proves some of their major properties, and gives some standard applications of them (e.g., that every countable group can be embedded in a 2-generator group).

ESSAY R

# Bass–Serre theory (to be written)

This is an essay on the topological version of Bass–Serre theory discussed in Scott's "Topological methods in group theory". The main example it works out in detail is $\mathrm{SL}_2(\mathbb{Z})$.

# Essays on additional technical topics

ESSAY S

# Smooth methods (to be written)

This essay explains how to use smooth manifold techiques to study things like the fundamental group. For smooth manifolds $M$ and $N$, it proves that continuous map $f\colon M \to N$ can be homotoped to smooth ones and that homotopic smooth maps $f, g\colon M \to N$ are smoothly homotopic. We already explain in the chapter on higher homotopy groups how this implies that $\pi_d(\mathbb{S}^n, x_0) = 0$ for $0 \leq d \leq n-1$. We will give some other applications here, e.g. that removing a submanifold of codimension at least 3 from a manifold does not change its fundamental group (note that this requires smoothness, and there are interesting counterxamples in the topological setting; I might sketch them).

ESSAY T

# Cofibrations (to be written)

Generalizing our earlier discussion of subspaces with mapping cylinder neighborhoods, this essay discusses cofibrations.

ESSAY U

# CW Complexes II (to be written)

Volume 1 contains an essay on the point-set topology of CW complexes. This essay continues that discussion by bringing in the tools from this volume. Here are some topics:

- CW complexes are locally contractible.
- Subcomplexes of CW complexes are deformation retracts of open neighborhoods.
- Gluing constructions and homotopies.
- the cellular approximation theorem.